

Title: PSI 2018/2019 - Quantum Field Theory I - Lecture 13

Date: Dec 03, 2018 10:00 AM

URL: <http://pirsa.org/18120000>

Abstract:

Gauge Theories: continued

- Rules Covariant derivatives & Action
- Perturbation theory & Gauge Fixing

Group $G = SU(2)$

generators $t_a = \frac{1}{2} \sigma_a \quad a=1,2,3$

σ_a Pauli Matrices

3 currents

3 vector fields $A_\mu^a(x) \quad a=1,2,3$ Adjoint Representation

$$A_\mu(x) = A_\mu^a(x) t_a$$

"Matter fields" $\phi = (\phi_i), \quad \psi = (\psi_i)$
complex

Fundamental representation
 $i=1,2$

Gauge transformations (infin't.)

$$g(x) = 1 + i \alpha^a(x) t_a$$

$$\phi_i(x) \rightarrow \phi_i(x) + i \alpha^a(x) t_a \cdot \phi_i(x) \quad \text{Fund.}$$

$$\phi_{Adj}(x) \rightarrow \phi_{Adj}(x) + i [\alpha(x), \phi_{Adj}(x)] \quad Adj$$

Group $G = SU(2)$

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"Matter fields" $\phi = (\phi_i)$, $\psi = (\psi_i)$ Fundamental representation
complex $i=1,2$

Gauge transformations (infinite) $\alpha(x) = \alpha^a(x) t_a$

$$g(x) = 1 + i \alpha^a(x) t_a \quad \text{local}$$

1, 2, 3 ϕ_F

$$\phi_i(x) \rightarrow \phi_i(x) + i \alpha^a(x) t_a \cdot \phi_i(x) \quad \text{Fund. Gauge transformations}$$

$$\phi_{Adj}(x) \rightarrow \phi_{Adj}(x) + i [\alpha(x), \phi_{Adj}(x)] \quad \text{Adj}$$

Covariant derivatives (w.r.t. local gauge transformations)

$$D_\mu \phi_F(x) = \frac{\partial}{\partial x^\mu} \phi_F(x) - i A_\mu(x) \phi_F(x) \quad \phi_F(x) = (\phi_i(x)) \quad \text{Fund.}$$

$$D_\mu \phi_{Adj}(x) = \frac{\partial}{\partial x^\mu} \phi_{Adj}(x) - i [A_\mu(x), \phi_{Adj}(x)] \quad \text{Adj}$$

Local

Gauge transformation for the gauge field

$SU(2)$ nonabelian

$$A_\mu(x) \rightarrow A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \partial_\mu \alpha(x) - i [A_\mu(x), \alpha(x)]$$

2] Field strength : covariant object $\in$ Adj Representation

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu(x), A_\nu(x)] = i[D_\mu, D_\nu]$$

antisymmetric tensor w.r to Lorentz Indices

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

can be written as $F_{\mu\nu}(x) = F_{\mu\nu}^a(x) t_a$ 3 components

In term of the structure constants of $SU(2)$ $f_{ab}^c \rightarrow \epsilon$ Levi-Civita symbol

$$[t_a, t_b] = f_{ab}^c t_c$$

Group $G = SU(2)$

generators $t_a = \frac{1}{2} \sigma_a$ $a=1,2,3$ ϕ_F

σ_a Pauli Matrices

3 currents

3 vector fields $A_\mu^a(x)$ $a=1,2,3$ Adjoint Representation

$$A_\mu(x) = A_\mu^a(x) t_a$$

α are Dirac Indices

"Matter fields" $\phi = (\phi_i)$, $\psi = (\psi_i^\alpha)$ Fundamental representation

complex
scalar

complex
spinor = Dirac

$i=1,2$

Local

Gauge transformations (infinite)

$$\alpha(x) = \alpha^a(x) t_a$$

Lorentz vector

$$g(x) = 1 + i \alpha^a(x) t_a \quad \text{local}$$

Spin 1

$$\phi_i(x) \rightarrow \phi_i(x) + i \alpha^a(x) t_a \cdot \phi_i(x) \quad \text{Fund. Gauge transformations} \quad \boxed{1}$$

$$\phi_{Adj}(x) \rightarrow \phi_{Adj}(x) + i [\alpha(x), \phi_{Adj}(x)] \quad \text{Adj}$$

Covariant derivatives (w.r.t. local gauge transformations)

$$D_\mu \phi_F(x) = \frac{\partial}{\partial x^\mu} \phi_F(x) - i A_\mu(x) \phi_F(x) \quad \phi_F^{(1)} = (\phi_i(x)) \quad \text{Fund.}$$

$$D_\mu \phi_{Adj}(x) = \frac{\partial}{\partial x^\mu} \phi_{Adj}(x) - i [A_\mu(x), \phi_{Adj}(x)] \quad \text{Adj}$$

SU(2) non abelian

Gauge transformation for the gauge field

$$A_\mu(x) \rightarrow A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \boxed{\partial_\mu \alpha(x)} - i [A_\mu(x), \alpha(x)]$$

↓

Fields

$$\boxed{F_{\mu\nu}}$$

antisymmetric

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

can be written

in terms of

$$[t_a, t_b]$$

2] Field strength : covariant object $\in$ Adj Representation

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu(x), A_\nu(x)] = i[D_\mu, D_\nu]$$

antisymmetric tensor w.r to Lorentz Indices

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

can be written as $F_{\mu\nu}(x) = F_{\mu\nu}^a(x) t_a$ 3 components

in term of the structure constants of $SU(2)$ $f_{ab}^c = \epsilon$ Levi-Civita symbol ϵ_{abc}

$$[t_a, t_b] = f_{ab}^c t_c$$

Similar to Maxwell = 0!!!

Gauge transformation

$$F_{\mu\nu}(x) \rightarrow F_{\mu\nu}(x) + i [\alpha(x), F_{\mu\nu}(x)]$$

$SU(2)$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{abc} A_\mu^b A_\nu^c$$

$a=1,2,3$

$$F_{\mu\nu}^a \rightarrow \vec{E}^a, \vec{B}^a \text{ electric \& magnetic fields}$$

$$A_\mu^a = (V^a, \vec{A}^a) \text{ Potential and vector potential}$$

example:

non linear term

$$\vec{E}^1 = \frac{\partial}{\partial t} \vec{A}^1 - \vec{\nabla} \cdot V^1 + \overbrace{V^2 \vec{A}^3 - V^3 \vec{A}^2}$$

$$\vec{B}^1 = \vec{\nabla} \times \vec{A}^1 + \vec{A}^2 \times \vec{A}^3$$

: covariant object $\in$ Adj Representation

$$\partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu(x), A_\nu(x)] = i [D_\mu, D_\nu]$$

non w.r to Lorentz Indices

non-linear term

$$F_{\mu\nu} = F_{\mu\nu}^a t_a \quad 3 \text{ components}$$

structure constants of $SU(2)$ $f_{ab}^c = \epsilon$ Levi-Civita symbol ϵ_{abc}

Gauge transformation

$$F_{\mu\nu}(x) \rightarrow F_{\mu\nu}(x) + i [\alpha(x), F_{\mu\nu}(x)]$$

$SU(2)$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon_{abc} A_\mu^b A_\nu^c$$

$a=1,2,3$

$$F_{\mu\nu}^a \rightarrow \vec{E}^a, \vec{B}^a \quad \text{electric \& magnetic}$$

$$A_\mu^a = (V^a, \vec{A}^a) \quad \text{Potential and vector}$$

$$F_{\mu\nu}(x)$$

example:

non linear term

$$\vec{E}^1 = \frac{\partial \vec{A}^1}{\partial t} - \vec{\nabla} \cdot \vec{V}^1 + \overbrace{V^2 A^3 - V^3 A^2}^{2 \rightarrow 3 \quad 3 \rightarrow 2}$$

$$\vec{B}^1 = \vec{\nabla} \times \vec{A}^1 + \vec{A}^2 \times \vec{A}^3$$

Action For the Gauge-Field
 $\Rightarrow$ "Maxwell Equations"

$$S_{\text{gauge}}[A] = -\frac{1}{2g^2} \int d^4x \operatorname{tr} [F_{\mu\nu}(x) F^{\mu\nu}(x)]$$

2x2 matrix

Adj. Repr

is invariant under
 local gauge transformations

0(1) gauge transformation

$$g(x) : M \rightarrow G = SU(2)$$

$$F_{\mu\nu}(x) \rightarrow g^{-1}(x) F_{\mu\nu}(x) g(x)$$

trace is ok for $SU(2)$ or $SU(N)$

General Group $\Rightarrow$ First Casimir $C_2(t_a, t_b)$

Dynamics at classical level $\rightarrow$ quantum

Interaction vertices and the propagator

Rescale $A_\mu \rightarrow g \bar{A}_\mu$, $\alpha \rightarrow g \bar{\alpha}$

$$\bar{A}_\mu \rightarrow \bar{A}_\mu + \partial_\mu \bar{\alpha} - i g [\bar{A}_\mu, \bar{\alpha}]$$

$$D_\mu \phi_F = \partial_\mu \phi_F - i g \bar{A}_\mu \cdot \phi_F$$

$$S_{\text{gauge}}[\bar{A}] = \int d^4x \mathcal{L}[\bar{A}] \quad \text{Lagrangian density}$$

$$\begin{aligned} \mathcal{L}[\bar{A}] = & -\frac{1}{4} (\partial_\mu \bar{A}_\nu^a - \partial_\nu \bar{A}_\mu^a) (\partial^\mu \bar{A}_a^\nu - \partial^\nu \bar{A}_a^\mu) \\ & - g \frac{1}{2} \epsilon_{abc} (\partial_\mu \bar{A}_\nu^a - \partial_\nu \bar{A}_\mu^a) \bar{A}_b^\mu \bar{A}_c^\nu \\ & - g^2 \frac{1}{4} (\bar{A}_\mu^a \bar{A}_\nu^b \bar{A}_a^\mu \bar{A}_b^\nu - \bar{A}_\mu^a \bar{A}_\nu^b \bar{A}_b^\mu \bar{A}_a^\nu) \end{aligned}$$

For $SU(2)$ $\text{tr}[t_a, t_b] = \frac{1}{2} \delta_{ab}$

g = "charge unit" or coupling constant

Interaction between gauge fields

Action for matter fields

replacing $\partial_\mu \rightarrow D_\mu$ (like Maxwell)
 $\uparrow$
 depends on A

$$\epsilon_{abc} \epsilon^{cde} = \delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd}$$

Dynamics at classical level $\rightarrow$ quantum

Interaction vertices and the propagator

Rescale $A_\mu = g \bar{A}_\mu$, $\alpha \rightarrow g \bar{\alpha}$

$$\bar{A}_\mu \rightarrow \bar{A}_\mu + \partial_\mu \bar{\alpha} - ig [\bar{A}_\mu, \bar{\alpha}]$$

$$D_\mu \phi_F = \partial_\mu \phi_F - ig \bar{A}_\mu \cdot \phi_F$$

$$S[A] = \int d^4x \mathcal{L}[A] \quad \text{Lagrangian density}$$

$a=1,2,3$ group indices

$$\mathcal{L}[A] = -\frac{1}{4} (\partial_\mu \bar{A}_\nu^a - \partial_\nu \bar{A}_\mu^a) (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)$$

← Maxwell

quadratic

$SU(3)$

$$-g \frac{1}{2} \epsilon_{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_b^\mu A_c^\nu$$

$$-g^2 \frac{1}{4} (A_\mu^a A_\nu^b A_a^\mu A_b^\nu - A_\mu^a A_\nu^b A_b^\mu A_a^\nu)$$

} new

cubic → polarizations & momentas

quartic → polarizations

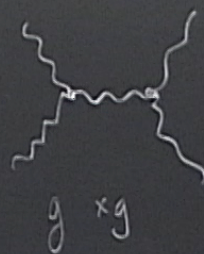
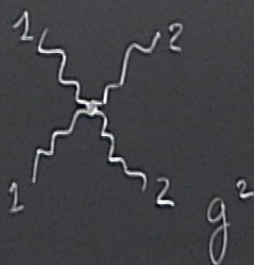
propagator

$\sim^a = \text{photon}$

interaction

$$k_1^\mu g \epsilon_{abc}$$

$g = \text{"unit of charge"}$
of gauge field



Gauge symmetry

⇒ interaction terms are not free

Matter fields : Dirac

$$S_{\text{Dirac}}[\bar{\Psi}, \Psi; A] = \int d^4x \bar{\Psi} (i \not{D} - m) \Psi = \int d^4x \bar{\Psi}_i^\alpha(x) \left[i (\gamma^\mu)^\beta_\alpha \partial_\mu - m \delta^\beta_\alpha \right] \Psi_\beta^i(x) \quad \rightarrow \quad \alpha$$

$$\Psi = \{ \Psi_\alpha^i(x) \} \quad \begin{array}{ll} i=1,2 & SU(2) \\ \alpha=1,4 & \text{Dirac} \end{array}$$

$$+ \bar{\Psi}_i^\alpha(x) A_\mu^a(x) \Psi_\beta^j(x) (\gamma^\mu)^\beta_\alpha t_a$$

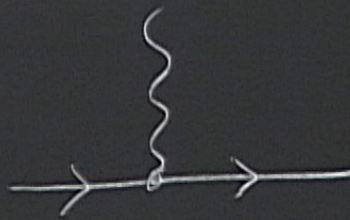
$\uparrow$ Dirac $\uparrow$ generators of $SU(2)$

$$\not{D} = \gamma^\mu \cdot \underline{D}_\mu = \gamma^\mu (\partial_\mu - i A_\mu)$$

$\begin{array}{ll} 4 \times 4 & 2 \times 2 \text{ matrix} \\ \text{Dirac} & \end{array}$

$$\sum_{\alpha} \int \psi_{\beta}^{\dagger}(x) \rightarrow \begin{array}{c} i \\ \alpha \end{array} \longrightarrow \begin{array}{c} i \\ \beta \end{array} \quad i \in \text{SU}(2) \text{ Dirac +} \\ \text{charge conservation}$$

$$(\gamma^{\mu})^B_{\alpha} \quad \begin{array}{c} \uparrow \\ \alpha \end{array} \quad \begin{array}{c} \uparrow \\ a \end{array} \\ \text{Dirac} \quad \uparrow \text{generator of} \\ \text{SU}(2)$$



Dirac

$$L = \int d^4x \bar{\Psi} (i \not{D} - m) \Psi = \int d^4x \bar{\Psi}_i^\alpha(x) \left[i (\gamma^\mu)^\beta_\alpha \partial_\mu - m \delta^\beta_\alpha \right] \Psi_\beta^i(x)$$

$i = 1, 2$ $SU(2)$

$\alpha = 1, 2$ Dirac

$$= \gamma^\mu (\partial_\mu - i A_\mu)$$

4×4 Dirac
 2×2 matrix

$$+ g \bar{\Psi}_i^\alpha(x) A_\mu^a(x) \Psi_\beta^j(x) (\gamma^\mu)^\beta_\alpha t_a$$

↑
Dirac

↑ generator of
 $SU(3)$

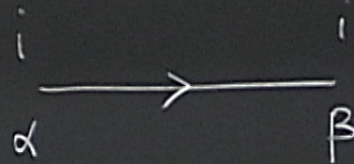
$$[i(\gamma^\mu)_\alpha^\beta \partial_\mu - m \delta_\alpha^\beta] \psi_\beta(x)$$

$$A_\mu^a(x) \psi_\beta(x) (\gamma^\mu)_\alpha^\beta (t_a)^{ij}$$

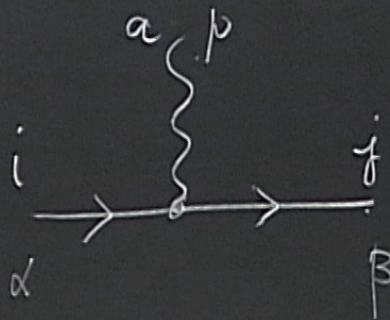
↑ Dirac

↑ generator of SU(2)

↑ Charge exchange



$i \leftarrow \text{SU}(2)$ Dirac + charge conservation



like in QED: spin
non trivial charge exchanges
unlike QED

Dirac +
charge conservation

QED - spin
charge exchanges
QED

Current

$$J_a^\mu = \underbrace{\bar{\Psi} \gamma^\mu t_a \Psi} + \frac{1}{g^2} \underbrace{\epsilon_{abc} F_{\mu\nu}^b A_\nu^c}$$

$a=1,3,2$

Quantization . More difficult than for $U(1)$

- Gauge Fixing . gauge invariance is not a "physical symmetry"
redundancy in the formulation of the physical theory
(useful)

- $A_\mu = (V, \vec{A})$ Axial gauge, $\vec{\nabla} \cdot \vec{A} = 0$, $\frac{\partial}{\partial t} V + \vec{\nabla} \cdot \vec{A} = 0$, . . .

- $SU(2)$,

Functional Integral

$$\int \underset{\substack{\uparrow \\ \text{measure}}}{D[A]} \exp(-S[A]) , \quad \text{gauge action} \swarrow$$
$$D[A] = \prod_{x \mu a} dA_\mu^a(x)$$

Space of all gauge field configurations A

$$\mathcal{A} = \{A\}, \text{ element } A = \{A_\mu^a(x)\}$$

Space of all local gauge transformations $\mathcal{G}$

$$\mathcal{G} = \{g\} \quad g = \{g(x) \in SU(2)\}$$

gauge transformation

$$A \rightarrow A_g$$

x) Lie Algebra of $\mathcal{G} = \text{Lie}(\mathcal{G}) = \{\alpha\} \quad \alpha = \{\alpha(x)\}$

$$g = \text{Id} + i\alpha$$

$$A \rightarrow A + D_A \alpha$$

very compact notations

ation A

$$A_\mu^a(x)$$

formations g

$$g(x) \in SU(2)$$

$$g = \text{Id} + i\alpha$$

$$A \rightarrow A + D_A \alpha$$

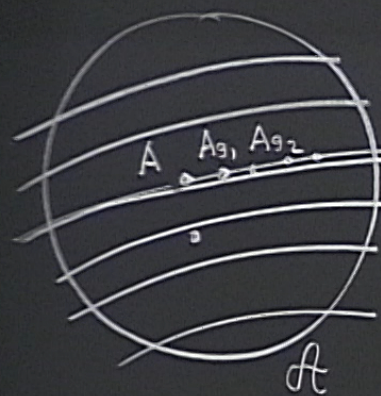
gauge transf
infinitesimal

very compact notations

$$A \rightarrow A_g = g A g^{-1} + i g \partial g^{-1}$$

$$) = \{\alpha\}$$

$$\alpha = \{\alpha(x)\}$$



Subspace of $A = \{A_g; g \in G\} = g(A) =$ space of all
gauge field config
physically equivalent to A
an orbit of G

Space of "physical configuration"
= space of orbits

G group acts
on A

$\mathcal{C} = A/G$ quotient space of
 A by the action
of G

$\mathcal{G} = \mathcal{G}(A)$ = space of all
gauge field config
physically equivalent to A

Quantize, keep the
gauge invariance

$$\int_{\mathcal{H}} D[A] \rightarrow \int_{\mathcal{G}} \text{which measure?}$$

space of
the action

- Gauge - Fixing
- Faddeev-Popov - Feynman & DeWitt procedure

which measure?