

Title: Free probability approach to the energy gap problem

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Abstract: <p>Finding a quantum phase transition between two even well-studied phases of matter can be a hard problem. Free probability theory suggests a new method which can give an answer if the Hamiltonians of two phases satisfy certain conditions. In the latter case, the spectral gap behavior can be calculated without solving the full Hamiltonian but using probabilistic estimation instead. As an example, I will consider generic artificial topological systems created by a periodic drive, including Floquet Majorana modes, and show how FPT can be used to predict and characterize disorder-driven phase transitions.

See also: Phys. Rev. Lett. 121, 126803</p>



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How probability theory helps to find the energy gap

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1. Introduction: quantum phase transitions
2. Method: free probability theory
3. Application: artificial quantum matter

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lem of sum of two Hamiltonians

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Quantum phase transition (QPT) problem

$$H = (1 - g)H_1 + gH_2, \quad g \in [0, 1]$$

- H_1 (H_2) represents phase 1 (phase 2)
- g is a fitting parameter
- transition happens at critical point $g = g_c$

Examples:

1. Second order phase transitions
2. Topological phase transitions

Connection between single-particle spectrum and QPT



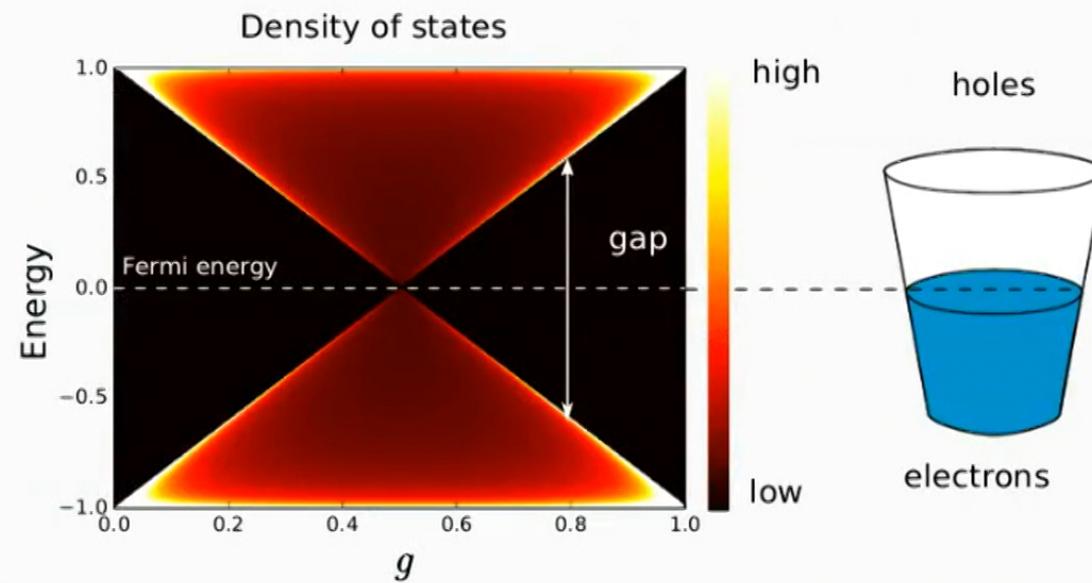
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Critical point is characterized by **vanishing of spectral gap**: $\Delta(g_c) = 0$



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Computation of the gap in generic systems

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Computing the gap is generally a **hard problem**.

Approximation methods

- Perturbation theory (g as small parameter)
- Direct diagonalization (limited size approx.)
- Quantum Monte Carlo and variational methods

My approach: to define a special class of *non-integrable* systems, where gap can be found exactly



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Method: free probability theory



Classical vs. Free Probability Theory (PT)



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Classical PT:

E_1 and E_2 have **distributions**
 $p_1(E)$ and $p_2(E)$

$$E = E_1 + E_2$$

If E_1 and E_2 are **independent** then
 $p(E)$ is **convolution**

$$p = p_1 * p_2$$

Method: Use additivity of
characteristic function $X_p(t)$

$$\log X_p(t) = \sum_{i=1,2} \log X_{p_i}(t)$$

Free PT:

H_1 and H_2 have **densities of states**
 $\rho_1(E)$ and $\rho_2(E)$

$$H = H_1 + H_2$$

If H_1 and H_2 are **freely independent**
then $\rho(E)$ is **free convolution**

$$\rho = \rho_1 \boxplus \rho_2$$

Method: Use additivity
of **R-transform** $R_\rho(w)$

$$R_\rho(w) = \sum_{i=1,2} R_{\rho_i}(w)$$



independence definition

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Definition. H_1 and H_2 are freely independent (or free) if

$$\frac{1}{\dim H} \langle \text{Tr} (\widetilde{H}_1^{k_1} \widetilde{H}_2^{m_1} \dots \widetilde{H}_1^{k_n} \widetilde{H}_2^{m_n}) \rangle = 0$$

where $\widetilde{H} = H - hI$ and $h = \langle \text{Tr } H \rangle / \dim H$.



independence definition

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where $\widetilde{H} = H - hI$ and $h = \langle \text{Tr } H \rangle / \dim H$.

Well known example: Wigner random matrices

$$\forall |\psi\rangle, |\phi\rangle : \quad \langle \psi | \phi \rangle \sim \frac{1}{\dim H}$$

where $|\psi\rangle$ ($|\phi\rangle$) is an eigenvalue of H_1 (H_2).

The class is much larger!

example: transverse Ising model

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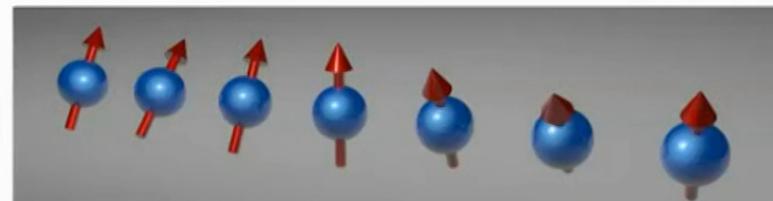
D transverse Ising spin chain

$$H = g \sum_n \sigma_n^x \sigma_{n+1}^x + (1-g) \sum_n \sigma_n^z$$

described by Bogoliubov-de Gennes Hamiltonian

$$H = g \sum_{r=1}^N \frac{1}{2} \left[(\tau_z + i\tau_y) \otimes |r\rangle\langle r+1| + \text{h.c.} \right] + (1-g) \sum_{r=1}^N \tau_z \otimes |r\rangle\langle r|$$

τ_y, τ_z are Pauli matrices in particle/hole (Nambu) space



example: proof of freeness

consider

$$H_1 = \sum_{r=1}^N \frac{1}{2} [(\tau_z + i\tau_y) \otimes |r\rangle\langle r+1| + \text{h.c.}], \quad H_2 = \sum_{x=1}^N \tau_z \otimes |r\rangle\langle r|$$

Is is easy to check that

$$H_1^2 = H_2^2 = I,$$

Therefore

$$\text{Tr} (\widetilde{H_1^{k_1}} \widetilde{H_2^{m_1}} \dots \widetilde{H_1^{k_n}} \widetilde{H_2^{m_n}}) = \begin{cases} 0, & \text{if any } k_i, m_i \text{ is even} \\ \text{Tr} (H_1 H_2)^n & \end{cases}$$

This expression vanishes because

$$(H_1 H_2)^n = \sum_{r=1}^N \frac{1}{2} [(I - \sigma_x) \otimes |r\rangle\langle r+n| + (I + \sigma_x) \otimes |r+n\rangle\langle r|]$$

Conclusion: H_1 and H_2 are free!

example: density of states

hamiltonian for transverse Ising model

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$$H = gH_1 + (1 - g)H_2$$

Density of states derivation (in 6 steps)

Step 1. Traced Green's function for gH_1

$$G_1(z) = \int_{-\infty}^{\infty} \frac{\rho(E)}{z - E} dE = \frac{1}{2} \left(\frac{1}{z - g} + \frac{1}{z + g} \right)$$

$z \in \mathbb{C}$, $\text{Im } z > 0$.

example: density of states

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Step 2. R-transform for gH_1

$$R_1(w) = \text{Inv}\{G_1\}(w) - \frac{1}{w} = \frac{1}{2w} \left(-1 \pm \sqrt{1 + 4g^2 w^2} \right)$$

$w \in \mathbb{C}$.



example: density of states

hamiltonian for transverse Ising model



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$w \in \mathbb{C}$.

Step 3. R-transform for H

$$R(w) = R_1(w) + R_2(w) = \frac{1}{2w} \left(\sqrt{1 + 4g^2 w^2} + \sqrt{1 + 4(1 - g)^2 w^2} \right) - \frac{1}{w}$$

example: density of states

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Step 4. Inverse G-function for H

$$\text{Inv}\{G\}(w) = R(w) + \frac{1}{w} = \frac{1}{2w} \left(\sqrt{1 + 4g^2 w^2} + \sqrt{1 + 4(1-g)^2 w^2} \right)$$

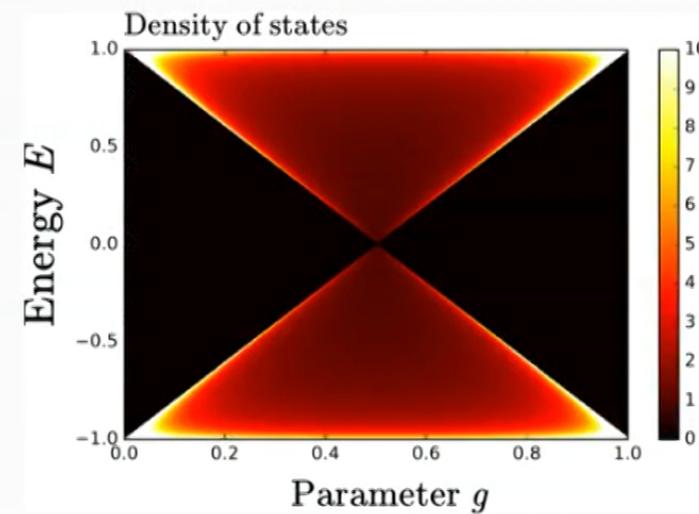
Step 5. G-function for H

$$w = G(z) = \frac{z}{\sqrt{(\Delta^2 - z^2)(1 - z^2)}}, \quad \Delta = |2g - 1|$$

Step 6. DOS for H

$$\begin{aligned} \rho(E) &= -\frac{1}{\pi} \text{Im} \lim_{\varepsilon \rightarrow 0} G(E + i\varepsilon) \\ &= \frac{1}{\pi} \frac{|E|}{\sqrt{(E^2 - \Delta^2)(1 - E^2)}} \end{aligned}$$

for $\Delta < |E| < 1$, $\rho(E) = 0$ elsewhere.



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Hamiltonian error analysis

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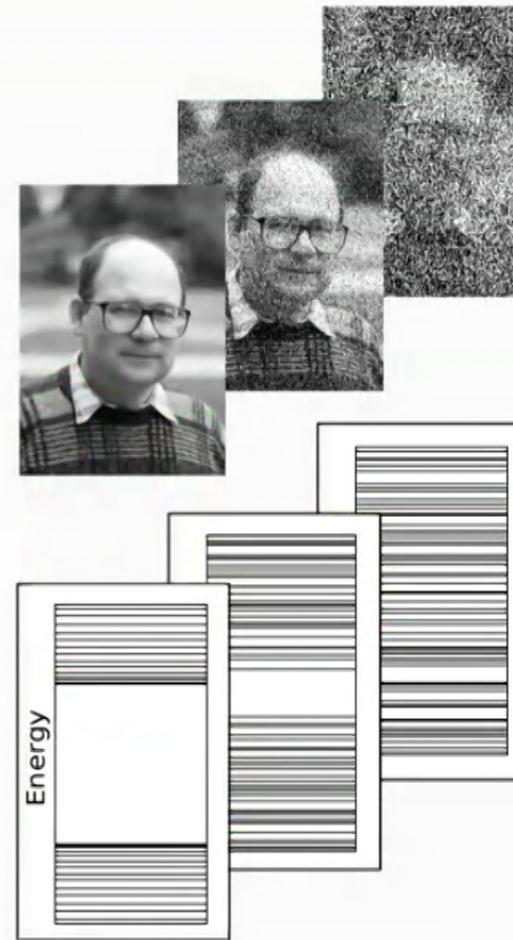
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Let us consider the gap of

$$H = H_0 + \lambda V$$

- H_0 is the idealized/solvable Hamiltonian of a gapped phase
- V is the “error” originated from disorder or interactions
- λ is the strength of the “error”



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example: proof of freeness

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$$H_1 = \sum_{r=1}^N \frac{1}{2} [(\tau_z + i\tau_y) \otimes |r\rangle\langle r+1| + \text{h.c.}], \quad H_2 = \sum_{x=1}^N \tau_z \otimes |r\rangle\langle r|$$

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This expression vanishes because

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Conclusion: H_1 and H_2 are free!



iltonian error analysis: the problem statement

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High-level assumptions:

1. H_0 and V are freely independent
2. DOS for H_0 and V are known

Questions:

- what is the gap behavior?
- where is the critical point λ_c ?
- what are the critical exponents?
- what happens with surface states?

Boltztonian error analysis: assumptions

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- DOS for H_0

$$\rho_1(E) \approx \frac{|E|}{\sqrt{E^2 - \Delta_0^2}} \rho_0, \quad |E| \geq \Delta_0$$

in the limit of small gap Δ_0

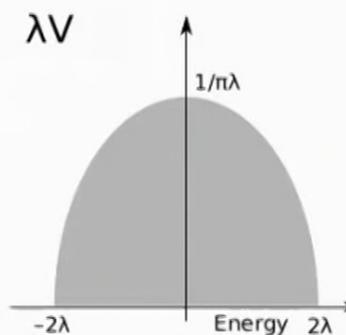
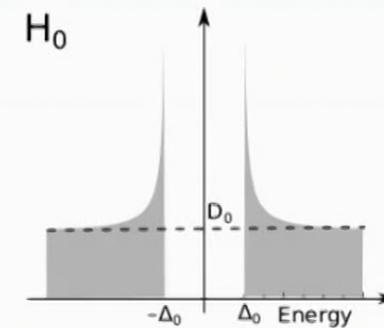
Examples: superconductors in any dimensions,
chiral materials

- DOS for V

$$\rho_2(E) = \frac{1}{2\pi\lambda^2} \sqrt{4\lambda^2 - E^2}, \quad |E| \leq 2\lambda$$

Example: Wigner random matrices

Key parameters are $\rho_0, \Delta_0, \lambda$



Wittenian error analysis: assumptions

- DOS for H_0

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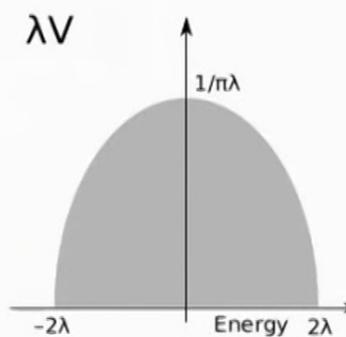
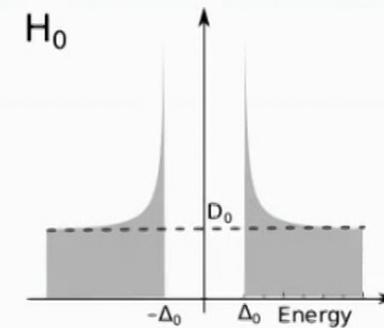
Examples: superconductors in any dimensions,
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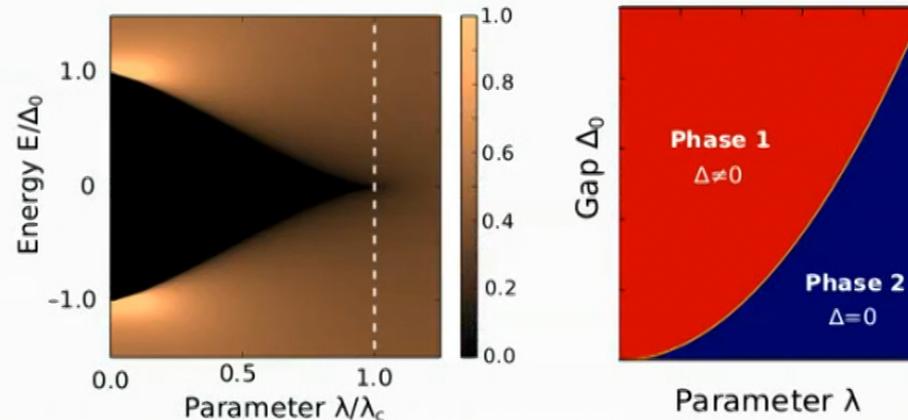
Example: Wigner random matrices

Key parameters are $\rho_0, \Delta_0, \lambda$





Bilstonian error analysis: result for bulk states



OS, R. Movassagh, Phys. Rev. Lett. **121**, 126803

1. The **spectral gap** as a function of strength λ

$$\Delta(\lambda) = \left(1 - (\lambda/\lambda_c)^{4/3}\right)^{3/2}$$

2. The **critical point**

$$\lambda_c = \sqrt{\Delta_0/\pi\rho_0}$$

3. The **critical exponents**

$$z\nu = 3/2$$

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Blochian error analysis: result for bulk for surface states

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New DOS with one surface state

$$\rho_1(E) = \underbrace{\frac{|E|}{\sqrt{E^2 - \Delta_0^2}} \rho_0}_{\text{bulk states}} + \underbrace{\frac{1}{L} \delta(E - E_0)}_{\text{surface state}},$$

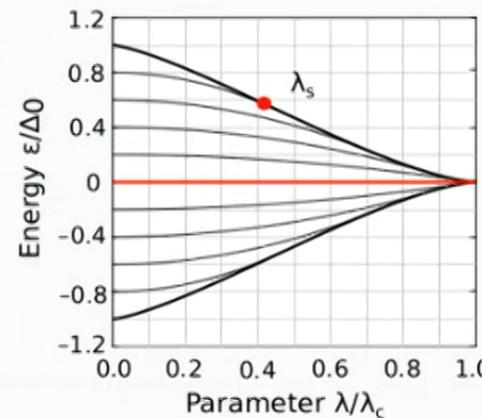
where L is the system size, E_0 is surface state energy.

The energy of the surface state as a function of λ

$$E(\lambda) = E_0 \left(1 - \frac{\pi \rho_0 \lambda^2}{\sqrt{\Delta_0^2 - E_0^2}} \right),$$

where $\lambda < \lambda_s$. λ_s can be derived from

$$E(\lambda_s) = \Delta(\lambda_s)$$



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Phys. Rev. Lett. **121**, 126803



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Applications: artificial quantum matter

Applications: Floquet engineering

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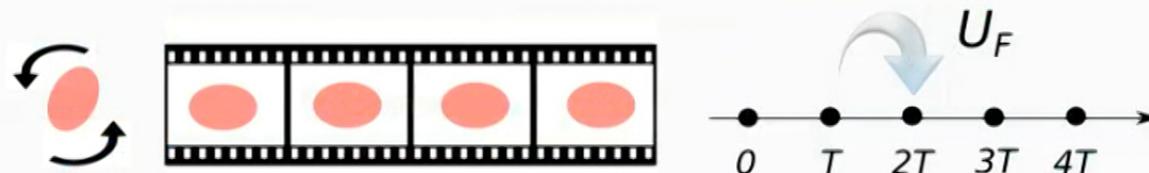
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Setup. Time-periodic Hamiltonian

$$H(t) = H_0 + F(t), \quad F(t) = F(t + T)$$

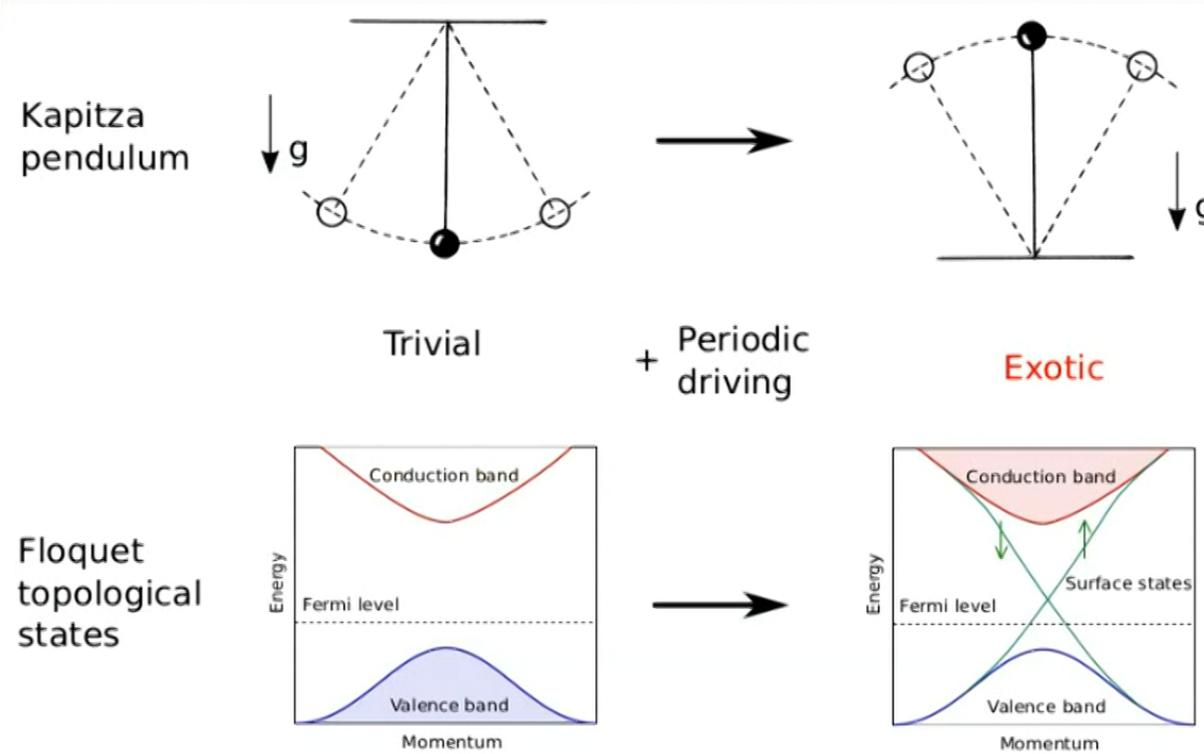
H_0 is trivial phase, V is external field, and T is driving period.



$$U_F = e^{-iH_F T} = \mathcal{T} \exp\left(-i \int_0^T H(t') dt'\right)$$

Applications: example

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Applications: Floquet Majorana modes

Driven Kitaev chain:

$$H(t) = \sum_i (J c_i^\dagger c_{i+1} + \text{h.c.}) + \mu(t) c_i^\dagger c_i - (\Delta c_i c_{i+1} + \text{h.c.})$$

Jiang et al - PRL 2011, Liu, Levchenko, Baranger - PRL 2013

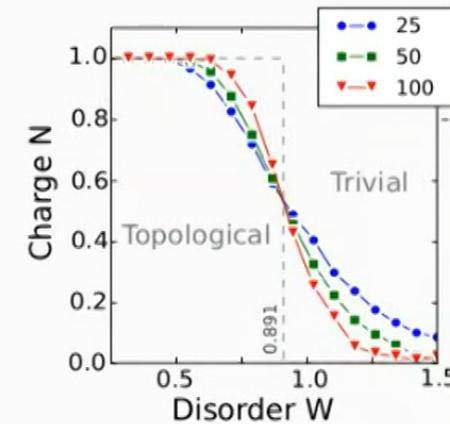
We consider a correction

$$\delta V = \sum_i u_i c_i^\dagger c_i$$

where u_i is a random variable, $u_i \in [-W, W]$, as a simple example of general disorder.

$$N = \text{sign}(\text{Pf}(iH_F))$$

where Pf is Pfaffian.



Applications: Floquet Majorana modes

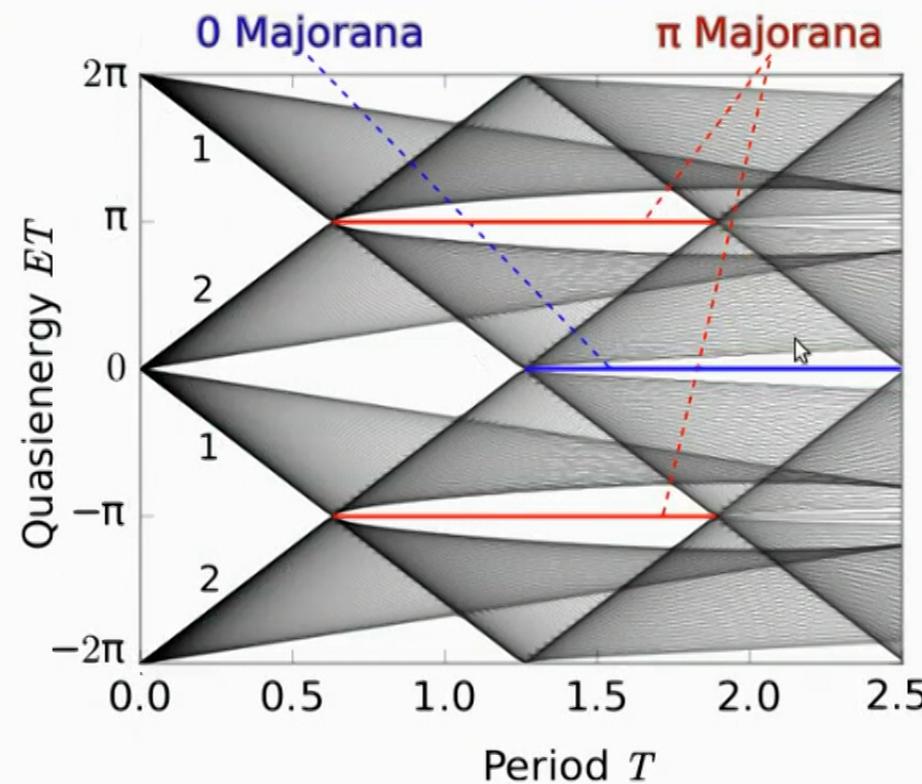


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Applications: why disorder is complex if Floquet systems



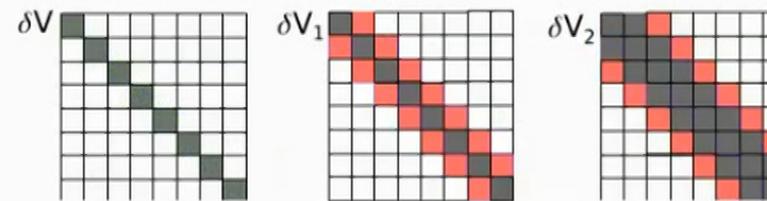
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Disorder induces corrections to Floquet Hamiltonian

$$H_0 + \delta V \rightarrow H'_F = H_F + \delta V_F,$$

High frequency expansion

$$\delta V_F = \delta V + \delta V_1 T + \delta V_2 T^2 + \dots$$



This series diverges at low frequencies!

tional material: freeness as an approximation

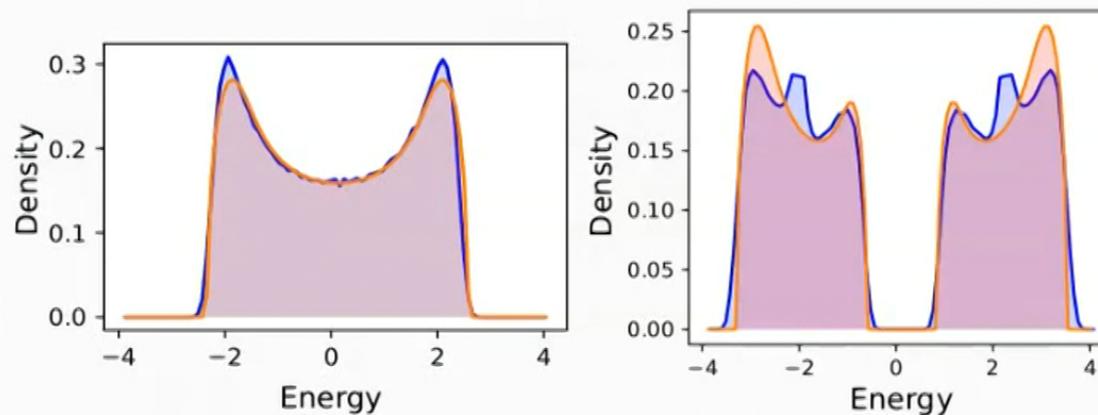


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generic local disorder model

$$H = \sum_{\vec{k}} \begin{pmatrix} E(\vec{k}) & \Delta(\vec{k}) \\ \Delta(\vec{k})^* & -E(-\vec{k}) \end{pmatrix} |\vec{k}\rangle\langle\vec{k}| + \sum_r u_r |\vec{r}\rangle\langle\vec{r}|$$



Left panel: Anderson insulator $E(\vec{k}) = 2J \cos \vec{k}$, $\Delta(\vec{k}) = 0$

Right panel: Disordered chiral superconductor
 $E(\vec{k}) = 2J \cos \vec{k}$, $\Delta(\vec{k}) = \Delta_0$



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- focus on “easily solvable” quantum phase transitions (free QPT)
- the density of states in these systems can be obtained using free probability theory: error analysis
- FQPT is relevant in physical systems subject to
 - specific symmetries (transverse Ising model)
 - complex disorder (e.g. periodically driven systems)

More details here: OS, R. Movassagh Phys. Rev. Lett **121**, 126803 (2018)

Thank you!

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