Title: No-free-information principle in general probabilistic theories

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Abstract: <p>In quantum theory, the no-information-without-disturbance and no-free-information principles express that those observables that do not disturb the measurement of another observable and those that can be measured jointly with any other observable must be trivial, i.e., coin tossing observables. We show that in the framework of general probabilistic theories these principles do not hold in general. In this way, we obtain characterizations of the probabilistic theories where these principles hold and we show that the two principles are not equivalent. </p>

No-free-information principle in general probabilistic theories

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The three principles

1. No broadcasting
   You can't make two coffes out of one.

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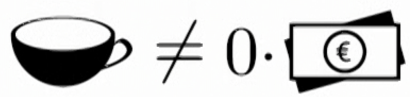
2. No information without disturbance
   You can't have the coffee for free.



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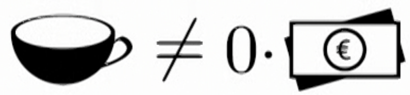
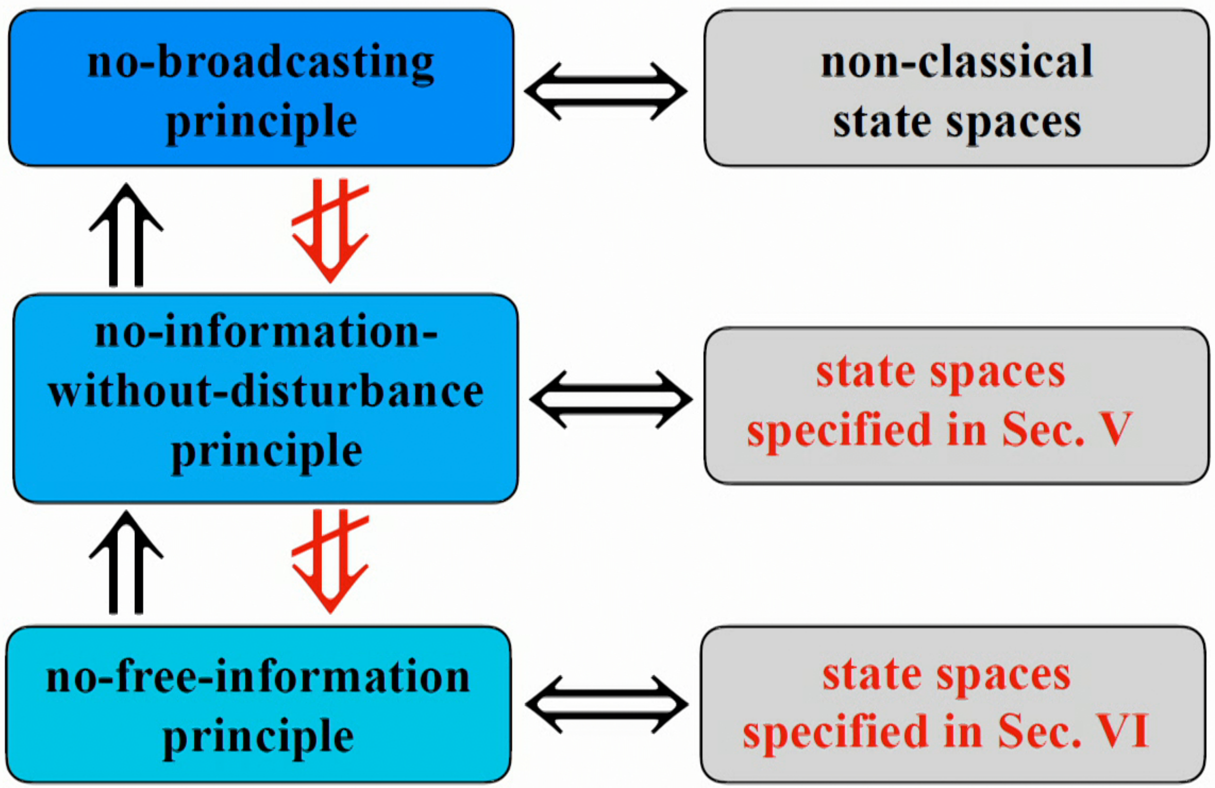


3. No free information
   You can't have the coffee for free even if you pay for the lunch.

The results



no-broadcasting principle
non-classical state spaces
no-information-without-disturbance principle
state spaces specified in Sec. V
no-free-information principle
state spaces specified in Sec. VI

GPT refresher

- K: state space, compact convex subset of $\mathbb{R}^n$
- $A(K)$: linear space of affine functions $f : K \to \mathbb{R}$
- $A(K)^+$: cone of positive affine functions, $f(x) \geq 0$, $\forall x \in K$
- $E(K)$: effect algebra of affine functions $f : K \to [0,1]$
- $A(K)^*$: dual vector space to $A(K)$
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In quantum theory:

- K: density matrices over finite dimensional Hilbert space
- $A(K)$: self-adjoint operators, $A(\rho) = \mathrm{Tr}(\rho A)$
- $A(K)^+$: cone of positive semi-definite matrices, $A \geq 0$
- $E(K)$: set of effects, $0 \leq A \leq \mathbb{1}$
- $A(K)^*$: self-adjoint operators
- $A(K)^{*+}$: cone of positive semi-definite matrices, $A \geq 0$

Measurements

Ω: sample space, finite measurable set of outcomes
$\mathcal{P}(\Omega)$: set of probability measures on Ω, simplex

Definition
Measurement is an affine map

$$m : K \to \mathcal{P}(\Omega).$$

Let $A \subset \Omega$, measurable
$m(x; A) =$ measure of the set A with respect to measure $m(x)$

Compatibility of measurements

Ω_1, Ω_2 sample spaces

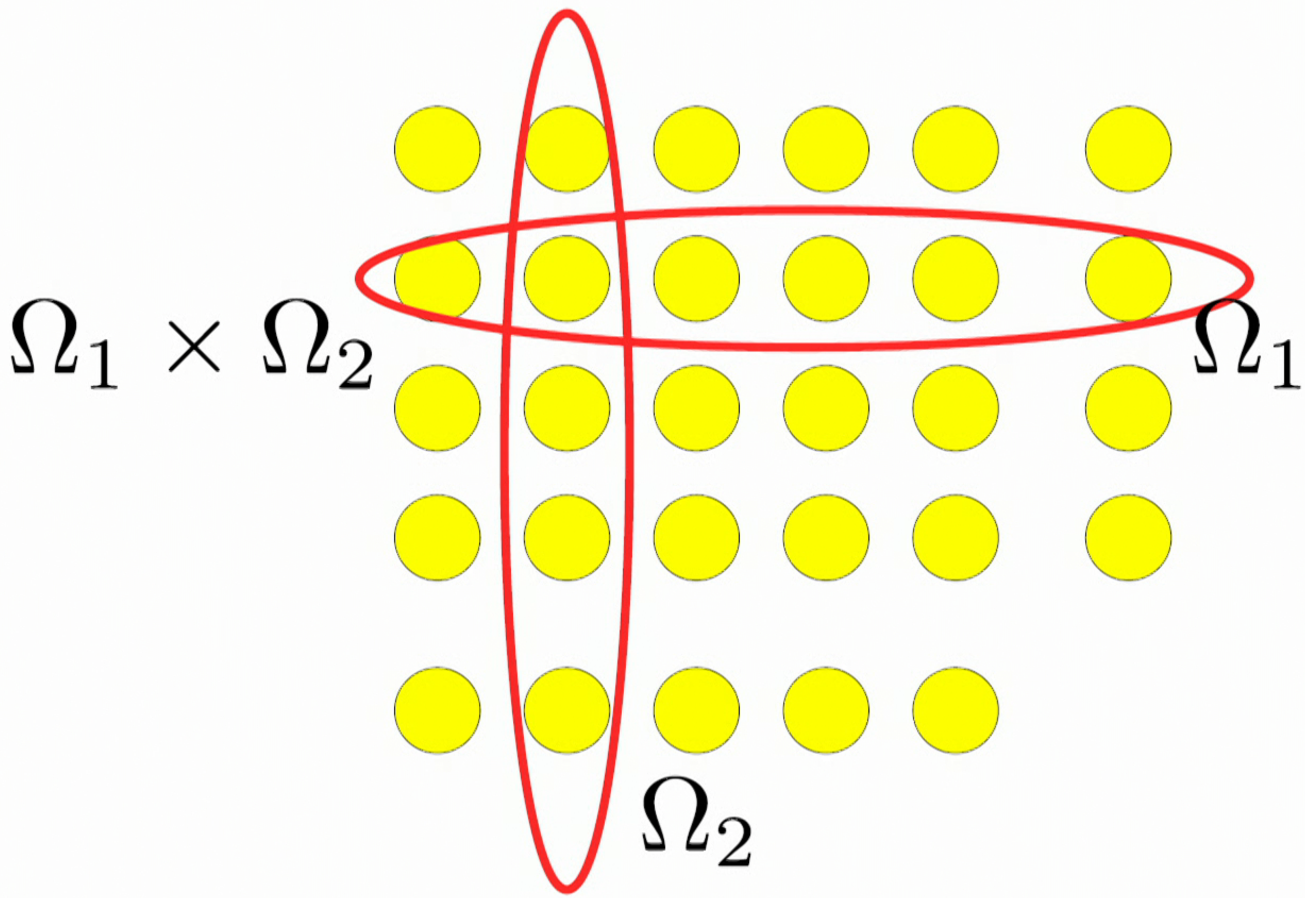
Definition
Measurements

$$m_1 : K \to \mathcal{P}(\Omega_1)$$
$$m_2 : K \to \mathcal{P}(\Omega_2)$$

are compatible if there exists $m : K \to \mathcal{P}(\Omega_1 \times \Omega_2)$ such that for all $A_1 \subset \Omega_1$, $A_2 \subset \Omega_2$ and $\forall x \in K$

$$m_1(x; A_1) = m(x; A_1 \times \Omega_2),$$
$$m_2(x; A_2) = m(x; \Omega_1 \times A_2).$$

Compatibility of measurements



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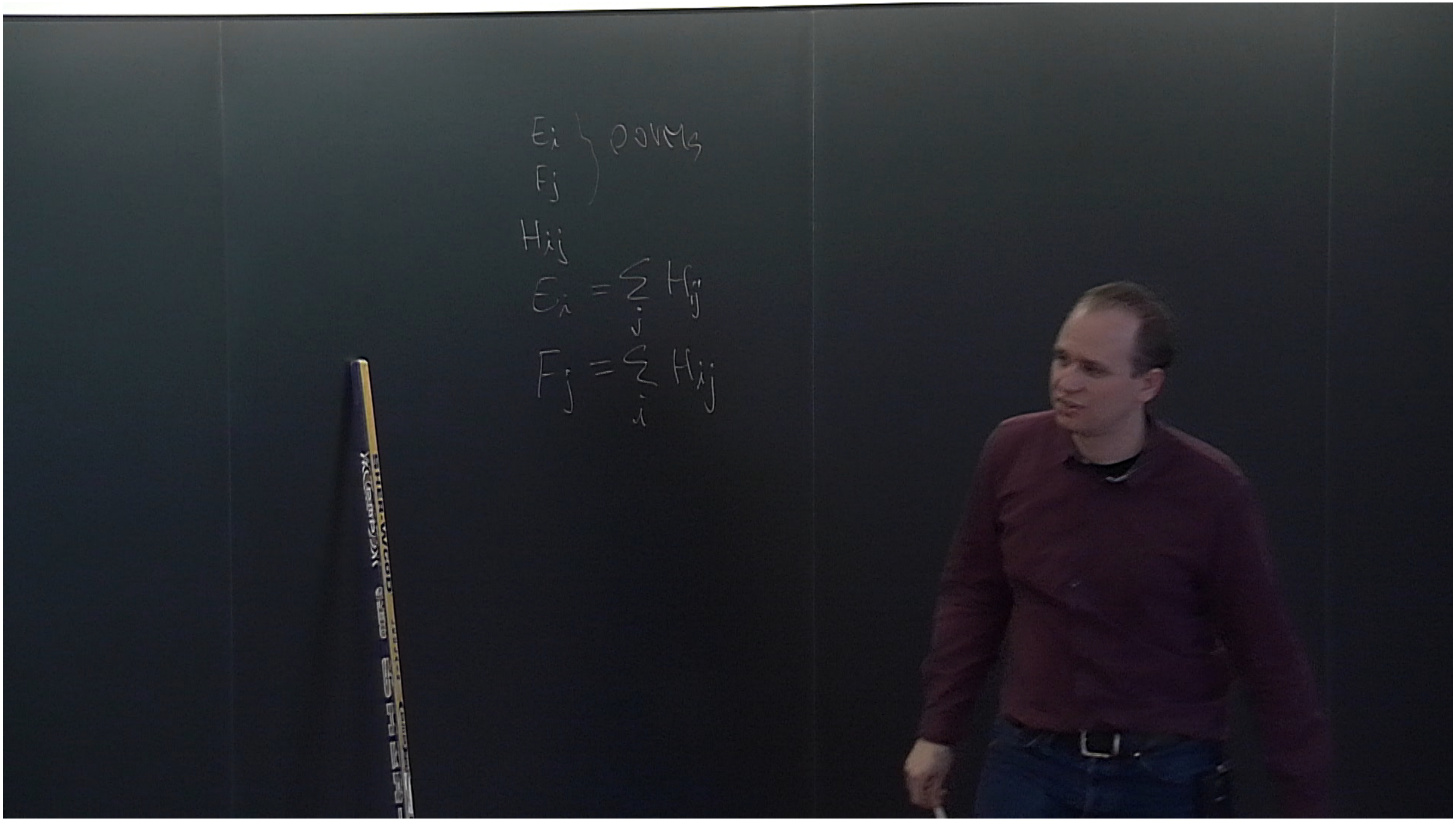
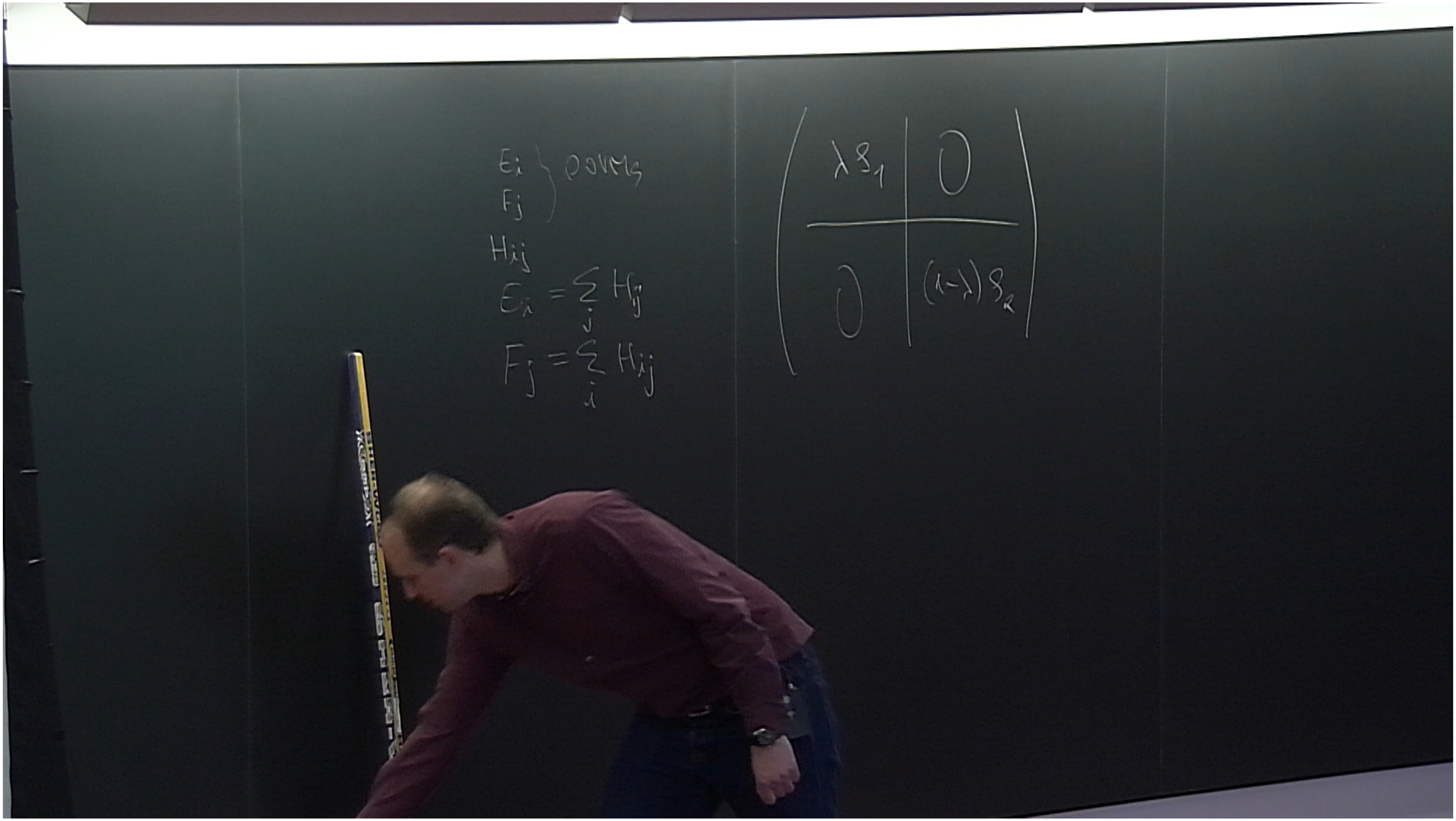

E_i, F_j } povms
H_{ij}
$E_i = \sum_j H_{ij}$
$F_j = \sum_i H_{ij}$

Direct convex hull

Definition

Let K_A, K_B be state spaces, then $K_A \oplus K_B$ is a state space of ordered and weighted pairs of states, i.e.

$$K_A \oplus K_B = \{(\lambda x, (1 - \lambda) y) : x \in K_A, y \in K_B, \lambda \in [0, 1]\}.$$


Ei } povms
Fj
Hij
Ei = Σ_j Hij
Fj = Σ_i Hij
λρ1 | 0
0 | (1-λ)ρ2

No infomation without disturbance

Theorem
There exists nontrivial observable compatible with the identity channel if and only

$$K = \oplus_{i=1}^{n} K_i.$$

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The main idea of the proof is that the measurement has to be constant on the sets K_i. □

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Corollary

Let τ be a coin-toss measurement, then m is compatible with the identity channel if and only if $\lambda m + (1-\lambda)\tau$ is compatible with the identity channel for all $\lambda \in [1, 0)$.

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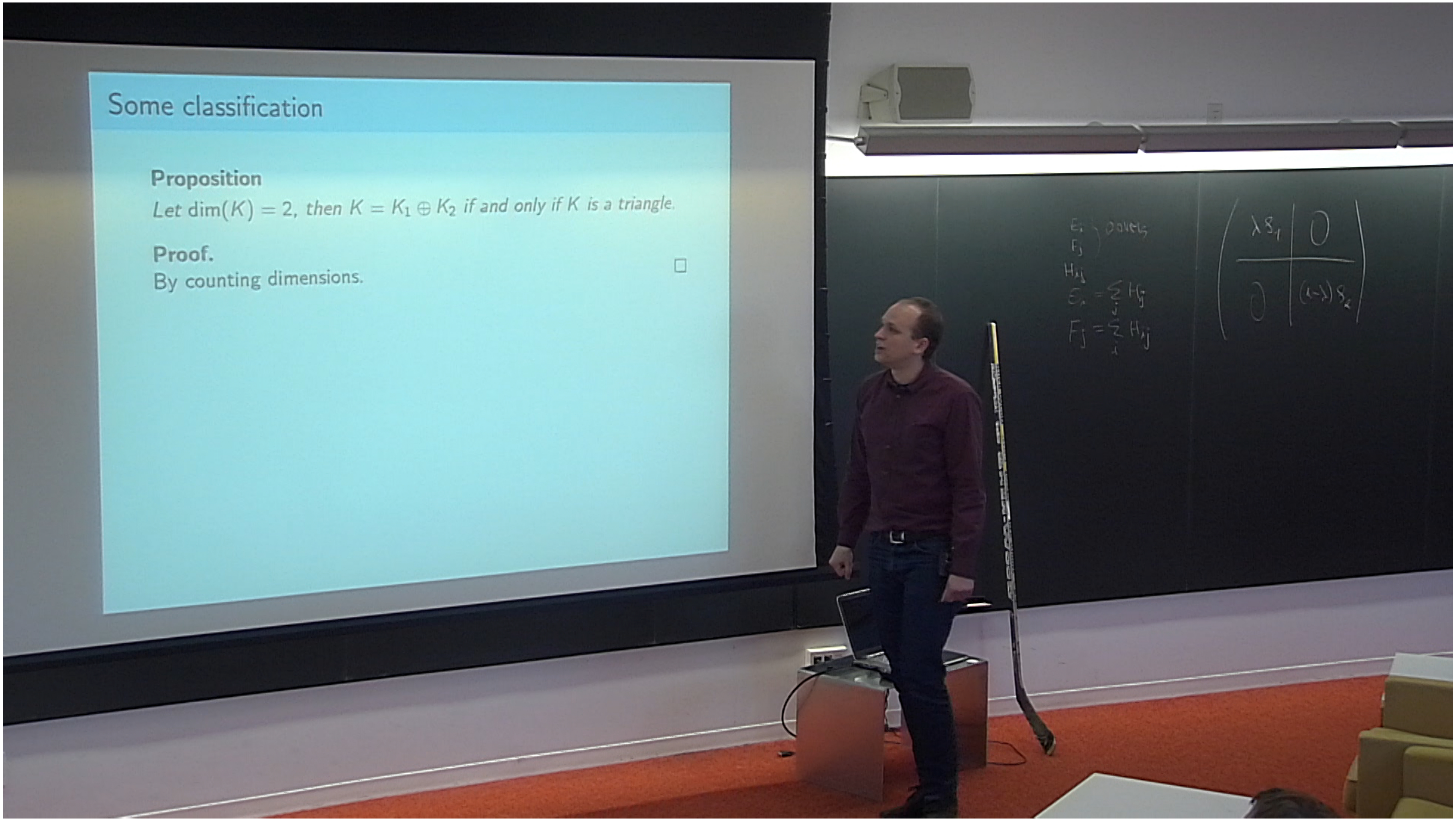
Noise does not help!


Some classification
Proposition
Let $\dim(K) = 2$, then $K = K_1 \oplus K_2$ if and only if K is a triangle.
Proof.
By counting dimensions.

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Let $\dim(K) = 2$*, then* $K = K_1 \oplus K_2$ *if and only if* K *is a triangle.*

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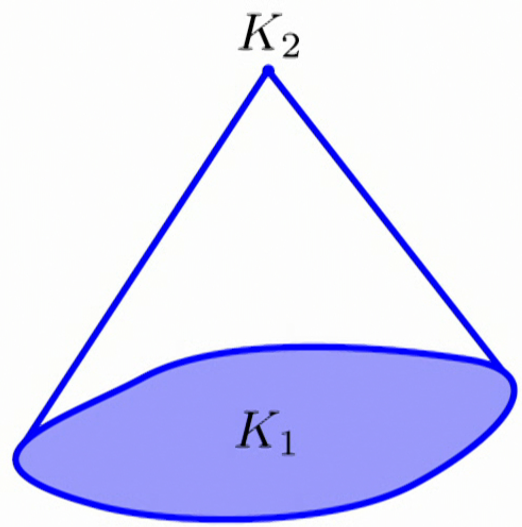
By counting dimensions. □

Proposition

Let $\dim(K) = 3$, *then* $K = K_1 \oplus K_2$ *if and only if* $K_2 = \{x\}$ *and* K *is pyramid-shaped.*



Post-processings and convex combinations

Let m_1, m_2 be measurements

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Post-processing is a map $\nu : \mathcal{P}(\Omega_2) \to \mathcal{P}(\Omega_1)$ that allows us to construct measurement

$$m_2' = \nu \circ m_2 : K \to \mathcal{P}(\Omega_1)$$

Post-processing is a form of order that gives rise to an equivalence.

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Definition
For $\lambda \in [0, 1]$ we define convex combination of measurements as the map

$$(\lambda m_1 + (1 - \lambda) m_2') : K \to \mathcal{P}(\Omega_1)$$

Simulability of measurements

Definition

Measurements $m_1, \ldots, m_k$ simulate a measurement m if there are

- post-processing $\nu_1, \ldots, \nu_k$
- numbers $\lambda_1, \ldots, \lambda_k$, s.t. $\lambda_i \in [0, 1]$ for $i = 1, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$

such that

$$m = \sum_{i=1}^{k} \lambda_i \nu_i \circ m_i$$

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Definition

We say that a measurement m is simulation irreducible if it can be simulated only by post-processing equivalent measurements.

No free information

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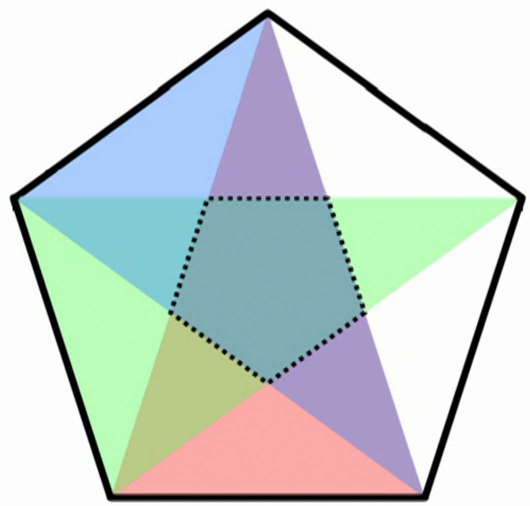

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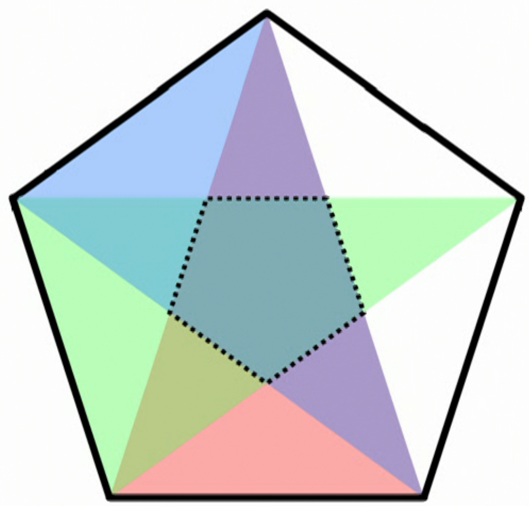
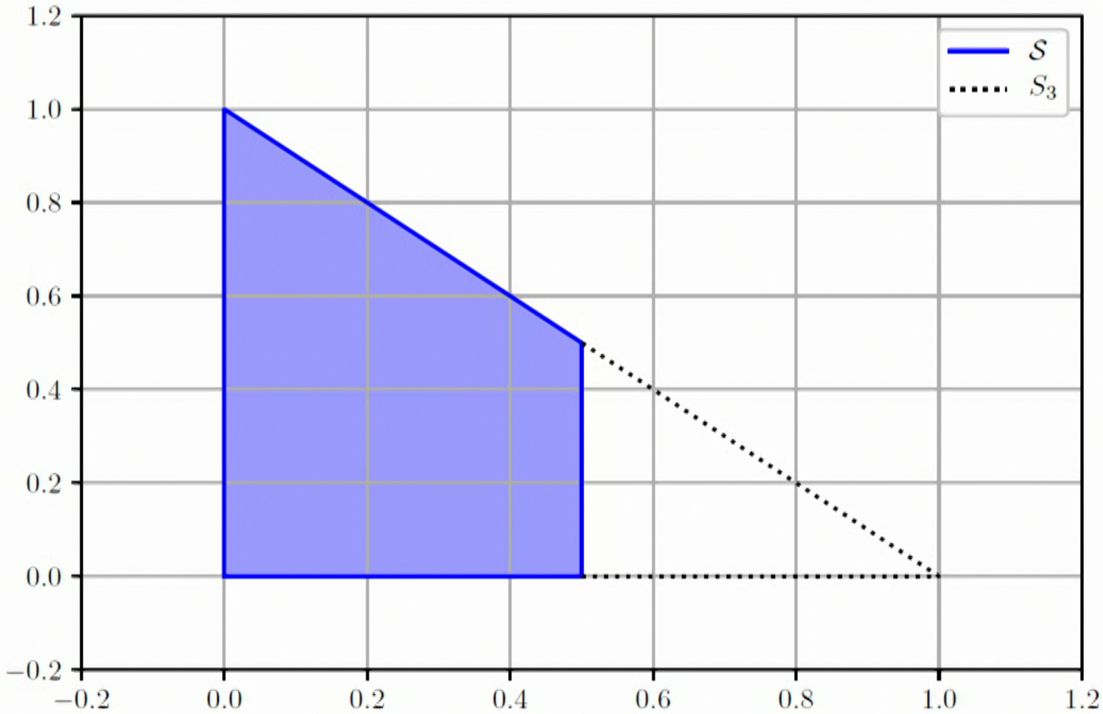


Noise helps! (sometimes)

Example: measurement compatible with all measurements

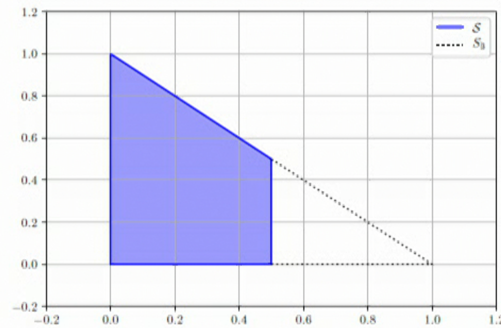


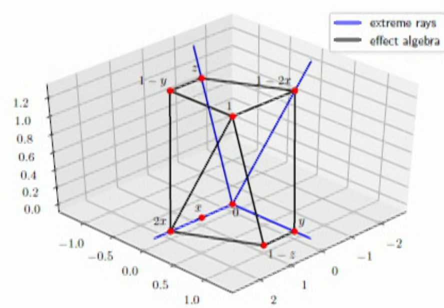
Example: measurement compatible with all measurements

Simulation irreducible measurements:

$$m_2 = 2x \otimes \delta_1 + (1 - 2x) \otimes \delta_2$$

$$m_3 = x \otimes \delta_1 + y \otimes \delta_2 + (1 - x - y) \otimes \delta_3$$





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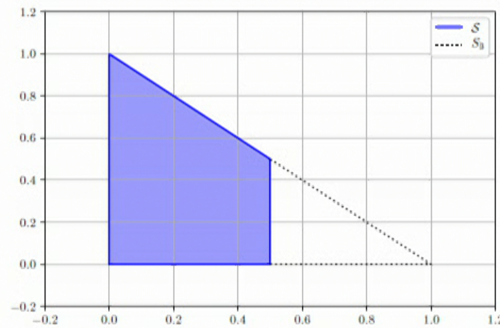

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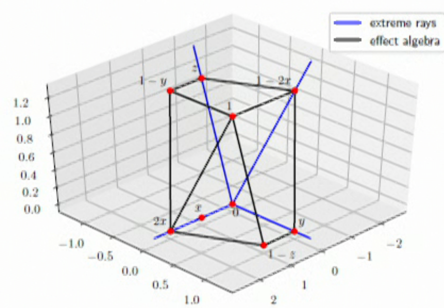

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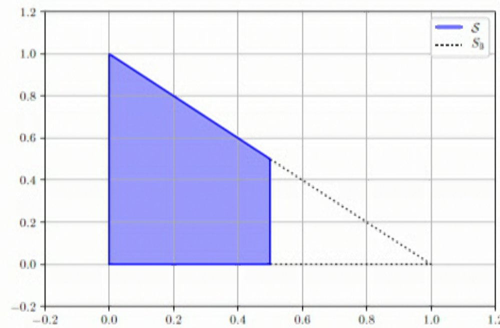

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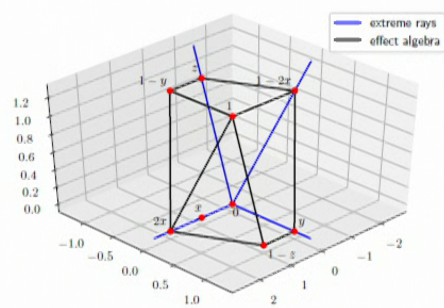
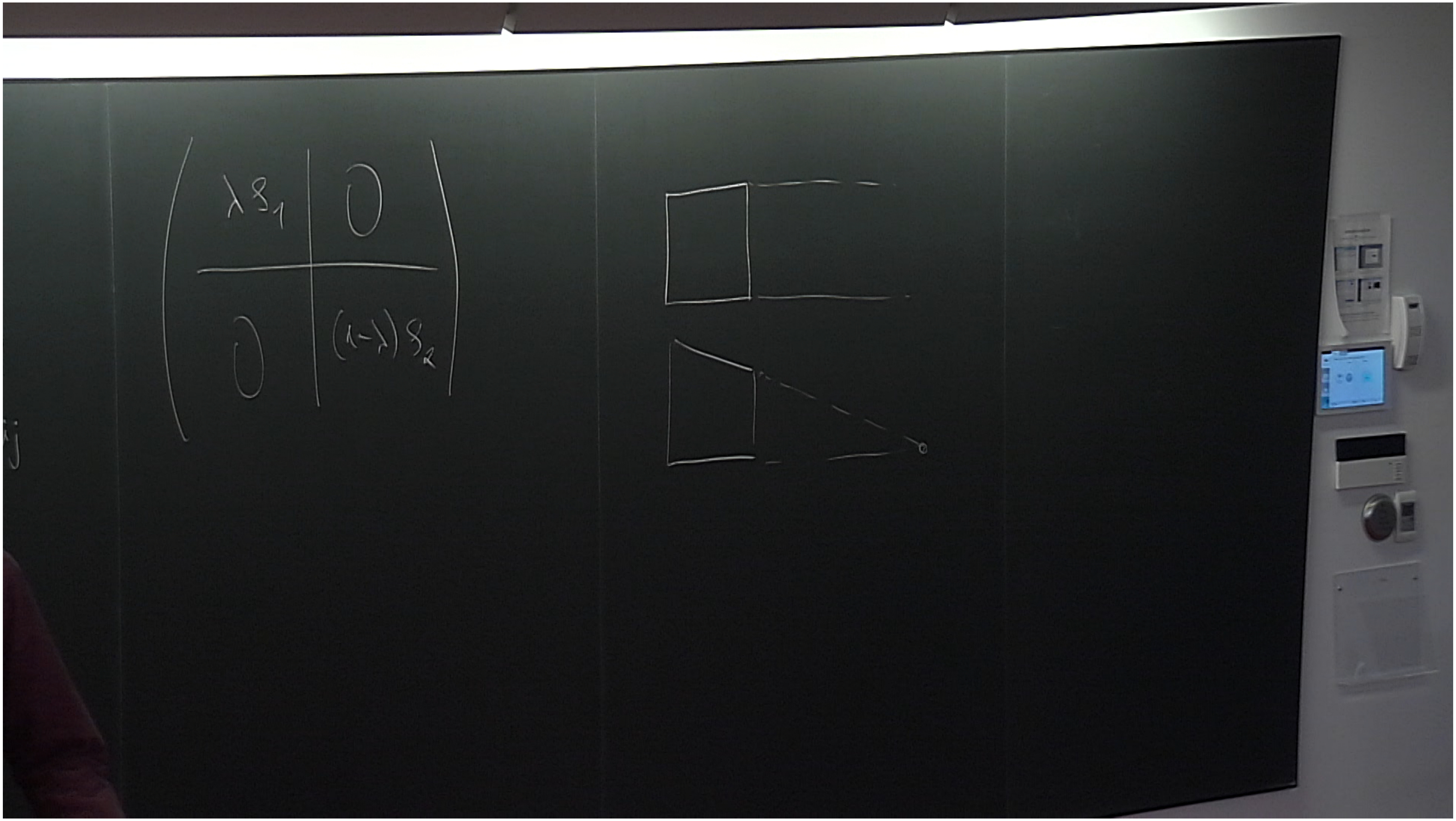
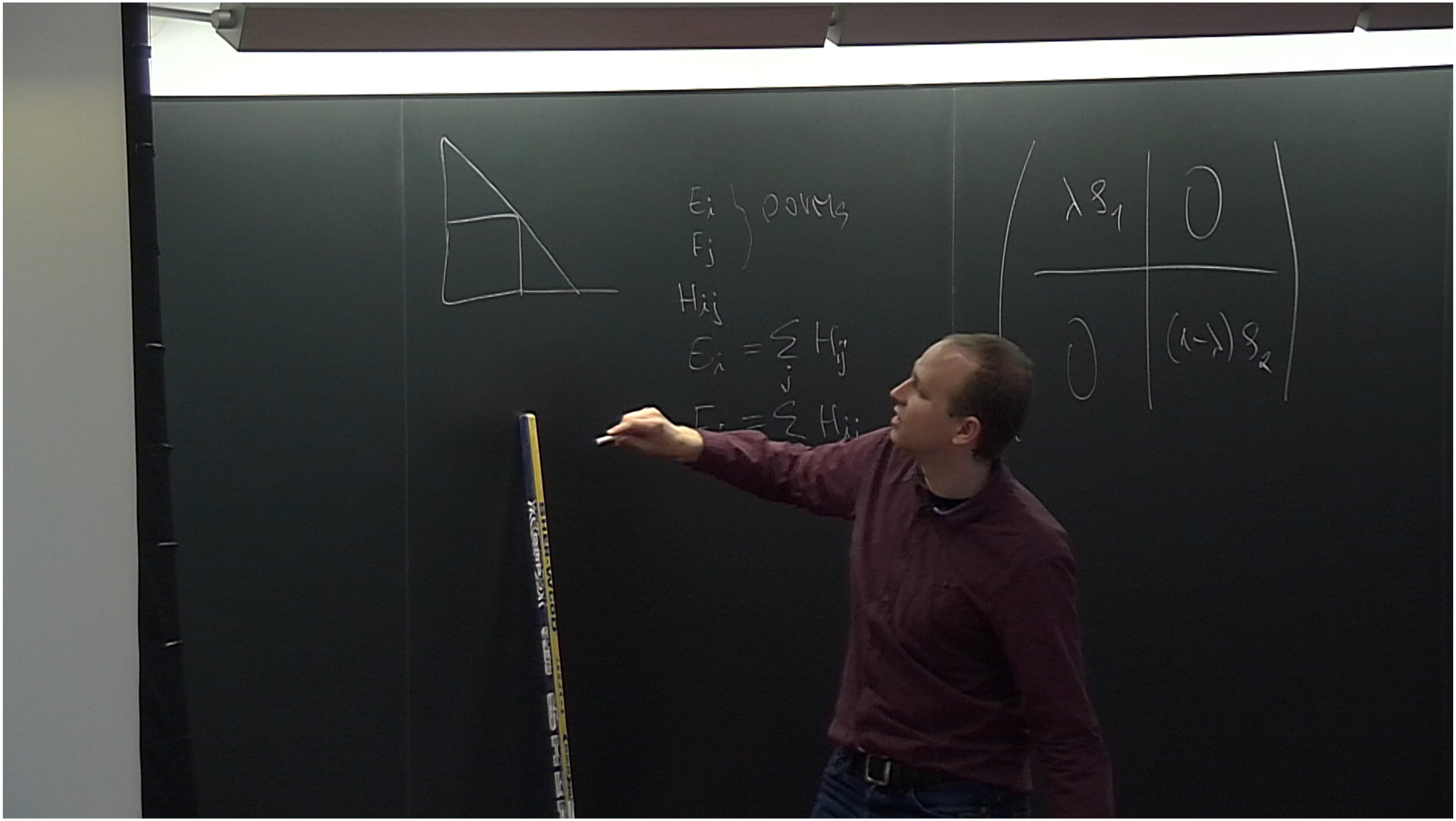
The joint measuremnet

$$m_2' = x \otimes \delta_1 + x \otimes \delta_2 + (1 - x) \otimes \delta_3$$




λϱ1
0
0
(1-λ)ϱ2



Ei } povms
Fj
Hij
Ei = Σj Hij
λρ1 | 0
0 | (1-λ)ρ2

Thank you for your attention!

Questions?
Comments?