Title: PSI 2018/2019 - Quantum Field Theory III - Lecture 2

Date: Jan 31, 2019 11:30 AM

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Abstract:

CFT's = QFT's w/ additional spacetime symmetries.

$\begin{cases} \underline{1} \text{ Kinematics.} \\ \underline{2} \text{ Dynamics. (consistency conditions)} \end{cases}$

Kinematics

- Geometry of conformal transformations.

1 Conformal transformations define a group G.

2 Continous transformations

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- Geometry of conformal transformations.

1 Conformal transformations define a group G.

2 For Continous transformations, the small transformations are completely encoded Lie algebra $[T_a, T_b] = i f_{ab}^{\ \ c} T_c$

CFT's = QFT's w/ additional spacetime symmetries.

1. Kinematics.
2. Dynamics. (consistency conditions)

Kinematics

- Geometry of conformal transformations.

1 Conformal transformations define a group G.

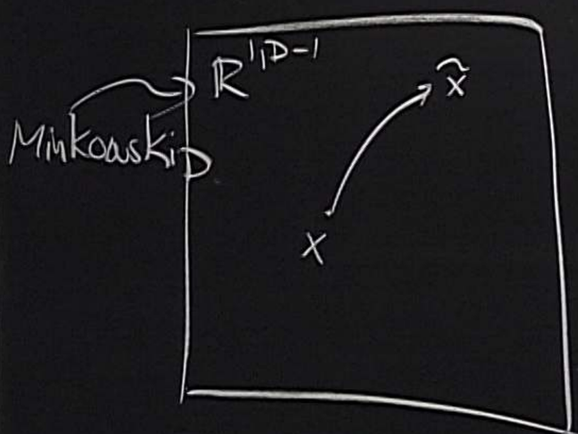
2 For Continous transformations, the small transformations are completely encoded by a Lie algebra $[T_a, T_b] = i f_{ab}^{c} T_c$ $\quad f_{ab}^{c}$: Jacobi identity obeyed

Conformal transformations:

1. Preserve angles between vectors (but not necessarily distances).

2. Preserve the lightcones

$$x^\mu \longrightarrow \tilde{x}^\mu(x)$$



$$ds^2 \longrightarrow e^{2\omega(x)}\, ds^2$$

$$x^\mu \longrightarrow \tilde{x}^\mu(x)$$

$$ds^2 \longrightarrow e^{2\omega(x)}\, ds^2$$

$$\boxed{\eta_{\rho\sigma}\, d\tilde{x}^\rho d\tilde{x}^\sigma = e^{2\omega(x)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

Minkowski metric

$$\partial \tilde{x}^\rho \quad \partial \tilde{x}^\sigma \qquad 2\omega(x)$$

...inkowski
metric

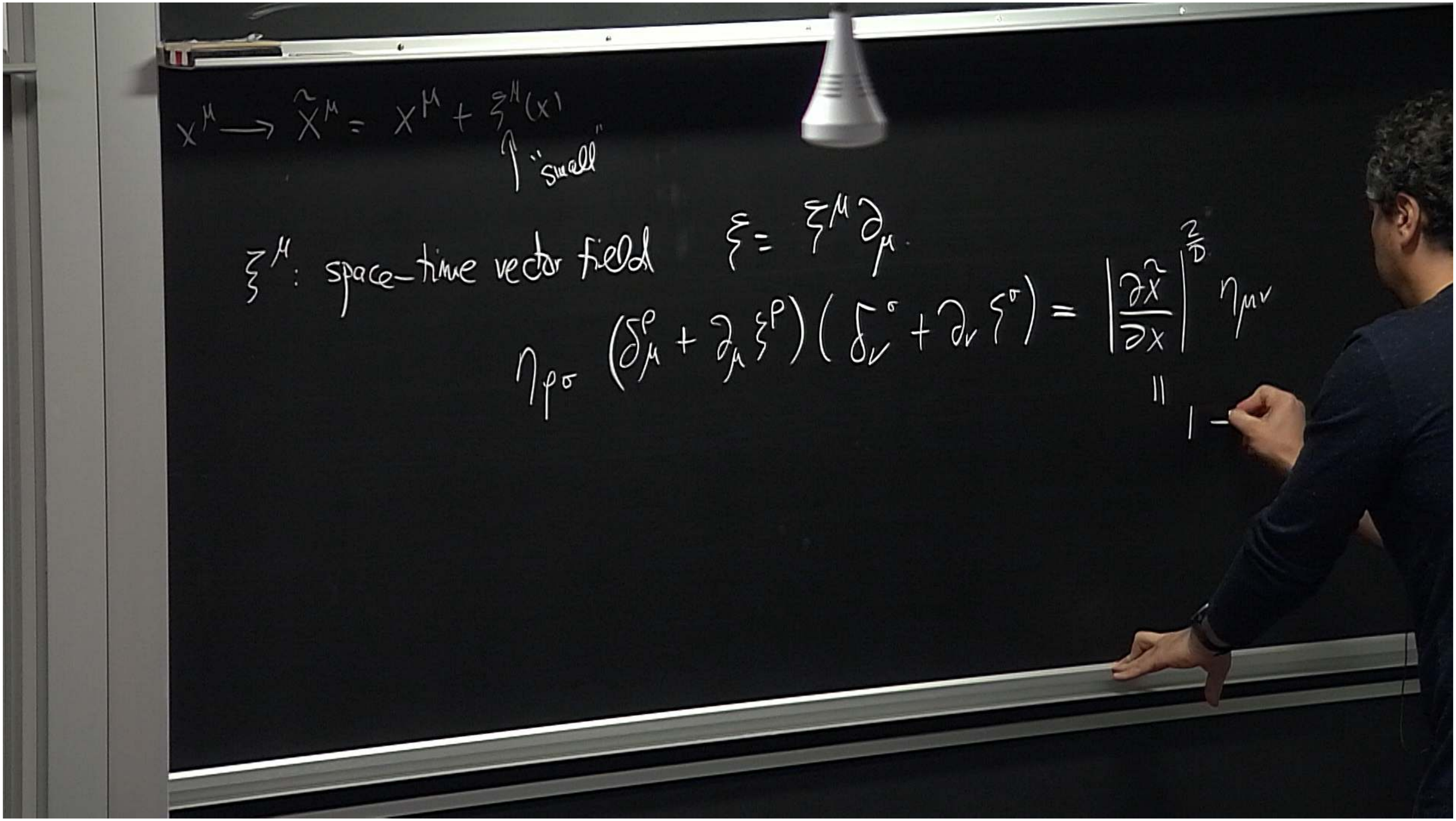
$$\eta_{\rho\sigma}\,\frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}}\,\frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}} = e^{2\omega(x)}\,\eta_{\mu\nu}$$

$$\overset{2}{\det}\left|\frac{\partial \tilde{x}}{\partial x}\right| = e^{2D\omega(x)}$$

$$\|$$

$$\left|\frac{\partial \tilde{x}}{\partial x}\right|$$

$$e^{\omega(x)} = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{1/D}$$


$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x)$
"small"
ξ^μ: space-time vector field $\xi = \xi^\mu \partial_\mu$
$\eta_{\rho\sigma} \left(\delta^\rho_\mu + \partial_\mu \xi^\rho\right)\left(\delta_\nu^\sigma + \partial_\nu \xi^\sigma\right) = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} \eta_{\mu\nu}$
$= 1 -$

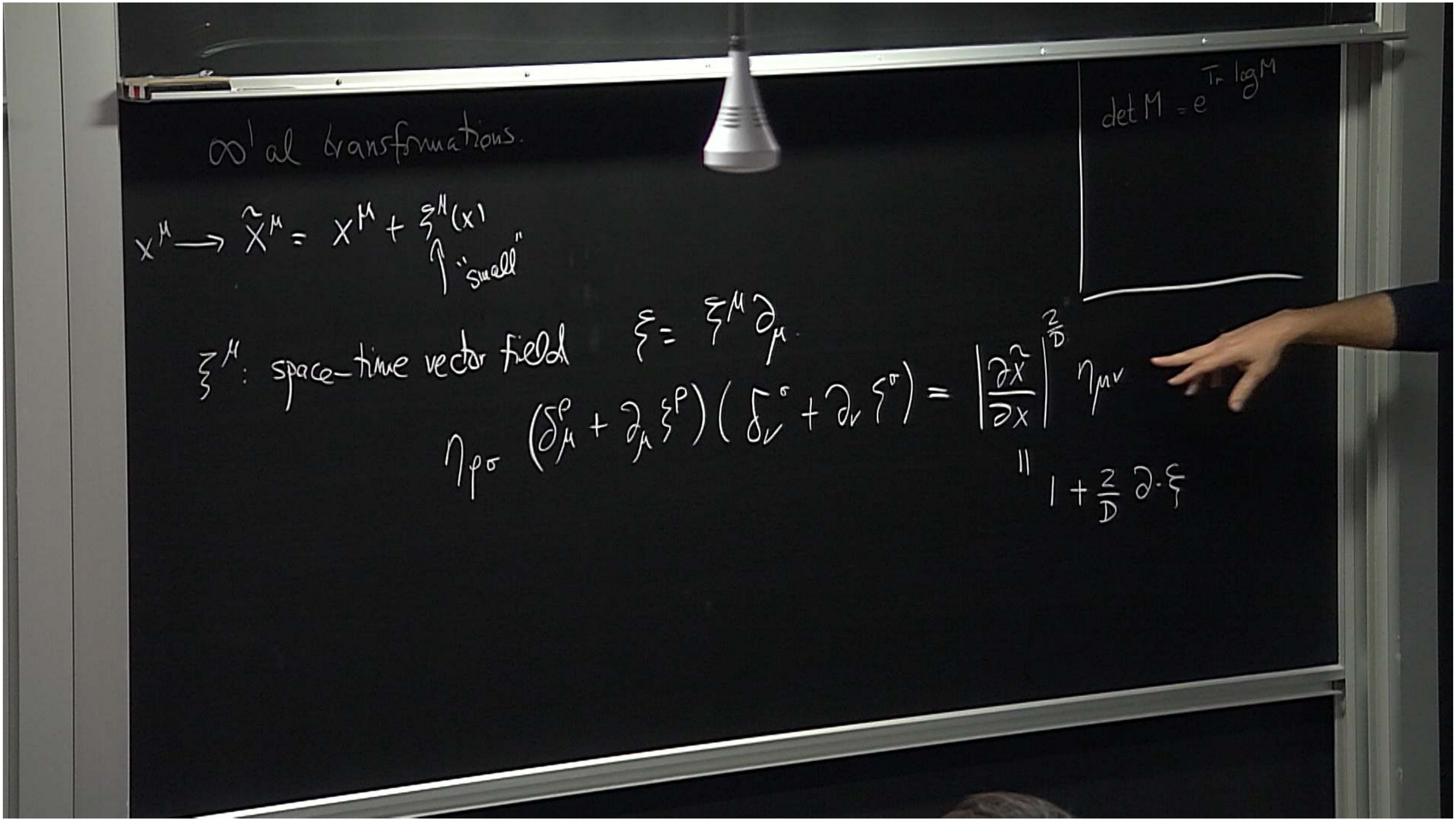
$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x)$$

"small"

ξ^μ: space-time vector field $\quad \xi = \xi^\mu \partial_\mu .$

$$\eta_{\rho\sigma}\left(\delta^\rho_\mu + \partial_\mu \xi^\rho\right)\left(\delta_\nu^{\sigma} + \partial_\nu \xi^\sigma\right) = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} \eta_{\mu\nu}$$

$$\left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} = 1 + \frac{2}{D}\,\partial\cdot\xi$$


∞'al transformations.
$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x)$
"small"
ξ^μ: space-time vector field $\xi = \xi^\mu \partial_\mu$.
$\eta_{\rho\sigma}(\delta^\rho_\mu + \partial_\mu \xi^\rho)(\delta_\nu^\sigma + \partial_\nu \xi^\sigma) = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} \eta_{\mu\nu}$
$= 1 + \frac{2}{D}\partial\cdot\xi$
$\det M = e^{\mathrm{Tr}\log M}$

"small"

ξ^μ: space-time vector field $\quad \xi = \xi^\mu \partial_\mu$.

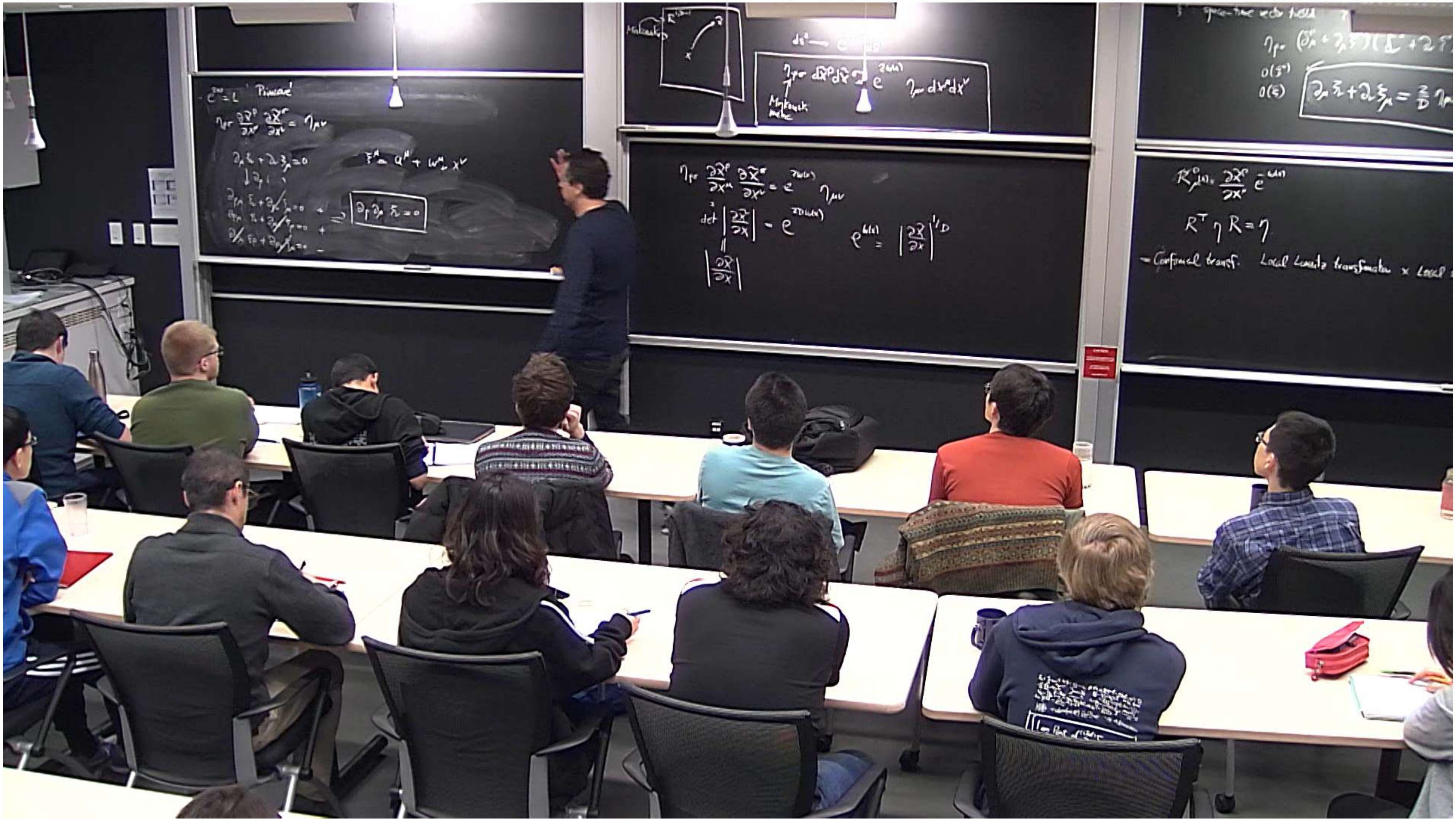
$$\eta_{\rho\sigma}\left(\delta^\rho_\mu + \partial_\mu \xi^\rho\right)\left(\delta_\nu^\sigma + \partial_\nu \xi^\sigma\right) = \left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} \eta_{\mu\nu}$$

$$\left|\frac{\partial \tilde{x}}{\partial x}\right|^{\frac{2}{D}} = 1 + \frac{2}{D}\,\partial\cdot\xi$$

$O(\xi^0)$ ✓

$O(\xi)$
$$\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}\,\eta_{\mu\nu}\,\partial\cdot\xi}$$

Conformal Killing vector equation



$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu$$

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$$

$$\downarrow \partial_\rho (\ldots)$$

$$\partial_{\rho\mu} \xi_\nu + \cancel{\partial_{\rho\nu} \xi_\mu} = 0 \quad +$$

$$\partial_{\rho\mu} \xi_\nu + \cancel{\partial_{\mu\nu} \xi_\rho} = 0 \quad +$$

$$\cancel{\partial_{\nu\mu} \xi_\rho} + \cancel{\partial_{\rho\nu} \xi_\mu} = 0 \quad -$$

$$\Rightarrow \boxed{\partial_\rho \partial_\mu \xi_\nu = 0}$$

$\partial_\mu \xi_\rho + \partial_\rho \xi_\mu = 0$

if ξ^μ is a CKV

$$\boxed{\partial_\mu \partial_\nu \partial_\rho \xi_\sigma = 0}$$

<u>D=2</u> :

$\mu = \nu = 1$ $\quad \partial_1 \xi_1 = \partial_2 \xi_2$

$\mu = 1, \nu = 2$ $\quad \partial_1 \xi_2 = - \partial_2 \xi_1$.

Cauchy Riemann equations.

$z = x^1 + i x^2$

$$\boxed{\partial_{\bar z} \xi = 0}$$

$$R^{\rho}{}_{\mu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} e^{-\omega(x)}$$

$$R^{T} \eta R = \eta .$$

$=$ Conformal transf. Local Lorentz transformation $\times$ Local scale transformations

$\partial_\nu \partial_\mu \xi_\rho + \partial_\rho \partial_\nu \xi_\mu = 0$

if ξ^μ is a CKV

$$\boxed{\partial_\mu \partial_\nu \partial_\rho \xi_\sigma = 0}$$

- Finite dim'al for $D>2$
- ∞' for $D=2$.

$\underline{D=2}$:

$\mu=\nu=1$ $\quad \partial_1 \xi_1 = \partial_2 \xi_2$

$\mu=1$, $\nu=2$ $\quad \partial_1 \xi_2 = -\partial_2 \xi_1$

Cauchy Riemann equations.

$z = x^1 + i x^2$

$$\boxed{\partial_{\bar z}\, \xi = 0}$$

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$$

$$D \qquad \frac{D(D-1)}{2} \qquad 1 \qquad D$$

$$P_\mu \qquad M_{\mu\nu} \qquad D \qquad K_\mu$$

dilatation

special conformal transformation

metric

$$\xi^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu b\cdot x$$

$$D \qquad \frac{D(D-1)}{2} \qquad 1 \qquad D$$

$\frac{(D+2)(D+1)}{2}$ dimensional conformal algebra.

$$P_\mu \qquad M_{\mu\nu} \qquad D \qquad K_\mu$$

D " dilatation

K_μ " special conformal transformation.

$$\delta(\xi) \quad \boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}\eta_{\mu\nu}\,\partial\cdot\xi}$$

Conformal Killing vector equation

$$\xi^\mu_{(1)}\,,\ \xi^\rho_{(2)}$$

$$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\sigma}M^{\mu\rho})$$

$$[M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu).$$

$$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho}K^\nu - \eta^{\nu\rho}K^\mu).$$

vector equation

Conformal algebra

$\xi^{\mu}_{(1)}, \xi^{\rho\sigma}_{(2)}$

$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$

$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} + \eta^{\nu\sigma} M^{\mu\rho})$

$[M^{\mu\nu}, P^{\rho}] = i(\eta^{\mu\rho} P^{\nu} - \eta^{\nu\rho} P^{\mu})$.

$[M^{\mu\nu}, K^{\rho}] = i(\eta^{\mu\rho} K^{\nu} - \eta^{\nu\rho} K^{\mu}) \leftarrow K^{\mu}$ is a vector

$[D, P^{\mu}] = -iP^{\mu} \quad [D, K^{\mu}] = iK^{\mu}$

$[P^{\mu}, K^{\nu}] = -2i(\eta^{\mu\nu} D + M^{\mu\nu})$.

$$X^M = (x^\mu, x^D, x^{D+1})$$

x^μ ∩ Minkowski

$$L_{MN} = \begin{cases} L_{\mu\nu} = M_{\mu\nu} \\ L_{D\,D+1} = \hat{D} \\ L_{\mu D} = \frac{1}{2}(P_\mu + K_\mu) \\ L_{\mu D+1} = \frac{1}{2}(P_\mu - K_\mu) \end{cases}$$

L_{MN} : anti-symmetric $(D+2)$-al matrix

$$[L_{MN}, L_{PQ}] = i\left(\hat{\eta}_{MP} L_{NQ} + \cdots\right)$$

$$\hat{\eta} = \begin{array}{c} \\ \\ \\ x^D \\ x^{D+1} \end{array} \begin{pmatrix} - & & & & \\ & + & & & \\ & & + & & \\ & & & \ddots & \\ & & & & + \\ & & & & & \boxed{\begin{matrix} + & \\ & - \end{matrix}} \end{pmatrix}$$

μ

$\Downarrow$

conformal algebra is $SO(2,D)$

$CFT_D \rightsquigarrow$

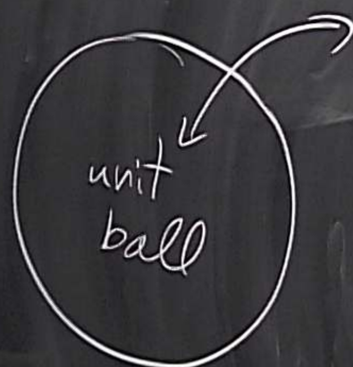
AdS_{D+1}

Conformal transformations not connected to identity:

I : $x^\mu \longrightarrow \dfrac{x^\mu}{x^2}$

1. It is a conformal transformation: $e^{2\omega(x)}$

2.



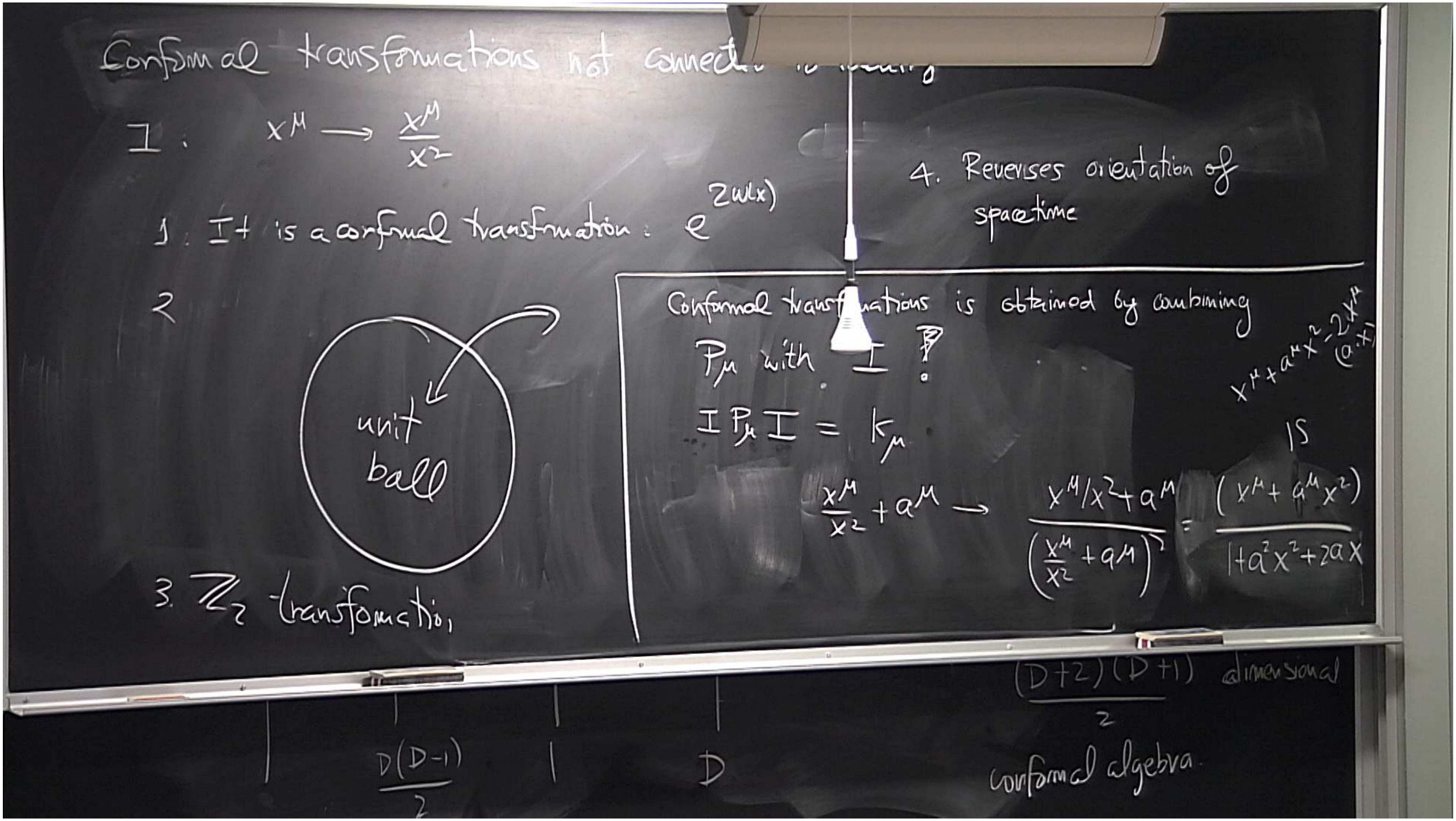
3. $\mathbb{Z}_2$ transformation

4. Reverses orientation of spacetime

Conformal transformations is obtained by combining P_μ with I ?

$I P_\mu I = K_\mu$

$$\frac{x^\mu}{x^2} + a^\mu \longrightarrow \frac{x^\mu/x^2 + a^\mu}{\left(\frac{x^\mu}{x^2} + a^\mu\right)^2} \qquad \frac{(x^\mu + a^\mu x^2)}{(x^\mu +}$$


Conformal transformations not connected to identity
I : $x^\mu \to \frac{x^\mu}{x^2}$
1. It is a conformal transformation : $e^{2w(x)}$
2
unit ball
3. $\mathbb{Z}_2$ transformation
4. Reverses orientation of spacetime
Conformal transformations is obtained by combining
P_μ with I ?
$I P_\mu I = K_\mu$
$\frac{x^\mu}{x^2} + a^\mu \to \frac{x^\mu/x^2 + a^\mu}{\left(\frac{x^\mu}{x^2} + a^\mu\right)^2} = \frac{(x^\mu + a^\mu x^2)}{1 + a^2 x^2 + 2a \cdot x}$
$x^\mu + a^\mu x^2 - 2x^\mu (a \cdot x)$
is
$\frac{D(D-1)}{2}$
D
$\frac{(D+2)(D+1)}{2}$ dimensional
conformal algebra.