Title: PSI 2018/2019 - Gravitational Physics - Lecture 4

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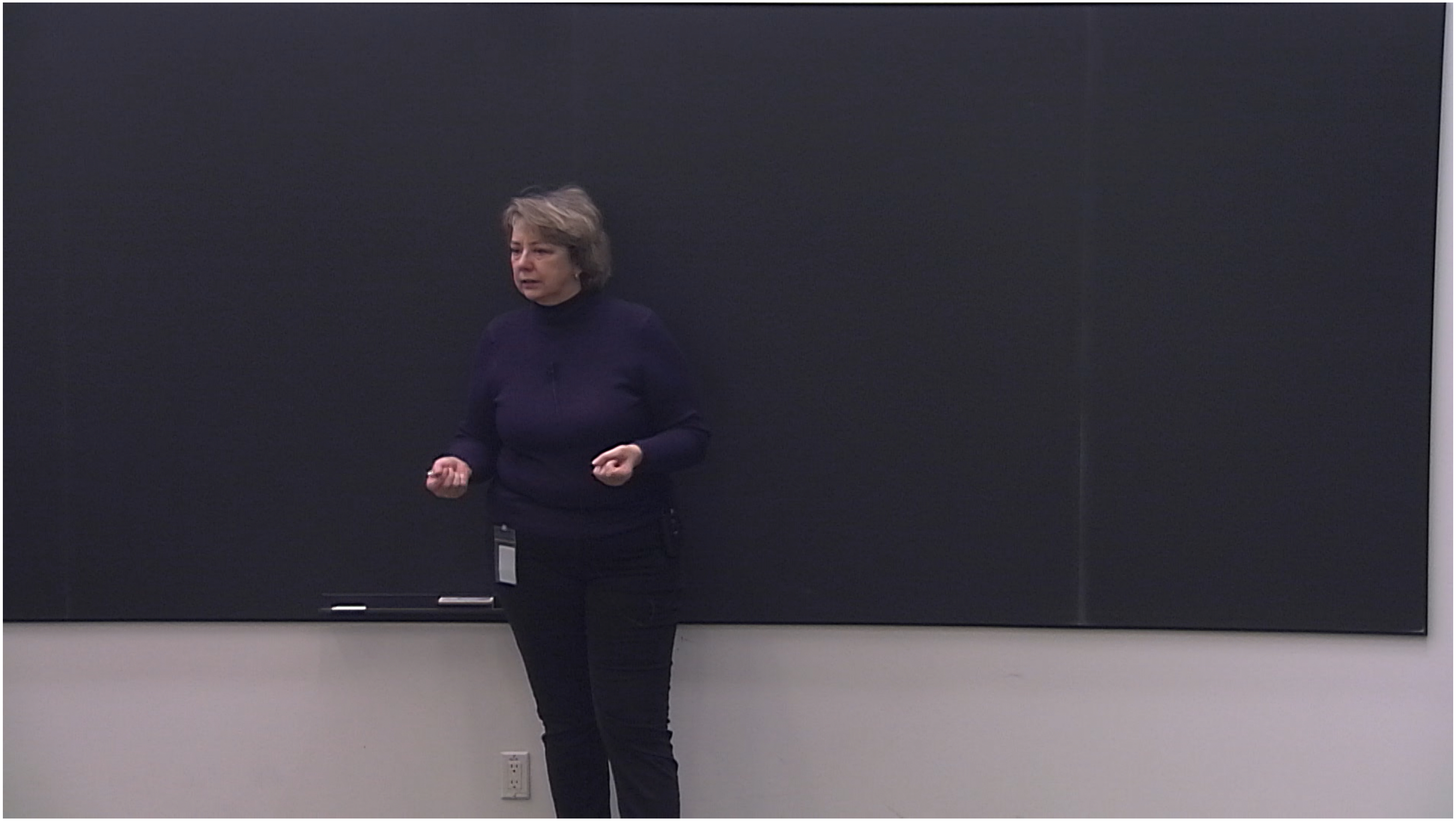
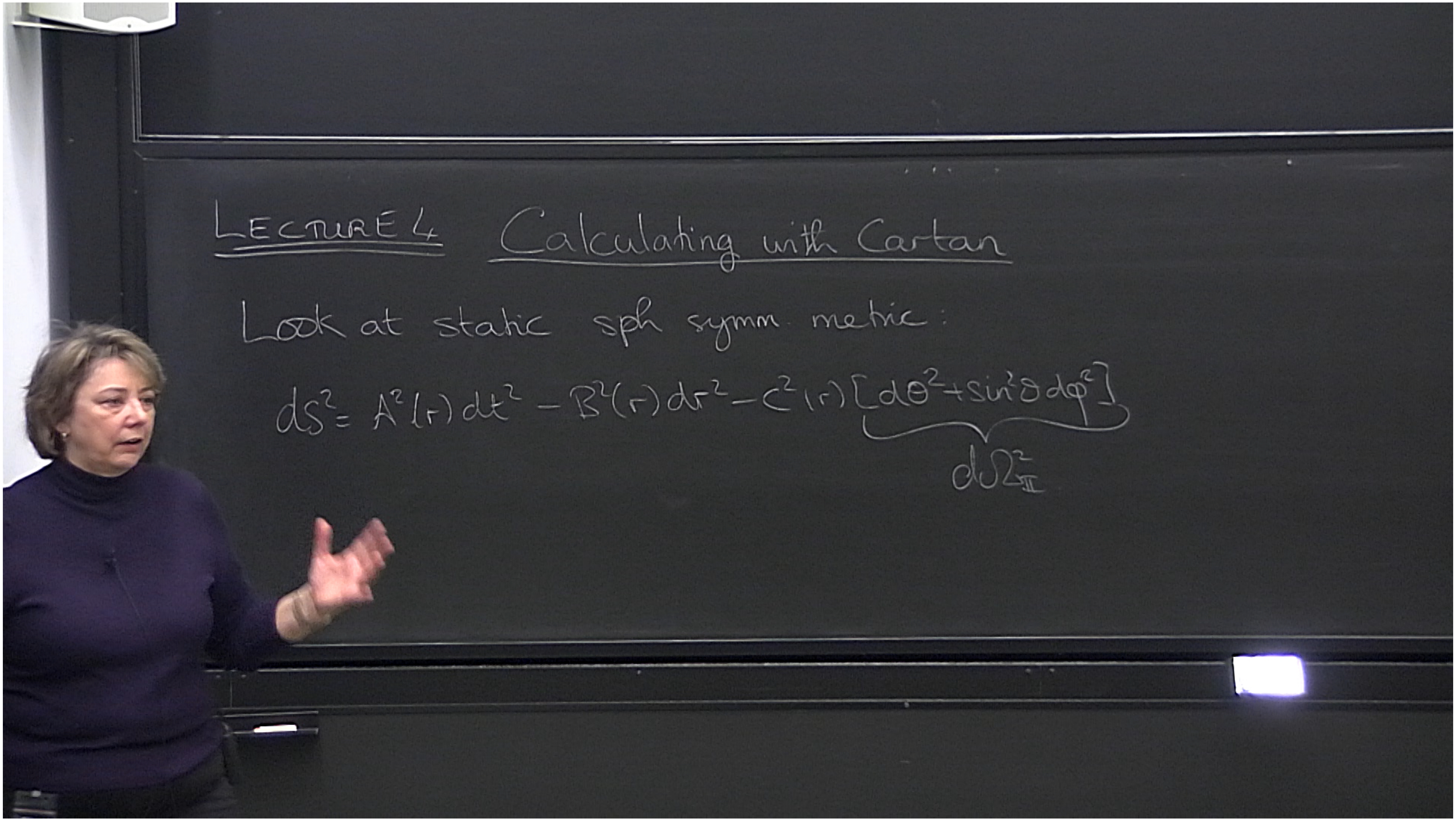
Abstract:




LECTURE 4
Calculating with Cartan
Look at static sph symm metric:
$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - C^2(r)\underbrace{[d\theta^2 + \sin^2\theta\, d\varphi^2]}_{d\Omega^2_{II}}$

LECTURE 4 Calculating with Cartan

Look at static sph symm metric:

$$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - C^2(r)\,\underbrace{[d\theta^2 + \sin^2\theta\, d\varphi^2]}_{d\Omega^2_{II}}$$

- o/n basis of 1-forms:

$$\underset{\sim}{\omega}^{\hat{t}} = A(r)\,\underline{d}t \,;\quad \underline{\omega}^{\hat{r}} = B\,\underline{d}r \,;\quad \underline{\omega}^{\hat{\theta}} = C\,\underline{d}\theta ;\quad \underline{\omega}^{\hat{\varphi}} = C \sin\theta\, \underline{d}\varphi .$$

Spin connection:

$$\underset{\sim}{d}\, \underline{\omega}^{\hat{t}} = A'\, \underline{d}r \wedge \underline{d}t = -\frac{A'}{AB}\, \underline{\omega}^{\hat{t}} \wedge \underline{\omega}^{\hat{r}} = -\underset{\sim}{\Theta}^{\hat{t}}{}_{\hat{a}} \wedge \underline{\omega}^{\hat{a}}$$

$$\rightarrow \underset{\sim}{\Theta}^{\hat{t}}{}_{\hat{r}} = \frac{A'}{AB}\, \underline{\omega}^{\hat{t}}$$

$$\underline{d}\,\underline{\omega}^{\hat{r}} = 0$$

$$\underline{d}\,\underline{\omega}^{\hat{\theta}} = C'\, \underline{d}r \wedge \underline{d}\theta$$

$$\Rightarrow \underset{\sim}{\Theta}^{\hat{\theta}}{}_{\hat{r}} = \frac{C'}{CB}\, \underline{\omega}^{\hat{\theta}}$$

$$\underline{d}\,\underline{\omega}^{\hat{\varphi}} = C' \sin\theta\, \underline{d}r \wedge \underline{d}\varphi + C \cos\theta\, \underline{d}\theta \wedge \underline{d}\varphi = -\underset{\sim}{\Theta}^{\hat{\varphi}}{}_{\hat{a}} \wedge \underline{\omega}^{\hat{a}}$$

$$\underset{\sim}{\Theta}^{\hat{\varphi}}{}_{\hat{r}} = \frac{C'}{CB}\, \underline{\omega}^{\hat{\varphi}}$$

$$\underset{\sim}{\Theta}^{\hat{\varphi}}{}_{\hat{\theta}} = \frac{1}{C} \cot\theta\, \underline{\omega}^{\hat{\varphi}}$$

$$ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - C^2(r)\underbrace{[d\theta^2 + \sin^2\theta\, d\varphi^2]}_{d\Omega^2_{\mathrm{II}}}$$

- o/n basis of 1-forms:

$$\underset{\sim}{d}\,\underset{\sim}{\omega}^{\hat t} = A'\,\underset{\sim}{d}r \wedge \underset{\sim}{d}t = -\frac{A'}{AB}\,\underset{\sim}{\omega}^{\hat t} \wedge \underset{\sim}{\omega}^{\hat r} = -\underset{\sim}{\Theta}^{\hat t}{}_{\hat a} \wedge \underset{\sim}{\omega}^{\hat a}$$

$$\rightarrow \underset{\sim}{\Theta}^{\hat t}{}_{\hat r} = \frac{A'}{AB}\,\underset{\sim}{\omega}^{\hat t}$$

$$\underset{\sim}{d}\,\underset{\sim}{\omega}^{\hat\theta} = C'\,\underset{\sim}{d}r \wedge \underset{\sim}{d}\theta \Rightarrow \underset{\sim}{\Theta}^{\hat\theta}{}_{\hat r} = \frac{C'}{CB}\,\underset{\sim}{\omega}^{\hat\theta}$$

$$= -\underset{\sim}{\Theta}^{\hat\varphi} \ldots$$

$$\underset{\sim}{\Theta}^{\hat\varphi}{}_{\hat r} = \ldots$$

$$\underset{\sim}{\Theta}^{\hat\varphi}{}_{\hat\theta} = \ldots$$

Aside:
$$\underset{\sim}{\Theta}^{\hat r}{}_{\hat\theta} = \eta^{\hat r\hat a}\,\underset{\sim}{\Theta}_{\hat a\hat\theta} = -\eta^{\hat r\hat a}\,\underset{\sim}{\Theta}_{\hat\theta\hat a} = -\eta^{\hat r\hat a}\,\eta_{\hat\theta\hat b}\,\underset{\sim}{\Theta}^{\hat b}{}_{\hat a} = -\underset{\sim}{\Theta}^{\hat\theta}{}_{\hat r}$$

- Curvature 2-form: $\underset{\sim}{R}^a{}_b = \underset{\sim}{d}\,\underset{\sim}{\Theta}^a{}_b + \underset{\sim}{\Theta}^a{}_c \wedge \underset{\sim}{\Theta}^c{}_b$

$$\underset{\sim}{R}^{\hat t}{}_{\hat r} = \underset{\sim}{d}\,\underset{\sim}{\Theta}^{\hat t}{}_{\hat r} = \underset{\sim}{d}\left(\frac{A'}{B}\,\underset{\sim}{d}t\right) = \left(\frac{A'}{B}\right)'\underset{\sim}{d}r \wedge \underset{\sim}{d}t = -\frac{1}{AB}\left(\frac{A'}{B}\right)'\,\underset{\sim}{\omega}^{\hat t} \wedge \underset{\sim}{\omega}^{\hat r}$$

$$= -\Theta^{\hat{\theta}}{}_{\hat{r}}$$

$$R^{\hat{\theta}}{}_{\hat{r}} = d\Theta^{\hat{\theta}}{}_{\hat{r}} + \cancel{\Theta^{\hat{\theta}}{}_{c} \wedge \Theta^{c}{}_{\hat{r}}}$$

$$= d\left(\frac{C'}{B}\, d\theta\right)$$

$$= -\left(\frac{C'}{B}\right)' \frac{1}{CB}\, \omega^{\hat{\theta}} \wedge \omega^{\hat{r}}$$

$$R^{\hat{\varphi}}$$

$$= \left(\frac{A'}{B}\right)' \, dr \wedge dt$$

$$= \frac{1}{AB}\left(\frac{A'}{B}\right)' \, \omega^{\hat t} \wedge \omega^{\hat r}$$

& $R^{\hat t}{}_{\hat\varphi} = -\frac{A'C'}{AB^2C}\, \omega^{\hat t} \wedge \omega^{\hat\varphi}$

$$R^{\hat\varphi}{}_{\hat r} = d\,\Theta^{\hat\varphi}{}_{\hat r} + \Theta^{\hat\varphi}{}_{\hat\theta} \wedge \Theta^{\hat\theta}{}_{\hat r}$$

$$= d\left(\frac{C'}{B}\sin\theta \, d\varphi\right) + \frac{1}{C}\cot\theta \, \omega^{\hat\varphi} \wedge \frac{C'}{CB}\, \omega^{\hat\theta}$$

$$= -\left(\frac{C'}{B}\right)' \frac{1}{CB}\, \omega^{\hat\varphi} \wedge \omega^{\hat r} + \frac{C'}{B}\cos\theta \, \cancel{d\theta \wedge d\varphi} + \frac{C'}{B}\cancel{\cos\theta \, d\varphi \wedge d\theta}.$$

$$
\mathcal{R}^{\hat{\varphi}}{}_{\hat{\theta}} = d\,\Theta^{\hat{\varphi}}{}_{\hat{\theta}} + \Theta^{\hat{\varphi}}{}_{\hat{r}} \wedge \Theta^{\hat{r}}{}_{\hat{\theta}}
$$

$$
= d\left(\cos\theta\, d\varphi\right) + \frac{C'}{CB}\,\omega^{\hat{\varphi}} \wedge \left(\frac{-C'}{CB}\right)\omega^{\hat{\theta}}
$$

$$
= -\frac{1}{C^2}\,\omega^{\hat{\theta}} \wedge \omega^{\hat{\varphi}} - \frac{C'^2}{C^2B^2}\,\omega^{\hat{\varphi}} \wedge \omega^{\hat{\theta}}
$$

$\hat{\theta}$

$\hat{\varphi} \wedge \left(-\frac{C'}{CB}\right) \omega^{\hat{\theta}}$

$\hat{\theta}$

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} = -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}}$$

$$R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} = -\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}}$$

$$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = \frac{1}{C^2} - \frac{C'^2}{C^2B^2}$$

$$\left.\begin{aligned}
R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} &= -\frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right) \\
R^{\hat{t}}{}_{\hat{\theta}\hat{t}\hat{\theta}} &= -\frac{A'C'}{AB^2C} = R^{\hat{t}}{}_{\hat{\varphi}\hat{t}\hat{\varphi}} \\
R^{\hat{\theta}}{}_{\hat{r}\hat{\theta}\hat{r}} &= -\frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB}\right) = R^{\hat{\varphi}}{}_{\hat{r}\hat{\varphi}\hat{r}} \\
R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} &= \frac{1}{C^2} - \frac{C'^2}{C^2B^2}
\end{aligned}\right\}$$

N.B.
Riemann
in
o/n basis

For coord basis

$$\underset{\sim}{R} = \underbrace{R^a{}_{bcd}}_{\text{components}} \underbrace{\underset{\sim}{e}_a \underset{\sim}{\omega}^b \underset{\sim}{\omega}^c \underset{\sim}{\omega}^d}_{\text{tensorial bit}}$$

Recall $\underset{\sim}{e}_a = e_a{}^\mu \dfrac{\partial}{\partial x^\mu}$

$$\underset{\sim}{\omega}^a = \omega^a{}_\mu dx^\mu$$

$$R^\mu{}_{\nu\lambda\rho} = e_a{}^\mu \, \omega^b{}_\nu \, \omega^c{}_\lambda \, \omega^d{}_\rho \, R^a{}_{bcd}$$

$$R^t{}_{rtr} = \underbrace{e_{\hat t}{}^t}_{1/A} \, \underbrace{\omega^{\hat r}{}_r}_{B} \, \underbrace{\omega^{\hat t}{}_t}_{A} \, \underbrace{\omega^{\hat r}{}_r}_{B.} \, R^{\hat t}{}_{\hat r \hat t \hat r}$$

$$R^{tr}{}_{tr} = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB}\right)$$

$$R^{\theta r}{}_{\theta r} = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{(CB)}\right) = R^{\varphi r}{}_{\varphi r}$$

$$R^{t\theta}{}_{t\theta} = R^{t\varphi}{}_{t\varphi} = \frac{A'C'}{AB^2C}$$

$$R^{\varphi\theta}{}_{\varphi\theta} = \frac{1}{C^2}\left(\frac{C'^2}{B^2} - 1\right)$$

$$R^t{}_t = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC}\right)$$

$$R^\theta{}_\theta = R^\varphi{}_\varphi = \frac{1}{B^2}\left(\frac{C''}{C} - \frac{C'B'}{CB} + \frac{A'C'}{AC} + \frac{C'^2}{C^2}\right) - \frac{1}{C^2}$$

$$R^r{}_r = \frac{1}{B^2}\left(\frac{A''}{A} + \frac{2C''}{C} - \frac{B'}{B}\left(\frac{A'}{A} + \frac{2C'}{C}\right)\right)$$

Gauge:

$C = r$ – area gauge

$B = 1$ – proper distance r

$A = 1/B$ – SCH gauge

$C = rB$ – 'ADM' gauge

Area gauge : $C = r$, $C' = 1$, $C'' = 0$

$$G^t{}_t = \frac{1}{r^2} - \frac{1}{r^2}\left(\frac{r}{B^2}\right)' = 8\pi G\, T^0{}_0$$

$$\Rightarrow \quad B^{-2} = 1 - \frac{2G}{r}\int \underbrace{4\pi r^2 T^0{}_0\, dr}_{\text{energy between } r \,\&\, r+dr}$$

$= 0$.

$$B^{-2} = 1 - \frac{2GM(r)}{r}$$

Vacuum soln, $T^0{}_0 = 0$.

$$B^{-2} = 1 - \frac{2GM}{r}$$

M an integration const

$\& \; \underline{R}^{t} \quad \hat{\varphi} = -\frac{A'C'}{AR^2C} \, \underline{\omega}^{t} \wedge \underline{\omega}^{\varphi}$

$= 0$

M an
integration
const

$$R^t_t - R^r_r = \frac{2}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \text{ in vac.}$$

$$\rightarrow \quad A \propto 1/B \quad \rightarrow \quad \text{SCH SOLN}$$

$$A^2 = B^{-2} = 1 - \frac{2GM}{r}$$

$$\Rightarrow B^{-2} = 1 - \frac{2G}{r}\int \underbrace{4\pi r^2 T^0_0 \, dr}_{\text{energy between } r \,\&\, r+dr}$$

$$B^{-2} = 1 - \frac{2GM}{r}$$ M an integration const

? What about Λ?

Cosmological constant

$$8\pi G T_{ab} = \Lambda g_{ab}$$

$R^t_t = R^r_r$ still

$A \propto 1/B$

$$\int 4\pi r^2 T^0_0 = \frac{\Lambda}{2G}\int r^2 = \frac{\Lambda r^3}{6G}$$

$$A^2 = B^{-2} = 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2$$

de-Sitter has $\Lambda > 0$, $\Lambda = 3/L^2$

get 2 horizons $r \simeq 2GM$, $r \simeq L$

– cosmological horizon at $r \sim L$

$M=0$ gives pure dS $A^2 = 1 - r^2/L^2$

dS is const curv spacetime

represented as a 4D hyperboloid

in 5D (flat) Mink. spacetime

$$X^2+Y^2+Z^2+U^2-T^2=L^2$$

For static slicing:

$$T = L\sqrt{1-\frac{r^2}{L^2}}\,\sinh t/L$$

$$U = L\sqrt{1-\frac{r^2}{L}}\,\cosh t/L$$

$$\underline{X} = r\,\underline{n}$$

FRW: global

$$T = L\sinh \tau/L$$

$$U = L\cosh \tau/L \cos\chi$$

$$\underline{X} = L\cosh\tau/L \sin\chi\,\underline{n}$$

$k=1$ cos.

STATIC $\tau = t + \frac{L}{2}\log\left(1-r^2/L^2\right)$

$\rightarrow$ FLAT $\rho = \dfrac{r e^{-t/L}}{\sqrt{1-r^2/L^2}}$