Title: PSI 2018/2019 - Foundations of Quantum Mechanics - Lecture 9

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Abstract:

Quantum Foundations
Lecture 9

PSI Review Class: 17th January 2019

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8) Ontological Models

1) Definitions
2) Examples
3) Excess Baggage
4) Contextuality
5) Ψ-ontology
6) Bell's Theorem
7) The Colbeck-Renner Theorem

Ontological Models

- With measurement independence and λ-mediation we have

$$\mathrm{Prob}(k|P,M) = \int_\Lambda \Pr(k|M,\lambda)\Pr(\lambda|P)\,d\lambda$$

- An ontological model is a model with ontic states satisfying: Single world realism, independence, measurement independence, and λ-mediation.

- In other words:
  - There is an ontic state space Λ
  - To each preparation we assign a probability distribution $\Pr(\lambda|P)$ over Λ
  - To each measurement we assign a probability distribution $\Pr(k|M,\lambda)$ over measurement outcomes that depends on λ
  - These should reproduce the operational predictions

$$\mathrm{Prob}(k|P,M) = \int_\Lambda \Pr(k|M,\lambda)\Pr(\lambda|P)\,d\lambda$$

8.3) Excess Baggage

- Lucien Hardy coined the term ontological excess baggage to refer to the fact that, even for a qubit Λ must be infinite.
- This is perhaps surprising because we can only reliably store and retrieve 1 bit of information in a qubit.
- Since Hardy's original proof, Montina has proved:
  - Λ must have the cardinality of the continuum.
  - Even if we allow the model to only approximately reproduce quantum theory, $|\Lambda| = O(e^d)$ where d is Hilbert space dimension
- Here we will prove Hardy's original result, see references in
  M. Leifer Quanta 3:67-155 (2014)
  D. Jennings and M. Leifer Contemp. Phys. 57:60-82 (2015)
  for references to later work.

A Useful Lemma

Lemma: Let P be a preparation of $|\psi\rangle$ and let M be a measurement in an orthonormal basis that includes $|\psi\rangle$

Let $\Lambda_\psi^P = \{\lambda \in \Lambda \mid Pr(\lambda|P) > 0\}$, $\Gamma_\psi^M = \{\lambda \in \Lambda \mid Pr(\psi|M,\lambda) = 1\}$

Then $\Lambda_\psi^P \subseteq \Gamma_\psi^M$ (up to measure-zero sets)

Proof:

$$1 = |\langle\psi|\psi\rangle|^2 = \int_\Lambda d\lambda\, Pr(\psi|M,\lambda)\, Pr(\lambda|P) = \int_{\Lambda_\psi^P} d\lambda\, Pr(\psi|M,\lambda)\, Pr(\lambda|P)$$

However, since $\int_{\Lambda_\psi^P} d\lambda\, Pr(\lambda|P) = 1$ and $Pr(\lambda|P) > 0$ on Λ_ψ^P

$Pr(\psi|M,\lambda)$ must be 1 everywhere on Λ_ψ^P (up to measure zero sets)

Hardy's Excess Baggage Theorem

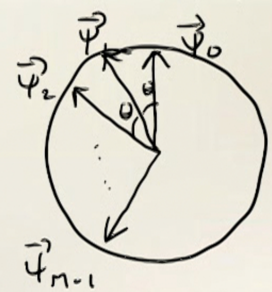
Theorem: Any ontological model that can reproduce the quantum predictions for orthonormal basis measurements on pure states in any Hilbert space dimension must have $|\Lambda| = \infty$.

Proof: Assume by contradiction that $|\Lambda| = N$ for some finite N.

Consider a 2d subspace spanned by $|0\rangle$ and $|1\rangle$ and the M states

$$|\psi_j\rangle = \cos\left(\frac{j\pi}{2M}\right)|0\rangle + \sin\left(\frac{j\pi}{2m}\right)|1\rangle \quad \text{for } j = 0, 1, 2, \ldots, M-1$$

We can choose M as large as we like



We have

$$|\langle\psi_k|\psi_j\rangle|^2 < 1 \quad \text{for all } j \neq k.$$

Hardy's Excess Baggage Theorem

Consider preparing the system in the state $|\psi_j\rangle$ and measuring it in a basis that includes $|\psi_k\rangle$ for $j \neq k$

Then $\sum_{\lambda \in \Lambda} \Pr(\psi_k|\lambda)\Pr(\lambda|\psi_j) < 1$ [Note: Naughty notation, so I'll use Λ_ψ for Λ_ψ^P also]

$\Rightarrow$ There must exist a $\lambda \in \Lambda_{\psi_j}$ s.t. $\Pr(\psi_k|\lambda) < 1$ otherwise sum would be 1

- Since $\Pr(\psi_k|\lambda) = 1$ everywhere on Λ_{ψ_k}, this means Λ_{ψ_j} and Λ_{ψ_k} must be different subsets of Λ.
- This applies to every pair $j \neq k$ so we must have M distinct subsets of Λ
- # distinct subsets of $\Lambda = 2^N \Rightarrow 2^N \geq M$ or $N \geq \log_2 M$
- But we can choose M as large as we like, so N is larger than any finite integer $\Rightarrow N = \infty$.

8.4) Contextuality

- We follow an approach to contextuality that is due to Rob Spekkens – Phys. Rev. A 71, 052108 (2005).
- The basic philosophy is based on Leibniz Principle of the Identity of Indiscernables:
  - No two distinct things exactly resemble each other.
- This principle is arguably very successful in physics:
  - e.g. Principle of relativity, Einstein's equivalence principle.
- The principle can also be thought of as a no fine tuning argument.
  - e.g. suppose objects A and B have some distinct physical property, but there is absolutely no measurement we can do to tell A and B apart. Then, our measurements must only reveal coarse-grained information that is fine-tuned in just such a way so as not to reveal the difference.
- Not all apparent fine tunings are evil, but they do require explanation.

Preparation Contextuality

- Define an equivalence relation on preparations in an operational theory:

$$P \sim Q \quad \Leftrightarrow \quad \mathrm{Prob}(k|P,M) = \mathrm{Prob}(k|Q,M) \text{ for all measurement-outcome pairs } (M,k).$$

- In particular, if $\rho_P = \rho_Q$ then $P \sim Q$.
- An ontological model is preparation noncontextual if,

$$P \sim Q \quad \Rightarrow \quad \Pr(\lambda|P) = \Pr(\lambda|Q).$$

- In words, whenever there is no observable distinction between two preparations, they are represented by the same epistemic state in the ontological model.
- A model that is not preparation noncontextual is called preparation contextual.

Mixing Preparations

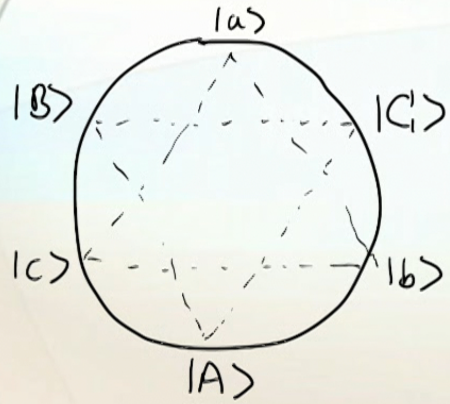
- If an operational theory contains preparations P and Q then we can construct a mixed preparation $pP + (1-p)Q$.
  - Physically this means, toss a coin with $p(\text{heads}) = p$, do P if it lands heads or Q if it lands tails, then forget the coin toss outcome.
- We will assume that the ontological model preserves mixtures:

$$\Pr(\lambda|pP + (1-p)Q) = p\Pr(\lambda|P) + (1-p)\Pr(\lambda|Q)$$

- This is actually an instance of preparation noncontextuality applied to the joint coin-system system. Conditioning on the outcome of the coin yields a preparation equivalent to P or Q.

Proof of Preparation Contextuality

① Consider the following 6 states on the equator of the Bloch sphere



We have $\langle a|A\rangle = \langle b|B\rangle = \langle c|C\rangle = 0$

This implies $\Lambda_a \cap \Lambda_A = \Lambda_b \cap \Lambda_B = \Lambda_c \cap \Lambda_C = \phi$

where $\Lambda_\psi = \{\lambda \,|\, Pr(\lambda|\psi) > 0\}$

Why?

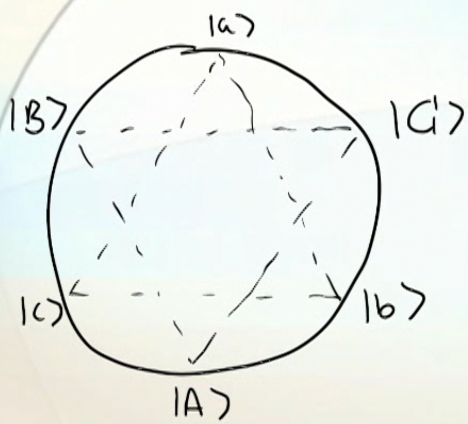
By lemma $\Lambda_\psi \subseteq \Gamma_\psi^M$ where $\Gamma_\psi^M = \{\lambda \,|\, Pr(\psi|M,\lambda) = 1\}$

But $Pr(a|M,\lambda) + Pr(A|M,\lambda) = 1$ for all $\lambda \in \Lambda$

so $Pr(a|M,\lambda) = 1 \Rightarrow Pr(A|M,\lambda) = 0$ and vice versa

$\therefore \Gamma_a^M \cap \Gamma_A^M = \phi \Rightarrow \Lambda_a \cap \Lambda_A = \phi$

Proof of Preparation Contextuality



We also have:

$$\frac{I}{2} = \frac{1}{2}\left(|a\rangle\langle a| + |A\rangle\langle A|\right)$$
$$= \frac{1}{2}\left(|b\rangle\langle b| + |B\rangle\langle B|\right)$$
$$= \frac{1}{2}\left(|c\rangle\langle c| + |C'\rangle\langle C'|\right)$$
$$= \frac{1}{3}\left(|a\rangle\langle a| + |b\rangle\langle b| + |c\rangle\langle c|\right)$$
$$= \frac{1}{3}\left(|A\rangle\langle A| + |B\rangle\langle B| + |C'\rangle\langle C'|\right)$$

So by preparation noncontextuality:

$$Pr(\lambda|\tfrac{I}{2}) = \frac{1}{2}\left(Pr(\lambda|a) + Pr(\lambda|A)\right)$$
$$= \frac{1}{2}\left(Pr(\lambda|b) + Pr(\lambda|B)\right)$$
$$= \frac{1}{2}\left(Pr(\lambda|c) + Pr(\lambda|C)\right)$$

$$Pr(\lambda|\tfrac{I}{2}) = \frac{1}{3}\left(Pr(\lambda|a) + Pr(\lambda|b) + Pr(\lambda|c)\right)$$
$$= \frac{1}{3}\left(Pr(\lambda|A) + Pr(\lambda|B) + Pr(\lambda|C')\right)$$

Proof of Preparation Contextuality

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Now, any given λ can only be in at most one of Λ_a or Λ_A, Λ_b or Λ_B, Λ_c or Λ_C.

Let's choose a λ that is not in Λ_a, not in Λ_b, and not in Λ_C. Then

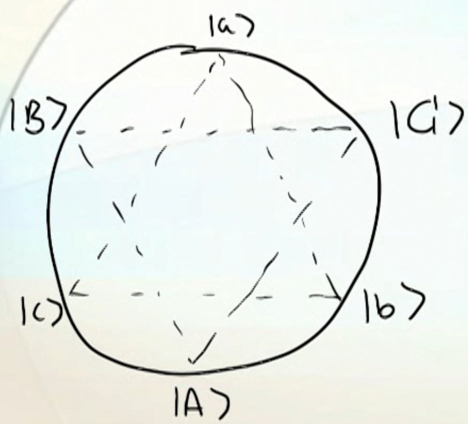
$$\Pr(\lambda|\tfrac{I}{2}) = \tfrac{1}{2}\Pr(\lambda|c) \qquad \Pr(\lambda|\tfrac{I}{2}) = \tfrac{1}{3}\Pr(\lambda|c)$$

$\Rightarrow 2\Pr(\lambda|\tfrac{I}{2}) = 3\Pr(\lambda|\tfrac{I}{2}) \Rightarrow \Pr(\lambda|\tfrac{I}{2}) = 0$ for this particular λ

We get a similar result for every choice of not in $\Lambda_{a/A}$, $\Lambda_{b/B}$, $\Lambda_{c/C}$

This exhausts $\Lambda \Rightarrow \Pr(\lambda|\tfrac{I}{2}) = 0$ everywhere, but this cannot be true for a probability distribution.

Proof of Preparation Contextuality



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$$Pr(\lambda|\tfrac{I}{2}) = \frac{1}{3}\left(Pr(\lambda|a) + Pr(\lambda|b) + Pr(\lambda|c)\right)$$
$$= \frac{1}{3}\left(Pr(\lambda|A) + Pr(\lambda|B) + Pr(\lambda|C')\right)$$

Measurement Contextuality

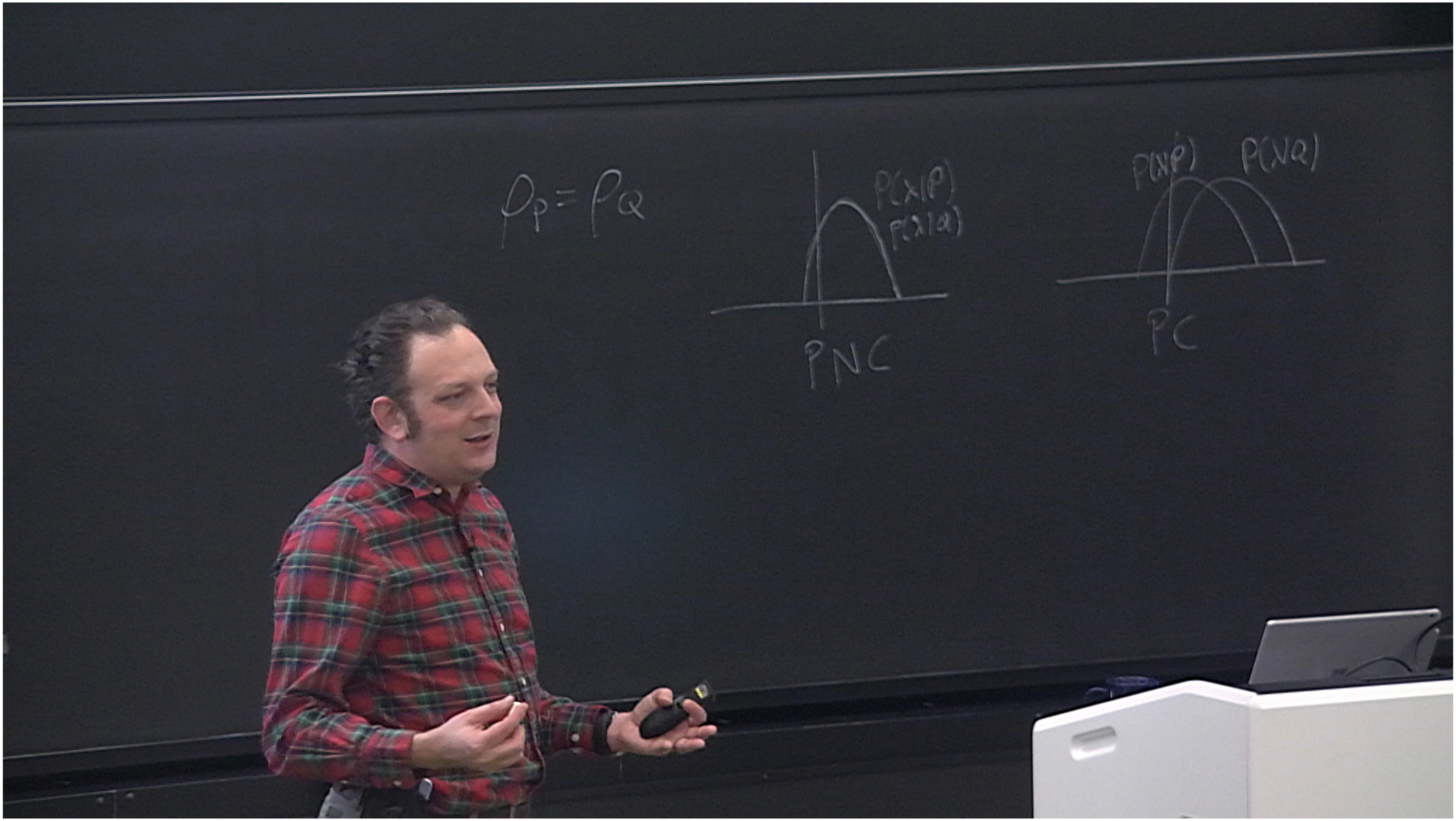
- Define an equivalence relation on measurement-outcome pairs in an operational theory:

$$(\mathrm{M},\mathrm{k}) \sim (N,l) \quad \Leftrightarrow \quad \mathrm{Prob}(k|P,M) = \mathrm{Prob}(l|P,N) \quad \text{for all preparations } P.$$

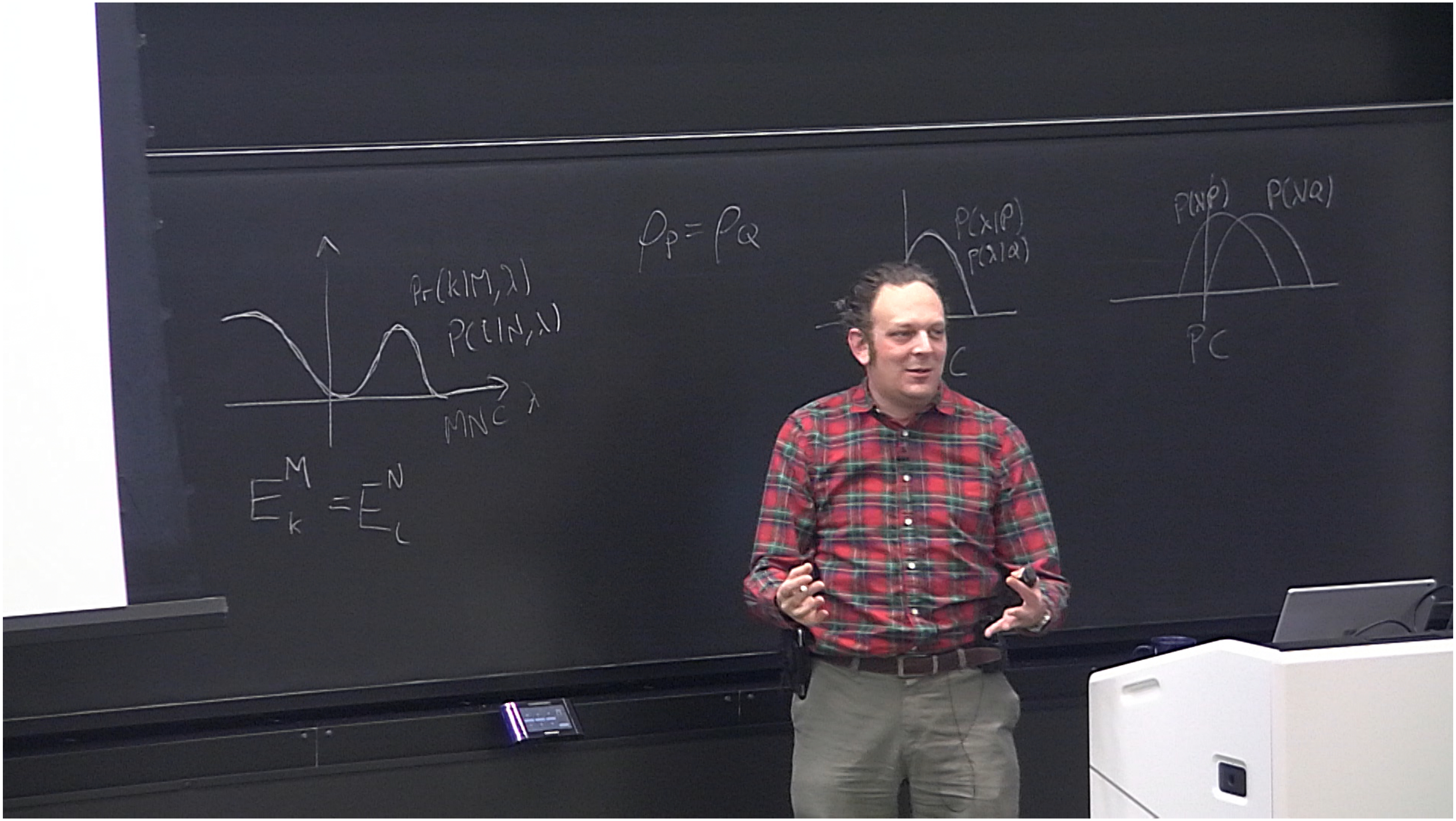
- In particular, if $E_k^M = E_l^N$ then $(\mathrm{M},\mathrm{k}) \sim (M,l)$.
- An ontological model is measurement noncontextual if,

$$(M,k) \sim (N,l) \quad \Rightarrow \quad \Pr(k|M,\lambda) = \Pr(l|N,\lambda).$$

- In words, whenever there is no observable distinction between two measurement-outcome pairs, they are represented by the same response function in the ontological model.
- A model that is not measurement noncontextual is called measurement contextual.



ρP = ρQ
P(λ|P)
P(λ|Q)
PNC
P(λ|P)
P(λ|Q)
PC


Pr(k|M,λ)
P(l|N,λ)
MNC
λ
$E^M_k = E^N_l$
$\rho_P = \rho_Q$
P(λ|P)
P(λ|Q)
PC

Kochen-Specker Contextuality

- Measurement noncontextual models exist:
  - e.g. Beltrametti-Bugajski: $\Pr(k|M,\lambda) = \mathrm{Tr}(\, E_k^M \, |\lambda\rangle\langle\lambda|)$.
- A Kochen-Specker (KS) noncontextual model is:
  - A model that only contains projective measurements.
  - Measurement noncontextual.
  - Outcome deterministic: $\Pr(\Pi|\lambda) = 0$ or 1 for all λ.
- We will prove in a later lecture that:

  $$\text{KS contextual} \;\Rightarrow\; \text{maximally } \psi\text{-epistemic} \;\Longrightarrow\; \text{preparation contextual}$$

  so KS contextuality is still worth proving.
- KS contextuality can only be proved in $d \geq 3$.
- By applying KS noncontextuality for projective measurements and measurement noncontextuality for POVMs, Spekkens obtained a proof in $d = 2$. We will focus on traditional KS proofs.

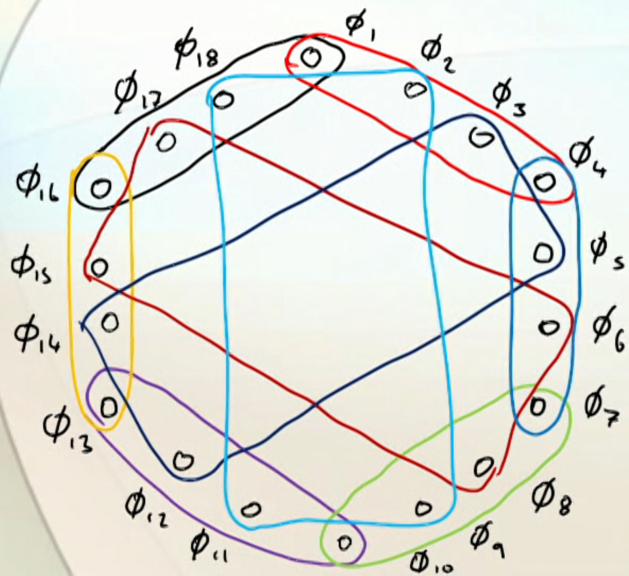
KS Contextuality and value assignments

- Due to the outcome determinism assumption, each λ determines a value function v_λ that assigns a value 0 or 1 to each projector.

$$v_\lambda(\Pi) = \Pr(\Pi|\lambda)$$

- Since probabilities must sum to 1, in each projective measurement $\{\Pi_k\}$, exactly one of the projectors must get value 1, the others getting value 0.
- Measurement noncontextuality then implies that the value assigned to Π_k does not depend on which measurement it is a part of.
- In particular, this applies to an orthonormal basis. For each basis $\{|\phi_k\rangle\}$, exactly one vector gets the value 1, the rest 0, and this value is the same for every basis that $|\phi_k\rangle$ appears in.
- In proving Kochen-Specker contextuality, we can focus on whether such a value function exists.

The 18-Ray Proof



- A. Cabello, J. Estebaranz, G. Garcia-Alcaine, Phys. Lett. A 212:183 (1996).
- In 4-dimensional quantum mechanics, we can find 18 states with the orthogonality structure depicted.
- Each hyperedge is an orthonormal basis.

ϕ_1	$(1,0,0,0)$	ϕ_{10}	$(0,1,0,-1)$
ϕ_2	$(0,1,0,0)$	ϕ_{11}	$(1,0,1,0)$
ϕ_3	$(0,0,1,1)$	ϕ_{12}	$(1,1,-1,1)$
ϕ_4	$(0,0,1,-1)$	ϕ_{13}	$(-1,1,1,1)$
ϕ_5	$(1,-1,0,0)$	ϕ_{14}	$(1,1,1-1)$
ϕ_6	$(1,1,-1,-1)$	ϕ_{15}	$(1,0,0,1)$
ϕ_7	$(1,1,1,1)$	ϕ_{16}	$(0,1,-1,0)$
ϕ_8	$(1,-1,1,-1)$	ϕ_{17}	$(0,1,1,0)$
ϕ_9	$(1,0,-1,0)$	ϕ_{18}	$(0,0,0,1)$

The 18-Ray Proof

Red	Blue	Green	Purple	Yellow	Black	Light Blue	Navy	Burgundy
ϕ_1	ϕ_4	ϕ_7	ϕ_{10}	ϕ_{13}	ϕ_{16}	ϕ_2	ϕ_3	ϕ_6
ϕ_2	ϕ_5	ϕ_8	ϕ_{11}	ϕ_{14}	ϕ_{17}	ϕ_9	ϕ_5	ϕ_8
ϕ_3	ϕ_6	ϕ_9	ϕ_{12}	ϕ_{15}	ϕ_{18}	ϕ_{11}	ϕ_{12}	ϕ_{15}
ϕ_4	ϕ_7	ϕ_{10}	ϕ_{13}	ϕ_{16}	ϕ_1	ϕ_{18}	ϕ_{14}	ϕ_{17}

- There are nine bases, and in each one, one of the $\phi_j's$ has to receive the value 1, the rest 0. So there will be 9 rays assigned the value 1 in total.
- However, each ϕ_j appears exactly two times in the table, so whichever of them are assigned the value 1, there will always be an even number of 1's in total. Contradiction!

KS Contextuality and value assignments

- We can also think of the value functions as assigning definite values to observables (self-adjoint operators) via

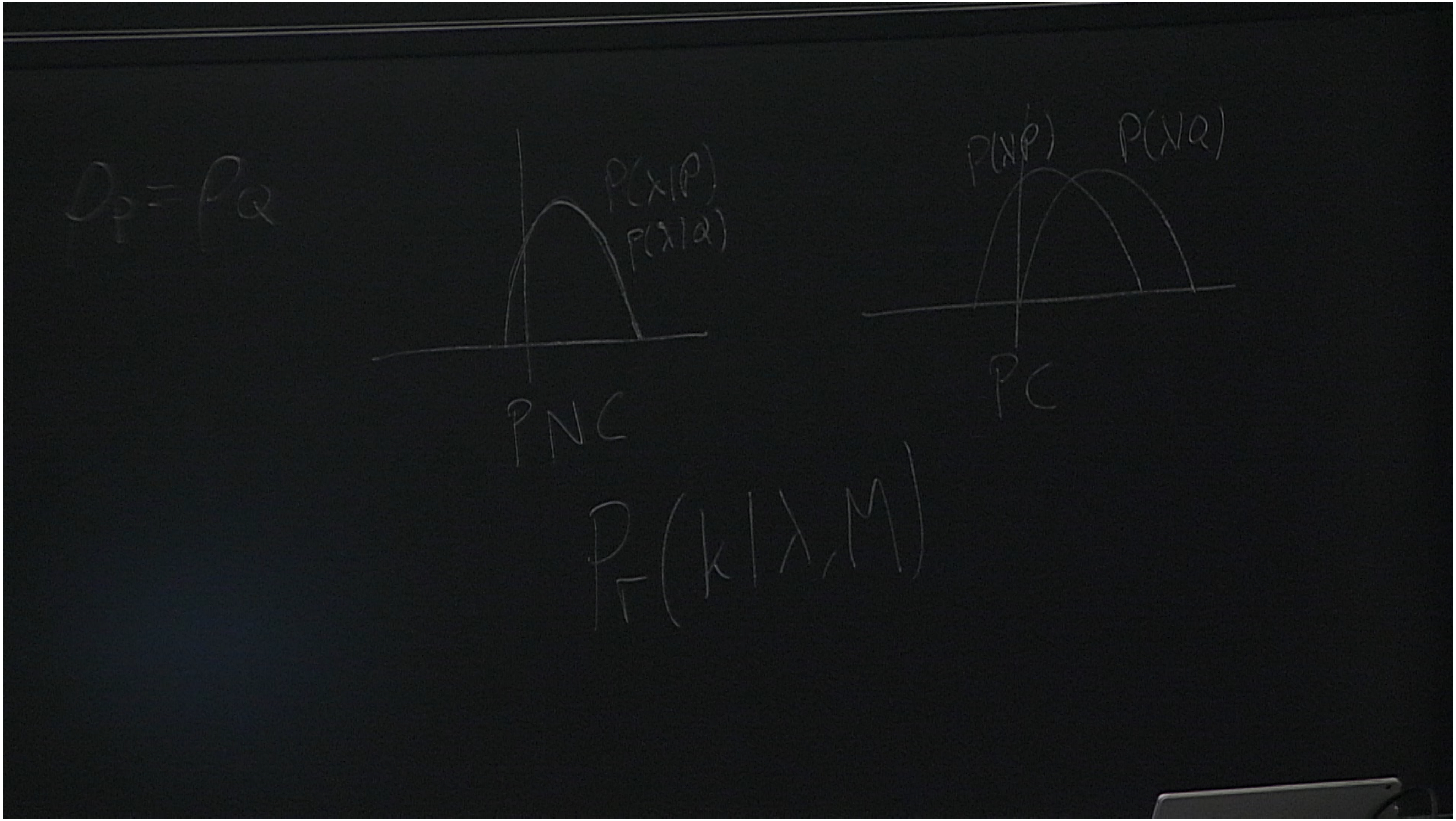
$$v(M) = \sum_j m_j \, v(\Pi_j)$$

- Now, if two observables M and N commute then they have a joint eigendecomposition.

$$M = \sum_j m_j \Pi_j \qquad N = \sum_j n_j \, \Pi_j$$

- And we will have:

$$MN = \sum_j m_j n_j \, \Pi_j \qquad M + N = \sum_j (m_j + n_j) \Pi_j$$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KS Contextuality and value assignments

- Since, in all of these decompositions, the same projector will get the value 1, whenever $[M, N] = 0$, the value functions will obey

$$v(MN) = v(M)v(N) \qquad\qquad v(M+N) = v(M) + v(N)$$

- If we define functions of operators by power series, this implies that whenever $M_1, M_2, \dots$ all mutually commute then

$$v\big(f(M_1, M_2, \dots)\big) = f(v(M_1), v(M_2), \dots)$$

- So another way of defining KS noncontextuality is: there exists a value function that assigns eigenvalues to observables that obeys $v\big(f(M_1, M_2, \dots)\big) = f(v(M_1), v(M_2), \dots)$ for mutually commuting observables.

The Peres-Mermin Square

⊙ Consider the following table of 9 two qubit observables:

			Row products
$\sigma_1 \otimes \sigma_1$	$\sigma_1 \otimes I$	$I \otimes \sigma_1$	I
$\sigma_3 \otimes \sigma_3$	$I \otimes \sigma_3$	$\sigma_3 \otimes I$	I
$\sigma_2 \otimes \sigma_2$	$\sigma_1 \otimes \sigma_3$	$\sigma_3 \otimes \sigma_1$	$-I$
I	I	I	
Column products			

⊙ Each observable has eigenvalues ± 1, so receives values ± 1.

⊙ Each row and column consists of mutually commuting observables.

⊙ The column products are all $+I$, which has value $+1$, so there must be an even number of -1's in each column, so an even number in total

⊙ However, one of the row products is $-I$, so there must be an odd number of -1's in that row, and an odd number in total $\Rightarrow$ contradiction.