Title: PSI 2018/2019 - Foundations of Quantum Mechanics - Lecture 3

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Abstract:

Quantum Foundations
Lecture 3

PSI Review Class: 9th January 2019

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Summary of Last Lecture

- Interference
  - Single quantum particles display interference phenomena. We cannot observe interference and also the path taken by the particle at the same time.
- The Measurement Problem
  - There are two ways of describing a measurement, via the measurement postulates or as a unitary interaction between two quantum systems.
  - In the orthodox interpretation, or any ψ-complete interpretation, these two descriptions are contradictory.
- The EPR Paradox
  - The orthodox interpretation, or any interpretation in which measurement outcomes are indeterminate before they are measured is necessarily *nonlocal* according to the EPR criterion.

3.4) The No-Cloning Theorem

- If $0 < |\langle\phi|\psi\rangle| < 1$ then there is no physical operation that outputs $|\psi\rangle \otimes |\psi\rangle$ when $|\psi\rangle$ is input and also $|\phi\rangle \otimes |\phi\rangle$ when $|\phi\rangle$ is input.

Proof:

- Physical operations must be unitary, so let $|\chi\rangle$ be a fixed state on the same Hilbert space as $|\psi\rangle$ and $|\phi\rangle$.
- A cloning unitary would satisfy

$$U|\psi\rangle \otimes |\chi\rangle = |\psi\rangle \otimes |\psi\rangle \qquad U|\phi\rangle \otimes |\chi\rangle = |\phi\rangle \otimes |\phi\rangle$$

- Unitaries preserve inner products, so

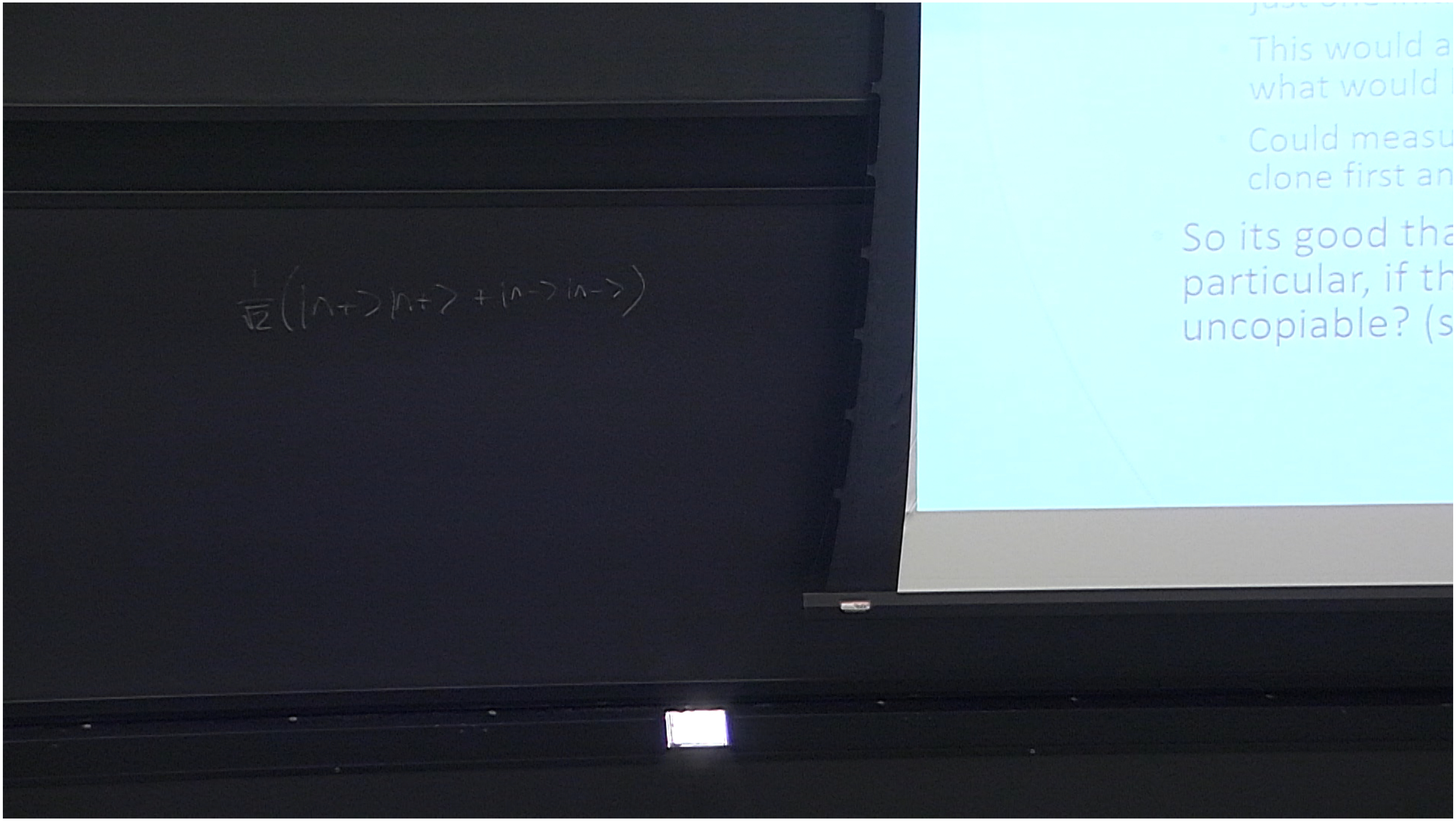
$$\langle\phi| \otimes \langle\chi| U^\dagger U |\psi\rangle \otimes |\chi\rangle = (\langle\phi| \otimes \langle\phi|)(|\psi\rangle \otimes |\psi\rangle)$$

$$\Rightarrow \langle\phi|\psi\rangle\langle\chi|\chi\rangle = \langle\phi|\psi\rangle^2$$

$$\Rightarrow \langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2 \quad \Rightarrow \quad |\langle\phi|\psi\rangle| = 0 \text{ or } \underline{1}$$

Comments on No-Cloning

- No-cloning is related to a number of other key features of quantum theory:
  - If we could perfectly clone, we could create an arbitrarily large number of copies of the initial state. Would allow us to determine the state exactly from just one initial copy.
  - This would allow us to signal superluminally in the EPR experiment (consider what would happen if we could clone state of B after measurement of A).
  - Could measure any observable without disturbing the state of the system (just clone first and put one copy to the side).
- So its good that no-cloning holds, but we should explain why. In particular, if the quantum state really exists then why should it be uncopiable? (suggests ψ-epistemic interpretation).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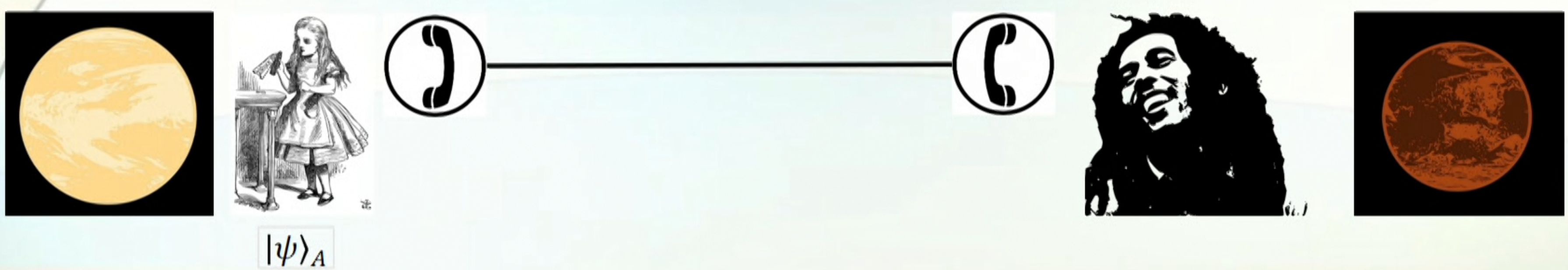
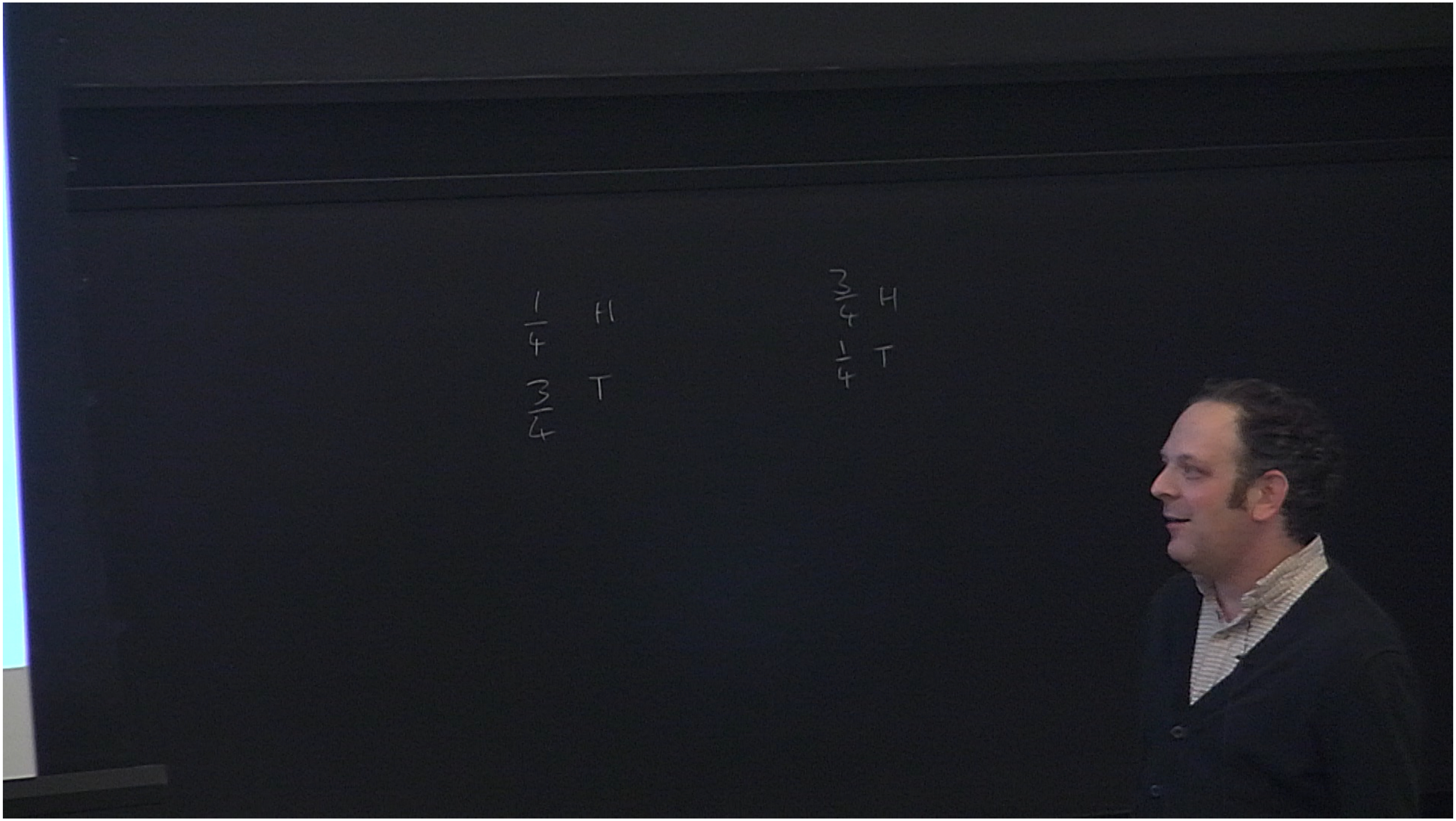
3.5) Quantum Teleportation



- Suppose Alice (on Venus) has a qubit in an unknown state that she wants to send to Bob (on Mars).
- The problem is they have no communication channel through which they can reliably send quantum systems.
- They only have an old fashioned telephone line, through which they can send classical data.
- Can Alice send the state to Bob?



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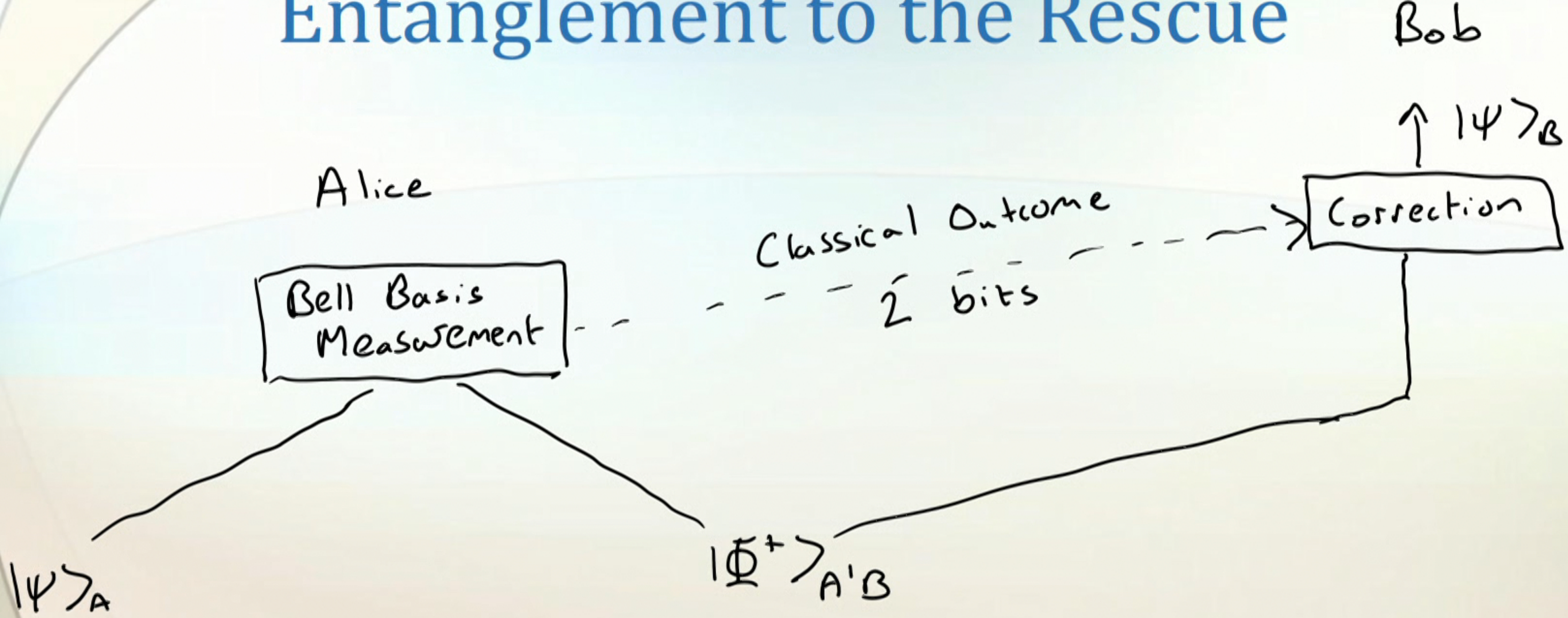
Quantum Teleportation

- At first sight, it seems that she obviously can't. As they only have a classical channel, she would have to convert the quantum state into classical information.
- As she does not know what the state is, she would have to measure it.
- But there is no measurement that will reliably tell her what the quantum state is (otherwise the no-cloning theorem would be violated).
- At best, she could send what she learns from the measurement, which would enable Bob to reconstruct a very unreliable approximation of $|\psi\rangle$.

Entanglement to the Rescue



- If Alice and Bob can do it if they pre-share two qubits in the entangled state

$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

Quantum Teleportation Protocol

1. Alice and Bob share two qubits in the entangled state
$$|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$
2. Alice performs a joint measurement of her system A in the unknown state $|\psi\rangle_A$ and A' in the Bell basis
$$|\Phi^\pm\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AA'} \pm |11\rangle_{AA'}\right)$$
$$|\Psi^\pm\rangle_{AA'} = \frac{1}{\sqrt{2}}\left(|01\rangle_{AA'} \pm |10\rangle_{AA'}\right)$$
3. Alice communicates the outcome to Bob. There are 4 possible outcomes, so 2 bits of communication.
4. Depending on the outcome, Bob applies one of four unitary operations to his qubit B. This transforms his system to $|\psi\rangle_B$.

Proving it Works

- Let $|\psi\rangle = a|0\rangle + b|1\rangle$. The initial state of the three systems is

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{A'B} = (a|0\rangle_A + b|1\rangle_A) \otimes \frac{1}{\sqrt{2}}\left(|00\rangle_{A'B} + |11\rangle_{A'B}\right)$$

$$= \frac{1}{\sqrt{2}}\left(a|000\rangle_{AA'B} + a|011\rangle_{AA'B} + b|100\rangle_{AA'B} + b|111\rangle_{AA'B}\right)$$

$$= \frac{1}{2\sqrt{2}}\left(|00\rangle_{AA'} + |11\rangle_{AA'}\right) \otimes (a|0\rangle_B + b|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|00\rangle_{AA'} - |11\rangle_{AA'}\right) \otimes (a|0\rangle_B - b|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|01\rangle_{AA'} + |10\rangle_{AA'}\right) \otimes (b|0\rangle_B + a|1\rangle_B)$$

$$+ \frac{1}{2\sqrt{2}}\left(|01\rangle_{AA'} - |10\rangle_{AA'}\right) \otimes (b|0\rangle_B - a|1\rangle_B)$$

Proving it Works

$$|\psi\rangle_A \otimes |\Phi^+\rangle_{A'B} = \frac{1}{2}\big(|\Phi^+\rangle_{AA'} \otimes U_0|\psi\rangle_B + |\Phi^-\rangle_{AA'} \otimes U_1|\psi\rangle_B$$
$$+|\Psi^+\rangle_{AA'} \otimes U_2|\psi\rangle_B + |\Psi^-\rangle_{AA'} \otimes U_3|\psi\rangle_B\big)$$

where

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Now, the partial inner product with $|\Phi^+\rangle_{AA'}$ is just $\frac{1}{2}U_0|\psi\rangle_B$, which has norm $\frac{1}{4}$, so Alice will get this outcome with probability $\frac{1}{4}$. If she does, Bob just has to apply $U_0^\dagger = U_0^T$ (which is the identity in this case so he does nothing) to obtain the state $|\psi\rangle_B$.
- The same is true for the other three outcomes, so, provided Bob knows the outcome of Alice's measurement, he can apply the appropriate unitary to obtain $|\psi\rangle_B$.

Is Teleportation Weird?

- Consider the ψ-ontic view of quantum states. The unknown qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ that Alice sends to Bob is specified by two complex numbers, which can take a continuum of values.
- It takes an infinite number of bits to specify these precisely, but if you are a ψ-ontologist you believe that the ontic state contains all this information, so this is really physically transmitted from Alice to Bob.
- But Alice only sends two bits of information to Bob, so how did this infinite amount of information get transmitted.
- Some people have suggested that it goes backwards in time!

Classical Teleportation

- On the ψ-epistemic view of quantum states (i.e. quantum states are something more like classical probability distributions) it is not so weird.
- A classical probability distribution $\begin{pmatrix} p \\ 1-p \end{pmatrix}$ for a bit is also specified by a continuous parameter, which takes an infinite number of bits to specify.
- But Alice can transmit it to Bob by sending just one bit to Bob, i.e. just send the bit itself.

Classical Teleportation

- In fact, there is a protocol that looks a lot like teleportation:
  1. Alice and Bob share two bits that are perfectly correlated. With probability $\frac{1}{2}$ they are both 0 and with probability $\frac{1}{2}$ they are both 1.
  2. Alice has another bit with an unknown probability distribution that she wants to send to bob.
  3. Alice checks whether her two bits are the same or different. Each possibility will happen with probability $\frac{1}{2}$.
  4. If they are the same, Bob does nothing. If they are different, Bob flips his bit. Bob's bit now has the same probability distribution as Alice's original bit.

Classical Teleportation

- This classical teleportation protocol has another name: the *one-time-pad* or *Vernam cipher*. It is a way for Alice to transmit information to Bob securely if they share correlated random bits.
- Since the bit Alice sends to Bob is uniformly random, it conveys no information about the bit Alice is trying to send to an eavesdropper who does not chare Alice and Bob's correlated bits.
- The same is true of quantum teleportation. An eavesdropper learns nothing about the quantum state Alice is sending to Bob.
- In any case, on the ψ-epistemic view, if the ontic state of a qubit contains only two-bits of information and the rest of the parameters of the quantum state only express knowledge about those bits then there would be no mystery.
- Unfortunately, there are many obstacles to this idea, as we shall see later in the course.

4) Abstract Tensors and String Diagrams

1) Vectors, Dual Vectors, Inner Products, and Tensor Products
2) Abstract Index Notation
3) Diagrammatic Notation
4) More Interesting Tensor Products
5) The Space of Linear Operators
6) Raising and Lowering Indices
7) Transpose, Conjugate and Duals
8) Trace and Partial Trace
9) Vector-Operator Correspondence
10) Application: Quantum Teleportation

4.1) Vectors, Dual Vectors, Inner Products, and Tensor Products

- In quantum mechanics, the (pure) states of a quantum system are vectors ("kets") $|\psi\rangle \in V$ in a vector space V.
- We can also define dual vectors $\langle g| \in V^\dagger$ as linear functions from V to $\mathbb{C}$.

$$\langle g|: V \to \mathbb{C}$$
$$\langle g|(a|\psi\rangle + b|\phi\rangle) = a\langle g|(|\psi\rangle) + b\langle g|(|\phi\rangle)$$

- If we define $(a\langle f| + b\langle g|)(|\psi\rangle) = a\langle f|(|\psi\rangle) + b\langle g|(|\psi\rangle)$ then $V^\dagger$ is a vector space called the *dual vector space*.

Inner Products and Dual Vectors

- In a Hilbert space, the inner product induces an isomorphism $\mathcal{H} \equiv \mathcal{H}^\dagger$
- Given a vector $|\psi\rangle \in \mathcal{H}$, we can define a dual vector $\langle\psi| \in \mathcal{H}^\dagger$ via

$$\langle\psi|\phi\rangle = (|\psi\rangle, |\phi\rangle)$$

  Linearity of inner product ensures this is a linear functional.
- Given a dual vector $\langle g| \in \mathcal{H}^\dagger$, let $\{|j\rangle\}$ be an orthonormal basis for $\mathcal{H}$ and define $g_j = \langle g|j\rangle$

  Then define $|g\rangle = \sum_j g^{\dagger j} |j\rangle$ with $g^{\dagger j} = g_j^*$

  Straightforward to prove that this is an isomorphism.

Tensor Products and Partial Inner Products

- Suppose $\mathcal{H}_A$ has an orthonormal basis $|j\rangle_A$ $\quad |\psi\rangle_A = \sum_j \psi^j |j\rangle_A$

  $\mathcal{H}_B$ " " " " $|k\rangle_B$ $\quad |\phi\rangle_B = \sum_k \phi^k |k\rangle_B$

- The tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ is the vector space spanned by $|j\rangle_A \otimes |k\rangle_B$

$$|\psi\rangle_{AB} = \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B$$

- It is a Hilbert space, inheriting its inner product from $\mathcal{H}_A$ and $\mathcal{H}_B$

  i.e. if $|\phi\rangle_{AB} = \sum_{jk} \phi^{jk} |j\rangle_A \otimes |k\rangle_B$

$$\langle\phi|\psi\rangle_{AB} = \sum_{jklm} \phi^\dagger_{jk} \psi^{lm} \langle j|l\rangle_A \langle k|m\rangle_B = \sum_{jk} \phi^\dagger_{jk} \psi^{lm} \quad \text{where } \phi^\dagger_{jk} = \phi^{jk*}$$

4.2) Abstract Index Notation

- It is cumbersome to keep track of long strings of bras/kets

e.g. $|j\rangle_A \otimes |k\rangle_B \otimes |l\rangle_C \otimes \ldots\ldots$

- We can develop an abstract index notation similar to that used in differential geometry and GR.

$$|\psi\rangle_A = \sum_j \psi^j |j\rangle_A \Rightarrow \psi^{j_A} \qquad {}_A\langle\phi| = \sum_j \phi_j \, {}_A\langle j| \Rightarrow \phi_{j_A}$$

$$\langle\phi|\psi\rangle_A = \phi_{j_A}\psi^{j_A}$$ ← summation convention for repeated indices

- It is necessary to include the label A of the Hilbert space $\mathcal{H}_A$ in the index j_A because Hilbert spaces may have different dimensions

Only upper A indices can be contracted with lower A indices.

Tensor Products and Partial Inner Products

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$$\langle\phi|\psi\rangle_{AB} = \sum_{\substack{jk\\lm}} \phi^\dagger_{jk} \psi^{lm} \langle j|l\rangle_A \langle k|m\rangle_B = \sum_{jk} \phi^\dagger_{jk} \psi^{lm} \quad \text{where } \phi^\dagger_{jk} = \phi^{jk*}$$

Abstract Index Notation

- For a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$, we would have

$$|\psi\rangle_{AB} = \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B \implies \psi^{j_A k_B}$$

$$_{AB}\langle\phi| = \sum_{jk} \phi_{jk}\, {}_A\langle j| \otimes {}_B\langle k| \implies \phi_{j_A k_B}$$

- The inner product is

$$\langle\phi|\psi\rangle_{AB} = \phi_{j_A k_B}\psi^{j_A k_B}$$

- However $\phi_{j_A k_B}\psi^{k_A j_B}$ is not a valid contraction

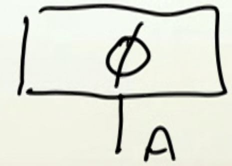
4.3) Diagrammatic Notation

- Even abstract tensors get tedious after a while, so it is useful to develop a way of representing them with diagrams.
  - A tensor is represented by a box.
  - A vector index is represented by an upward directed line with Hilbert space label.
  - A dual vector index is represented by a downward directed line with a Hilbert space label.
  - Contraction (taking inner products) is represented by joining lines.

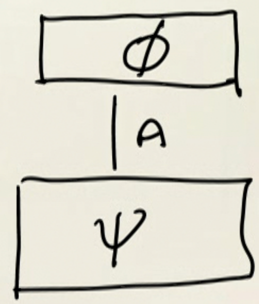
Examples

$$|\psi\rangle_A = \sum_j \psi^j |j\rangle_A \iff \psi^{j_A} \iff$$

$${}_A\langle\phi| = \sum_j \phi_j \, {}_A\langle j| \iff \phi_{j_A} \iff$$

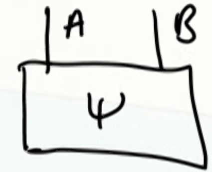


$$\langle\phi|\psi\rangle_A \iff \phi_{j_A}\psi^{j_A} \iff$$

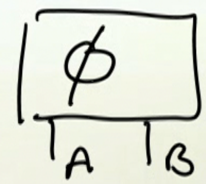


Examples

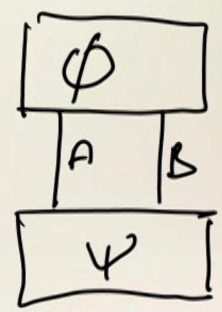
$$|\psi\rangle_{AB} = \sum_{jk} \psi^{jk} |j\rangle_A \otimes |k\rangle_B \Longleftrightarrow \psi^{j_A k_B} \Longleftrightarrow$$



$${}_{AB}\langle\phi| = \sum_{jk} \phi_{jk}\, {}_A\langle j| \otimes {}_B\langle k| \Longleftrightarrow \phi_{j_A k_B} \Longleftrightarrow$$

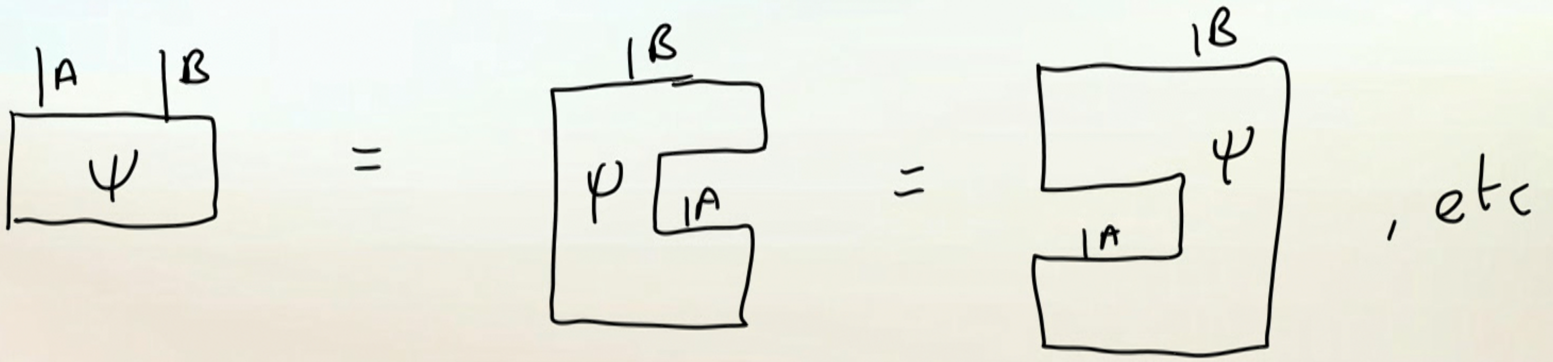


$$\langle\phi|\psi\rangle_{AB} \Longleftrightarrow \phi_{j_A k_B}\psi^{j_A k_B} \Longleftrightarrow$$



Diagrammatic Notation

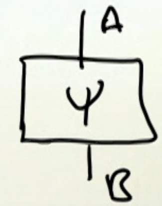
- Note: The shape of a box does not matter. Only the direction of the lines coming out of it matters, e.g.



4.4) More Interesting Tensor Products

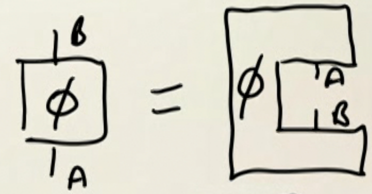
- $\mathcal{H}_A$ and $\mathcal{H}_B^\dagger$ are both Hilbert spaces, so there is no reason why we can't form the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$
- This would be the vector space of objects of the form

$$\sum_{jk} \psi^{j}{}_{k}\, |j\rangle_A \otimes {}_B\langle k| \iff \psi^{j_A}{}_{k_B} \iff$$

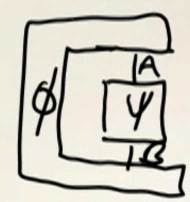


- The dual space to $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ is $(\mathcal{H}_A \otimes \mathcal{H}_B^\dagger)^\dagger = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B$

$$\sum_{jk} \phi_{j}{}^{k}\, {}_A\langle j| \otimes |k\rangle_B \iff \phi^{k_B}{}_{j_A} \iff$$



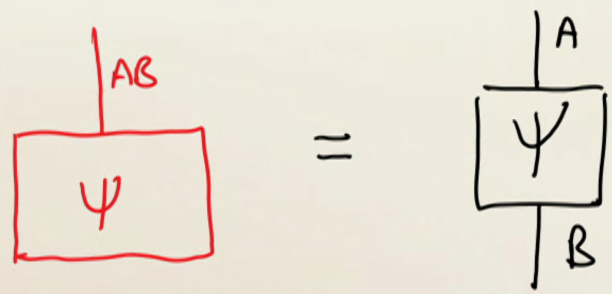
- The inner product is given by $\phi^{k_B}{}_{j_A}\, \psi^{j_A}{}_{k_B} \iff$



More Interesting Tensor Products

- An object like $\sum_{jk} \psi^j_k |j\rangle_A \otimes {}_B\langle k|$ is neither a ket nor a bra.
- However $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ is still just a Hilbert space, like any other.
- Sometimes it will be useful to think of $\mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ as a space of "kets" and its dual $\mathcal{H}_A^\dagger \otimes \mathcal{H}_B$ as a space of "bras".
- When doing so, I will use red brakets and red diagrams

$$|\psi\rangle_{AB} = \sum_{jk} \psi^j_k |j\rangle_A \otimes {}_B\langle k|$$

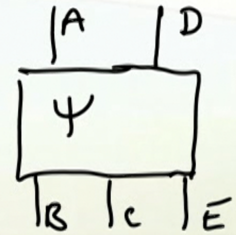


More Interesting Tensor Products

- Clearly, we can iterate this construction and consider complicated tensor products

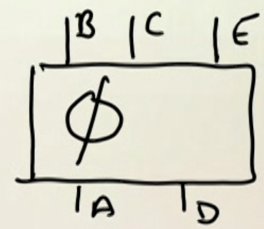
e.g. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B^\dagger \otimes \mathcal{H}_C^\dagger \otimes \mathcal{H}_D \otimes \mathcal{H}_E^\dagger$

$$\sum_{jklmn} \psi^{jm}_{kln} |j\rangle_A \otimes {}_B\langle k| \otimes {}_C\langle l| \otimes |m\rangle_D \otimes {}_E\langle n| \iff \psi^{j_A m_D}_{k_B l_C n_E} \iff$$



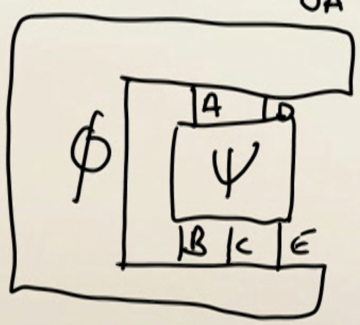
- It's dual space is $\mathcal{H}^\dagger = \mathcal{H}_A^\dagger \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D^\dagger \otimes \mathcal{H}_E$

$$\sum_{jklmn} \phi^{kln}_{jm}\, {}_A\langle j| \otimes |k\rangle_B \otimes |l\rangle_C \otimes {}_D\langle m| \otimes |n\rangle_E \iff \phi^{k_B l_C n_E}_{j_A m_D} \iff$$



- Inner product:

$$\phi^{k_B l_C n_E}_{j_A m_D}\, \psi^{j_A m_D}_{k_B l_C n_E} \iff$$

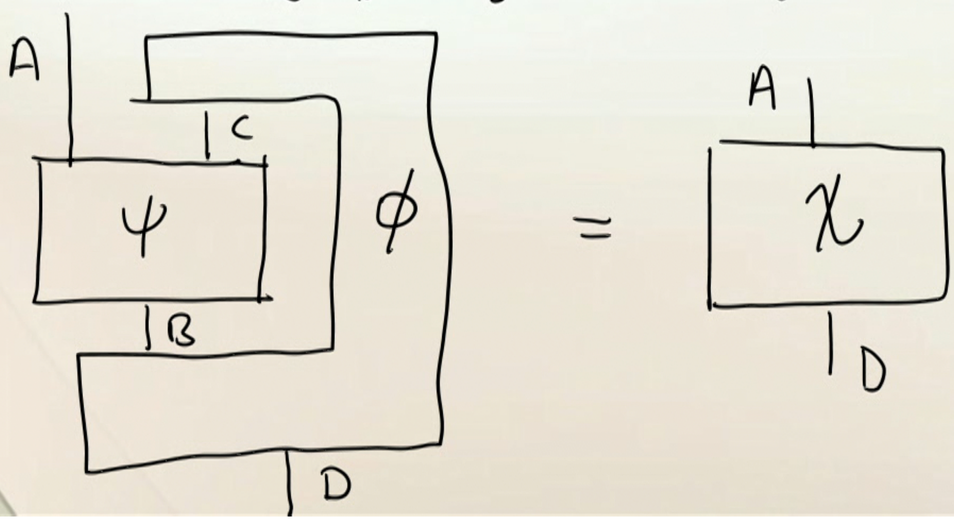


Partial Inner Products

- We can also define partial inner products where we only contract over some of the indicies

e.g. $\psi^{j_A l_C}_{k_B} \in \mathcal{H}_A \otimes \mathcal{H}_B^\dagger \otimes \mathcal{H}_C \qquad \phi^{k_B}_{l_C m_D} \in \mathcal{H}_B \otimes \mathcal{H}_C^\dagger \otimes \mathcal{H}_D^\dagger$

$$\phi^{k_B}_{l_C m_D} \psi^{j_A l_C}_{k_B} = \chi^{j_A}_{m_D}$$



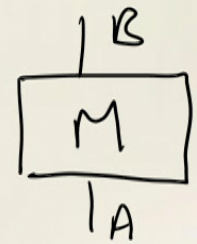
4.5) The Space of Linear Operators

- Now let's consider the space of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ denoted $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)$ [just $\mathcal{L}(\mathcal{H}_A)$ if $\mathcal{H}_A = \mathcal{H}_B$]
- In Dirac notation, we know that an operator can be written in terms of its matrix elements

$$M = \sum_{jk} M^j_k \, |j\rangle_B \, {}_A\langle k| \qquad \text{where} \qquad M^j_k = {}_B\langle j | M | k \rangle_A$$

- But this looks just like an object in $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger$

$$\sum_{jk} M^j_k \, |j\rangle_B \otimes {}_A\langle k| \iff M^{j_B}_{k_A} \iff$$

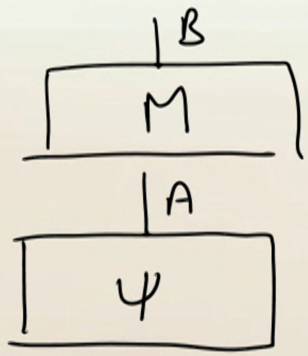


The Space of Linear Operators

- If we treat M as an element of $\mathcal{H}_B \otimes \mathcal{H}_A^\dagger$ then the action of M on a vector $|\psi\rangle_A \in \mathcal{H}_A$ is just partial inner product

$$M = \sum_{jk} M^j_k \, |j\rangle_B \otimes {}_A\langle k| \qquad |\psi\rangle_A = \sum_l \psi^l \, |l\rangle_A$$

$$M|\psi\rangle_A = \sum_{jk} M^j_k \psi^k \, |j\rangle_B \qquad \text{or} \qquad M^{j_A}_{k_B} \psi^{k_B}$$



The Space of Linear Operators

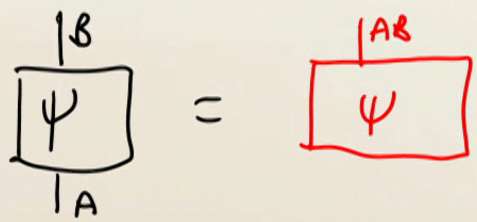
- In general, the space of linear operators from $\mathcal{H}_A$ to $\mathcal{H}_B$ is (isomorphic to)

$$\mathcal{H}_B \otimes \mathcal{H}_A^{\dagger} \equiv \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B)$$

output space. (for $\mathcal{H}_B$)   dual of input space (for $\mathcal{H}_A^{\dagger}$)

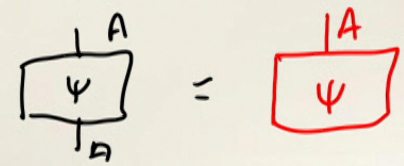
- Everything can be done with tensor products and partial inner products!
- Note, the space $\mathcal{H}_B \otimes \mathcal{H}_A^{\dagger} \equiv \mathcal{L}(\mathcal{H}_A \rightarrow \mathcal{H}_B)$ is just what we introduced the red kets and diagrams for

$|\psi\rangle_{AB} \in \mathcal{H}_B \otimes \mathcal{H}_A^{\dagger}$



If $\mathcal{H}_A = \mathcal{H}_B$, I will abbreviate

$|\psi\rangle_A \in \mathcal{H}_A \otimes \mathcal{H}_A^{\dagger}$



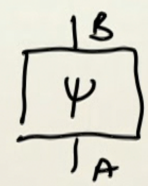
Duperators

- The dual of an operator is a linear functional from operators to scalars (a "duperator")
- $(\mathcal{H}_B \otimes \mathcal{H}_A^\dagger)^\dagger = \mathcal{H}_A \otimes \mathcal{H}_B^\dagger$ so $\mathcal{L}(\mathcal{H}_A \to \mathcal{H}_B)^\dagger = \mathcal{L}(\mathcal{H}_B \to \mathcal{H}_A)$

  i.e. The duperators from A to B are the operators from B to A

Operator from A to B:

$$|\psi\rangle_{AB} = \sum_{jk} \psi_j^k \, |j\rangle_A \otimes {}_B\langle k| \iff \psi_{j_A}^{k_B} \iff$$



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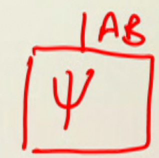
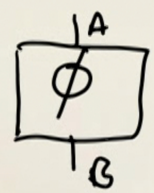


Duperator from A to B:

$${}_{AB}\langle\phi| = \sum_{jk} \phi_k^j \, {}_A\langle j| \otimes |k\rangle_B \iff \phi_{k_B}^{j_A} \iff$$



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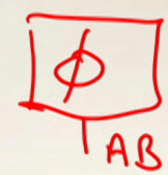
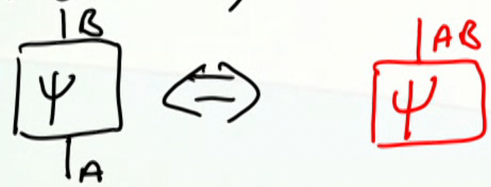


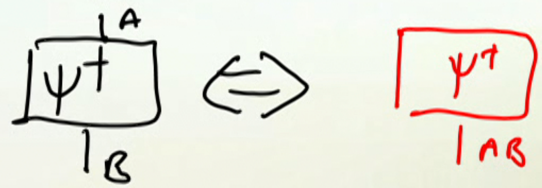
Inner Products of Operators

- Using the general correspondence between vectors and duals, we have

$$|\psi\rangle_{AB} = \sum_{jk} \psi_j^k \, |j\rangle_A \otimes {}_B\langle k| \iff \psi_{j_A}^{k_B} \iff$$

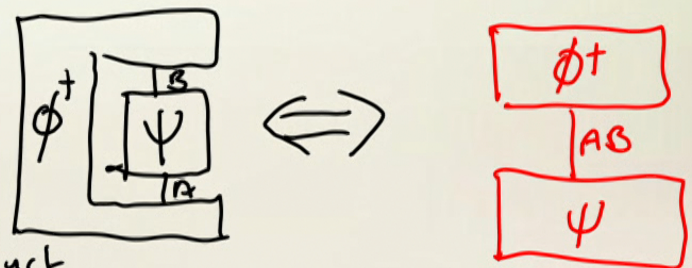


$${}_{AB}\langle\psi| = \sum_{jk} \psi^{\dagger}{}^{j}_{k} \, {}_A\langle j| \otimes |k\rangle_B \iff \psi^{\dagger}{}^{j_A}_{k_B} \iff$$



- The inner product is then:

$$\langle\phi|\psi\rangle_{AB} = \mathrm{Tr}(\phi^\dagger\psi) = \phi^{\dagger}{}^{j_A}_{k_B} \, \psi^{k_B}_{j_A} \iff$$



- This is called the "Hilbert-Schmidt" inner product.

4.6) Raising and Lowering Indices

- Consider the vector $|\delta\rangle_{AA} = \sum_j |j\rangle_A \otimes |j\rangle_A = \sum_{jk} \delta^{jk} |j\rangle_A \otimes |k\rangle_A$

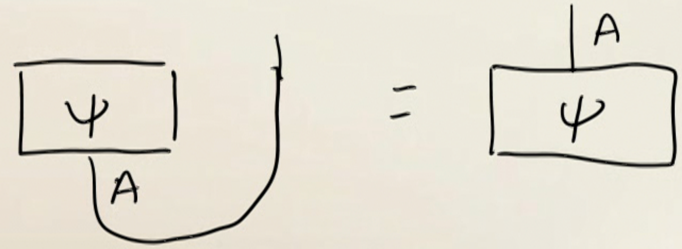
  where $\delta^{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$. This lives in $\mathcal{H}_A \otimes \mathcal{H}_A$

- Abstract index notation : $\delta^{j_A k_A}$

- Diagramatic notation :

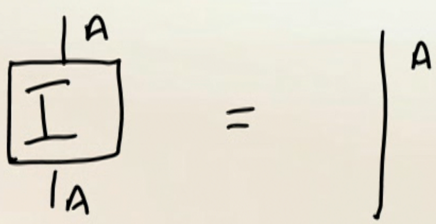
- A partial inner product with this turns a bra into a ket

  $\delta^{j_A k_A} \psi_{k_A} = \psi^{j_A}$



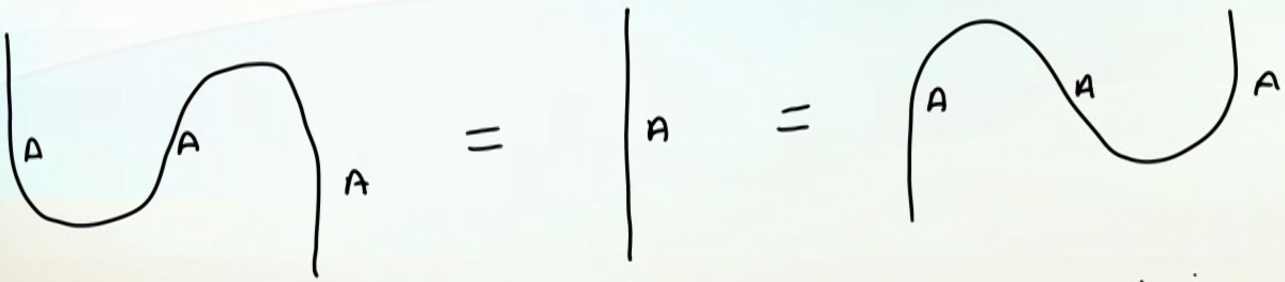
The Identity Operator

- The identity operator is $I_A = \sum_j |j\rangle_A \otimes {}_A\langle j| = \sum_{jk} \delta^j_k |j\rangle_A \otimes {}_A\langle k|$
- In abstract index notation, this is just $\delta^{j_A}_{k_A}$
- And as a diagram it is just a vertical piece of wire



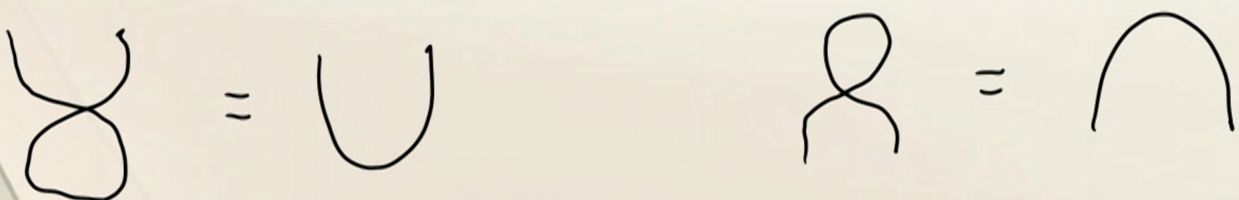
The Yanking Axioms

- The various δ tensors satisfy the following properties



$$\delta^{j_A k_A}\delta_{k_A m_A} = \delta^{j_A}_{m_A} = \delta_{m_A k_A}\delta^{k_A j_A}$$

The yanking axioms allow us to prove lots of things using just diagrams.

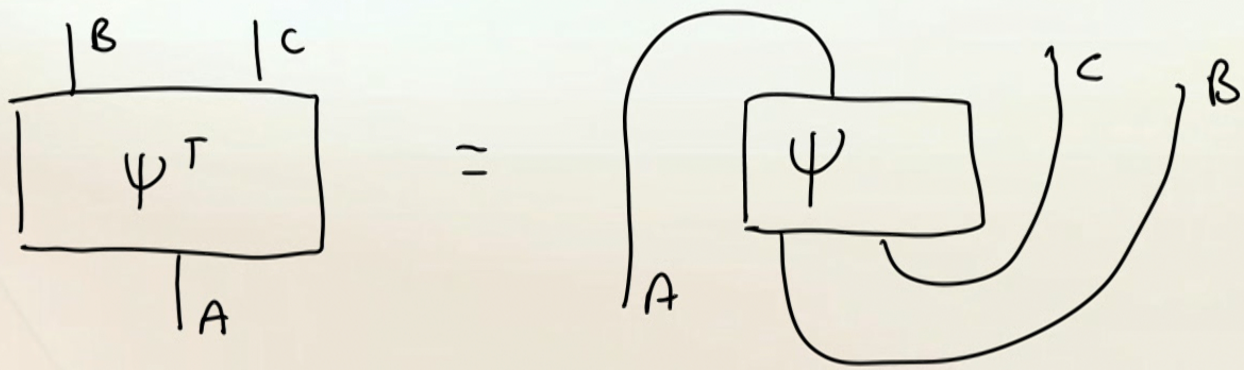
- 

Just expresses the fact that order of indices is unimportant in abstract index notation.

4.7) Transpose, Conjugate, Duals, and Trace

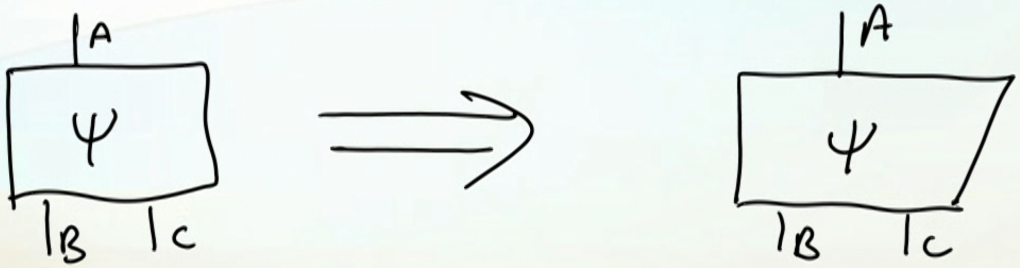
- The transpose is defined as

$$\psi^{T\,k_B l_C}{}_{j_A} = \psi^{k_B l_C}{}_{j_A} = \delta_{j_A m_A}\,\delta^{k_B n_B}\,\delta^{l_C r_C}\,\psi^{m_A}{}_{n_B l_C}$$



A Bit of Diagrammatic Trickery

- We can make more intuitive diagrams if we introduce a bit of asymmetry to our boxes.



- Then we can represent transpose by 180° rotation