Title: PSI 2018/2019 - Foundations of Quantum Mechanics - Lecture 4

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Abstract:

Quantum Foundations
Lecture 4

PSI Review Class: 10th January 2019

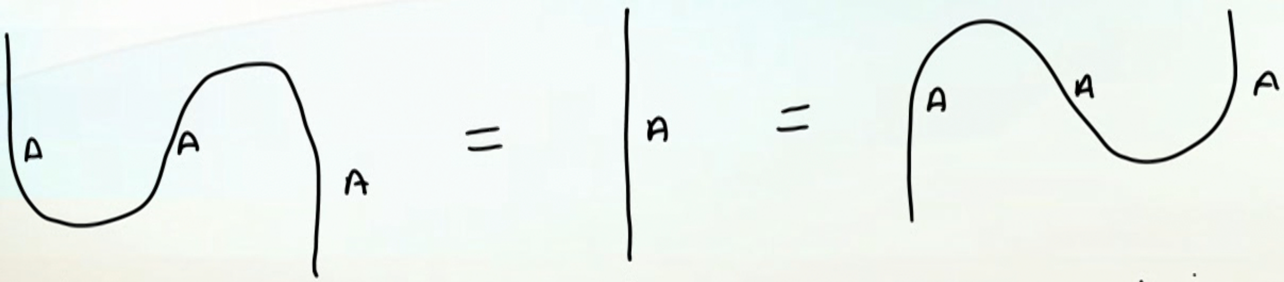
Instructor: Matthew Leifer

leifer@chapman.edu

PI Office: 353

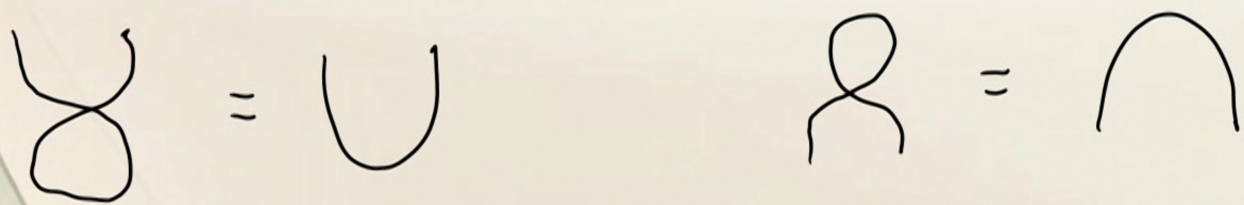
The Yanking Axioms

- The various δ tensors satisfy the following properties



$$\delta^{j_A k_A}\,\delta_{k_A m_A} = \delta^{j_A}_{m_A} = \delta_{m_A k_A}\,\delta^{k_A j_A}$$

The yanking axioms allow us to prove lots of things using just diagrams.

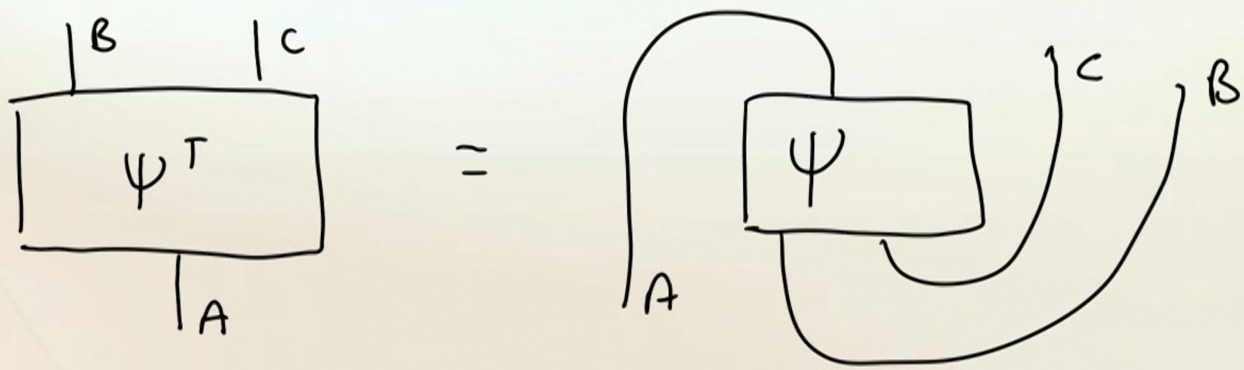
- 

Just expresses the fact that order of indices is unimportant in abstract index notation.

4.7) Transpose, Conjugate, Duals, and Trace

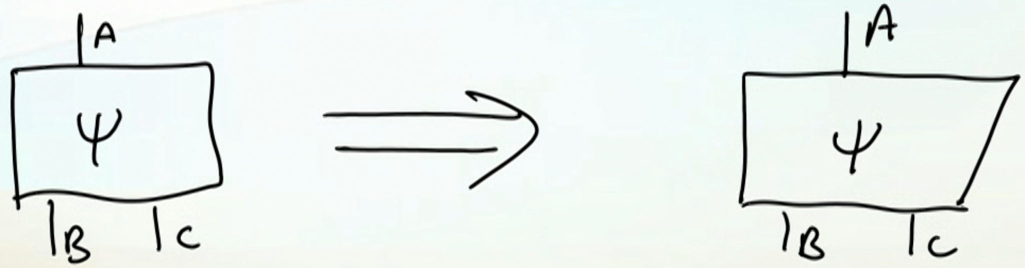
- The transpose is defined as

$$\Psi^{T\,k_B l_C}{}_{j_A} = \Psi^{k_B l_C}{}_{\dot{j}_A} = \delta_{j_A m_A}\,\delta^{k_B n_B}\,\delta^{l_C r_C}\,\Psi^{m_A}{}_{n_B l_C}$$

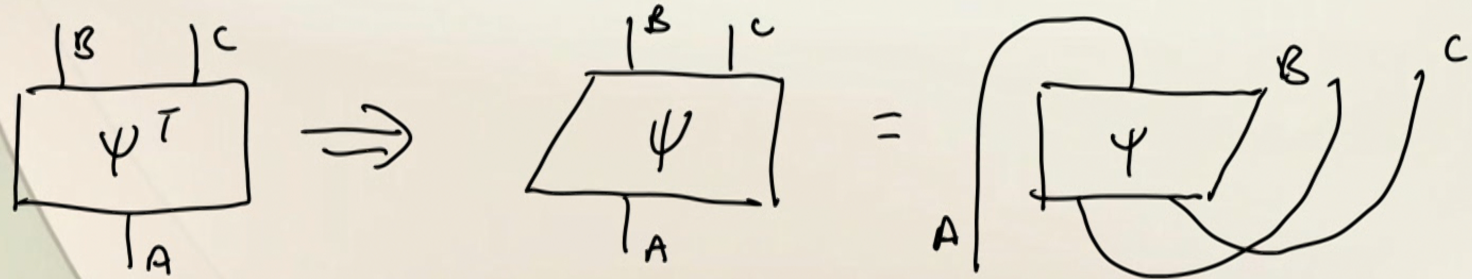


A Bit of Diagrammatic Trickery

- We can make more intuitive diagrams if we introduce a bit of asymmetry to our boxes.

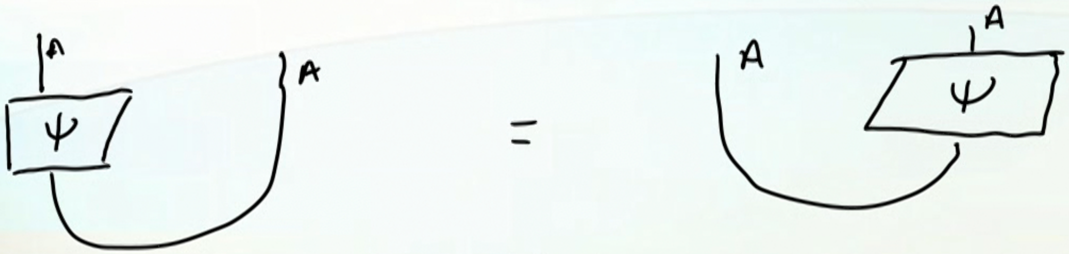


- Then we can represent transpose by 180° rotation

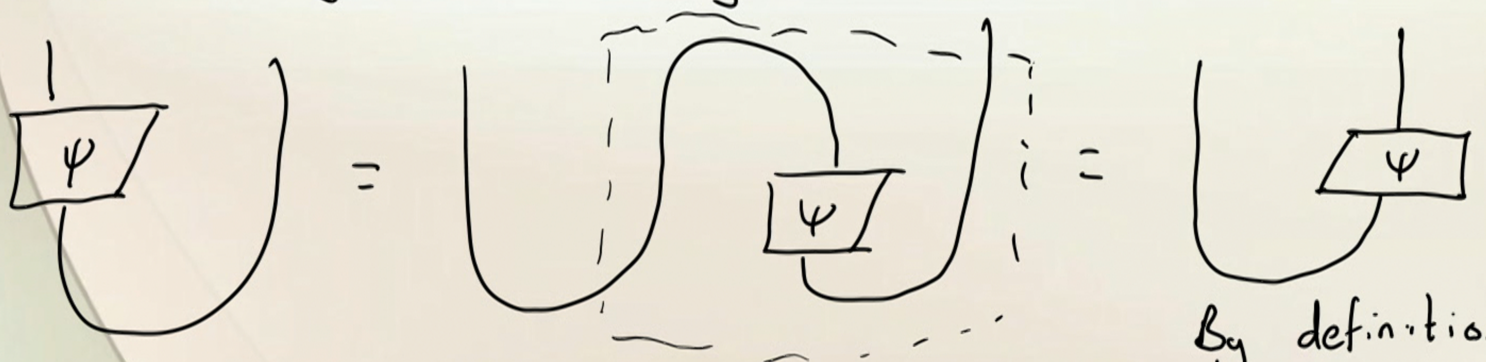


Fun With Diagrams

- Now let's actually prove something with diagrams



- Proof: Using the Yanking axioms



By definition of transpose

The Yanking Axioms

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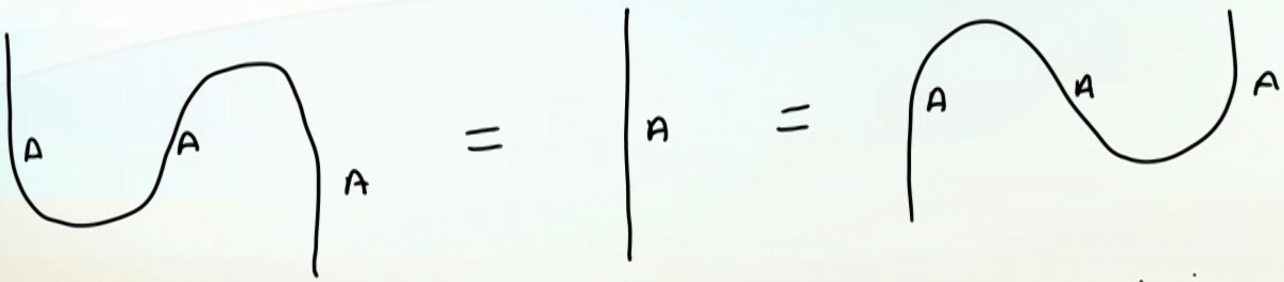
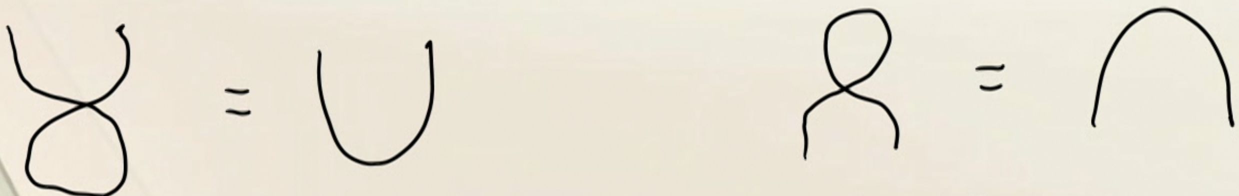


$$\delta^{j_A k_A}\delta_{k_A m_A} = \delta^{j_A}_{m_A} = \delta_{m_A k_A}\delta^{k_A j_A}$$

The yanking axioms allow us to prove lots of things using just diagrams.

- 

Just expresses the fact that order of indices is unimportant in abstract index notation.

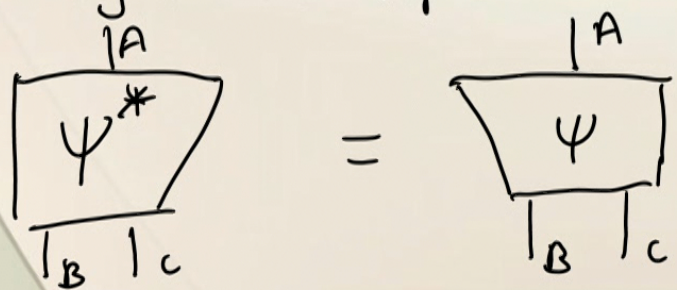
Conjugate

- The conjugate of a tensor is just the tensor you obtain by taking the complex conjugate of all of its components

$$\sum_{jkl} \psi^{j}_{kl} |j\rangle_A \otimes {}_B\langle k| \otimes {}_C\langle l| \xleftrightarrow{\;*\;} \sum_{jkl} (\psi^{j}_{kl})^* |j\rangle_A \otimes {}_B\langle k| \otimes {}_C\langle l|$$

$$(\psi^*)^{j_A}_{k_B l_C} = (\psi^{j_A}_{k_B l_C})^*$$

- In a diagram we represent it by reflecting in a vertical axis



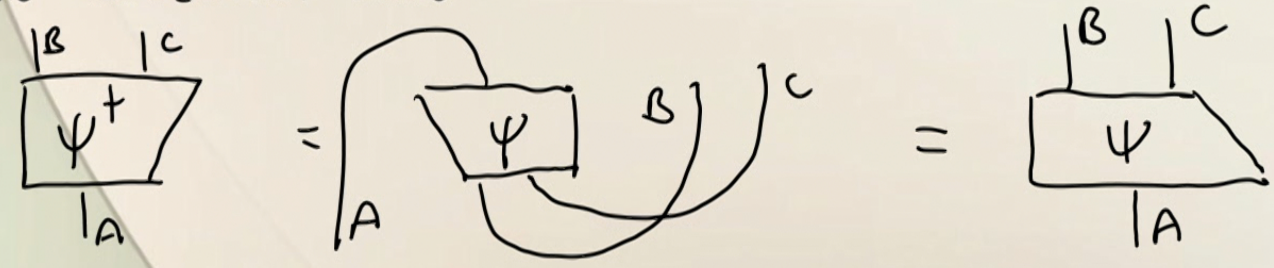
Dual/Adjoint

- The <u>dual</u> or <u>adjoint</u> is defined as taking the complex conjugate, followed by the transpose, or vice versa.

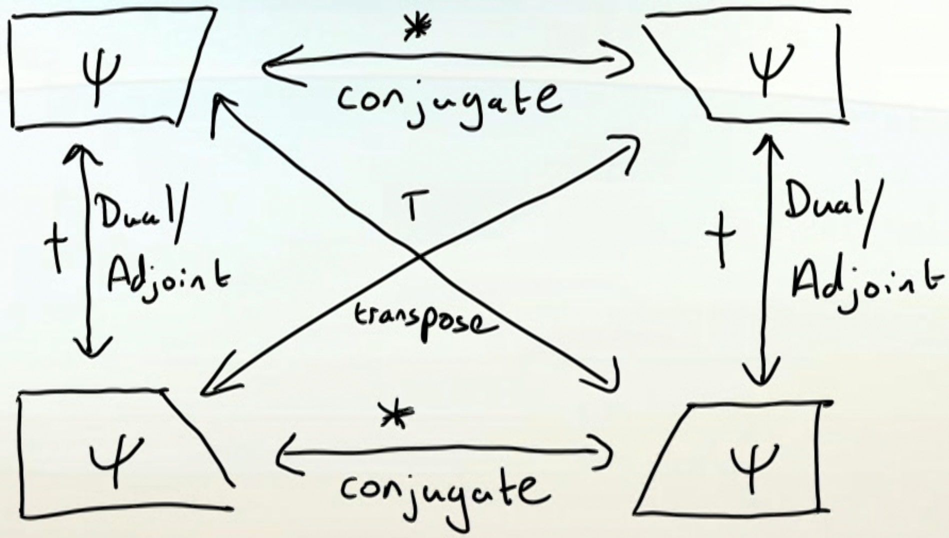
$$\sum_{jkl} \psi^{j}_{kl} \, |j\rangle_A \otimes {}_B\langle k| \otimes {}_C\langle l| \xleftrightarrow{\dagger} \sum_{jkl} \psi^{*\,kl}_{j} \, {}_A\langle j| \otimes |k\rangle_B \otimes |l\rangle_C$$

$$(\psi^\dagger)^{k_B l_C}_{j_A} = \delta_{j_A m_A} \delta^{k_B n_B} \delta^{l_C r_C} (\psi^*)^{m_A}_{n_B r_C}$$

- For obvious reasons, in diagrams it is represented by a reflection in the horizontal axis



Summary



- Diagrams with the yanking axioms, conjugate, transpose, and adjoint are known as *string diagrams*.
- Everything you can do by imagining the wires are made of string, rotating boxes, and sliding them along the strings corresponds to a valid mathematical proof.

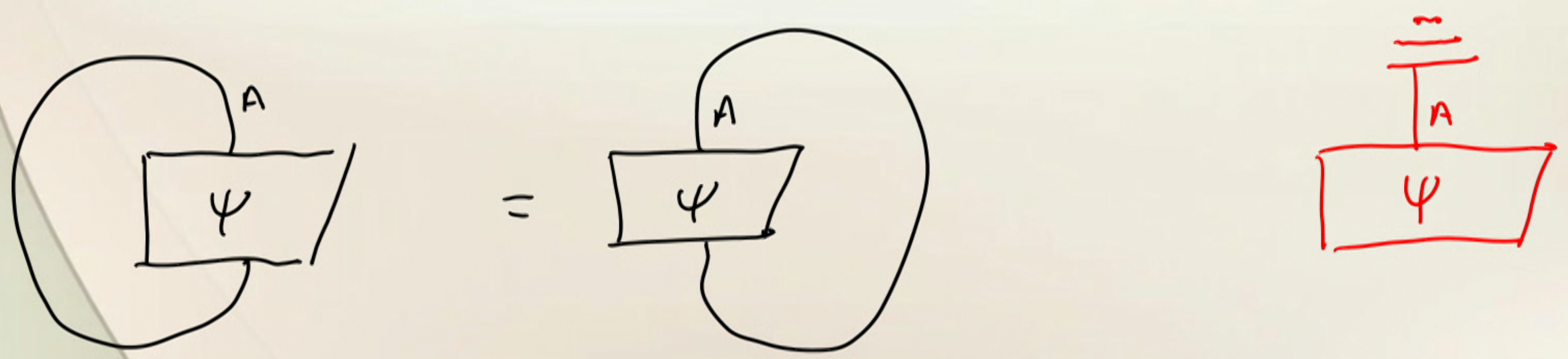
4.8) Trace and Partial Trace

- The <u>trace</u> of an operator $\sum_{jk} \psi^j_k |j\rangle_A \otimes {}_A\langle k|$ is defined as

$$\sum_j \psi^j_j$$

- In abstract index notation $\psi^{j_A}_{j_A} = \delta_{j_A m_A} \psi^{m_A}_{k_A} \delta^{j_A k_A} = \delta_{m_A k_A} \psi^{m_A}_{j_A} \delta^{j_A k_A}$

In red diagrams, we will use

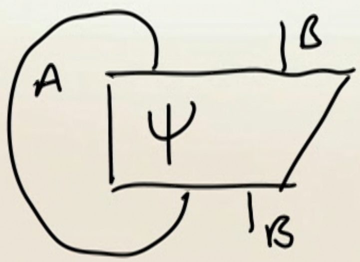


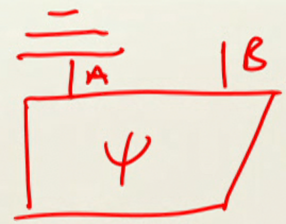
Partial Trace

- We can obviously contract any indices that have the same system label. This is called a partial trace in quantum theory

e.g. $$\mathrm{Tr}_A\left(\sum_{jklm} \psi^{jl}_{km}\, |j\rangle_A \otimes {}_A\langle k| \otimes |l\rangle_B \otimes {}_B\langle m|\right)$$

$$= \sum_{jlm} \psi^{jl}_{jm}\, |l\rangle_B \otimes {}_B\langle m|$$

$$\psi^{j_A l_B}_{j_A m_B} =$$ 

$=$ 

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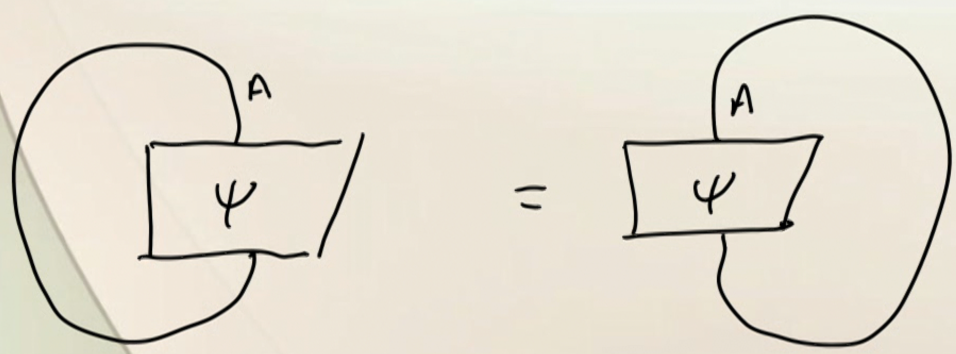

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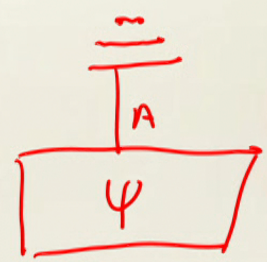
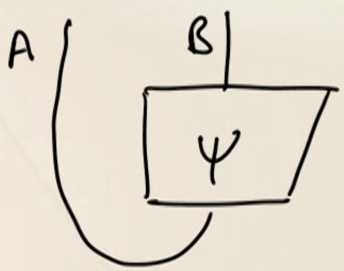


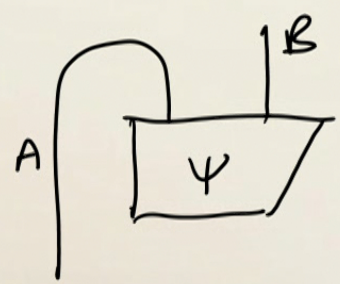
4.9) Vector Operator Correspondence

○ Raising and lowering indices induces a correspondence between operators in $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B) = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B$ and vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$

$$\sum_{jk} \psi_j^{\ k} |k\rangle_B \otimes {}_A\langle j| \longleftrightarrow \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B$$

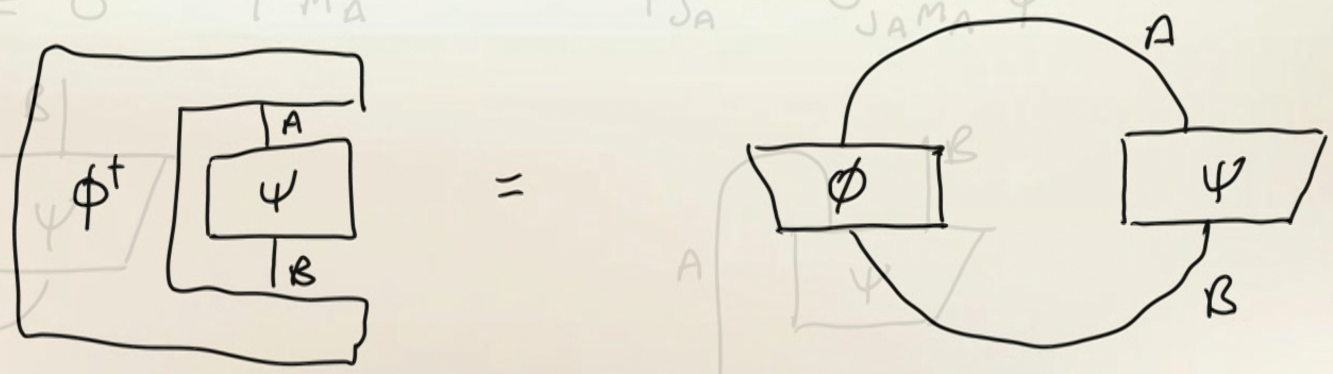
$$\psi^{j_A k_B} = \delta^{j_A m_A} \psi_{m_A}^{\ \ k_B} \qquad \psi_{j_A}^{\ \ k_B} = \delta_{j_A m_A} \psi^{j_A j_B}$$



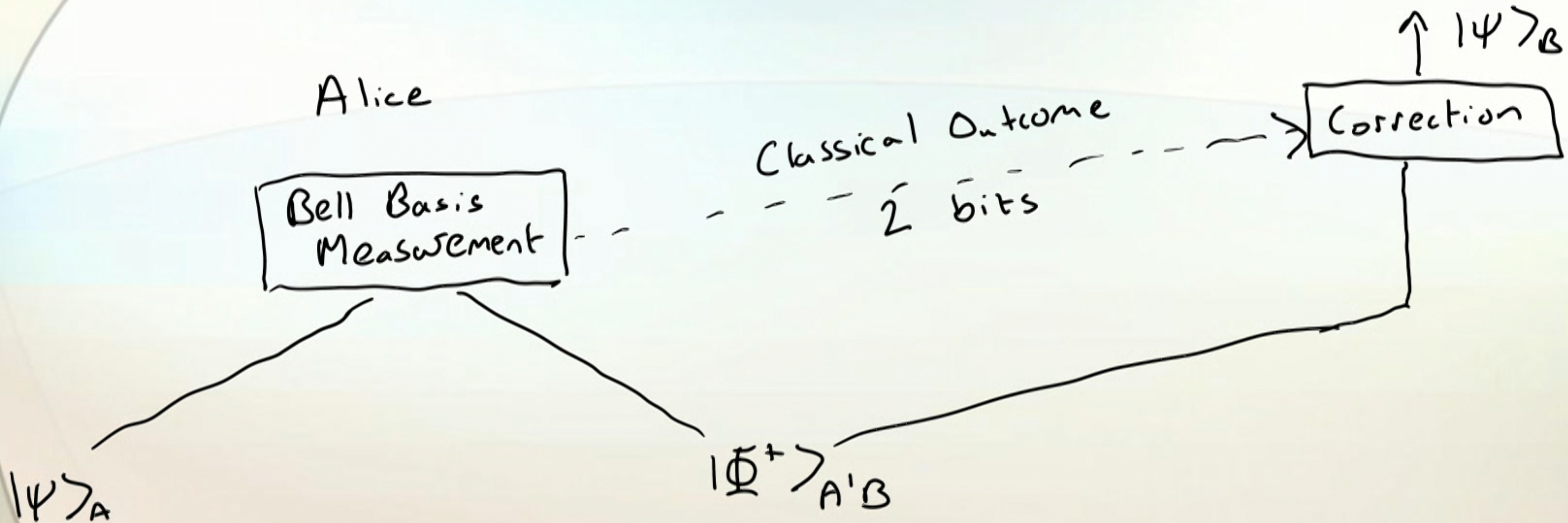


Getting Rid of Awkward Boxes

- For vectors in mixed tensor products of bra and ket spaces e.g. $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$, we had to use awkwardly shaped boxes to express the inner product as a diagram
- Using transposes and conjugates, we can now get rid of the awkward boxes



4.10) Quantum Teleportation



- If Alice and Bob can do it if they pre-share two qubits in the entangled state

$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

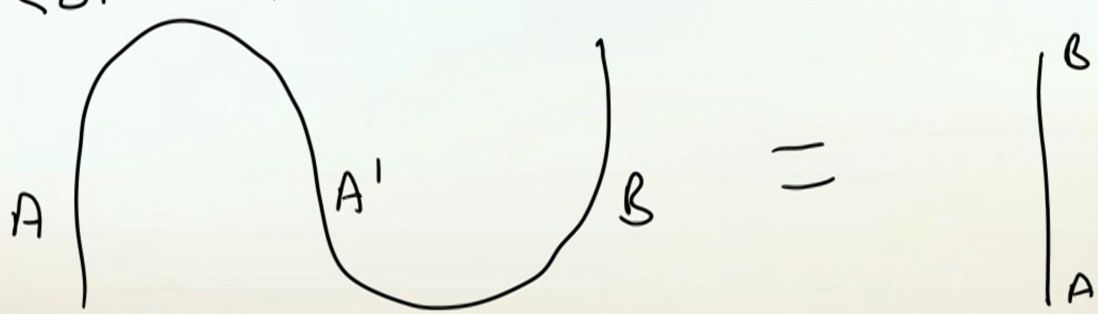
Quantum Teleportation Protocol

1. Alice and Bob share two qubits in the entangled state
$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$
2. Alice performs a joint measurement of her system A in the unknown state $|\psi\rangle_A$ and A' in the Bell basis
$$\left|\Phi^\pm\right\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AA'} \pm |11\rangle_{AA'}\right)$$
$$\left|\Psi^\pm\right\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|01\rangle_{AA'} \pm |10\rangle_{AA'}\right)$$
3. Alice communicates the outcome to Bob. There are 4 possible outcomes, so 2 bits of communication.
4. Depending on the outcome, Bob applies one of four unitary operations to his qubit B. This transforms his system to $|\psi\rangle_B$.

Proving it Works Using Diagrams

- Recall the Yanking axiom

$\langle \delta | = \langle 00 | + \langle 11 |$



$|\delta\rangle = |00\rangle + |11\rangle$

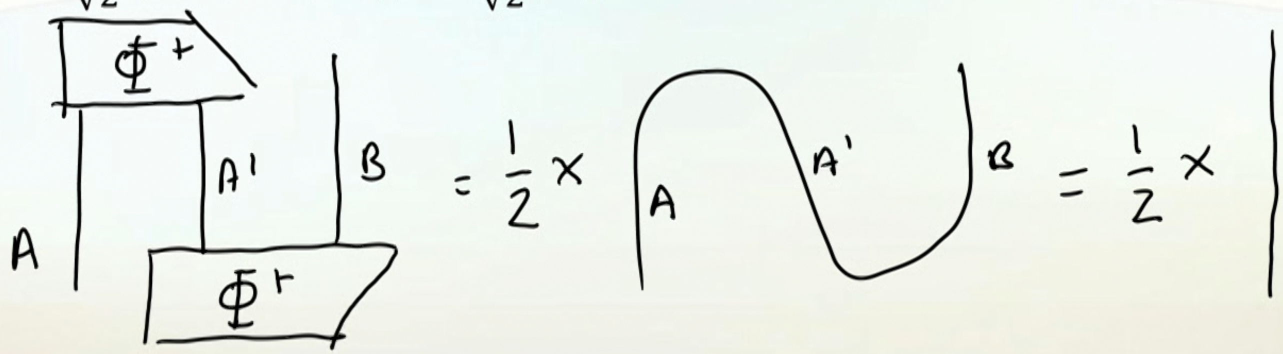
Identity $I = |0\rangle\langle 0| + |1\rangle\langle 1|$

So this axiom says

$${}_{AA'}\langle \delta | \delta \rangle_{A'B} = I_{A \to B}$$

Proving it Works Using Diagrams

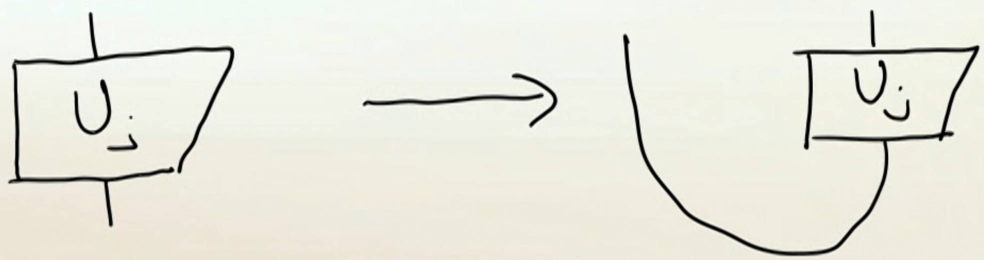
- Now $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}|\delta\rangle$, so this axiom tells us that:



- In other words, if you prepare $|\Phi^+\rangle_{AB}$ and get the outcome $|\Phi^+\rangle_{AA'}$ in a measurement, you'll get an identity channel from A to B, up to a factor of $\frac{1}{2}$.
- The factor $\frac{1}{2}$ is just a scalar in front of the output state. It's modulus squared is the probability of this outcome happening, which is $\frac{1}{4}$.

Proving it Works Using Diagrams

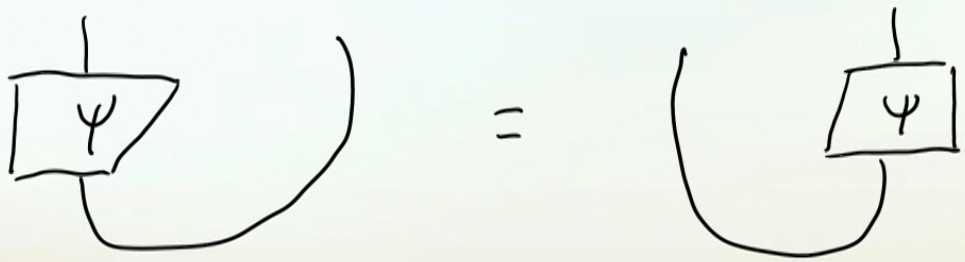
- What about the other 3 outcomes. Well, the four unitary matrices U_0, U_1, U_3 are orthogonal according to the Hilbert-Schmidt inner product
$$\mathrm{Tr}\left(U_j^T U_k\right) = 2\delta_{jk}$$
- We can convert them into states using the vector-operator correspondence:



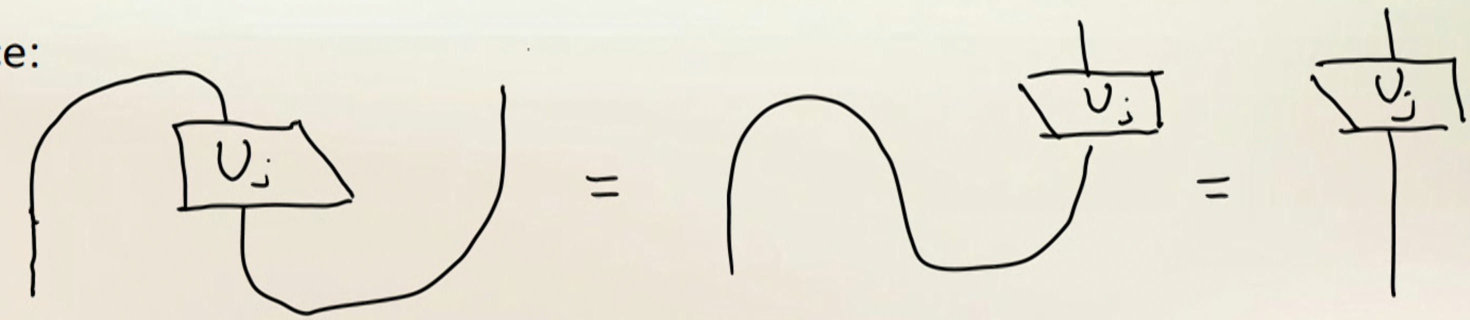
- This correspondence preserves inner products, so we will get an orthogonal basis.
- Would you believe that these are just the four states in the Bell basis (up to normalization)?

Proving it Works Using Diagrams

- So, let's see what happens when we measure in this basis. We already proved:

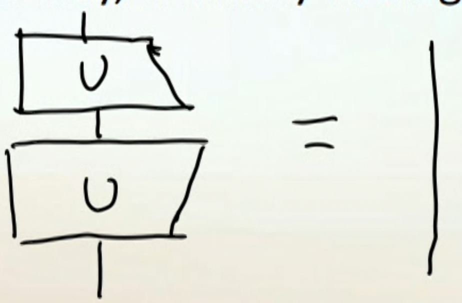


- Hence:

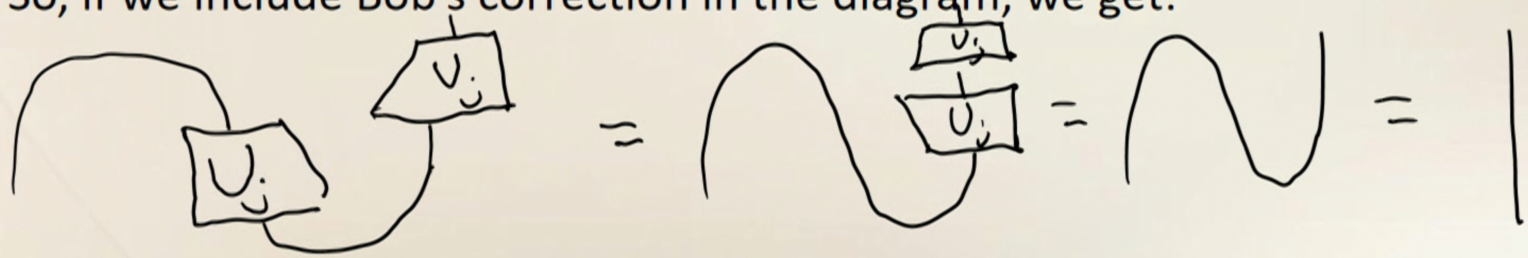


Proving it Works Using Diagrams

- So this shows that the total effect is to perform the transformation $U_j^* = U_j$ to the system, which Bob can undo with $\left(U_j^*\right)^\dagger = U_j^T$.
- By the way, unitarity in diagrams is expressed as



- So, if we include Bob's correction in the diagram, we get:



Proving it Works Using Diagrams

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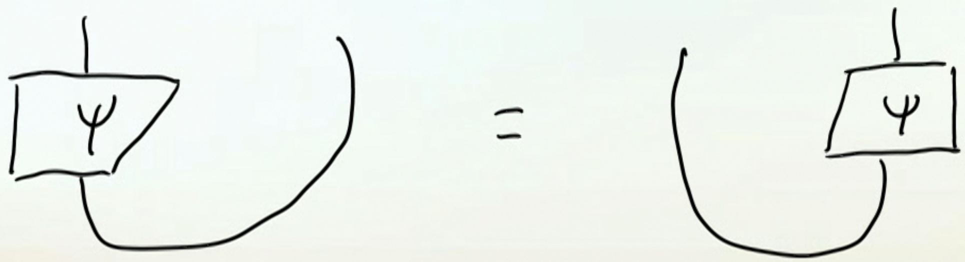

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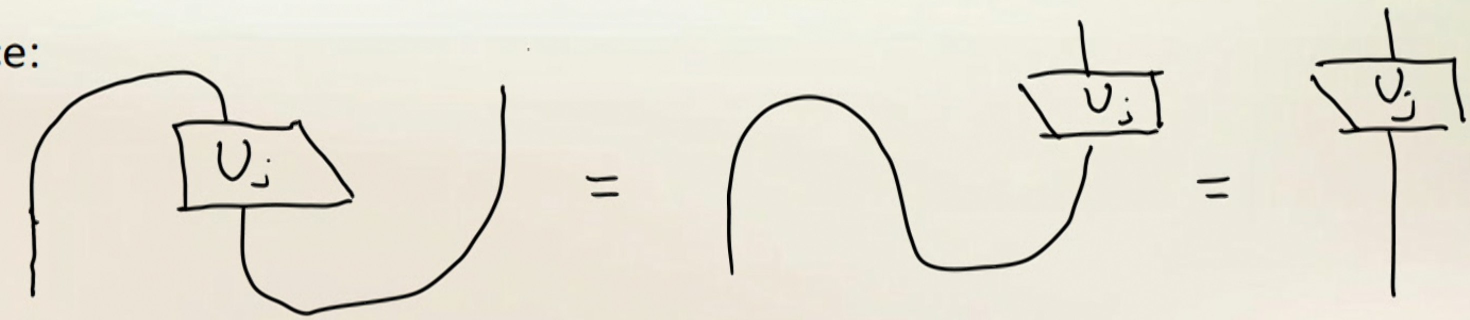


5) The Generalized Formalism

1) The Two Churches of Quantum Theory
2) The Hilbert Space of Hermitian Matrices
3) Density Operators
4) Positive Operator Valued Measures (POVMs)
5) Completely Positive Trace Preserving (CPT) Maps
6) The Lindblad Equation

5.1) The Two Churches of Quantum Theory

- The Church of The Larger Hilbert Space:
  - Quantum theory is a dynamical theory, akin to a classical field theory, but with a weirder object called the wavefunction in place of a classical field.
  - All is to be derived from a quantum state (of the universe in principle) evolving unitarily according to the Schrödinger equation.
  - Today, we will allow projective measurements as well, but see lecture on Everett/many-worlds for how to derive them.
- The Church of The Smaller Hilbert Space:
  - Something strange has happened to our physical variables: they have become noncommutative.
  - Quantum theory is the only consistent probability theory for such variables.
- In this section, we will give both churches views on each construction.

5.2) The Hilbert Space of Hermitian Matrices

- As it is a Hilbert space $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ must have multiple orthonormal bases.
- The <u>standard basis</u> that we have been using is just $|jk\rangle_{AB} = |j\rangle_B \otimes {}_A\langle k|$

$$\langle jk|lm\rangle_{AB} = \left(|j\rangle_B \otimes {}_A\langle k|\right)^\dagger \left(|l\rangle_B \otimes {}_A\langle m|\right) = \langle j|l\rangle_B \langle m|k\rangle_A = \delta_j^l \, \delta_m^k$$

- Clearly, $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ has dimension $d_A \times d_B$
- But there are other bases, e.g. consider $\mathcal{L}(\mathcal{H}_A)$ with $\mathcal{H}_A = \mathbb{C}^2$ and let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $S_j = \frac{1}{\sqrt{2}} \sigma_j$ is an orthonormal basis as $\mathrm{Tr}(S_j^\dagger S_k) = \delta_{jk}$

- Consequently, every 2×2 operator can be written as $M = \frac{1}{2} \sum_j m_j \sigma_j$ with $m_j = \mathrm{Tr}(\sigma_j M)$

The Space of Hermitian Matrices

- If input and output spaces are the same, we can have Hermitian matrices

$$M^\dagger = M \quad \text{with} \quad M = \sum_{jk} M_{jk} |j\rangle_A \otimes {}_A\langle k| \qquad M^\dagger = \sum_{jk} M^*_{jk} |k\rangle_A \otimes {}_A\langle j|$$

- The set of Hermitian matrices on $\mathcal{H}_A$, denoted $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space over $\mathbb{R}$.

i.e. if $M^\dagger = M$ $N^\dagger = N$ then $(\alpha M + \beta N)^\dagger = \alpha M + \beta N$

so long as $\alpha, \beta \in \mathbb{R}$

and $\mathrm{Tr}(M^\dagger N) \in \mathbb{R}$

- The dimension of this space is d^2

d real parameters + $(d-1)d$ real parameters

$= d^2$ real parameters

Hermitian Bases

- The matrices $|j\rangle\langle k|$ are not Hermitian, but there must be a basis of d^2 Hermitian matrices.
- We already saw $\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

  which is a Hermitian basis for operators on $\mathbb{C}^2$
- In general, you can take

$$\left.\begin{array}{l}\frac{1}{\sqrt{2}}\left(|j\rangle\langle k| + |k\rangle\langle j|\right)\\ \frac{1}{\sqrt{2}}\left(i|j\rangle\langle k| - i|k\rangle\langle j|\right)\end{array}\right\}$$

These are not all different and not all nonzero (e.g. take $j=k$)

Proper counting gives d^2 orthonormal matrices.

Hermitian Matrices are Self-Dual

- Because $\mathcal{S}(\mathcal{H}_A) \subseteq \mathcal{L}(\mathcal{H}_A) \cong \mathcal{H}_A \otimes \mathcal{H}_A^\dagger$, the dual $\mathcal{S}(\mathcal{H}_A)^\dagger$ won't be all of $\mathcal{L}(\mathcal{H}_A)$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a real Hilbert space, $\mathcal{S}(\mathcal{H}_A)^\dagger$ consists of linear functionals from $\mathcal{S}(\mathcal{H}_A)$ to $\mathbb{R}$, not $\mathbb{C}$
- Because $\mathcal{S}(\mathcal{H}_A)$ is a Hilbert space, the inner product still induces an isomorphism $\mathcal{S}(\mathcal{H}_A) \cong \mathcal{S}(\mathcal{H}_A)^\dagger$

  Write $M \in \mathcal{S}(\mathcal{H}_A)$ in terms of a self-adjoint basis

  $$M = \sum_j m_j N_j$$

  real coefficients

  Then $M^\dagger = \sum_j m_j N_j^\dagger = \sum_j m_j N_j$ so a vector in $\mathcal{S}(\mathcal{H}_A)$ is its own dual vector

5.3) Density Operators

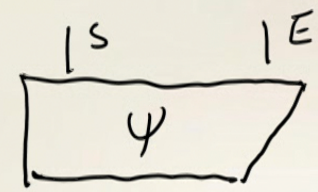
- According to the larger church, the universe always has a pure state vector $|\psi\rangle$.
- If any other mathematical object is used for a quantum state, it must be because we are looking at a subsystem.
- State space is $\mathcal{H}_S \otimes \mathcal{H}_E$

System we are interested in

Environment

and we have $|\psi\rangle_{SE} = \sum_{jk} \psi^{jk} |j\rangle_S \otimes |k\rangle_E$ or $\psi^{j_S k_E}$

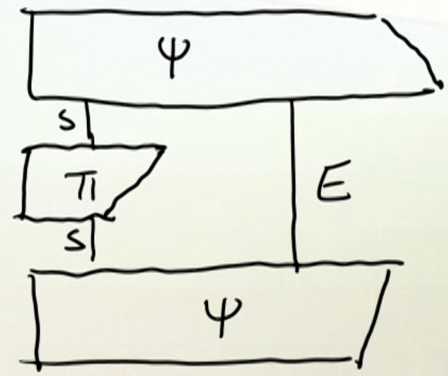
or

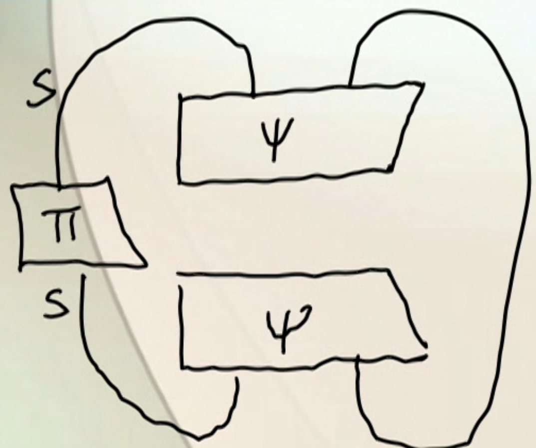


The View from the Larger Church

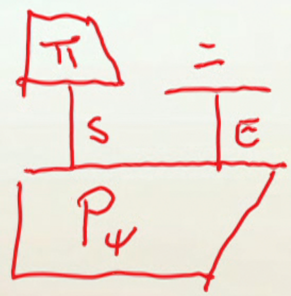
- Suppose we make a projective measurement on system S alone. The probability of getting outcome corresponding to projector Π is

$$ {}_{SE}\langle \psi | \Pi_S \otimes I_E | \psi \rangle_{SE} = \psi^{\dagger}_{j_S k_E} \, \Pi^{j_S}_{l_S} \, \psi^{l_S k_E} = $$





$E =$

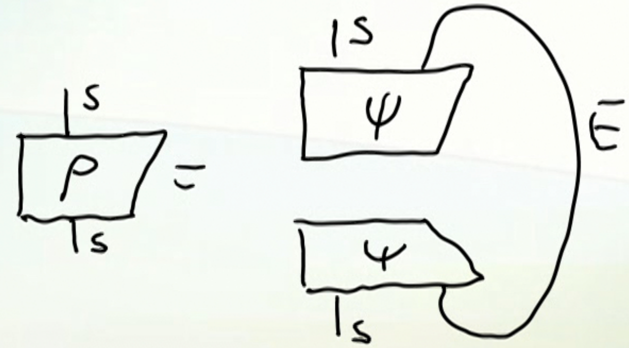


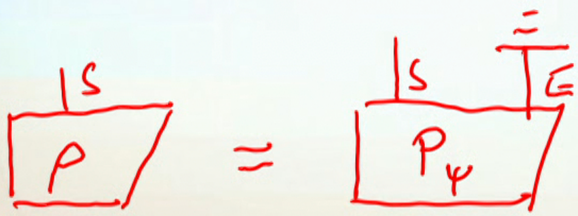
where $P_\psi = |\psi\rangle\langle\psi|$

The View From The Larger Church

⊙ If we define the object

$\rho_S = \mathrm{Tr}_{\bar{E}}(|\psi\rangle\langle\psi|)$ $\quad \rho^{i_S}_{j_S} = \psi^{i_S k_E}\,\psi^\dagger_{j_S k_E}$





then the probability is $\mathrm{Prob}(\Pi) = \mathrm{Tr}(\Pi_S \rho_S) = \Pi^{k_S}_{j_S}\,\rho^{j_S}_{k_S} =$

⊙ ρ lives in the space $\mathcal{H}_S \otimes \mathcal{H}_S^\dagger \equiv \mathcal{L}(\mathcal{H}_S) \equiv \mathcal{L}(\mathcal{H}_S)^\dagger$

so it is both an operator and a duperator

⊙ We normally call it a <u>density operator</u> (although we use it as a duperator)

An aside on positive operators

- A positive operator $M \in \mathcal{L}(\mathcal{H}_A)$ is an operator that satisfies

$$ {}_A\langle\psi|M|\psi\rangle_A \geq 0 \quad \text{for all} \quad |\psi\rangle_A \in \mathcal{H} $$

- Theorem: An operator is positive iff it is self-adjoint and has positive (≥ 0) eigenvalues

Proof:

Let $|\psi\rangle = |\phi\rangle + i|\chi\rangle$

$$ \Rightarrow \langle\phi|M|\phi\rangle - i\langle\chi|M|\phi\rangle + i\langle\phi|M|\chi\rangle + \langle\chi|M|\chi\rangle \geq 0 \quad (1) $$

Let $|\psi\rangle = |\phi\rangle - i|\chi\rangle$

$$ \langle\phi|M|\phi\rangle + i\langle\chi|M|\phi\rangle - i\langle\phi|M|\chi\rangle + \langle\chi|M|\chi\rangle \geq 0 \quad (2) $$

$\frac{(2)-(1)}{2i}$: $\langle\chi|M|\phi\rangle = \langle\phi|M|\chi\rangle$ which is the definition of self-adjoint.

An aside on positive operators

A self adjoint operator has real eigenvalues

Let $M|\phi\rangle = \lambda|\phi\rangle$

By positivity $\langle\phi|M|\phi\rangle \geq 0 \Leftrightarrow \lambda \geq 0$

Conversely, if $M = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ with $\lambda_j \geq 0$ then

$$\langle\psi|M|\psi\rangle = \sum_j \lambda_j \langle\psi|\phi_j\rangle\langle\phi_j|\psi\rangle = \sum_j \lambda_j |\langle\phi_j|\psi\rangle|^2 \geq 0.$$

⊙ Theorem: An operator $M \in \mathcal{L}(\mathcal{H}_A)$ is positive iff it can be written

as $M = N^\dagger N$ where $N \in \mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$

Proof: If $M = N^\dagger N$ then $\langle\psi|M|\psi\rangle = \langle\psi|N^\dagger N|\psi\rangle = \| N|\psi\rangle \|^2 \geq 0$

Conversely, if $M = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$ then let $M^{1/2} = \sum_j \sqrt{\lambda_j}\, |\psi_j\rangle\langle\psi_j|$

and then $M = N^\dagger N$ for $N = M^{1/2}$

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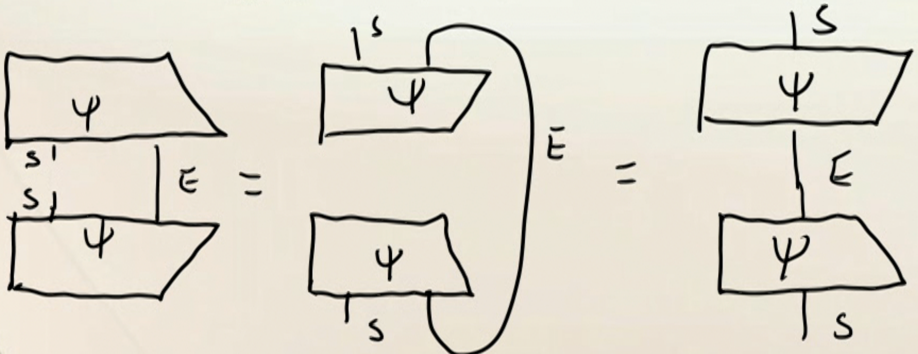
Properties of the density operator

- Given that $\psi^{j_s k_E}$ is normalized $\psi^{j_s k_E}\psi^{\dagger}_{j_s k_E} = 1$

  $\therefore \mathrm{Tr}(\rho) = \rho^{j_s}_{j_s} = \psi^{j_s k_E}\psi^{\dagger}_{j_s k_E} = 1$

- We can also write $\rho^{k_s}_{j_s} = \psi^{k_s}{}_{l_E}\,\psi^{\dagger\, l_E}{}_{j_s}$

Proof: $\psi^{k_s l_E}\psi^{\dagger}_{j_s l_E} = \psi^{k_s}{}_{m_E}\,\delta^{m_E l_E}\,\delta_{n_E l_E}\,\psi^{\dagger\, n_E}{}_{j_s} = \psi^{k_s}{}_{m_E}\,\delta^{m_E}_{n_E}\,\psi^{\dagger\, n_E}{}_{j_s} = \psi^{k_s}{}_{n_E}\,\psi^{\dagger\, n_E}{}_{j_s}$

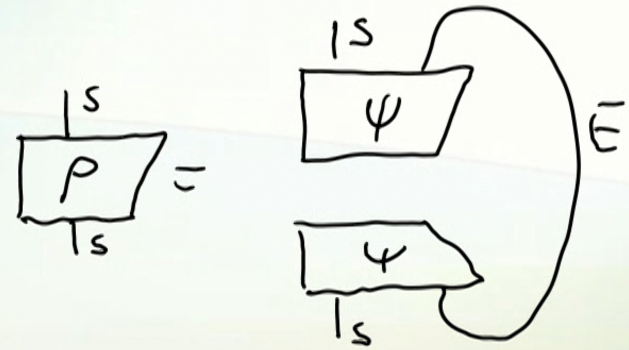


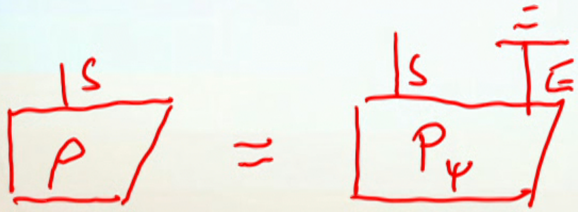
(Don't have a great diagrammatic notation for a partial transpose)

The View From The Larger Church

- If we define the object

$$\rho_S = \mathrm{Tr}_{\bar{E}}\left(|\psi\rangle\langle\psi|\right) \qquad \rho^{i_S}_{j_S} = \psi^{i_S k_E}\,\psi^{\dagger}_{j_S k_E}$$





then the probability is $\mathrm{Prob}(\Pi) = \mathrm{Tr}(\Pi_S \rho_S) = \Pi^{k_S}_{j_S}\,\rho^{j_S}_{k_S} =$

- ρ lives in the space $\mathcal{H}_S \otimes \mathcal{H}_S^{\dagger} \equiv \mathcal{L}(\mathcal{H}_S) \equiv \mathcal{L}(\mathcal{H}_S)^{\dagger}$

  so it is both an operator and a duperator

- We normally call it a <u>density operator</u> (although we use it as a duperator)

Properties of the density operator

- Given that $\psi^{j_S k_E}$ is normalized $\psi^{j_S k_E}\,\psi^\dagger_{j_S k_E} = 1$

  $\therefore \mathrm{Tr}(\rho) = \rho^{j_S}_{j_S} = \psi^{j_S k_E}\,\psi^\dagger_{j_S k_E} = 1$

- We can also write $\rho^{k_S}_{j_S} = \psi^{k_S}_{l_E}\,\psi^{\dagger\, l_E}_{j_S}$

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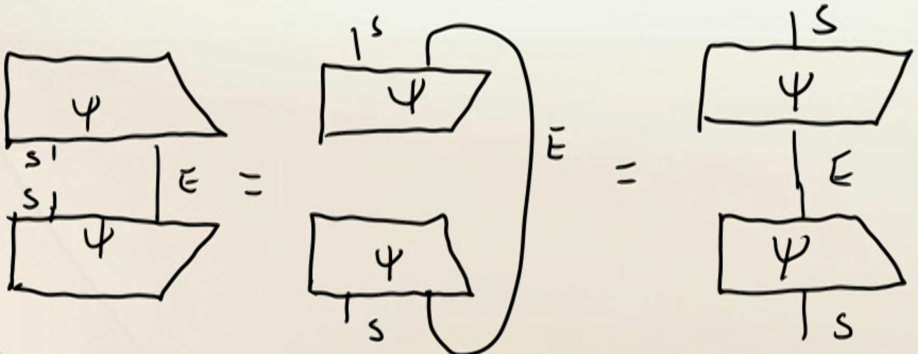


(Don't have a great diagrammatic notation for a partial transpose)

Properties of the density operator

- $\rho^{k_s}_{j_s} = \psi^{\dagger\, l_E}_{j_s} \psi^{k_s}_{l_E} = \psi^{k_s}_{l_E} \psi^{\dagger\, l_E}_{j_s}$ is of the form $N^\dagger N$
  with $N = \psi^{\dagger\, l_E}_{j_s} \in \mathcal{L}(\mathcal{H}_S \to \mathcal{H}_E)$
  so $\rho^{k_s}_{j_s}$ is a positive operator.
- In summary, density operators must be positive and have Trace $=1$.
- Can any positive, trace 1 operator arise from ignoring the environment for some $|\psi\rangle_{SE}$.

  Yes Define $|\psi_\rho\rangle_{SS'} = \rho^{1/2}_S |\delta\rangle_{SS'}$ or $\psi_\rho^{\,j_s k_s} = (\rho^{1/2})^{j_s}_{l_s} \delta^{l_s}_{k_s}$

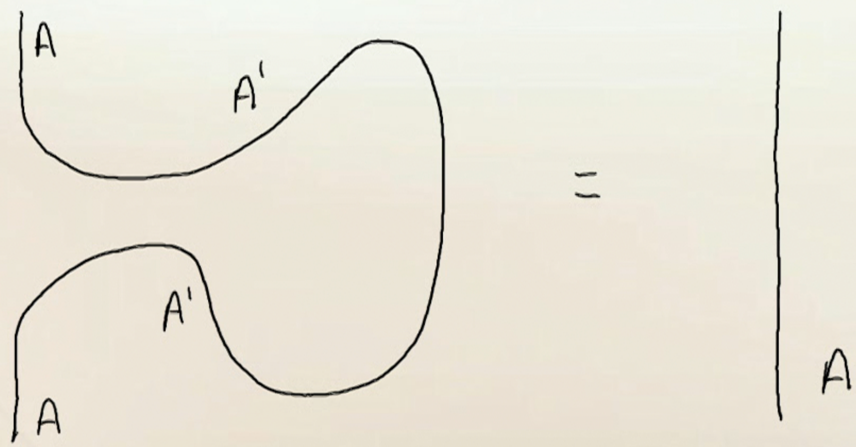
  This is an example of a purification of a density operator.

Yet More Fun With Diagrams

- The partial trace of a maximally entangled state is maximally mixed

$$|\Phi^+\rangle_{AA'} = \frac{1}{\sqrt{2}}\big(|00\rangle_{AA'} + |11\rangle_{AA'}\big) \qquad \mathrm{Tr}_{A'}(|\Phi^+\rangle\langle\Phi^+|_{AA'}) = \frac{I}{2}$$

- Since $|\delta\rangle_{AA'} = \sqrt{2}|\Phi^+\rangle_{AA'}$, this is equivalent to $\mathrm{Tr}_{A'}(|\delta\rangle\langle\delta|_{AA'}) = I$



The View from the Smaller Church

- According to the smaller church, a quantum state should be <u>any</u> consistent way of assigning probabilities to observables.
- We can view a quantum state as a functional that assigns expectation values to observables

$$\rho : \mathcal{S}(\mathcal{H}_A) \longrightarrow \mathbb{R}$$

- When we apply it to projection operators, we should get probabilities.
- Classically, expectation values behave linearly

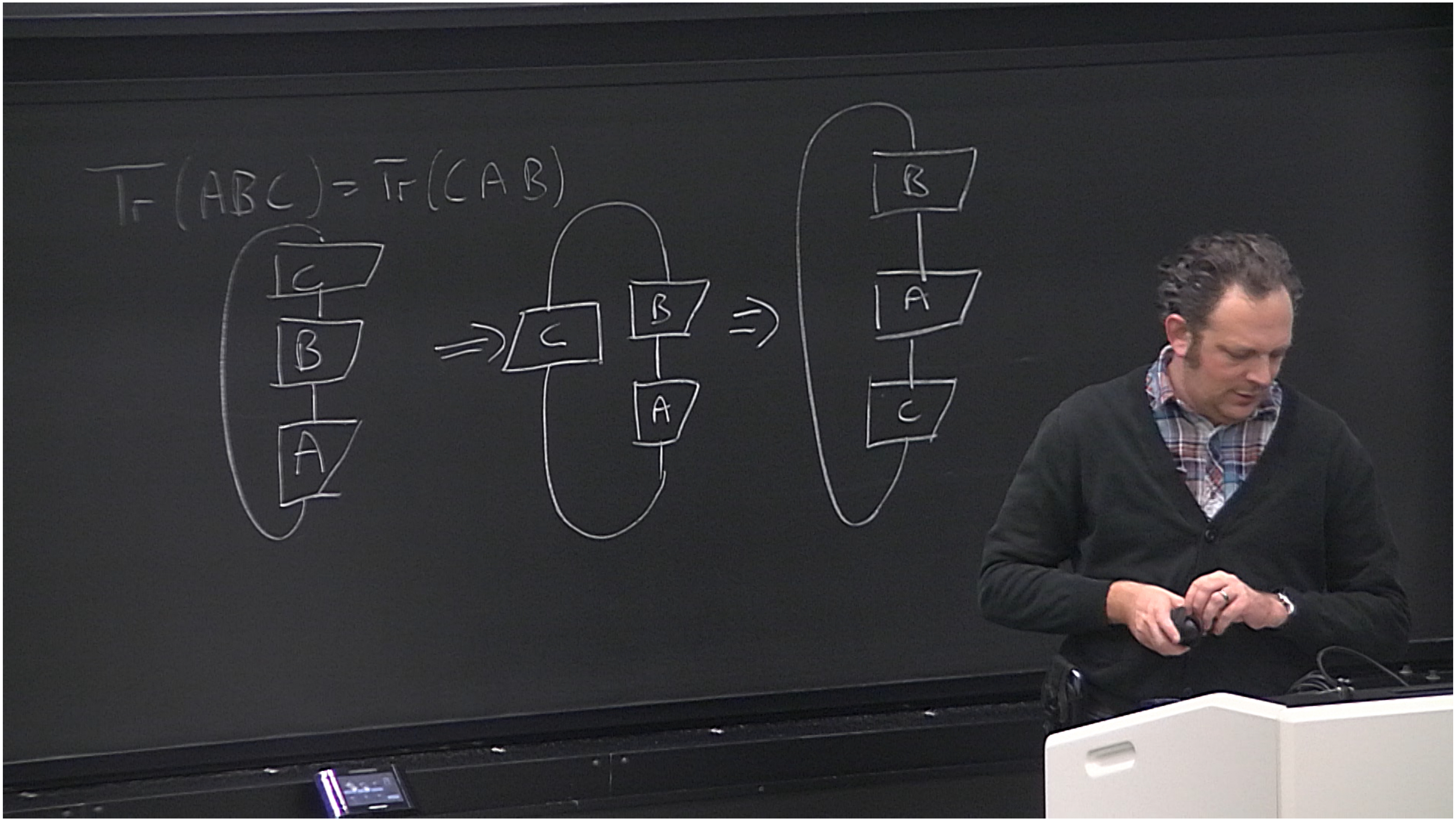
$$\langle \alpha X + \beta Y \rangle = \alpha \langle X \rangle + \beta \langle Y \rangle$$

- We will impose this for quantum observables too (but can remove this later) $\rho(\alpha M + \beta N) = \alpha \rho(M) + \beta \rho(N)$

$$U_A |\Phi^+\rangle_{AA'} = U_{A'}^T |\Phi^+\rangle_{AA'}$$

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad U_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\langle M | N \rangle = \mathrm{Tr}(M^\dagger N) \qquad U_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$


Tr(ABC) = Tr(CAB)
C
B
A
C
B
A
B
A
C