Title: Koszul Duality Seminar

Date: Dec 12, 2018 02:30 PM

URL: http://pirsa.org/18120023

Abstract: Being held in the Skyroom.

dg. over a field of char = 0

Last time, maybe:

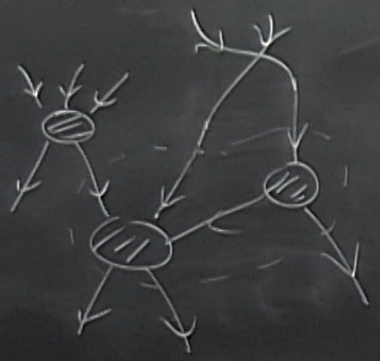
Defn: operads / rooted trees = dioperads / directed trees

A dg dioperad P is a bunch of dg vector spaces $P(m,n)$ — "m-to-n operations"

and

- $P(m,n)$ is a S_m-S_n bimodule
- Compositions parameterized by directed trees.

e.g.



$\rightsquigarrow$ comp map

$$P(3,2) \otimes P(1,2) \otimes P(3,2)$$

$$\downarrow \qquad \searrow P(3,2) \otimes P(3,3)$$

$$P(5,4) \swarrow$$

assoc

A rep'n of P aka "P-alg":

is a vspace V and actual n-to-n maps realizing P.

i.e. $V \rightsquigarrow End(V)(n,n) := hom(V^{\otimes n} \to V^{\otimes n})$

vector spaces $\longrightarrow$ algebras

A rep'n of P is hom $P \to End(V)$.

Eg:

- operads $\subseteq$ dioperads.

$O(-) \longmapsto P(m,n) := \begin{cases} O(m) & \text{if } n=1 \\ \emptyset & \text{if } n \neq 1 \end{cases}$

- com Frobenius: $P(m,n) := \mathbb{K}$.

$\mathrm{Frob}_{\mathcal{Q}}$ — $P(m,n) = \mathbb{K}[\text{Bijections of } \mathcal{Q}_{m,n}]$

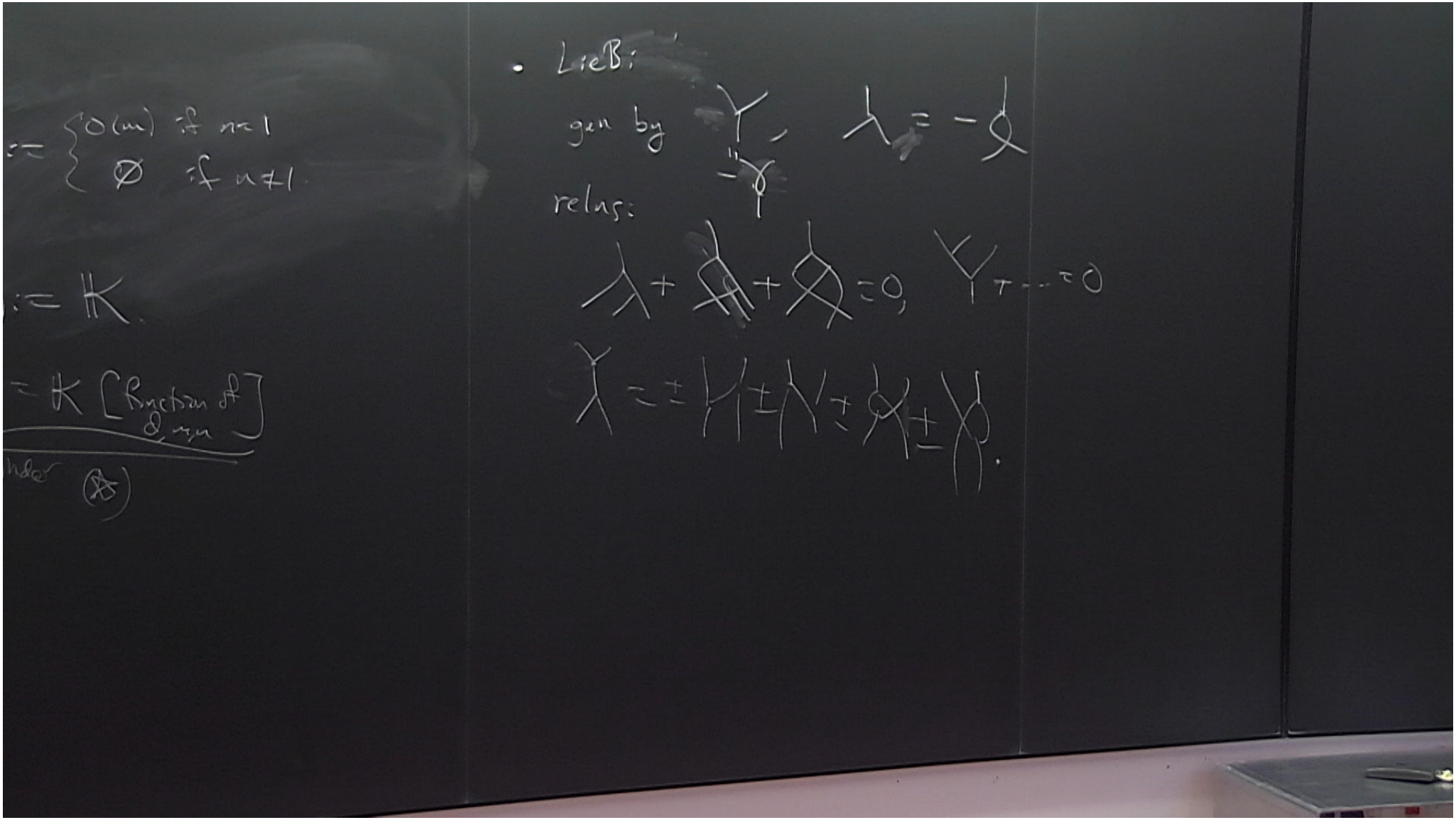
↑

com Frob algs where Δ has degree $\mathcal{Q}$.

$P(1,2) \otimes P(3,2)$

$\to P(3,2) \otimes P(3,3)$

$\leftarrow$


:= { O(n) if n=1
∅ if n≠1
:= K
= K [Functions of
Under (*)
• Lie B:
gen by
relns:

Ex:

- operads $\subseteq$ dioperads.

$O(-) \longmapsto P(m,n) := \begin{cases} O(m) & \text{if } n=1 \\ \emptyset & \text{if } n \neq 1 \end{cases}$

- LieB:

gen by

relns:



- com Frobenius: $P(m,n) := \mathbb{K}$.

Frob_Δ — $P(m,n) = \mathbb{K}[\text{Sym}_{\Delta,m,n}]$

↑

com Frob algs where Δ has degree Δ

$\otimes P(1,2) \otimes P(3,2)$

$\to P(3,2) \otimes P(3,3)$

$\leftarrow$

Under $\circledast$, if P is

Quasifree ← free if you forget the Δ if...

Model cat technicalities

request $P(0,n) = P(n,0) = \emptyset$

$(\circledast)$ $P(1,1) = \Delta \cdot \mathbb{K}$.

then $\hom(P, \mathcal{E}nd_Q(V)) \simeq \hom(P, \mathcal{E}nd_Q(W))$

if $V \cong W$ is ... ← equiv of Δ-sets

if $P \to Q$ $\hom(Q, \mathcal{E}nd_Q(V))$

Defn:

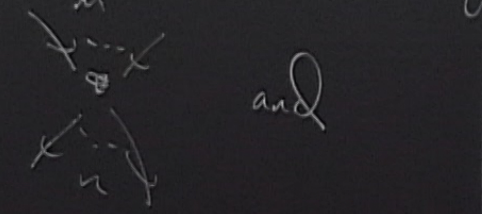
A homotopy Lie Bialg is an alg for any cofree resolution of {Lie Bi.}

$$hLB := \int$$

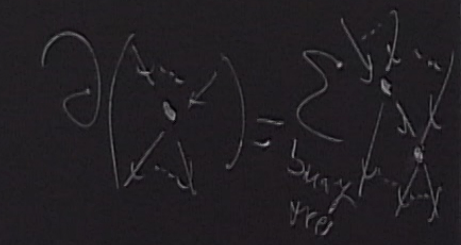
- $hLB \xrightarrow{\sim} LB$ is a coFib. resln
- hLB poly = Poisson alg. Pointed mfds

E.g.

Look at quasifree dbap. gen by a single operation of each arity



and



Defn:

An arrowed diperad

V-spaces parameterized by 4-tuples of $\mathbb{N}$s.

- $\mathcal{P}(m_1, m_2; n_1, n_2)$
- distinguished element "f" $\in \mathcal{P}(1,0;0,1)$
- compositions parameterized by two-colored directed trees.

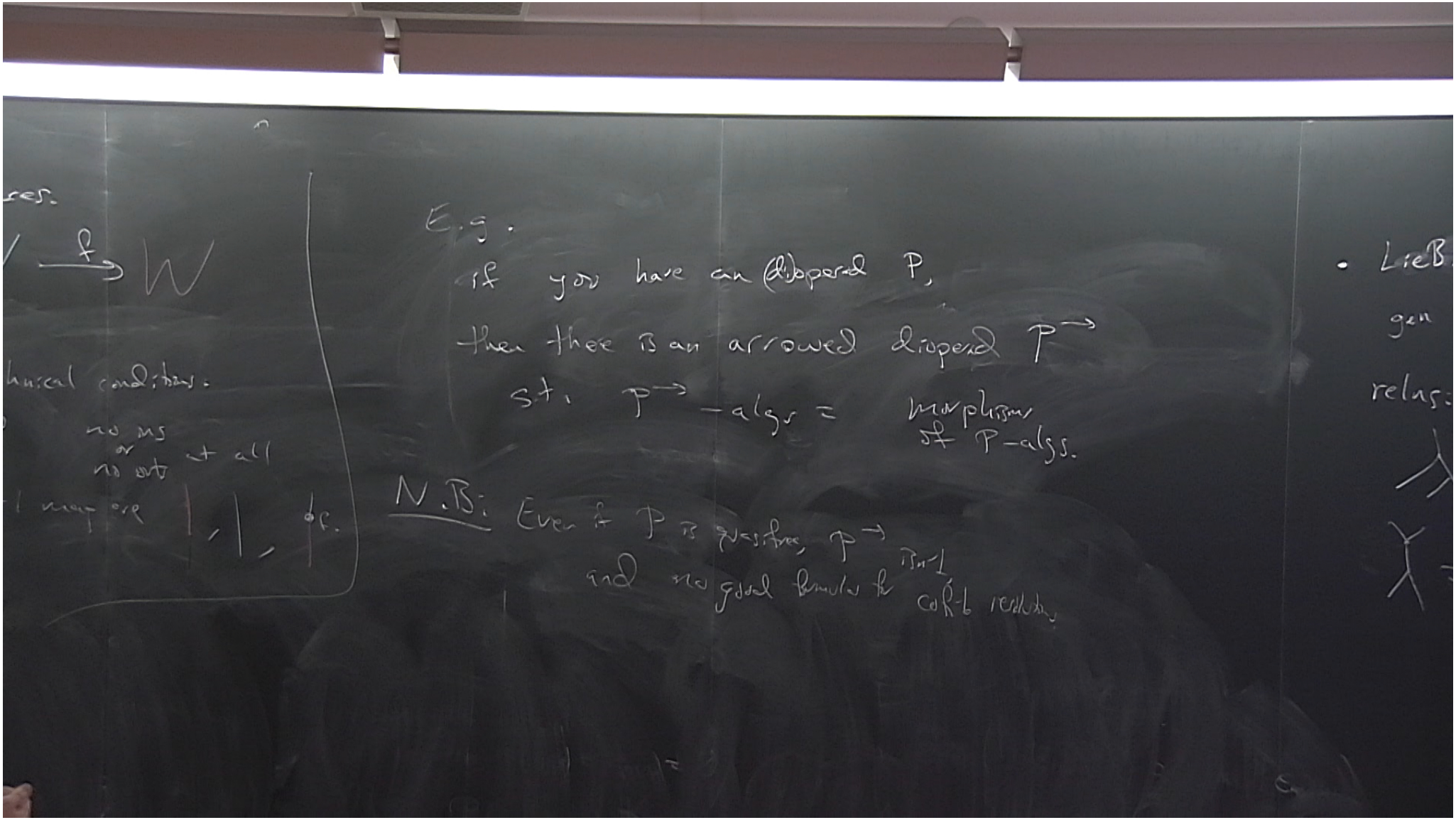
diperads act on V-spaces.

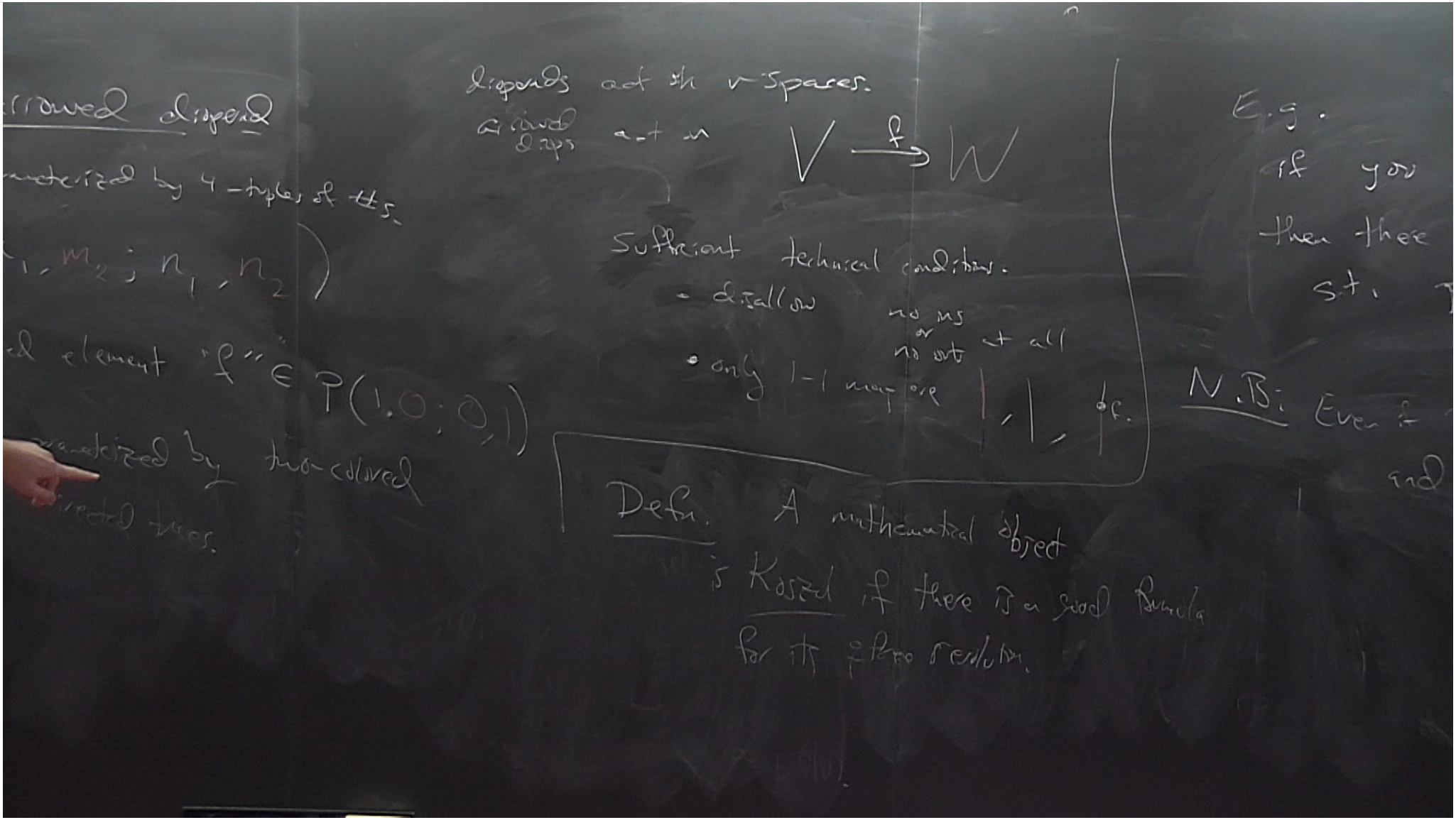
arrowed diops act on

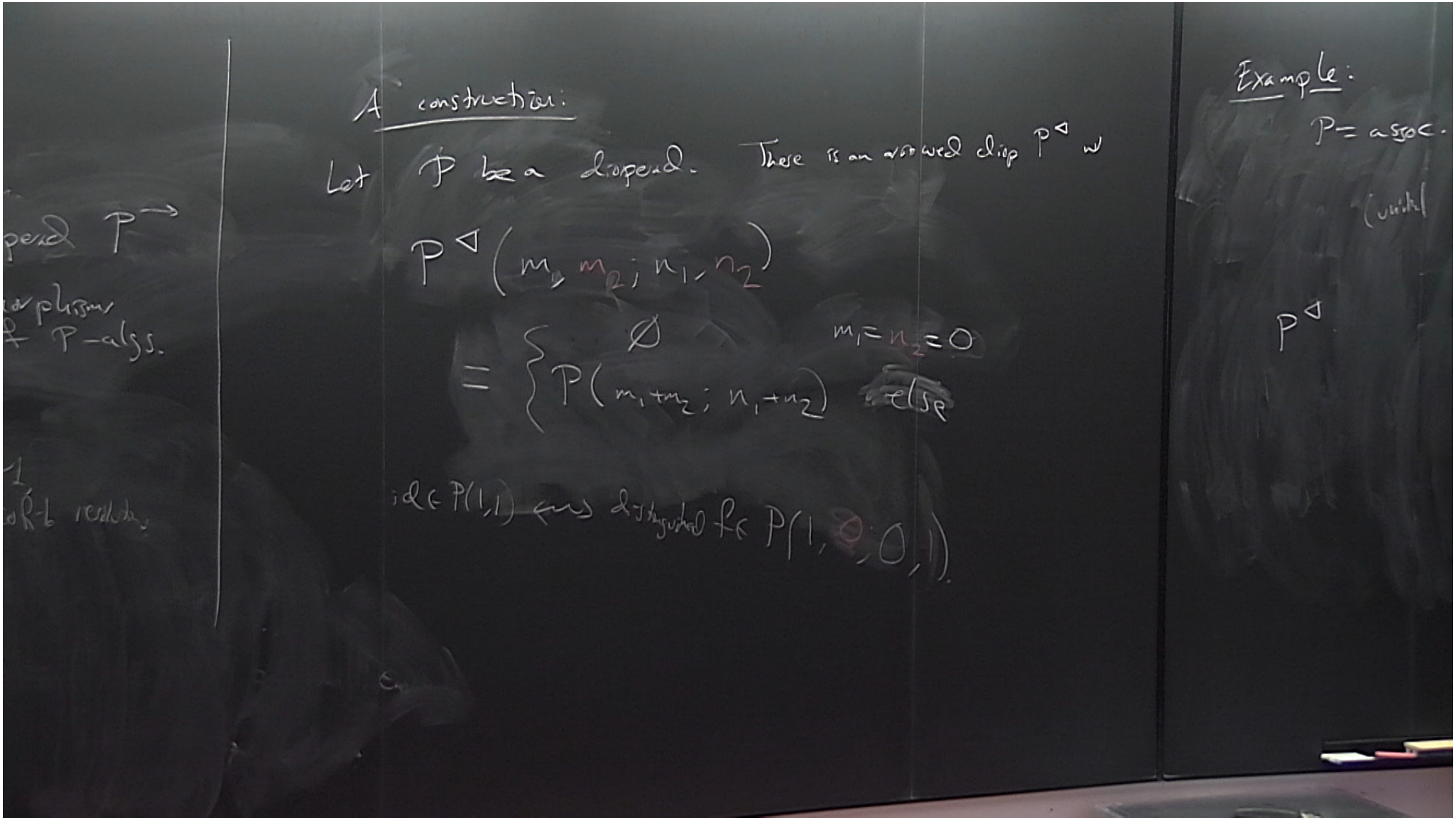
$V \xrightarrow{f} W$

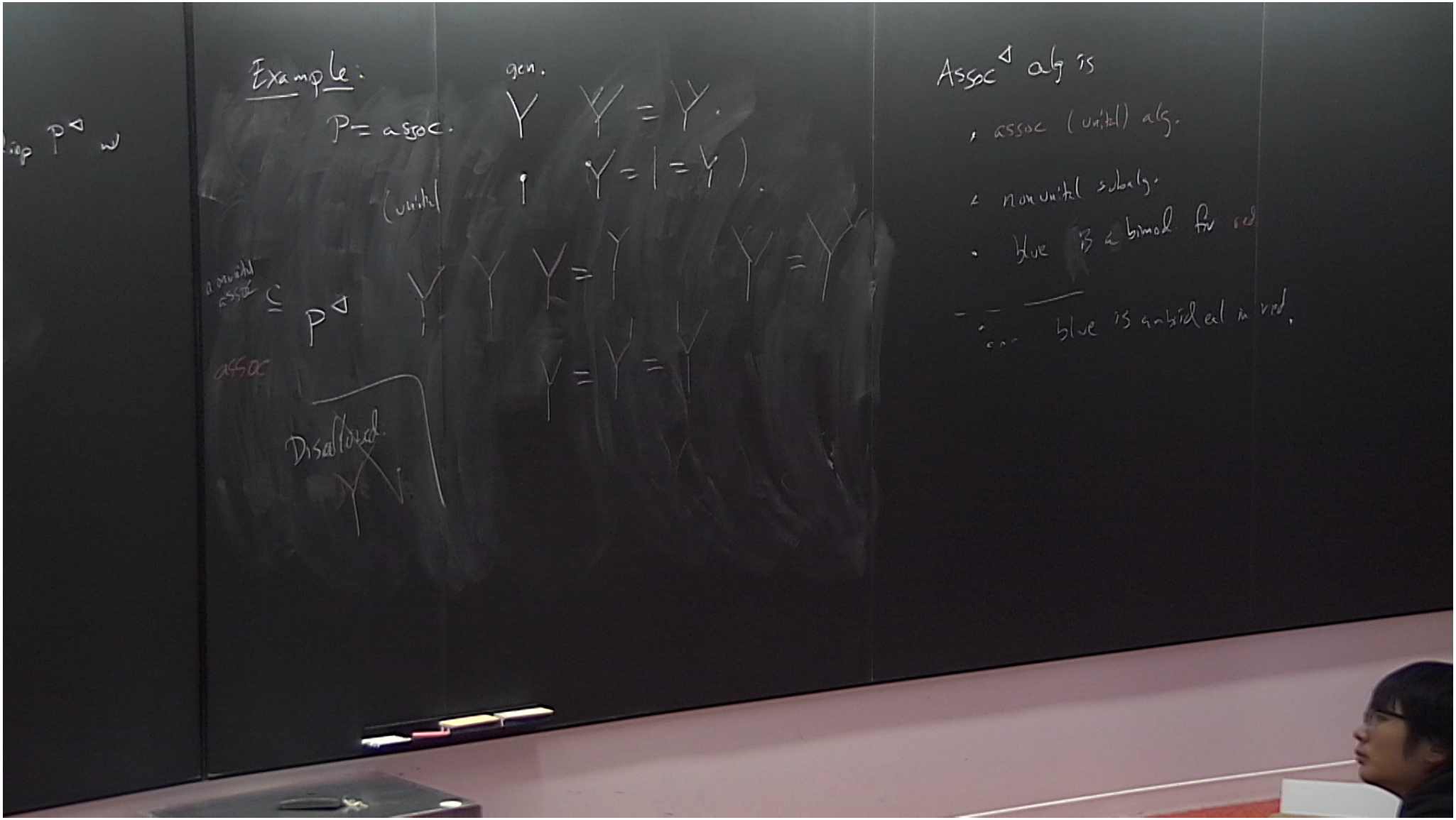
Sufficient technical conditions:

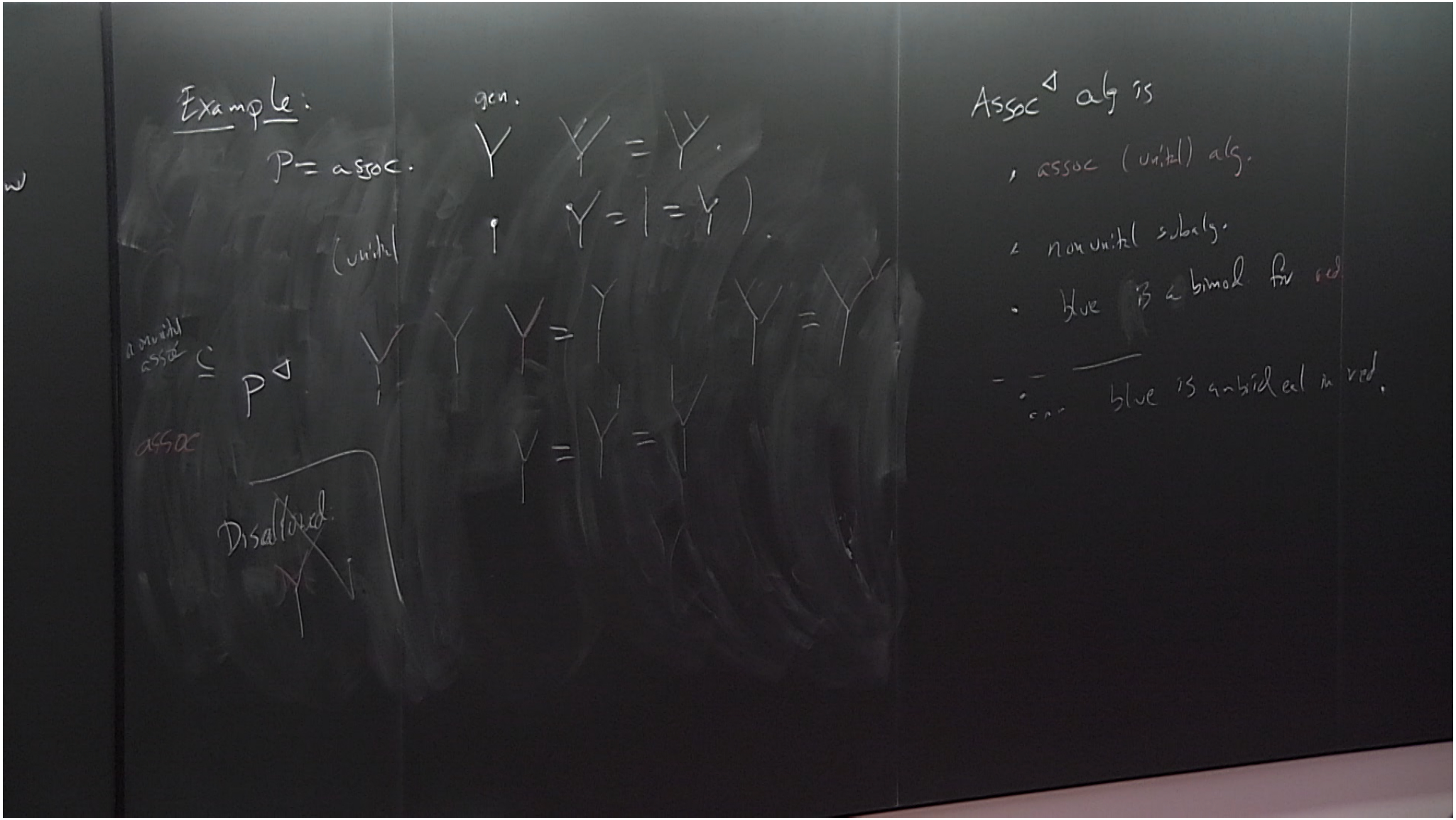
- disallow no IN or no OUT at all
- only 1-1 maps of f.


E.g.
if you have an (∞,1)-operad P,
then there is an arrowed (∞,1)-operad P→
s.t. P→-algs = morphisms of P-algs.
N.B.: Even if P is quasi-free, P→
and no good formula for its
cofib resolution


rrowed diagrams
acterized by 4-tuples of ℕ's.
$m_1, m_2; n_1, n_2$)
ed element "f" $\in P(1,0;0,1)$)
acterized by two-colored
irected trees.
diagrams act on k-spaces.
a fixed diagram acts on
$V \xrightarrow{f} W$
Sufficient technical conditions.
- disallow no ins or no outs at all
- only 1-1 maps
f.
E.g.
if you
then there
s.t.
N.B: Even if
and
Defn. A mathematical object is Kosz if there is a fixed formula for its fibre resolution.


A construction:
Let $\mathcal{P}$ be a dioperad. There is an associated dioperad $\mathcal{P}^{\triangleleft}$ w
$\mathcal{P}^{\triangleleft}(m_1, m_2; n_1, n_2)$
$= \begin{cases} \emptyset & m_1 = n_2 = 0 \\ \mathcal{P}(m_1+m_2; n_1+n_2) & \text{else} \end{cases}$
$id \in \mathcal{P}(1,1)$ $\leftrightarrow$ distinguished $\in \mathcal{P}(1,0;0,1)$
Example:
$\mathcal{P}$ = assoc.
$\mathcal{P}^{\triangleleft}$


Example:
gen.
P = assoc.
(unital
Assoc^◁ alg is
, assoc (unital) alg.
< non-unital subalg.
blue is a bimod for red
∴ blue is an ideal in red.
Disallowed.


Example:
gen.
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Assoc alg is
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nonunital assoc
assoc
Disallowed.

A construction:

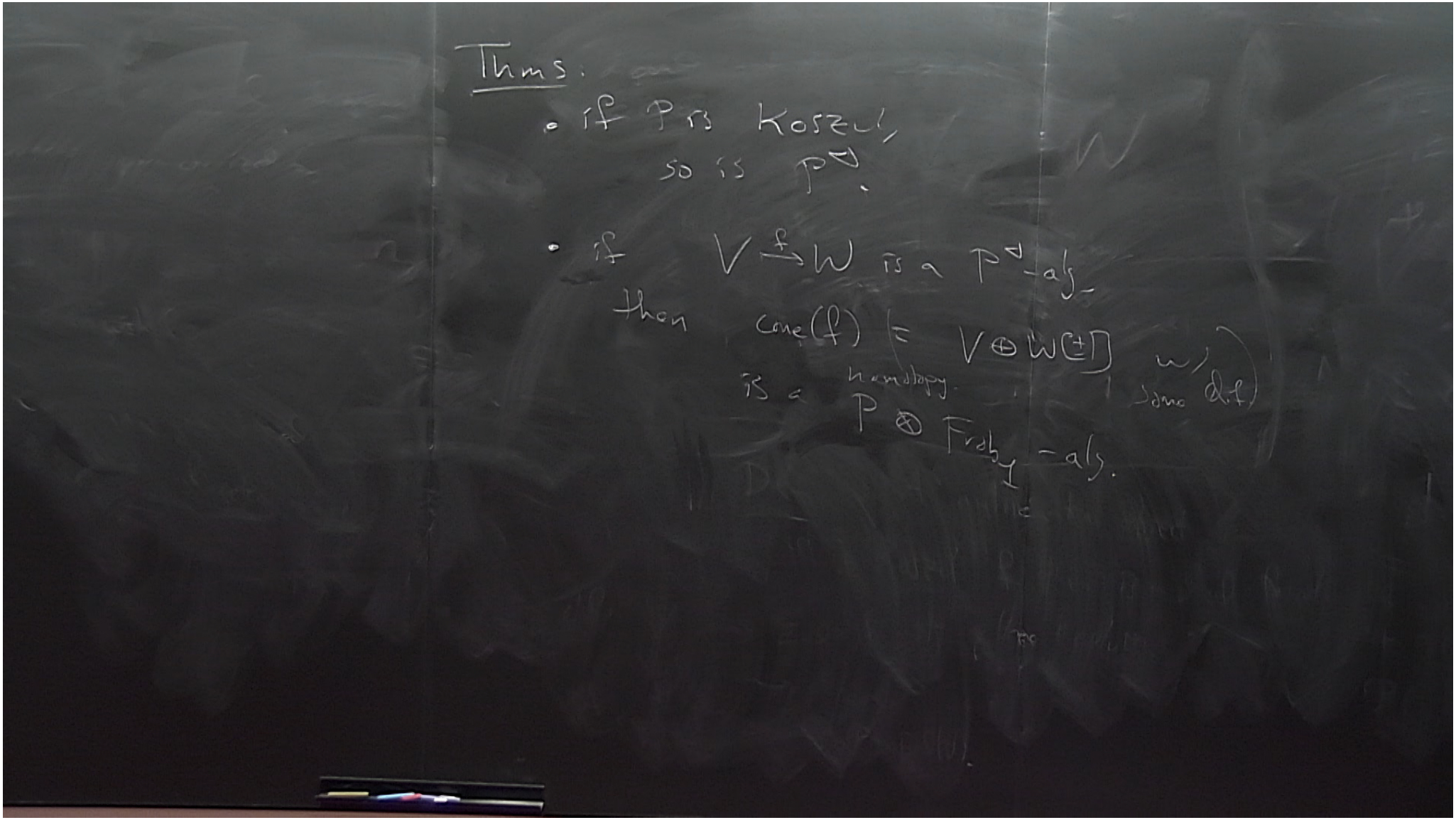
Let $\mathcal{P}$ be a dioperad. There is an augmented dioperad $\mathcal{P}^{\triangleleft}$ w

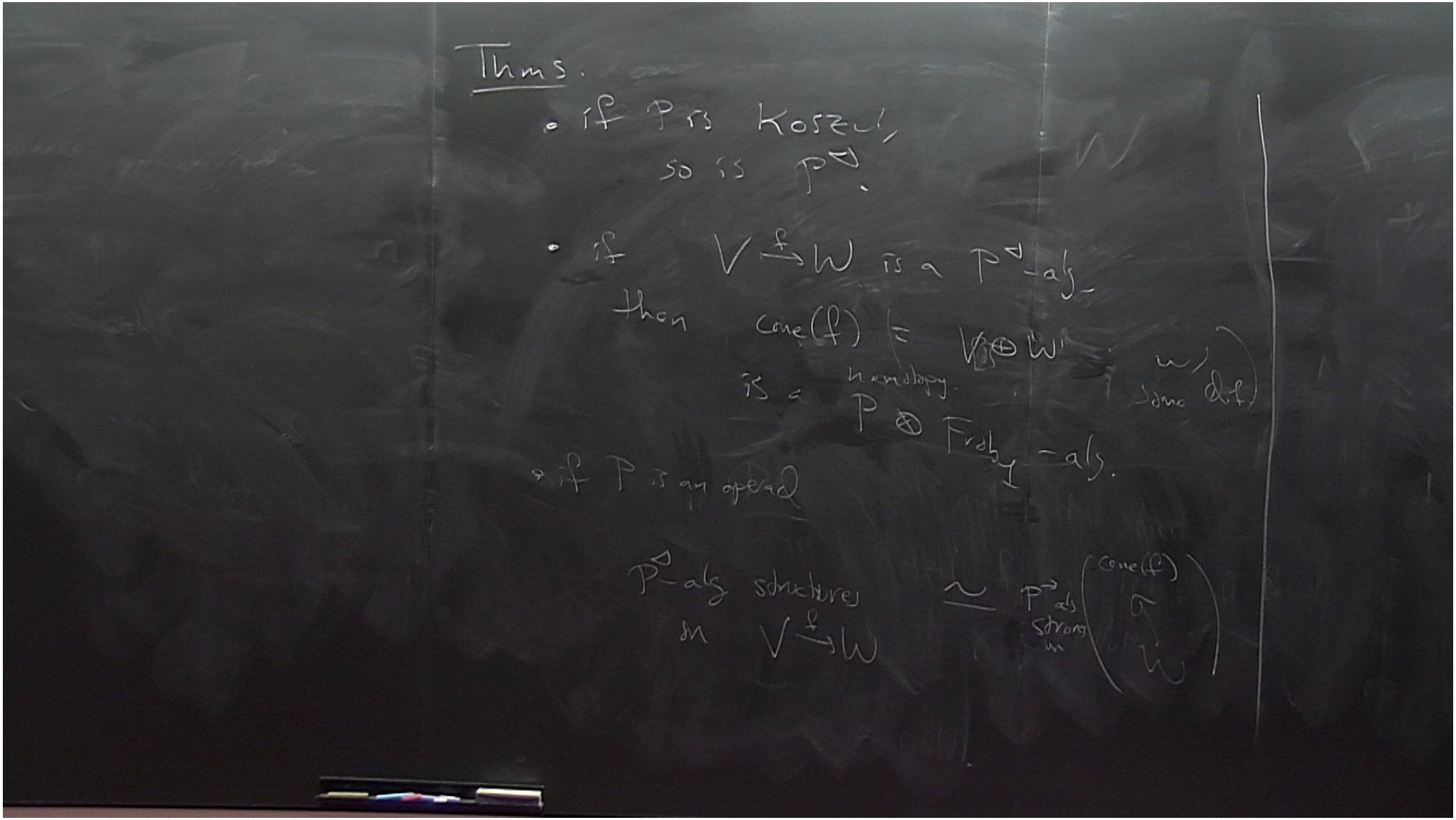
$$\mathcal{P}^{\triangleleft}(m_1, m_2; n_1, n_2)$$

$$= \begin{cases} \emptyset & m_1 = n_2 = 0 \\ \mathcal{P}(m_1 + m_2; n_1 + n_2) & \text{else} \end{cases}$$

$id \in \mathcal{P}(1,1) \iff$ distinguished $\in \mathcal{P}(1, 0; 0, 1)$.

$\mathcal{P}^{\triangleleft}$-algs $\cong$ $\mathcal{P}$-ideals


Thms.
• if P is Koszul,
so is $P^!$.
• if $V \xrightarrow{f} W$ is a $P^!$-alg.
then $\mathrm{cone}(f) = V \oplus W[\pm 1]$ w/
is a homotopy. some diff
$P \otimes \mathrm{Frob}_I$-algs.


Thms.
• if P is Koszul, so is P^{∇}.
• if $V \xrightarrow{f} W$ is a P^{∇}-alg. then cone(f) = $V \oplus W$ (with some diff) is a homotopy $P \otimes \mathrm{Frob}_1$-alg.
• if P is an operad
P^{∇}-alg structures on $V \xrightarrow{f} W$ $\simeq$ $P^{\rightarrow}$-alg strong on (cone(f), V, W)

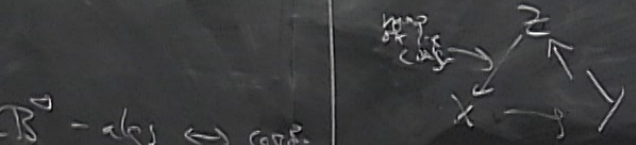
Lie algs ⟷ manifolds
inf. points

LB ⟷ Poisson manifolds.

I want to convince you that

LB^{∇} - algs ↔ cdgas.

Z is Poisson,
a cdga on Z should be:

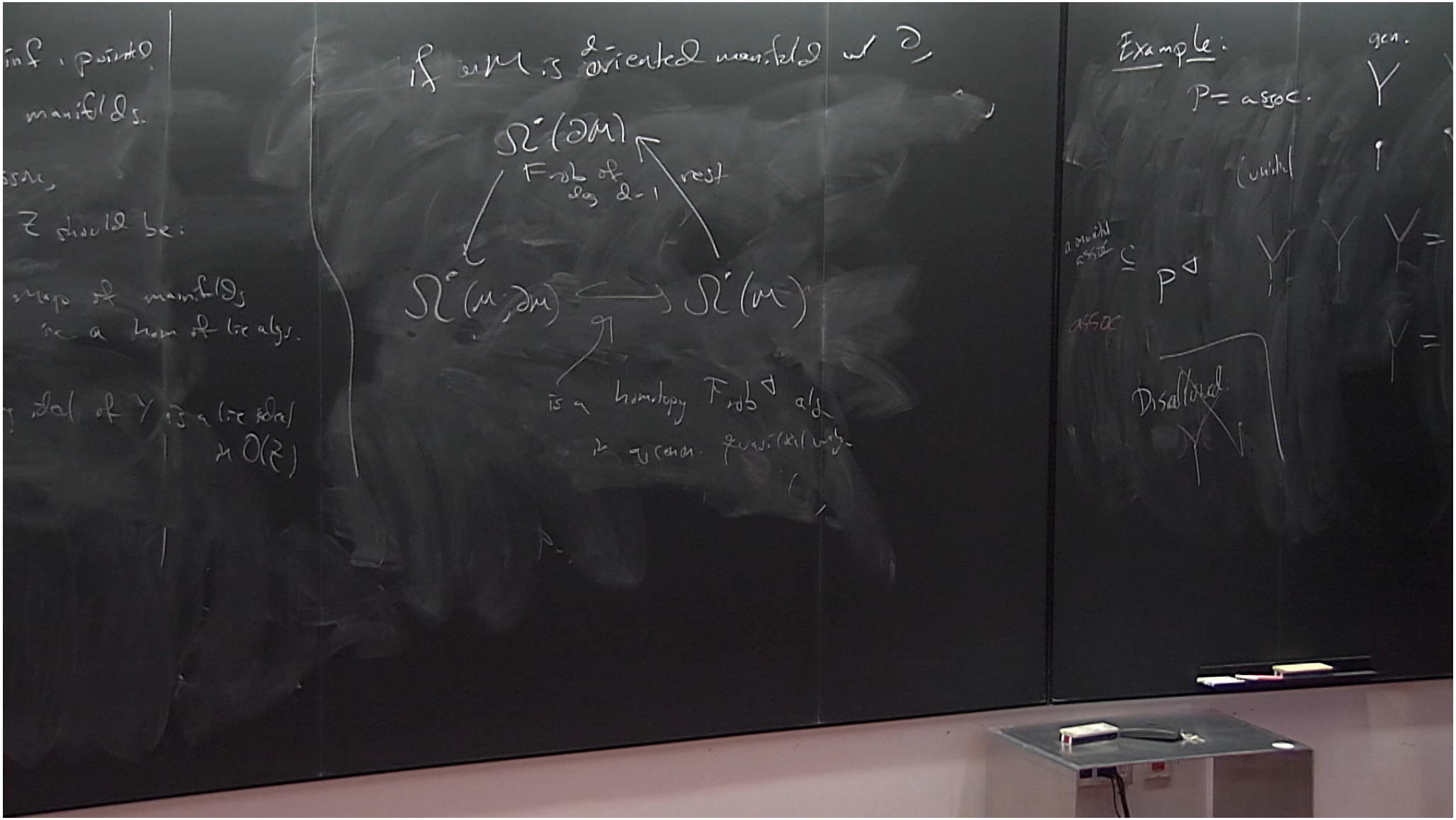


map of lie algs

maps of manifolds
ie. a map of lie algs.

also vanishing ideal of Y is a lie ideal in $\mathcal{O}(Z)$

... unpack


if ∂M is oriented manifold w ∂
$\Omega^\cdot(\partial M)$
Fr ob of dg $d-1$
rest
$\Omega^\cdot(M, \partial M)$
$\Omega^\cdot(M)$
is a homotopy Fr ob ∇ alg
Example:
gen.
P = assoc.
P^∇
Disallowed.