Title: Quantum Epidemiology: Operator Growth, Thermal Effects, and SYK

Date: Nov 26, 2018 11:00 AM

URL: http://pirsa.org/18110101

Abstract: In many-body chaotic systems, the size of an operator generically grows in Heisenberg evolution, which can be measured by certain out-of-time-ordered four-point functions. However, these only provide a coarse probe of the full underlying operator growth structure. We develop a methodology to derive the full growth structure of fermionic systems, that also naturally introduces the effect of finite temperature. We then apply our methodology to the SYK model, which features all-to-all q-body interactions. We derive the full operator growth structure in the large q limit at all temperatures. We see that its temperature dependence has a remarkably simple form consistent with the slowing down of scrambling as temperature is decreased. Furthermore, our finite-temperature scrambling results can be modeled by a modified epidemic model, where the thermal state serves as a vaccinated population, thereby slowing the overall rate of infection.

Quantum Epidemiology

Operator Growth, Thermal Effects, and SYK

Xiao-Liang Qi[a] Alexandre Streicher[ab]

[a]Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA
[b]Department of Physics, University of California, Santa Barbara, CA

November 26, 2018

Operator Evolution

- Heisenberg evolution of an operator

$$W(t) = e^{iHt} W e^{-iHt} = W + it[H, W] - \frac{t^2}{2}[H, [H, W]] + ...$$

- What happens? Consider a chaotic spin chain

$$H = \sum_i \sigma_z^i \sigma_z^{i+1} - \frac{21}{20}\sigma_x^i + \frac{1}{2}\sigma_z^i$$

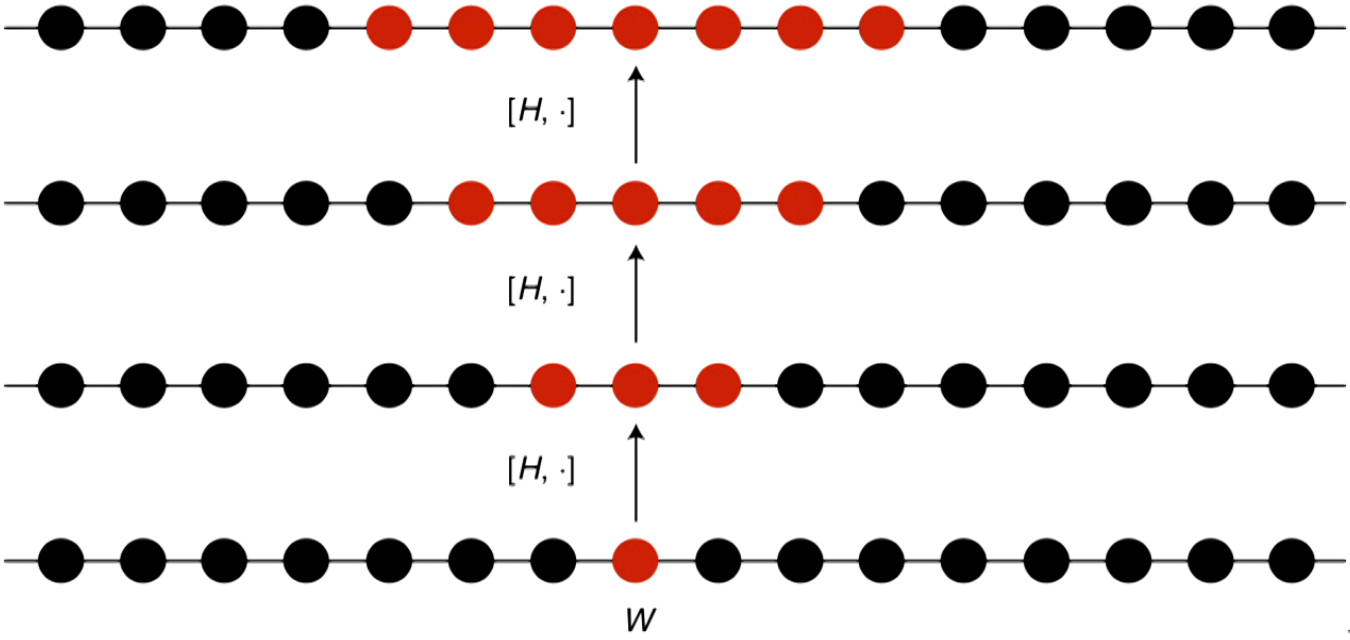
- Then we see that operator evolution gives

$$\begin{aligned}\sigma_x^j(t) = \sigma_x^j &- t\left(2\sigma_z^{j-1}\sigma_y^j + 2\sigma_y^j\sigma_z^{j+1} + \sigma_y^j\right) \\ &+ t^2 \,(\text{terms on three or less sites}) \\ &+ t^3 \,(\text{terms on four or less sites}) + ...\end{aligned}$$

Operator Spreading

As time evolves, operators grow

$$W(t) = e^{iHt} W e^{-iHt} = W + it[H, W] - \frac{t^2}{2}[H, [H, W]] + ...$$



[1]Image Credit: Brian Swingle

Spatial Growth of Operators

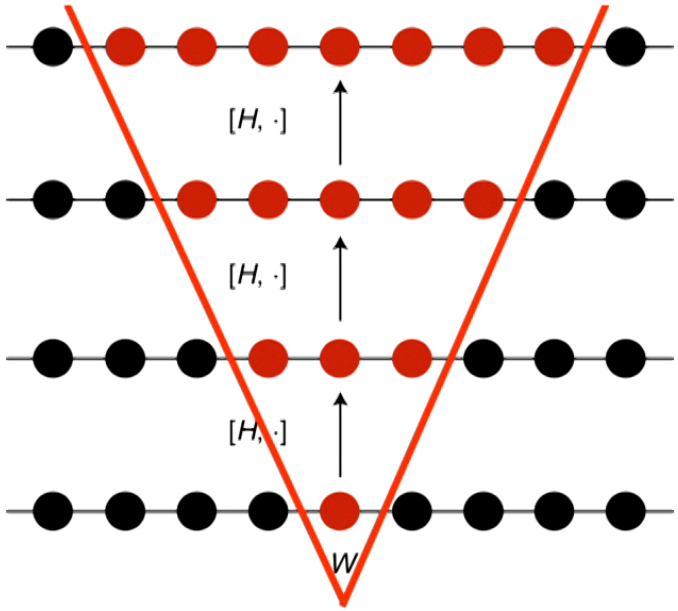
- Geometrically Local Growth

$$\frac{d}{dt} Vol\left[W\left(t\right)\right] \propto S.Area\left[W\left(t\right)\right]$$

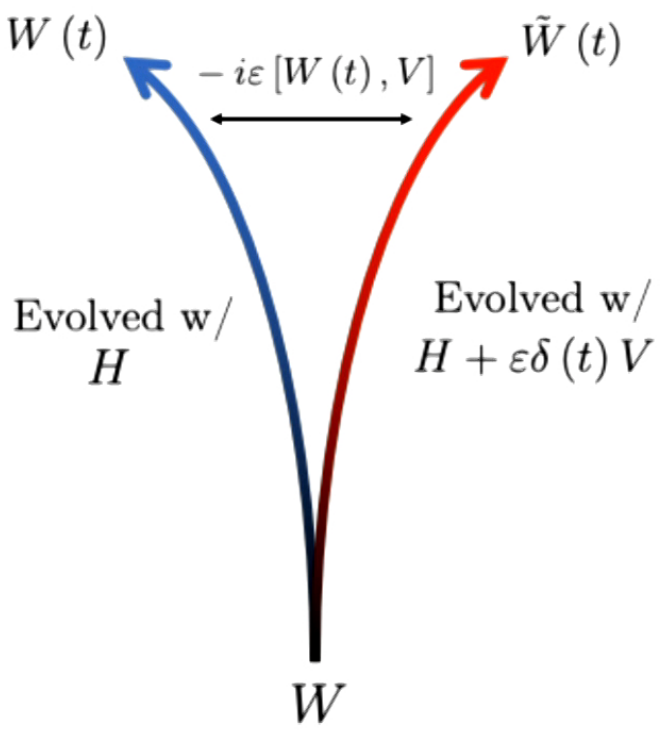
- Bound [Lieb & Robinson 1972]

$$\left\|\left[W_{x_1}\left(t\right), V_{x_2}\right]\right\| \leq \frac{v\left|t\right|}{\left|x_1 - x_2\right|}$$

- Need to wait until $t \approx {}^{v}/_{|x_1 - x_2|}$ for operators at x_1 to be affected by operators at x_2



Another Coarse Probe of Operator Growth



- Rather than the commutator norm, use the finite temperature commutator squared

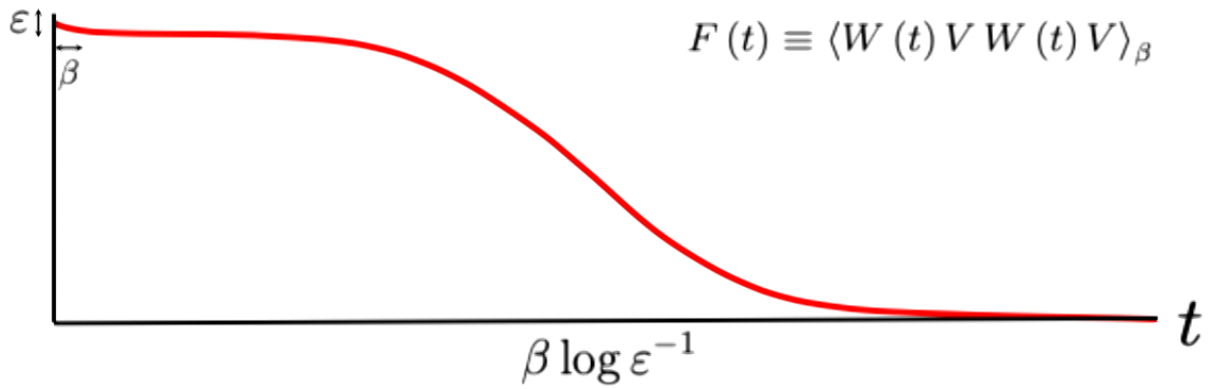
$$C(t) \equiv \left\langle \left| \frac{\delta W(t)}{\delta V(0)} \right|^2 \right\rangle_\beta = \left\langle |[W(t),V]|^2 \right\rangle_\beta$$

- Out-of-time-order correlators (OTOC)

$$F(t) \equiv \langle W(t)\, V\, W(t)\, V \rangle_\beta$$

A Bound on Chaos

- The commutator squared grows and the OTOC decays sigmoidally (a.k.a. slow→fast→slow)

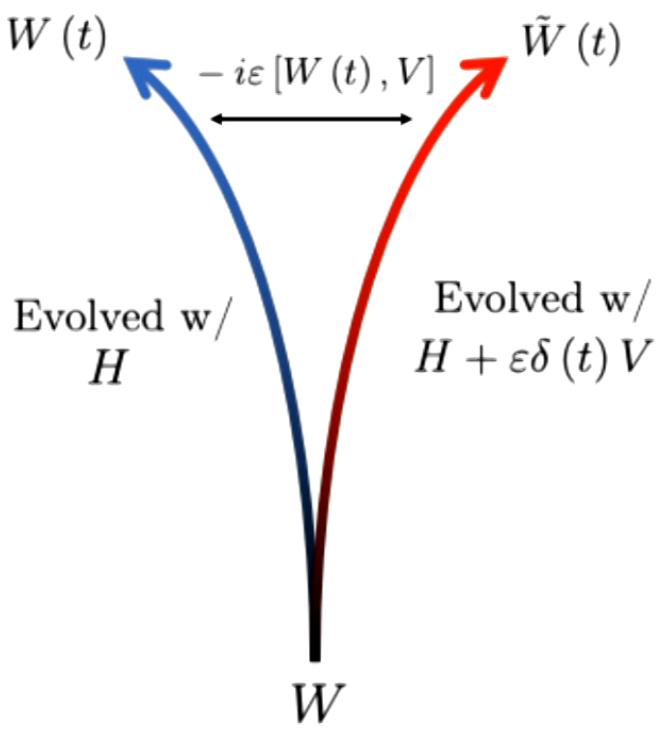


- This behavior is often bounded [Maldacena, Shenker, & Stanford 2015]

$$C(t) \sim \left| 1 - \frac{F(t)}{F(0)} \right| \leq \varepsilon e^{\lambda_L t} \quad \lambda_L \leq 2\pi T$$

where λ_L is the quantum Lyapunov exponent and $\varepsilon \propto \hbar$ or $\varepsilon \propto N^{-1}$

Another Coarse Probe of Operator Growth



- Rather than the commutator norm, use the finite temperature commutator squared

$$C(t) \equiv \left\langle \left| \frac{\delta W(t)}{\delta V(0)} \right|^2 \right\rangle_\beta = \left\langle |[W(t),V]|^2 \right\rangle_\beta$$

- Out-of-time-order correlators (OTOC)

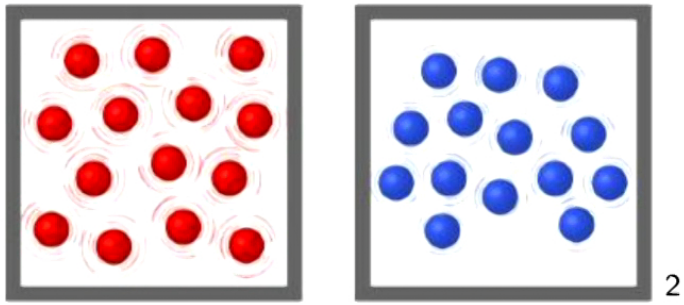
$$F(t) \equiv \langle W(t)\, V\, W(t)\, V \rangle_\beta$$

Finite Temperature Effects

Puzzle

$W(t)$ always evolves the same way, so why does the operator growth slow down at lower temperatures/energies?

- Because of dispersion relations if (quasi)particles



[2]

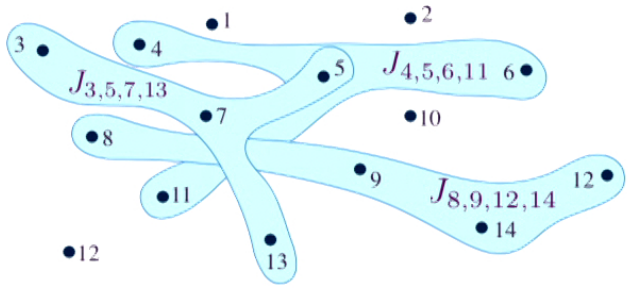
- But... this happens even without (quasi)particles
  - QFTs at strong coupling ($\mathcal{N} = 4$ SYM)
  - "Strange Metals"

[2] Image Credit: LibreTexts

The Sachdev-Ye-Kitaev (SYK) Model

All-to-all q-body interactions of N Majorana fermions

$$H = i^{q/2} \sum_{1 \le i_1 \ldots \le i_q \le N} J_{i_1 \ldots i_q} \psi_{i_1} \ldots \psi_{i_q} \qquad \{\psi_i, \psi_j\} = \delta_{ij}$$



[3]

Random couplings with non-trivial (t'Hooft/planar) large N limit

$$\left\langle J^2_{i_1 \ldots i_q} \right\rangle = \frac{J^2}{\binom{N-1}{q-1}} = \frac{\mathcal{J}^2}{2q\binom{N-1}{q-1}}$$

[3] Image Credit: S. Sachdev

A First Look at SYK Operator Growth

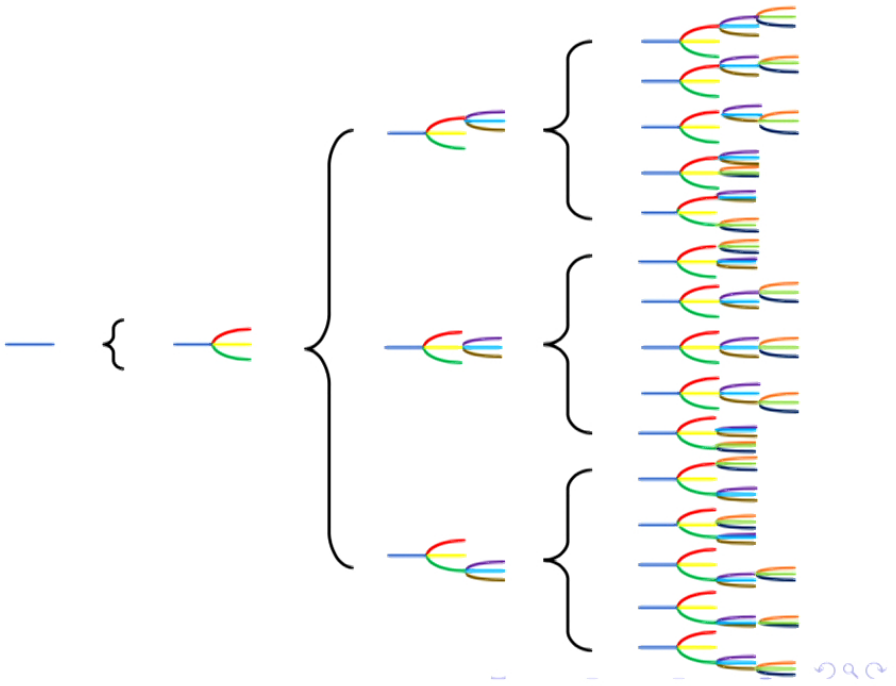
Let's study the first couple terms in evolving a simple operator

$$\psi_1(t) = \psi_1 + it[H,\psi_1] - \frac{t^2}{2}[H,[H,\psi_1]] - i\frac{t^3}{6}[H,[H,[H,\psi_1]]] + ...$$

- We notice what seems like exponential growth

$$\frac{d}{dt}Vol \propto Vol$$
$$\frac{d}{dt}Size \propto Size$$

- We need a more precise notion of $Size$



Operator Growth Methodology

Orthogonal Basis of Definite Size Operators

Ordered Products of Fermions

$$\Gamma_I \equiv \Gamma_{i_1 i_2 ... i_k} \propto \psi_{i_1} ... \psi_{i_k} \qquad 1 \leq i_1 < i_2 < ... < i_k \leq N$$

- Have definite size: $Size\left[\Gamma_I\right] = |I| = k$
- Are orthogonal and span all operators: $\mathrm{Tr}\left(\Gamma_I^\dagger \Gamma_J\right) \propto \delta_{IJ}$

- We "insert a complete set of operators" to study size growth (i.e. how $\mathcal{O}(t)$ becomes supported along larger products of fermions)

$$\mathcal{O}\left(t\right) = \sum_I \Gamma_I \mathrm{Tr}\left(\Gamma_I^\dagger \mathcal{O}\left(t\right)\right) = \sum_i c_i\left(t\right)\psi_i + i \sum_{i<j<k} c_{ijk}\left(t\right)\psi_i\psi_j\psi_k$$

A Useful Maximally-Entangled State

It is useful to introduce a doubled Hilbert space and map single-sided operators to doubled states. Key ingredient is a maximally entangled state

Bell Pair Vacuum

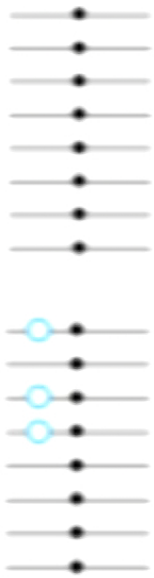
$$c_i |0\rangle \propto \left(\psi_i^L + i\psi_i^R\right)|0\rangle = 0 \Longleftrightarrow |0\rangle = \sum_n |n\rangle_L |n\rangle_R$$

- Acting with any fermion creates an entangled bit

$$\psi_1^L |0\rangle = -i\psi_1^R |0\rangle = \frac{\psi_1^L - i\psi_1^R}{2}|0\rangle \propto c_1^\dagger |0\rangle = |100...0\rangle$$

- Operator basis is equivalent to entangled bit basis

$$\Gamma_I^L |0\rangle = |\Gamma_I\rangle \Longleftrightarrow \langle \Gamma_I | \Gamma_J \rangle_{LR} \propto \delta_{IJ}$$



Mapping Operators and States

Thus, the aforementioned operator basis coefficients are also doubled wavefunctions coefficients

$$\mathcal{O}(t) = \sum_I \Gamma_I \mathrm{Tr}\left(\Gamma_I \mathcal{O}(t)\right) \equiv \sum_I \Gamma_I c_I(t)$$

$$\mathcal{O}^L(t)|0\rangle \equiv |\mathcal{O}(t)\rangle = \sum_I |\Gamma_I\rangle \langle \Gamma_I|\mathcal{O}(t)\rangle = \sum_I |\Gamma_I\rangle c_I(t)$$

The projection coefficients are the same due to the properties of a maximally entangled state

$$\langle 0|\Gamma_I^L \mathcal{O}^L(t)|0\rangle = \sum_{m,n} \langle m|\Gamma_I^L \mathcal{O}^L(t)|n\rangle_L \langle m|n\rangle_R = \mathrm{Tr}\left(\Gamma_I \mathcal{O}(t)\right)$$

Operator Growth = Doubled State Evolution

Understanding how the operator $\mathcal{O}(t)$ grows across different product fermions Γ_I is exactly the same as understanding how the doubled state $|\mathcal{O}(t)\rangle \equiv \mathcal{O}^L(t)|0\rangle$ grows across different entangled bits $|\Gamma_I\rangle \equiv \Gamma_I^L|0\rangle$.

The Doubled Theory has a Size Operator

One of the main reasons it is easier to think about operator growth in the doubled theory is due the presence of the size operator

Number (Size) Operator

In the doubled theory, there exists an operator

$$n \equiv \sum_{j=1}^{N} n_j \equiv \sum_{j=1}^{N} c_j^\dagger c_j = \sum_{j=1}^{N} \frac{1 + i\psi_j^L \psi_j^R}{2}$$

which counts the number of bits (size of the operator)

$$\langle \Gamma_I | n | \Gamma_J \rangle = |J| \langle \Gamma_I | \Gamma_J \rangle = Size\left[\Gamma_J\right] \langle \Gamma_I | \Gamma_J \rangle$$

The OTOC Calculates the Expectation Value of Size

We are now equipped to show (with a few lines of algebra) that the symmetric OTOC is related to the expectation value of size for the thermal operator $\psi_1(t)\,\rho^{1/2}$

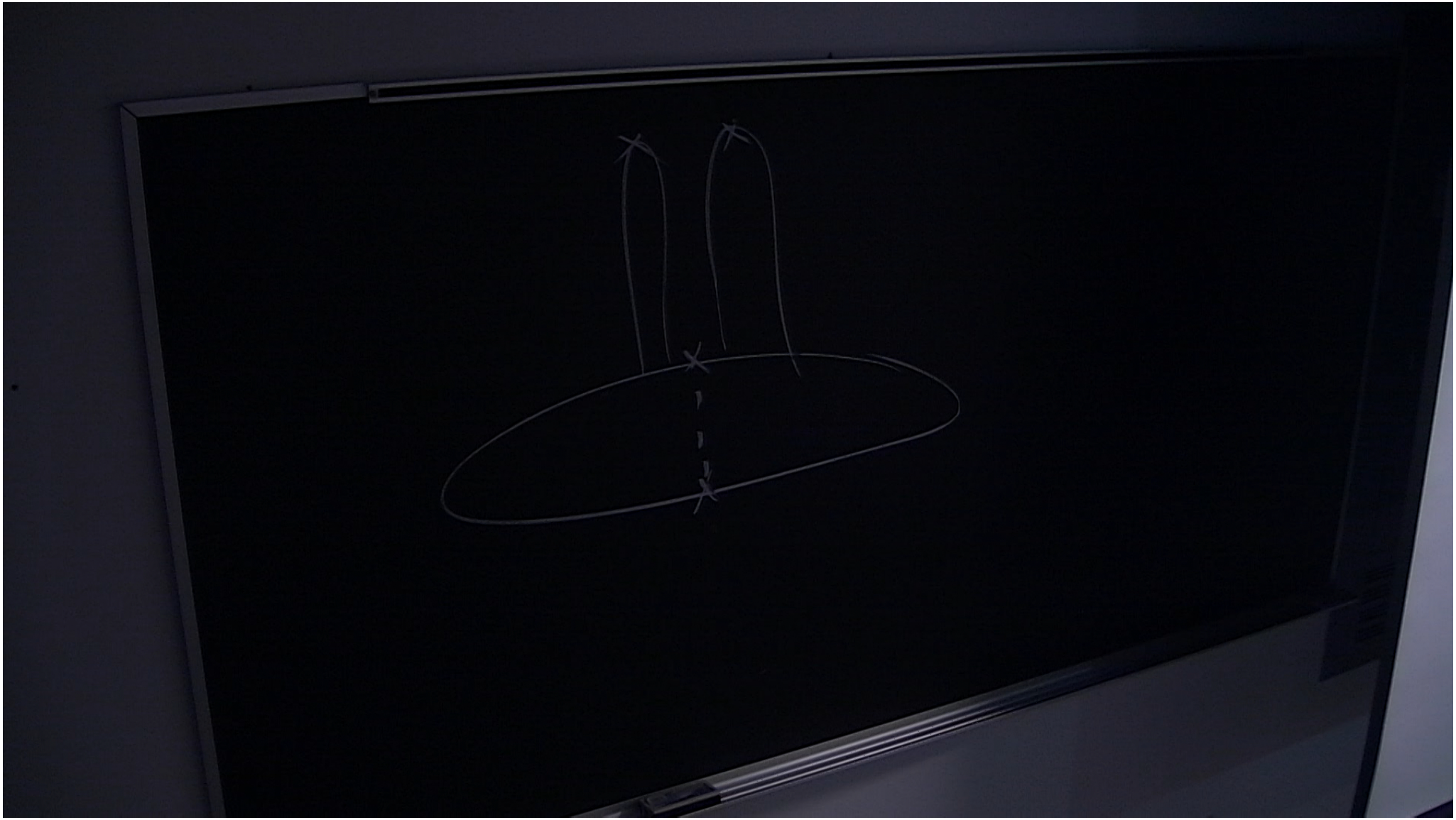
$$-\frac{1}{N}\sum_{j=1}^{N}\mathrm{Tr}\left(\rho^{1/2}\psi_1(t)\,\psi_j\psi_1(t)\,\rho^{1/2}\psi_j\right)=1-\frac{2}{N}n\left[\psi_1(t)\,\rho^{1/2}\right]$$

Note that the state $\left|\rho^{1/2}\right\rangle$ is the familiar thermofield double state since

$$\left|\rho^{1/2}\right\rangle=\rho_L^{1/2}\left|0\right\rangle=\rho_L^{1/2}\sum_n\left|n\right\rangle_L\left|n\right\rangle_R=\sum_n\frac{e^{-\beta E_n/2}}{\sqrt{Z(\beta)}}\left|n\right\rangle_L\left|n\right\rangle_R\equiv\left|TFD\right\rangle$$

Precursor State

$$\left|\psi_1(t)\,\rho^{1/2}\right\rangle=\psi_1^L(t)\left|\rho^{1/2}\right\rangle=\psi_1^L(t)\left|TFD\right\rangle$$



Thermal Operator Growth = Precursor State Evolution

Understanding how the thermal operator $\psi_1(t)\rho^{1/2}$ grows across different product fermions Γ_I is exactly the same as understanding how the precursor state $\left|\psi_1(t)\rho^{1/2}\right\rangle = \psi_1^L(t)|TFD\rangle$ grows across different entangled bits $|\Gamma_I\rangle$

- The average size of this state is given by the symmetric OTOC

$$-\frac{1}{N}\sum_{j=1}^{N}\mathrm{Tr}\left(\rho^{1/2}\psi_1(t)\psi_j\psi_1(t)\rho^{1/2}\psi_j\right) = 1-\frac{2}{N}n\left[\psi_1(t)\rho^{1/2}\right]$$

Learning More than the Average Size

- Let us insert a size "chemical potential" in order to obtain to full distribution rather than just the average size calculated by the OTOC

$$\mathcal{Z}_\mu \left[\psi_1(t)\,\rho^{1/2}\right] \equiv \left\langle \rho^{1/2} \middle| \psi_1^L(t)\, e^{-\mu n} \psi_1^L(t) \middle| \rho^{1/2} \right\rangle$$

- This generates the size distribution of $\psi_1(t)\,\rho^{1/2}$:

$$\mathcal{Z}_\mu \left[\psi_1(t)\,\rho^{1/2}\right] = \sum_{n=0}^{\infty} P_n \left[\psi_1(t)\,\rho^{1/2}\right] e^{-\mu n}$$

- Looks like an unnormalized two-point function. We may write

$$\mathcal{Z}_\mu \left[\psi_1(t)\,\rho^{1/2}\right] = \mathcal{G}_\mu \left(\frac{\beta^+}{4} + it, \frac{\beta^-}{4} + it\right) \mathcal{Z}_\mu \left[\rho^{1/2}\right]$$

where we have defined the "twisted" two-point function

$$\mathcal{G}_\mu(\tau_a, \tau_b) = \frac{\langle 0|\, \mathcal{T}\left[\rho_L^{1/2} \rho_R^{1/2} e^{-\mu n(\beta/4)} \psi_1^L(\tau_a) \psi_1^L(\tau_b)\right] |0\rangle}{\left\langle \rho^{1/2} \middle|\, e^{-\mu n} \middle| \rho^{1/2} \right\rangle}$$

Twisted Boundary Conditions

- Thinking about what happens as operators approach $e^{-\mu n}$, we see that this term simply twists the left and right fields into each-other

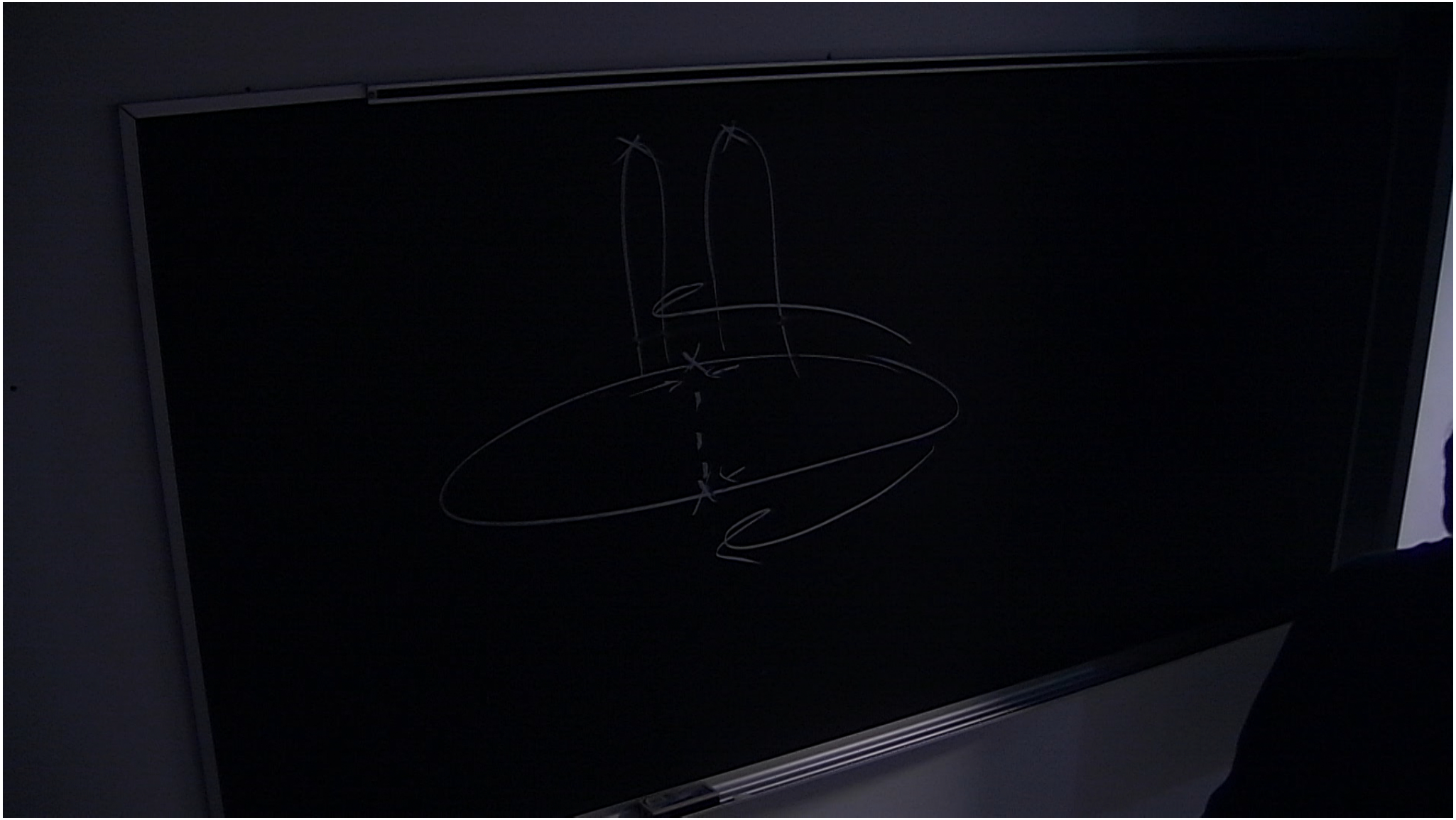
$$\begin{pmatrix} \psi^L \\ i\psi^R \end{pmatrix} e^{-\mu n} = e^{-\mu n} \begin{pmatrix} \cosh(\mu) & -\sinh(\mu) \\ -\sinh(\mu) & \cosh(\mu) \end{pmatrix} \begin{pmatrix} \psi^L \\ i\psi^R \end{pmatrix}$$

- We may then "de-purify" the system using the field redefinition

$$\psi_i(\tau) = \begin{cases} \psi_i^L(\tau) & 0 \leq \tau \leq \beta/2 \\ i\psi_i^R(\beta - \tau) & \beta/2 \leq \tau < \beta \end{cases}$$

which gives us the twisted boundary conditions in a more understandable form

$$\begin{pmatrix} \psi(\beta/4^+) \\ \psi(3\beta/4^-) \end{pmatrix} = \begin{pmatrix} \cosh(\mu) & -\sinh(\mu) \\ -\sinh(\mu) & \cosh(\mu) \end{pmatrix} \begin{pmatrix} \psi(\beta/4^-) \\ \psi(3\beta/4^+) \end{pmatrix}$$



Uncovering the Growth Structure

Since $\mathcal{Z}_\mu\left[\psi_1(t)\,\rho^{1/2}\right] = \mathcal{G}_\mu\left(\frac{\beta}{4}^{+} + it, \frac{\beta}{4}^{-} + it\right)\mathcal{Z}_\mu\left[\rho^{1/2}\right]$, we have

$$P_n\left[\psi_1(t)\rho^{1/2}\right] = \left(K^\beta\left[\psi_1(t)\right] * P\left[\rho^{1/2}\right]\right)_n = \sum_{m=0}^{n} K_m^\beta\left[\psi_1(t)\right] P_{n-m}\left[\rho^{1/2}\right]$$

where the twisted two-point function gives us the "growth distribution"

$$\mathcal{G}_\mu\left(\frac{\beta^+}{4} + it, \frac{\beta^-}{4} + it\right) = \sum_{m=0}^{N} e^{-m\mu} K_m^\beta\left[\psi_1(t)\right]$$

Finding the Full Growth Structure of the Thermal Operator $\psi_1(t)\rho^{1/2}$

Calculate the Euclidean two-point function in your original theory with the μ-dependent twisted boundary conditions. Analytically continue the times, then power expand in $e^{-\mu}$ to read off the growth distribution $K^{\beta}[\psi_1(t)]$.

- The average size of this distribution will be the connected component of the OTOC.
- Using this methodology, we were able to produce a new compact expression for the SYK OTOC
- Our result interpolates between known expressions in two different regimes.

The Thermal State is Heavy and Heavily Peaked

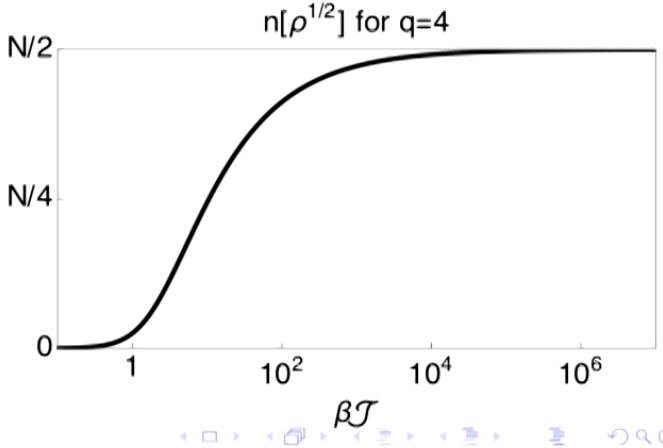
We find the average size of the thermal state to be heavily peaked around

$$n\left[\rho^{1/2}\right] = \frac{N}{2}\left(1 - \left(\frac{\lambda_L}{\lambda_L^{\beta=0}}\right)^{2/q}\right) = \frac{N}{2}\left(1 - \left(\frac{\lambda_L}{2\mathcal{J}}\right)^{2/q}\right)$$

where the Lyapunov exponent λ_L is given for large/small $\beta\mathcal{J}$ by

$$\lambda_L = \frac{2\pi}{\beta}\left(1 - \frac{\pi}{\beta\mathcal{J}} + \mathcal{O}\left(\frac{1}{\beta^2\mathcal{J}^2}\right)\right) = 2\mathcal{J}\left(1 - \frac{\beta^2\mathcal{J}^2}{8} + \mathcal{O}\left(\beta^4\mathcal{J}^4\right)\right)$$

- The operator $\rho^{1/2} \propto e^{-\beta H/2}$ becomes larger in size at colder temperatures, as expected.

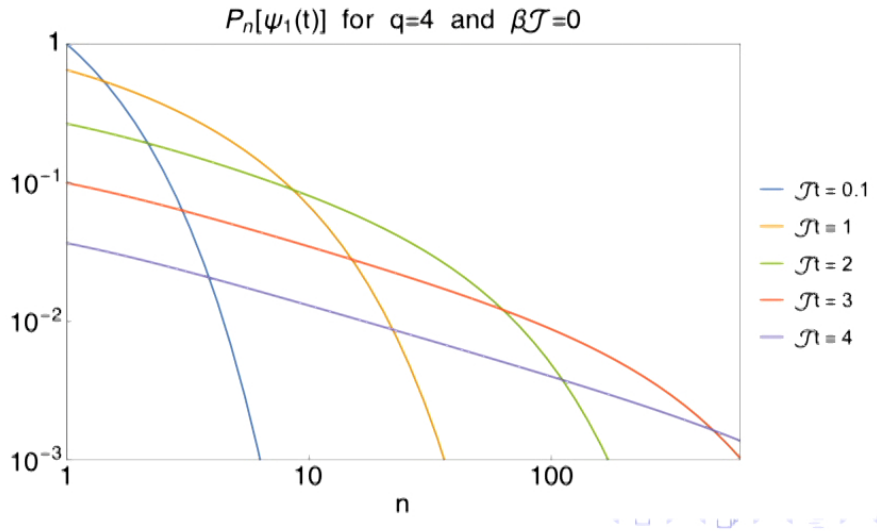


Infinite Temperature Growth Distribution

Our analysis reproduces previous results [Roberts, Stanford, & A.S. 2018]

$$P_{1+n}\left[\psi_1(t)\right] = (-1)^{n/q}\binom{-2/q}{n/q}\left(\frac{\tanh^n\left(\lambda_L^{\beta=0}t/2\right)}{\cosh^2\left(\lambda_L^{\beta=0}t/2\right)}\right)^{2/q}$$

$$\approx \frac{2}{n}\exp\left(\frac{2}{q}\left(\log n - \lambda_L^{\beta=0}t - 2e^{\log n - \lambda_L^{\beta=0}t}\right)\right)$$



The Effect of Temperature on the Growth Distribution

SYK Operator Growth Universality

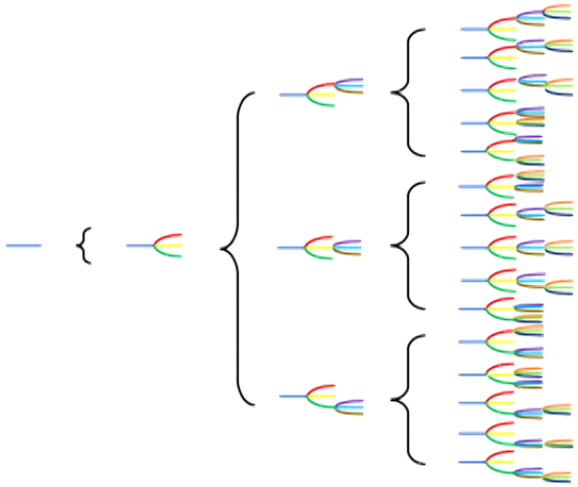
In the Lyapunov regime, the growth distribution takes a universal form

$$K_{1+n}^{\beta}\left[\psi_1\left(t\right)\right] = \left(-1\right)^{n/q}\binom{-2/q}{n/q}\left(\frac{\tanh^n\left(\lambda_L t/2\right)}{\cosh^2\left(\lambda_L t/2\right)}\right)^{2/q}$$

where the effect of temperature is simply to slow the rate, while preserving the infinite self-repeating tree structure

Epidemics

- Intuition in $q = 4$ SYK: $[H, \psi_1]$ generates terms with 3 flavors, but then $[H, [H, \psi_1]]$ generates three times as many terms as $[H, \psi_1]$ generated, and so on.
- We expect an infection model to apply
  - the rate of growth should be proportional to the current size
  - it should also be sensitive to the finiteness of the N, slowing down at late time as we are bottlenecked by competition for flavors



SYK Epidemiology

- The operator $\psi_1(t)$ grows as

$$\frac{dn\left[\psi_1(t)\right]}{dt} \approx \lambda_L^{\beta=0} n\left[\psi_1(t)\right]$$

- The thermal operator $\psi_1(t)\rho^{1/2}$ grows as

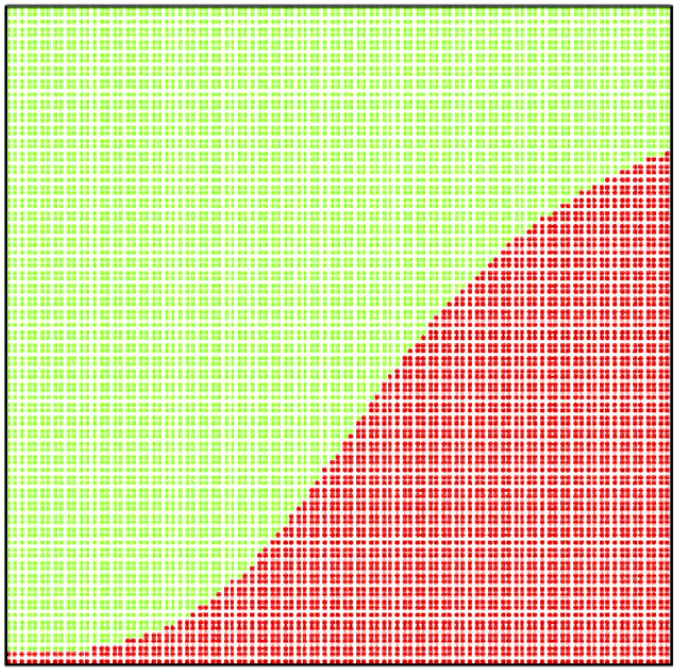
$$\frac{dn\left[\psi_1(t)\rho^{1/2}\right]}{dt} \approx \lambda_L^{\beta=0}\left(1-\frac{n\left[\rho^{1/2}\right]}{n_*}\right)^{q/2}\left(n\left[\psi_1(t)\rho^{1/2}\right]-n\left[\rho^{1/2}\right]\right)$$

- Aside from the power of $q/2$, the first term is expected, as there is now a small infectable population.
- However, it appears that only the "connected" population $n\left[\psi_1(t)\rho^{1/2}\right]-n\left[\rho^{1/2}\right]$ possess the ability to infect others

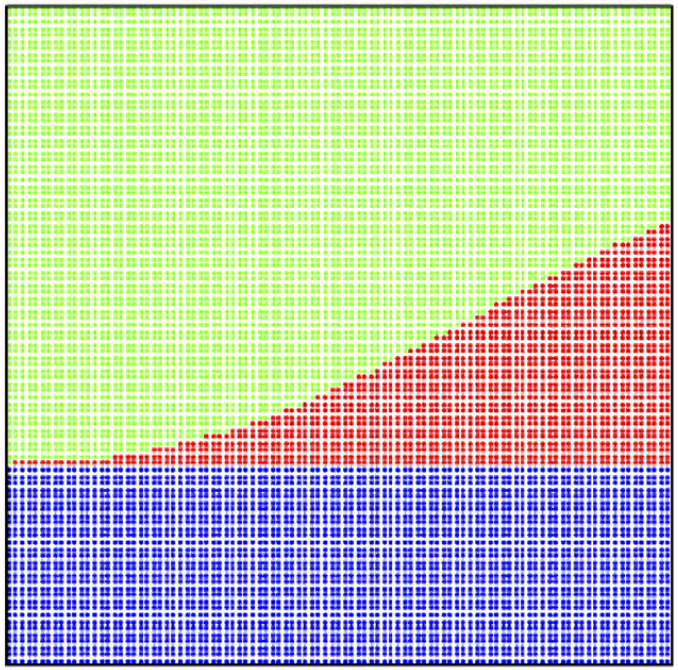
Thermal Epidemic Model

Leads us to conclude that the population $n\left[\rho^{1/2}\right]$ effectively serves as a vaccinated population

Scrambling at Infinite Temperature



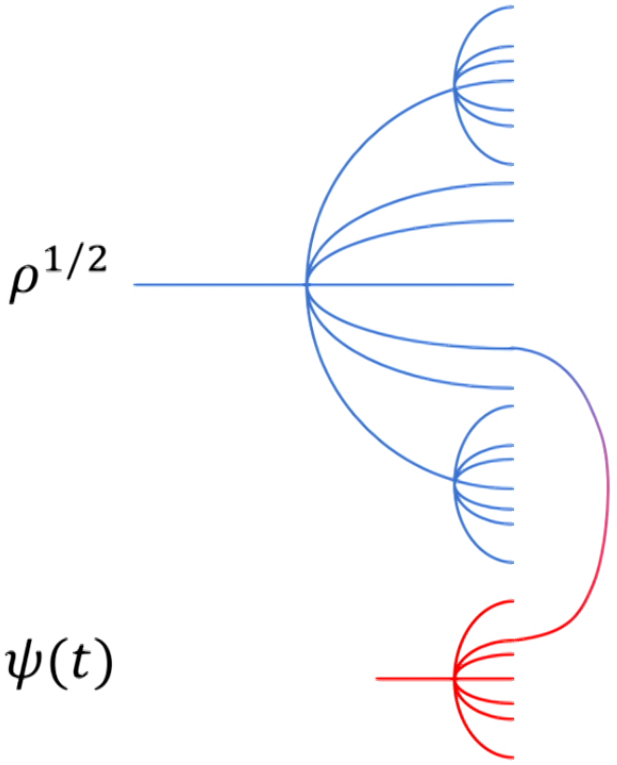
Scrambling at Finite Temperature



Operator Collisions/Contractions

- At lower temperatures, $\rho^{1/2}$ has increasing ability to collide/contract with the growth terms in $\psi_1(t)$, thereby slowing down the growth
- Thus, it makes sense that the growth rate is related to the size of $\rho^{1/2}$

$$\lambda_L = \lambda_L^{\beta=0}\left(1 - \frac{n\left[\rho^{1/2}\right]}{n_*}\right)^{q/2}$$



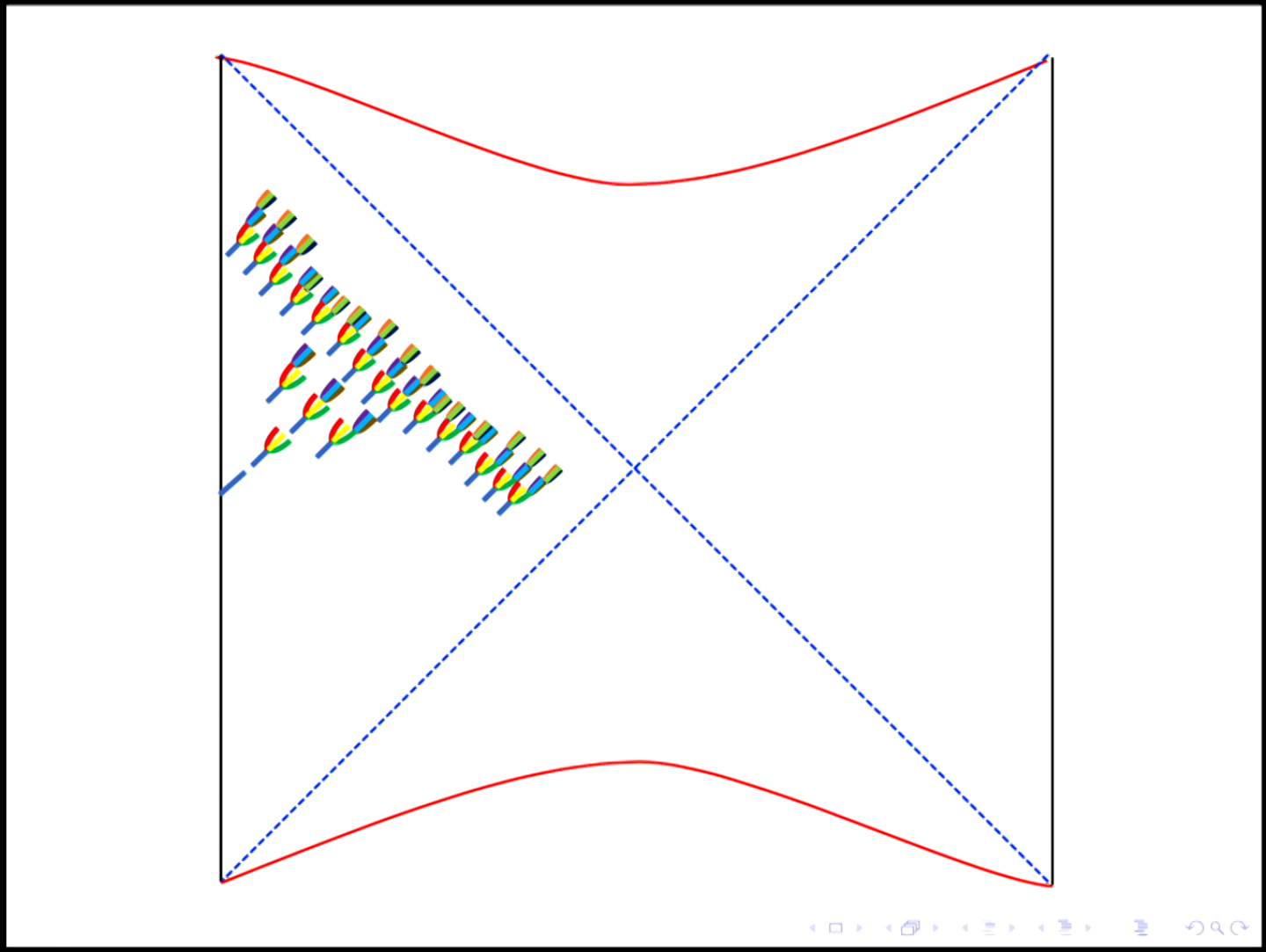
Connections with Gravity

Size-Momentum Correspondence

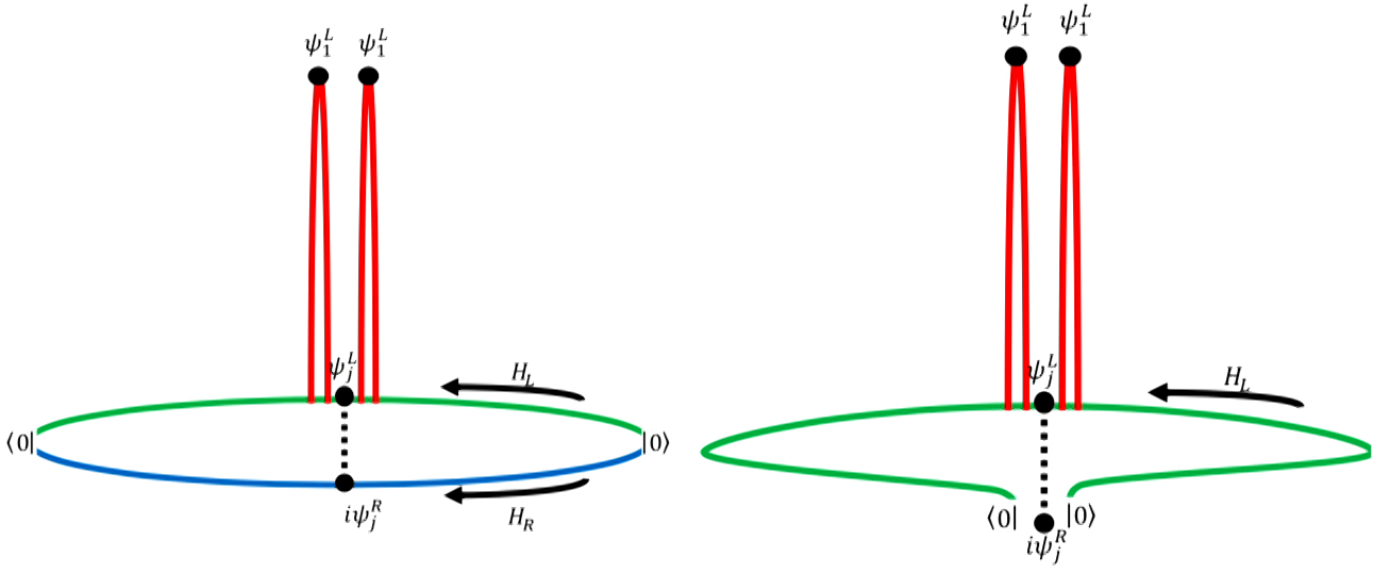
- Low energy sector of SYK is dual to the near-horizon Rindler and AdS_2 throat region of a near-extremal charged black hole
- It was shown that the growth of momentum for a particle falling from the boundary of the throat matched the average growth of the operator $\psi_1(t)\,\rho^{1/2}$ [Brown, Gharibyan, A. S., Susskind, Thorlacius, & Zhao 2018]
- WIP: Even more interesting is that our precursor wavefunction squared, which is essentially given by the universal form of the growth distribution

$$(-1)^{n\Delta}\binom{-2\Delta}{n\Delta}\left(\frac{\tanh^n(t/2)}{\cosh^2(t/2)}\right)^{2\Delta}$$

  shows up in the square of transforms of AdS_2 black hole bulk-to-boundary propagators.
- WIP: It appears as though the wavefunction of the SYK thermal fermion is similar to that of a particle falling into an AdS_2 black hole



Finite T OTOC: Removing the Evolution of the Right



$$F(t) = \langle 0|\rho_L^{1/4}\rho_R^{1/4}\psi_1^L(t)\,\delta\,\psi_1^L(t)\,\rho_L^{1/4}\rho_R^{1/4}|0\rangle = \left\langle \psi_1^L(t)\,\delta\,\psi_1^L(t)\right\rangle_{TFD}$$

$$= \langle 0|\rho_L^{1/2}\psi_1^L(t)\,\delta\,\psi_1^L(t)\,\rho_L^{1/2}|0\rangle = \delta\left[\psi_1(t)\,\rho^{1/2}\right] = 1 - \frac{2}{N}n\left[\psi_1(t)\,\rho^{1/2}\right.$$