Title: PSI 2018/2019 - Foundations of Quantum Mechanics - Lecture 5

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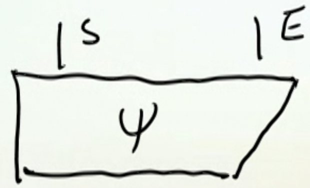
Abstract:

Summary From Last Lecture

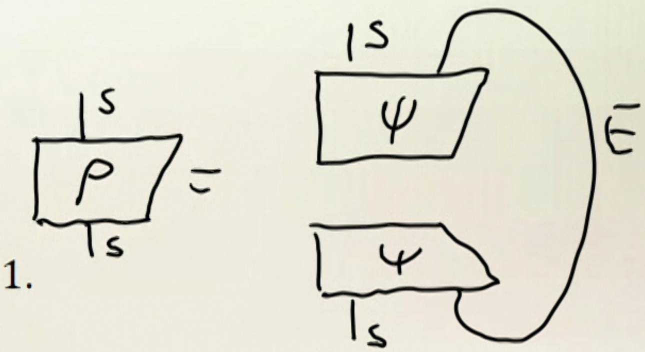
- Larger Hilbert Space view of density operators
  - Divide universe into system and environment $\mathcal{H}_{SE} = \mathcal{H}_S \otimes \mathcal{H}_E$
  - Universe is in a pure state

$|\psi\rangle_{SE} = \sum_{jk} \psi^{jk} |j\rangle_S \otimes |k\rangle_E$ or $\psi^{j_S k_E}$ or



  - System S can be described by a density operator ρ_S on $\mathcal{H}_S$

$\rho_S = \mathrm{Tr}_E(|\psi\rangle\langle\psi|_{SE})$ or $\rho^{j_S}_{k_S} = \psi^{j_S m_E} \psi^{\dagger}_{k_S m_E}$ or



  - Density operators are positive $\langle\phi|\rho_S|\phi\rangle \geq 0$ and $\mathrm{Tr}_S(\rho_S) = 1$.

The View from the Smaller Church

- According to the smaller church, a quantum state should be <u>any</u> consistent way of assigning probabilities to observables.
- We can view a quantum state as a functional that assigns expectation values to observables

$$\rho : \mathcal{S}(\mathcal{H}_A) \longrightarrow \mathbb{R}$$

- When we apply it to projection operators, we should get probabilities.
- Classically, expectation values behave linearly

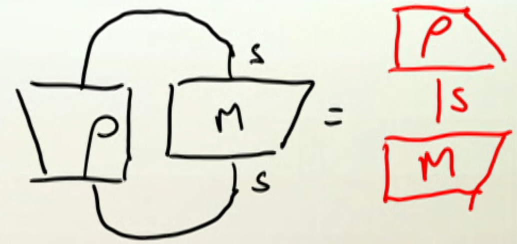
$$\langle \alpha X + \beta Y \rangle = \alpha \langle X \rangle + \beta \langle Y \rangle$$

- We will impose this for quantum observables too (but can remove this later) $\rho(\alpha M + \beta N) = \alpha \rho(M) + \beta \rho(N)$

The View from the Smaller Church

- A linear functional from $\mathcal{S}(\mathcal{H}_A)$ to $\mathbb{R}$ is the definition of $\mathcal{S}(\mathcal{H}_A)^\dagger$, so ρ must be a duperator.
- However, we already saw that $\mathcal{S}(\mathcal{H}_A)$ is self-dual, so we get for free that ρ is a self-adjoint operator

$$\rho(M) = \rho^{j_s}_{k_s} M^{k_s}_{j_s} = \mathrm{Tr}(\rho M) =$$

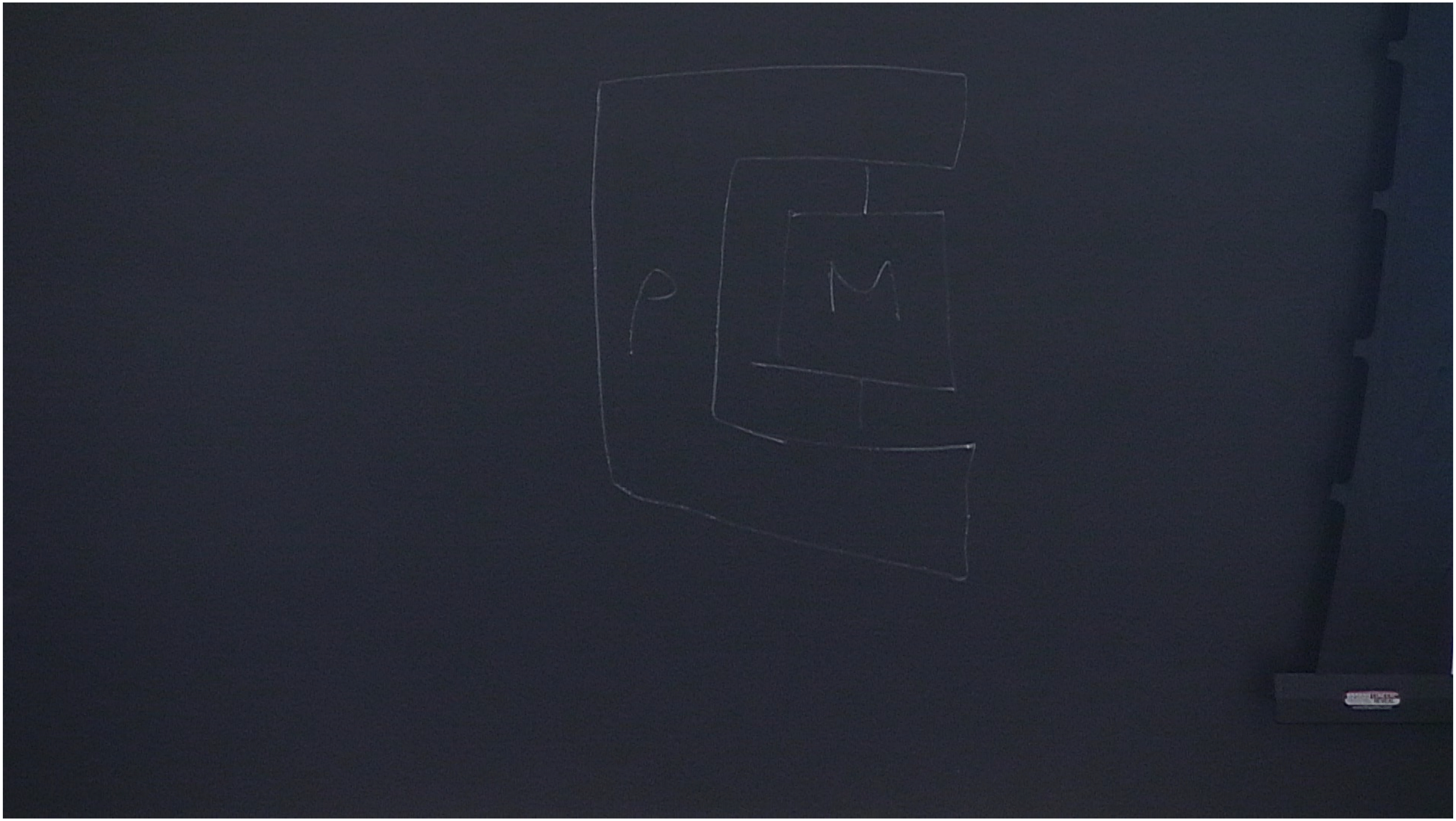


- Since projectors must get assigned probabilities

$$\mathrm{Tr}(\rho \Pi) \geq 0 \qquad \rho^{j_s}_{k_s} \Pi^{k_s}_{j_s} \geq 0$$

- Let Π be a 1-dimensional projector $\quad \Pi^{k_s}_{j_s} = \psi^\dagger_{j_s} \psi^{k_s}$

Then $\psi^\dagger_{j_s} \rho^{j_s}_{k_s} \psi^{k_s} = \langle \psi | \rho | \psi \rangle \geq 0$ which is positivity.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Removing the Linearity Condition

- The linearity condition $\rho(M+N) = \rho(M) + \rho(N)$ is not operationally meaningful when M and N do not commute

  We can't measure $M+N$ by measuring M at the same time as N and then adding the results.

- Fortunately it can be removed.

- Gleason's Theorem (which is hard to prove) states that:

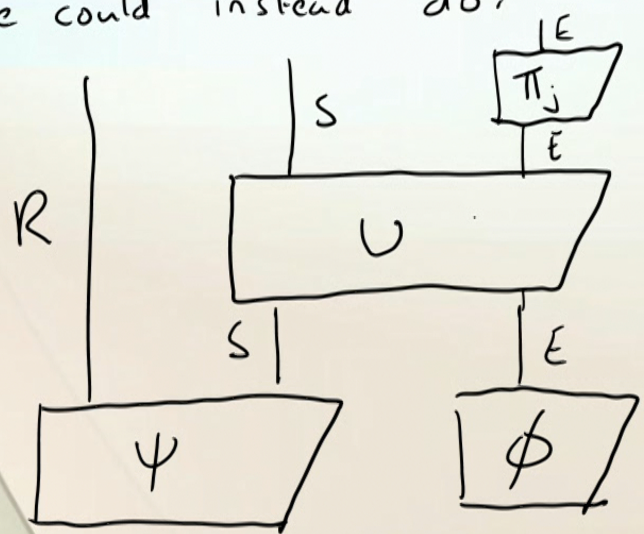
  For Hilbert space dimension ≥ 3 <u>any</u> function from projectors to $\mathbb{R}$ that satisfies $f(\Pi) \geq 0$, $f(\Pi_1 + \Pi_2) = f(\Pi_1) + f(\Pi_2)$ if $\Pi_1 \Pi_2 = 0$, $f(I) = 1$ is of the form $f(\Pi) = \mathrm{Tr}(\rho \Pi)$ for some density operator ρ.

Summary

- A general quantum state on a system A is a density operator $\rho_A \in \mathcal{L}(\mathcal{H}_A)$ satisfying
  - Positivity $\langle\psi|\rho_A|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}_A$.
  - $\mathrm{Tr}(\rho_A) = 1$.
- This can be derived from
  - There is a pure state $|\psi\rangle_{AE} \in \mathcal{H}_{AE}$ and ignoring the environment
    $$\rho_A = \mathrm{Tr}_E(|\psi\rangle\langle\psi|)$$
  - A state must be something that assigns consistent probabilities to all projective measurements on the system.

5.4) Postivie Operator Valued Measures (POVMs)

- We have (temporarily) allowed projective measurements as a primitive in the larger church, but we don't need to make a measurement directly on the system.
- We could instead do:



and view the projective measurement on the environment as a measurement of the system

- Can we describe the probabilities for the measurement outcomes in terms of operators acting on $\mathcal{H}_S$ alone?

The View from the Larger Church

$$\mathrm{Prob}(\Pi_j) = \mathrm{Tr}_{SE}\left(\Pi_j^E U_{SE}\, \rho_S \otimes |\phi\rangle_E\langle\phi| U_{SE}^\dagger\right)$$

$$= \mathrm{Tr}_{SE}\left(|\phi\rangle_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE}\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left(\sum_k {}_E\langle k|\phi\rangle_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |k\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left({}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} \left[\sum_k |k\rangle_E\langle k|\right] |\phi\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left({}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |\phi\rangle_E\, \rho_S\right)$$

$$= \mathrm{Tr}_S\left(E_j \rho_S\right)$$

where $E_j := {}_E\langle\phi| U_{SE}^\dagger \Pi_j^E U_{SE} |\phi\rangle_E \in \mathcal{L}(\mathcal{H}_S)$

The View from the Larger Church

○ Properties of the E_j operators

1) E_j is a positive operator, i.e. ${}_S\langle\psi|E_j|\psi\rangle_S \geq 0 \quad \forall\, |\psi\rangle_S \in \mathcal{H}_S$.

$${}_S\langle\psi|E_j|\psi\rangle_S = \left({}_S\langle\psi| \otimes {}_E\langle\phi|\right) U_{SE}^\dagger \Pi_j^E U_{SE} \left(|\psi\rangle_S \otimes |\phi\rangle_E\right)$$

This is a special case of

$${}_{SE}\langle\psi| U_{SE}^\dagger \Pi_j^E U_{SE} |\psi\rangle_{SE}$$ and $U_{SE}^\dagger \Pi_j^E U_{SE}$ is of the form $N^\dagger N$ with $N = \Pi_j^E U_{SE}$ so is positive

2) $\sum_j E_j = I_S$

$$\sum_j E_j = {}_E\langle\phi| U_{SE}^\dagger \left(\sum_j \Pi_j^E\right) U_{SE} |\phi\rangle_E = {}_E\langle\phi| U_{SE}^\dagger U_{SE} |\phi\rangle = {}_E\langle\phi| I_{SE} |\phi\rangle_E$$
$$= I_S \langle\phi|\phi\rangle_E = I_S$$

It can be shown that any set of operators $\{E_j\}$ satisfying 1) and 2) can arise in this way.

The View from the Smaller Church

- According to the smaller church any consistent way of assigning probabilities to measurement outcomes should be a valid description of a measurement.
- A measurement outcome should be described by an affine functional from density operators to $\mathbb{R}$

$$f_j(p\rho_A + (1-p)\sigma_A) = p\, f_j(\rho_A) + (1-p)\, f_j(\sigma_A)$$

- We can extend this to a linear functional on $S(\mathcal{H}_A)$ by defining

$$f_j(-\rho_A) := -f_j(\rho_A)$$

- $S(\mathcal{H}_A)^\dagger = S(\mathcal{H}_A)$ and the inner product is $\mathrm{Tr}(MN)$, so we know that

$$f_j = \mathrm{Tr}(E_j \rho_A) \quad \text{for some self adjoint operator } E_j \in S(\mathcal{H}_A)$$

The View from the Smaller Church

- It is easy to see that E_j has to be positive. $\mathrm{Tr}(E_j \rho_A) \geq 0$ because it is a probability. Let $\rho_A = |\psi\rangle_A\langle\psi|$.

$$0 \leq \mathrm{Tr}\left(E_j \, |\psi\rangle_A\langle\psi|\right) = {}_A\langle\psi| E_j |\psi\rangle$$

  and further, all density operators can be written as $\rho_A = \sum_j p_j |\psi_j\rangle_A\langle\psi_j|$ so this is sufficient for $\mathrm{Tr}(E_j \rho_A) \geq 0$ for all ρ_A

- Secondly, we must have

$$\sum_j \mathrm{Tr}(E_j \rho_A) = 1 \quad \text{for all } \rho_A$$

$$\Rightarrow \mathrm{Tr}\left[\left(\sum_j E_j\right)\rho_A\right] = 1$$

$$\Rightarrow {}_A\langle\psi| \left(\sum_j E_j\right) |\psi\rangle_A = 1 \quad \text{for all pure states } |\psi\rangle_A$$

We already proved that this implies $\sum_j E_j = I_A$

$$\langle\psi|A|\psi\rangle = \langle\psi|B|\psi\rangle \quad \forall|\psi\rangle \Rightarrow A = B$$

$$E_j(\rho_A) = \mathrm{Tr}(E_j\rho_A)$$

$$\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$$

$$E_i(\rho_A) = \mathrm{Tr}(E_i \rho_A)$$

$$\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \langle\psi|A|\psi\rangle$$

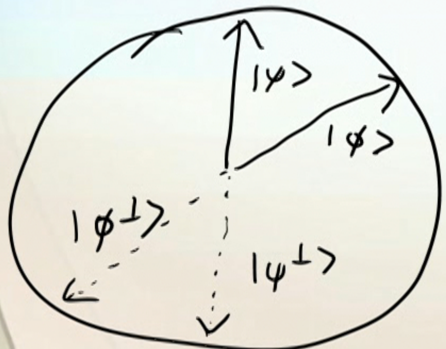
$$1 = \langle\psi|\psi\rangle = \langle\psi|I|\psi\rangle$$

Summary

- A general measurement is described by a set $\{E_j\}$ of positive operators such that $\sum_j E_j = I$.
- This is called a positive operator valued measure.
- The probability rule is $\mathrm{Prob}(E_j) = \mathrm{Tr}(E_j \rho)$
- This can be derived from:
  - A system interacts with an initially uncorrelated environment. We perform a projective measurement on the environment.
  - The requirement that measurements should assign well-defined probability distributions to all density operators.

Example

- You shouldn't always think of a POVM as a noisy version of a projective measurement. Sometimes it is the optimal measurement to do.
- E.g. Unambiguous state discrimination



$$E_\phi = a|\psi^\perp\rangle\langle\psi^\perp| \qquad E_\psi = b|\phi^\perp\rangle\langle\phi^\perp|$$

$$E_? = I - E_\phi - E_\psi$$

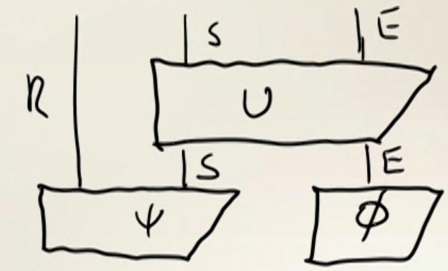
Maximise a and b such that $E_?$ is a positive operator

$$P_{succ} = 1 - |\langle\phi|\psi\rangle|$$

5.5) Completely Positive Maps

- The dynamics of an isolated system is unitary, but in general a system might interact with its environment. How do we keep track of the state (density operator) of the system on its own?
- According to the larger church, the system and environment generally start in a (possibly entangled) pure state $|\psi\rangle_{SR} \in \mathcal{H}_S \otimes \mathcal{H}_R$
- However, the description to be given here only works if the system interacts with a part of the environment it is initially uncorrelated with, so we assume $|\psi\rangle_{SR} \otimes |\phi\rangle_E \in \mathcal{H}_S \otimes \mathcal{H}_R \otimes \mathcal{H}_E$ and the dynamics is $|\tilde{\psi}\rangle_{SRE} = U_{SE}\, |\psi\rangle_{SR} \otimes |\phi\rangle_E$

  where $|\tilde{\psi}\rangle_{SRE}$ is the final state of SRE.



The View from the Larger Church

- We are only interested in keeping track of the density operator of S.

Initially: $\rho_S = \mathrm{Tr}_R \left(|\psi\rangle_{SR}\langle\psi| \right)$

After U: $\tilde{\rho}_S = \mathrm{Tr}_{RE} \left(|\tilde{\psi}\rangle_{SRE}\langle\tilde{\psi}| \right)$

$$= \mathrm{Tr}_{RE} \left(U_{SE} |\psi\rangle_{SR} |\phi\rangle_E \; {}_{SR}\langle\psi| \, {}_E\langle\phi| U_{SE}^\dagger \right)$$

$$= \mathrm{Tr}_E \left(U_{SE} \, \rho_S \otimes |\phi\rangle_E\langle\phi| \, U_{SE}^\dagger \right)$$

$$= \sum_j {}_E\langle j| U_{SE} |\phi\rangle_E \, \rho_S \, {}_E\langle\phi| U_{SE}^\dagger |j\rangle_E$$

$$= \sum_j M^{(j)} \rho_S M^{(j)\dagger} \leftarrow \text{This is called the operator sum decomposition}$$

where $M^{(j)} = {}_E\langle j| U_{SE} |\phi\rangle_E$ are called Kraus operators

The View from the Larger Church

- The Kraus operators have to satisfy

$$\sum_j M^{(j)\dagger} M^{(j)} = \sum_j {}_E\langle\phi| U_{SE}^\dagger |j\rangle_E \langle j| U_{SE} |\phi\rangle_E = {}_E\langle\phi| U_{SE}^\dagger U_{SE} |\phi\rangle_E$$

$$= {}_E\langle\phi| I_{SE} |\phi\rangle_E = {}_E\langle\phi|\phi\rangle_E \, I_S = I_S$$

- Do they have to satisfy any other constraints?

No. For any set of operators $M^{(j)} \in \mathcal{L}(\mathcal{H}_S)$ s.t. $\sum_j M^{(j)\dagger} M^{(j)} = I_S$ you can construct a unitary U_{SE}

$$M^{(j)} = {}_E\langle j| U_{SE} |\phi\rangle_E$$

(see e.g. Nielsen and Chuang for proof)

The View from the Smaller Church

- According to the smaller church, dynamics should be any mapping of states to states that leads to well-defined probabilities for all observables at the output.
- This turns out to be remarkably subtle.
- Firstly, we will allow the output Hilbert space $\mathcal{H}_B$ to be different from the input Hilbert space $\mathcal{H}_A$

  We may add a new subsystem or discard part of the system during the dynamics.
- So we need some sort of map $\mathcal{E}_{B|A}$ from $\mathcal{L}(\mathcal{H}_A)$ to $\mathcal{L}(\mathcal{H}_B)$ that maps density operators to density operators.

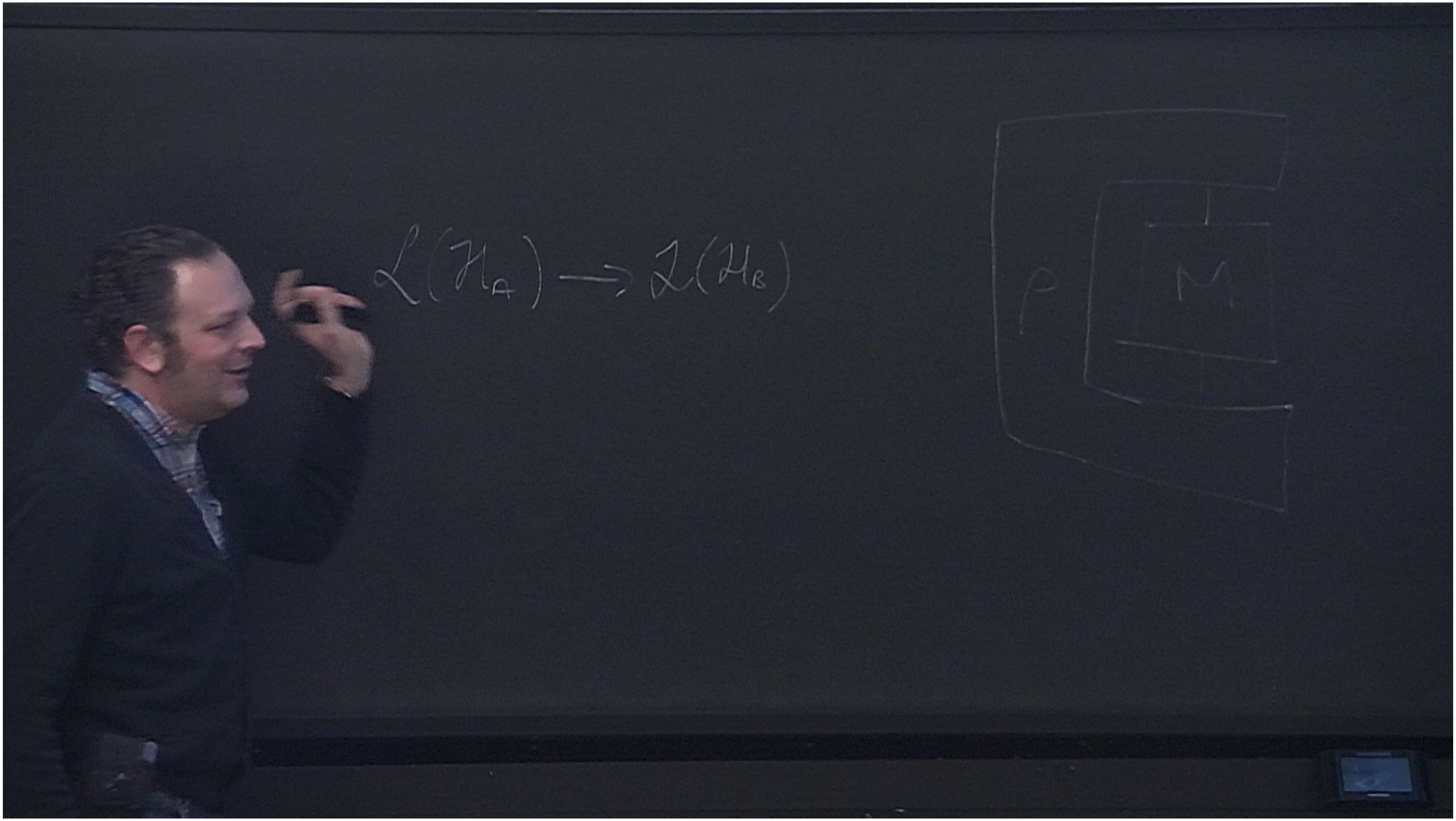
The View from the Smaller Church

- We will demand that $\mathcal{E}_{B|A}$ is linear. Why?

  If we prepare ρ_A with probability p
  or σ_A with probability $(1-p)$

  Then $\mathcal{E}_{B|A}(p\rho_A + (1-p)\sigma_A) = p\,\mathcal{E}_{B|A}(\rho_A) + (1-p)\,\mathcal{E}_{B|A}(\sigma_A)$
- Strictly speaking, this only means that $\mathcal{E}_{B|A}$ has to be affine, i.e. acts linearly on positive linear combinations.
- But you can always extend an affine map to a linear one just by defining $\mathcal{E}_{B|A}(-\rho_A) = -\mathcal{E}_{B|A}(\rho_A)$
- So, we will have a linear operator from linear operators to linear operators

  $\mathcal{E}_{B|A} \in \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B))$ Sometimes called a superoperator.


$\mathcal{L}(\mathcal{H}_A) \longrightarrow \mathcal{L}(\mathcal{H}_B)$
ρ
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The View from the Smaller Space

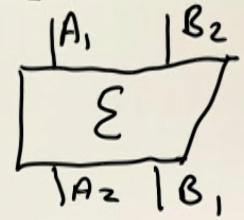
⊙ Now comes the fun part:

$$\mathcal{L}(\mathcal{H}_A) = \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^{\dagger} \qquad \mathcal{L}(\mathcal{H}_B) = \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger}$$

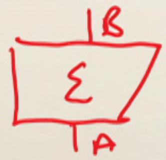
These are both $\mathcal{H}_A$ but it helps to keep track of which is the input and which is the output

$$\therefore\ \mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)) = \mathcal{L}(\mathcal{H}_{A_2} \otimes \mathcal{H}_{A_1}^{\dagger} \to \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger}) = \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_1}^{\dagger} \otimes \mathcal{H}_{A_2}^{\dagger} \otimes \mathcal{H}_{A_1}$$

$$\therefore\ \mathcal{E}_{B|A} = \sum_{jklm} \mathcal{E}_{kl}^{jm}\, |j\rangle_{A_1} \otimes {}_{A_2}\langle k| \otimes {}_{B_1}\langle l| \otimes |m\rangle_{B_2} \qquad \mathcal{E}_{k_{A_2} l_{B_1}}^{j_{A_1} m_{B_2}}$$

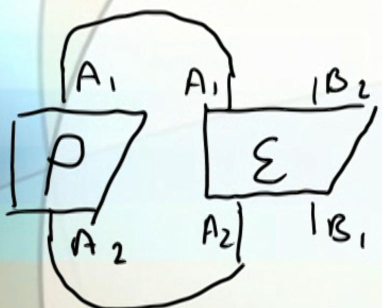


$$\mathcal{E}_{B|A} = \sum_{(jk)(lm)} \mathcal{E}_{(lm)}^{(jk)}\, |jk\rangle_A \otimes {}_B\langle lm|$$



The View from the Smaller Church

- The action of $\mathcal{E}_{B|A}$ on a density operator ρ_A is going to be



$$\mathcal{E}_{B|A}(\rho_A) = \sum_{jklm} \mathcal{E}^{jm}_{kl}\, {}_{A_2}\langle k|\rho_A|j\rangle_{A_1} \otimes |m\rangle_{B_2}\,{}_{B_1}\langle l|$$
$$= \mathcal{E}^{j_{A_1} m_{B_2}}_{k_{A_2} l_{B_1}}\, \rho^{k_{A_2}}_{j_{A_1}}$$

- The space $\mathcal{L}(\mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)) = \mathcal{H}_{A_1} \otimes \mathcal{H}^\dagger_{A_2} \otimes \mathcal{H}^\dagger_{B_1} \otimes \mathcal{H}_{B_2}$ can be decomposed in a different way, which will end up giving us the operator-sum decomposition.

$$\left(\mathcal{H}_{A_1} \otimes \mathcal{H}^\dagger_{B_1}\right) \otimes \left(\mathcal{H}_{B_2} \otimes \mathcal{H}^\dagger_{A_2}\right) = \mathcal{L}(\mathcal{H}_{B_1} \to \mathcal{H}_{A_1}) \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2})$$
$$= \mathcal{L}(\mathcal{H}_{A_1} \to \mathcal{H}_{B_1})^\dagger \otimes \mathcal{L}(\mathcal{H}_{A_2} \to \mathcal{H}_{B_2})$$
$$= \mathcal{L}(\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B))$$

The View from the Smaller Church

- On this decomposition, we would write the action of $\mathcal{E}_{B|A}$ as

$$\mathcal{E}_{B|A}(\rho_A) = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, |m\rangle_{B_2} \, {}_{A_2}\langle k| \rho_A |j\rangle_{A_1} \, {}_{B_1}\langle l|$$

- If we view $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ as a space of kets $|jk\rangle_{AB} = |j\rangle_B \, {}_A\langle k|$
and $\mathcal{L}(\mathcal{H}_B \to \mathcal{H}_A)$ as a space of bras $\langle jk|_{AB} = |k\rangle_A \, {}_B\langle j|$

Then $\mathcal{E}_{B|A}$ has the form

$$\mathcal{E}_{B|A} = \sum_{jklm} \mathcal{E}^{jm}_{kl} \, |mk\rangle_{AB} \langle lj|$$

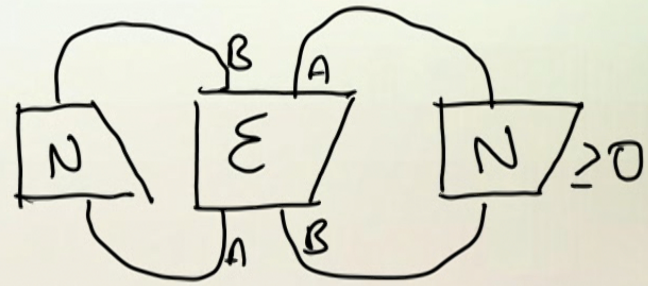
The View from the Smaller Church

① If I can show that $\mathcal{E}_{B|A}$ is a positive operator on this space

i.e. $\forall\ N \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$

$$\langle N|\mathcal{E}_{B|A}|N\rangle = \sum_{jklm} \mathcal{E}^{jm}_{kl} \langle N|mk\rangle\langle lj|N\rangle \geq 0$$

which means $\quad N^{\dagger\, k_A}_{m_B}\, \mathcal{E}^{j_A m_B}_{k_A l_B} N^{l_B}_{j_A} \geq 0$



Then we will have an eigenoperator decomposition with positive eigenvalues

$$\mathcal{E}_{B|A} = \sum_j \lambda_j |R^{(j)}\rangle\langle R^{(j)}| = \sum_j |M^{(j)}\rangle\langle M^{(j)}| \quad \text{where } M^{(j)} = \sqrt{\lambda_j}\, R^{(j)}$$

The View from the Smaller Church

- Now if $\mathcal{E}_{B|A} = \sum_j |M^{(j)}\rangle\langle M^{(j)}|$

  then $\mathcal{E}_{B|A}(\rho_A) = \sum_j M^{(j)} \rho_A M^{(j)\dagger}$ so we'll have an operator sum decomposition

- We'll prove that the required positivity holds soon.
- However, if the operator sum decomposition holds, we can prove that

$$\sum_j M^{(j)\dagger} M^{(j)} = I$$

- If $\mathcal{E}_{B|A}(\rho_A)$ is a density operator then $\mathrm{Tr}(\mathcal{E}_{B|A}(\rho_A)) = \mathrm{Tr}(\rho_A) = 1$

  for any input density operator ρ_A, i.e. $\mathcal{E}_{B|A}$ is trace preserving

$$\mathrm{Tr}_B\left(\mathcal{E}_{B|A}(\rho_A)\right)$$

$$= \mathrm{Tr}_B\left(\sum_j M^{(j)} \rho M^{(j)\dagger}\right)$$

$$= \mathrm{Tr}\left(\left(\sum_j M^{(j)\dagger} M^{(j)}\right)\rho\right) = \mathrm{Tr}(\rho)$$