Title: PSI 2018/2019 - Gravitational Physics - Lecture 3

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Abstract:

L3 Connections & Cartan

The covariant deriv works by adding structure (the connection) to the tgt bundle.

Cartan

...orks by
...nection } to

$$\underset{\sim}{\nabla}\, \underline{e}_b = \Gamma^c{}_{ab}\, \underline{e}_c \otimes \underline{\omega}^b$$

derivation — basis vector — connection components - scalars — dual basis to $\{\underline{e}_a\}$.

or $\Gamma^a_{bc} = \langle \omega^a | \nabla_b\, \underline{e}_c \rangle$

"$\underline{e}_b \cdot \underline{\nabla}$"

Dual basis: $\langle \underline{\omega}^a | \underline{e}_b \rangle = \delta^a{}_b$.

Def...

Defn $\underline{\nabla}$ is a derivation that

- commutes with contractions
- Leibnitzian
- Reduces to $\underline{d}$ on fns.

$$\underset{\sim}{\nabla}\, \underline{e}_b = \Gamma^{c}_{a\,b}\, \underline{e}_c \otimes \underset{\sim}{\omega}^{a}$$

derivation — basis vector — connection components - scalars — dual basis to $\{\underline{e}_a\}$.

$$\text{or } \Gamma^{a}_{bc} = \langle \omega^a | \nabla_b\, \underline{e}_c \rangle$$

" $\underline{e}_b \cdot \underset{\sim}{\nabla}$

Dual basis: $\langle \underset{\sim}{\omega}^a | \underline{e}_b \rangle = \delta^a_b$.

Defn

ω^a

dual basis to $\{e_a\}$

$= \omega^a{}_b$

<u>Defn</u> $\underline{\nabla}$ is a derivation that

- commutes with contractions
- Leibnitzian
- Reduces to $\underline{d}$ on fns.

Used to:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_\mu{}^\nu{}_\lambda V^\lambda$$

For a vector:

$$\underset{\sim}{\nabla}\, \underset{\sim}{V} = \underset{\sim}{\nabla}(V^a \underset{\sim}{e}_a)$$

$T^*(M) \otimes T(M)$

$$\overset{②}{=} (\underset{\sim}{\nabla} V^a)\underset{\sim}{e}_a + (\underset{\sim}{\nabla}\underset{\sim}{e}_a) V^a$$

$$\overset{③}{=} \underset{\sim}{d} V^a \underset{\sim}{e}_a + \Gamma^b_{ca} \underset{\sim}{e}_b \underset{\sim}{\omega}^c V^a$$

$$= \underline{(V^a{}_{,c} + \Gamma^a_{c\,b} V^b)} \underset{\sim}{e}_a \underset{\sim}{\omega}^c$$

$e_b\,\omega^c\,V^a$

$e_a\,\omega^c$

In G.R. we use Levi-Civita connection: 'symmetric', $\nabla g = 0$

Torsion $\quad T(u,v) = \nabla_u v - \nabla_v u - [u,v]$

almost the anti-symmetric part of connection

$$T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} - C^a_{\ bc}$$

$\nabla_{\underline{v}} \underline{u} - [\underline{u}, \underline{v}]$

of connection

$C^{a}_{\ bc}$

where

$$C^{a}_{\ bc} = \langle \underline{\omega}^{a} | [\underline{e}_b, \underline{e}_c] \rangle$$

are the structure constants of the basis $\{\underline{e}_a\}$.

Defn The connection 1-forms or spin-connection are defined as

$$\underset{\sim}{\theta}^{a}{}_{c} = \Gamma_{b}{}^{a}{}_{c}\, \underline{\omega}^{b}$$

Allows us to differentiate spinors

Cartan relates spin connection
(& torsion) to derivatives of $\{\underline{\omega}^a\}$

- For metric connection $\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\Theta}_{ab} + \underset{\sim}{\Theta}_{ba}$

__Proof:__ $\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\nabla} g_{ab} = \underset{\sim}{\nabla} \langle g | \underline{e}_a, \underline{e}_b \rangle$

$= \langle g | \underline{e}_a, \underset{\sim}{\nabla} \underline{e}_b \rangle + \langle g | \underset{\sim}{\nabla} \underline{e}_a, \underline{e}_b \rangle$

connection

of $\{\underset{\sim}{\omega}^a\}$

$\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\theta}_{ab} + \underset{\sim}{\theta}_{ba}$

$\underline{\nabla} \langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$

$\langle g | \underset{\sim}{e}_a, \underline{\nabla} \underset{\sim}{e}_b \rangle + \langle g | \underline{\nabla} \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle$

$$= \Gamma^c_{da} \langle g | \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle \underset{\sim}{\omega}^d$$

$$+ \Gamma^c_{db} \langle g | \underset{\sim}{e}_a, \underset{\sim}{e}_c \rangle \underset{\sim}{\omega}^d$$

$$= g_{cb}\, \underset{\sim}{\theta}^c{}_a + g_{ac}\, \underset{\sim}{\theta}^c{}_b$$

$$= \underset{\sim}{\theta}_{ba} + \underset{\sim}{\theta}_{ab}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$\underset{\sim}{\omega}^r = \underset{\sim}{d} r$$

$$\underset{\sim}{\omega}^\theta = r \underset{\sim}{d} \theta$$

$$\underset{\sim}{\omega}^\varphi = r \sin\theta \, \underset{\sim}{d} \varphi$$

$$g(\underset{\sim}{e}_a, \underset{\sim}{e}_b) = \delta_{ab}$$

$$\underline{V} = V_{\hat r} \underline{e}_r + V_\theta \underline{e}_\theta$$

$$\|\underline{V}\| = \sqrt{V_{\hat r}^2 + V_{\hat\theta}^2}$$

$$\underline{e}_r = \frac{\partial}{\partial r} \qquad (1,0,0) \text{ in coord basis}$$

$$\underline{e}_\theta = \frac{1}{r}\frac{\partial}{\partial \theta} \qquad \left(0, \frac{1}{r}, 0\right) \text{ "}$$

$$\underline{e}_\varphi = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} = \left(0, 0, \frac{1}{r\sin\theta}\right) \text{ "}$$

$$\|\underline{e}_r\| = 1$$

$$\|\underline{e}_\theta\| = 1$$

$$= \frac{1}{2}\left(T^a_{\ bc} + C^a_{\ bc}\right)\underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \underset{\approx}{T}^a + \frac{1}{2}\langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c$$

torsion tensor $\frac{1}{2} T^a_{\ bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$

$$\delta^a_c \qquad \delta^a_b$$

but $\langle \underline{d}\,\underline{\omega}^a | \underline{e}_b, \underline{e}_c \rangle = \underline{e}_b\left(\cancel{\langle \underline{\omega}^a | \underline{e}_c \rangle}\right) - \underline{e}_c\left(\cancel{\langle \underline{\omega}^a | \underline{e}_b \rangle}\right)$

$- \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$ from identity

So $\underset{\sim}{T}^a = \underset{\sim}{\theta}^a{}_c \wedge \underline{\omega}^c - \frac{1}{2}\langle \underline{\omega}^a | [e_b, e_c] \rangle \underline{\omega}^b \wedge \underline{\omega}^c$

$= \underset{\sim}{\theta}^a{}_c \wedge \underline{\omega}^c + \frac{1}{2}\langle \underline{d}\underline{\omega}^a | e_b, e_c \rangle \underline{\omega}^b \wedge \underline{\omega}^c$

$$\boxed{\underset{\sim}{T}^a = \underline{d}\underline{\omega}^a + \underset{\sim}{\theta}^a{}_c \wedge \underline{\omega}^c}$$ (C1)

If torsion vanishes, then $\underset{\sim}{\theta}^a{}_c \wedge \underline{\omega}^c = -\underline{d}\underline{\omega}^a$

& $\underset{\sim}{\theta}_{(ab)} = 0$

Curvature: defined as commutator of derivs:

$$\underline{R}(\underline{u},\underline{v})\,\underline{w} = [\underline{\nabla}_u \underline{\nabla}_v - \underline{\nabla}_v \underline{\nabla}_u - \underline{\nabla}_{[\underline{u},\underline{v}]}]\,\underline{w}$$

$$\underline{R}: T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$$

Physically represents tidal forces.

Curvature: defined as commutator of derivs.

$$\underline{R}(\underline{u},\underline{v})\,\underline{w} = \left[\underline{\nabla}_u\underline{\nabla}_v - \underline{\nabla}_v\underline{\nabla}_u - \underline{\nabla}_{[\underline{u},\underline{v}]}\right]\underline{w}$$

$$\underline{R}: T_p(M)\times T_p(M)\times T_p(M)\longrightarrow T_p(M)$$

Physically represents tidal forces.

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$$

Cartan's 2^{nd} structural eqn

Curvature 2-form is:

$$\underset{\sim}{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$$

(C2) $$\boxed{\underline{R}^a{}_b = \underline{d}\,\Theta^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b}$$

structural eqn

form is:

$\frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$

$$\boxed{= \underline{d}\,\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b}$$

e.g. $\mathbb{R}^2$ $\quad dr^2 + r^2 d\theta^2$

$\underline{\omega}^r = \underline{d}r \qquad \underline{\omega}^\theta = r\,\underline{d}\theta$

$\underline{d}\,\underline{\omega}^r = 0 \qquad \underline{d}\,\underline{\omega}^\theta = \underline{d}r \wedge \underline{d}\theta$

$= \frac{1}{2} \underline{e}_r \wedge \underline{e}_\theta$

(1) $\quad \underline{d}\,\underline{\omega}^a = -\underline{\Theta}^a{}_b \wedge \underline{\omega}^b$

$\underline{\Theta}^\theta{}_r = \frac{1}{r} \underline{\omega}^\theta \quad$ no $\underline{R}^a{}_b$

$$S^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 .$$

$$\underline{\omega}^\theta = \underline{d}\theta \qquad \underline{\omega}^\varphi = \sin\theta \, \underline{d}\varphi$$

$$\underline{d}\underline{\omega}^\theta = 0 . \qquad \underline{d}\underline{\omega}^\varphi = \cot\theta \, \underline{\omega}^\theta \wedge \underline{\omega}^\varphi$$

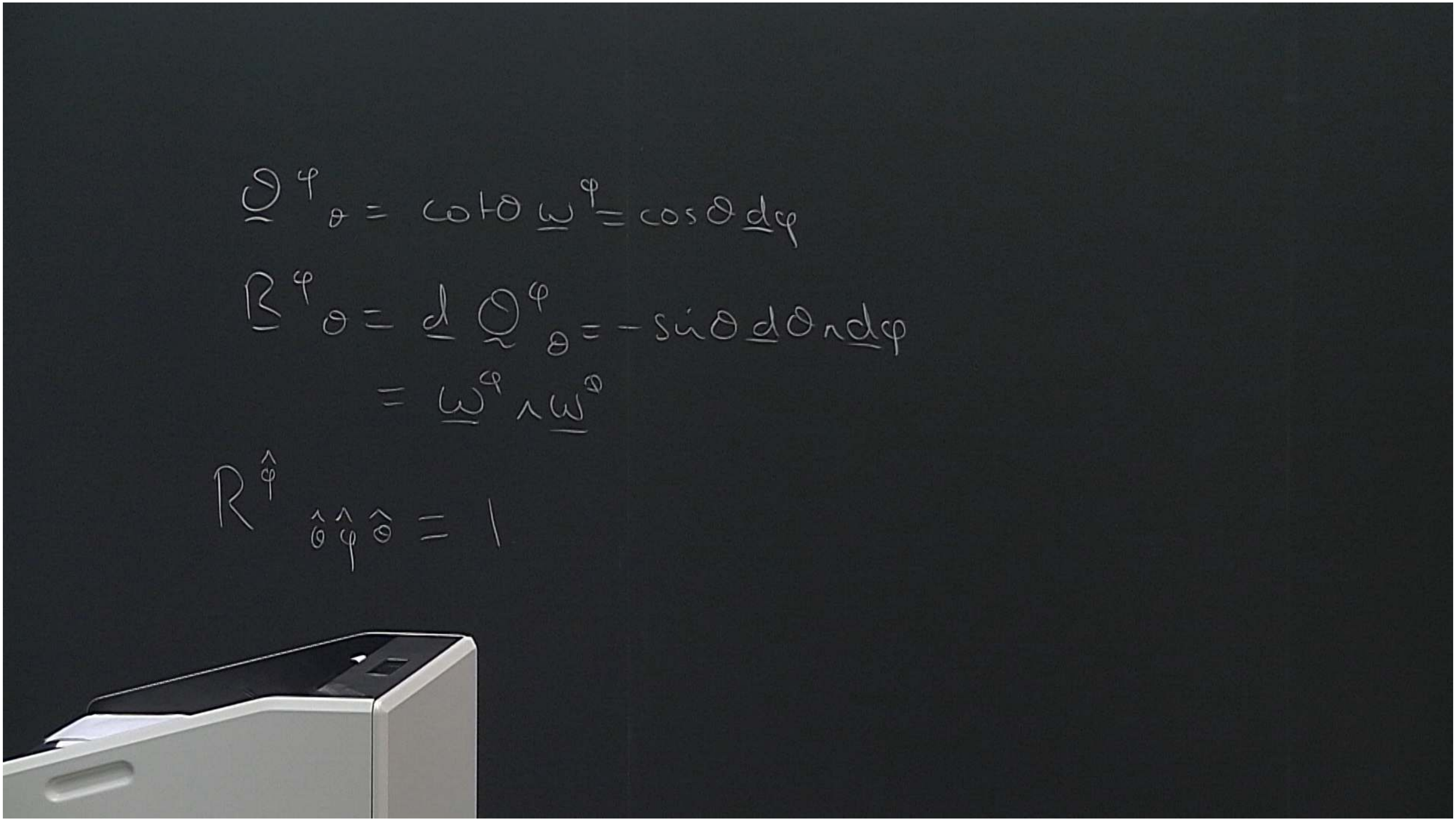
$d\theta^2$

$= r \, \underline{d}\theta$

$= \underline{d}r \wedge \underline{d}\theta$

$= \frac{1}{2} \underline{e}_r \wedge \underline{e}_\theta$

$\underline{R}^a{}_b$


$\underset{\sim}{\omega}^{\varphi}{}_{\theta} = \cot\theta\, \underline{\omega}^{\varphi} = \cos\theta\, \underline{d}\varphi$
$\underset{\sim}{R}^{\varphi}{}_{\theta} = \underline{d}\, \underset{\sim}{\omega}^{\varphi}{}_{\theta} = -\sin\theta\, \underline{d}\theta \wedge \underline{d}\varphi$
$= \underline{\omega}^{\varphi} \wedge \underline{\omega}^{\theta}$
$R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = 1$