Title: Free and Interacting Short-Range Entangled Phases of Fermions: Beyond the Ten-Fold Way

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Abstract: <p>It is well-known that sufficiently strong interactions can destabilize some SPT phases of free fermions, while others remain stable even in the presence of interactions. It is also known that certain interacting phases cannot be realized by free fermions. In this talk, we will study both of these phenomena in low dimensions and determine the map from free to interacting SPT phases for an arbitrary unitary symmetry. We will also describe how to compute invariants characterizing interacting phases for free band Hamiltonians with symmetry (in any dimension) using only representation theory.</p>

Free and Interacting Phases of Short-Range Entangled Fermions: Beyond the Ten-Fold Way

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The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$

- Fermions $a_j^A, (a_j^A)^\dagger$, indexed by $j =$ 1D lattice site, $A = 1, \ldots, n$ species
- Real fermions

$$\Gamma_{2j-1}^I = a_j^A + (a_j^A)^\dagger, \qquad \Gamma_{2j}^A = -i(a_j^A - (a_j^A)^\dagger), \qquad \{\Gamma_I^A, \Gamma_J^B\} = 2\delta_{IJ}\delta^{AB}$$

- Time-reversal symmetry

$$T^2 = 1, \quad T a_J T^{-1} = -a_J$$

- Local translation-invariant free (quadratic) fermion Hamiltonian

$$\hat{H} = \frac{i}{2} \sum_j \left(u_{AB} \Gamma_{2j-1}^A \Gamma_{2j}^B + v_{AB} \Gamma_{2j}^A \Gamma_{2j+1}^B \right)$$

- Interested in values of parameters u, v such that $\hat{H}$ is gapped and T-symmetric.

The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$

- Stable deformation classes ("phases")
  1. tensor by an ancilla system with product state ground state

$$\hat{H}_0 = \sum_j \left(a_j^\dagger a_j - \frac{1}{2} \right) = \frac{-i}{2} \sum_j \Gamma_{2j-1}\Gamma_{2j}, \qquad |\psi_{g.s.}\rangle = \otimes_j |0\rangle_j, \quad a_j|0\rangle_j = 0$$

  2. continuously deform the parameters while preserving the gap and symmetry
- Nontrivial Majorana chain

$$\hat{H}_1 = \frac{-i}{2} \sum_j \Gamma_{2j}\Gamma_{2j+1} = \frac{1}{2} \sum_j \left(-a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j \right)$$

- Free classification: $n \in \mathbb{N}$ (boundaries = difference classes $n - m \in \mathbb{Z}$)

$$\hat{H}_n = \sum_A^n \hat{H}_1^A = \frac{-i}{2} \sum_{j,A}^{A=n} \Gamma_{2j}^A \Gamma_{2j+1}^A$$

  - These exhaust all invertible / SRE (ie nondegenerate) phases.

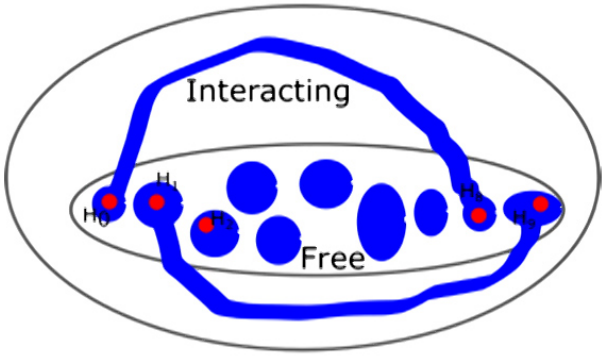
The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$

- Turning on local interactions

$$\hat{H} = \frac{i}{2}\sum_j \left(u_{AB}\Gamma^A_{2j-1}\Gamma^\beta_{2j} + v_{AB}\Gamma^A_{2j}\Gamma^B_{2j+1} \right) + t\Gamma\Gamma\Gamma\Gamma + \cdots$$

- In this larger parameter space, $\hat{H}_8$ is destabilized by interactions: $\hat{H}_8 \sim \hat{H}_0$.[1]

$$\mathbb{Z} \to \mathbb{Z}/8$$



- If T-asymmetric terms are permitted, $\hat{H}_2 \sim \hat{H}_0$ at the quadratic level:

$$\mathbb{Z}/2 \to \mathbb{Z}/2$$

[1] Fidkowski-Kitaev, 2009

The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$

- The K-theory[2] and spin cobordism classifications of invertible phases yield similar results in all dimensions and 10-fold way symmetry classes.

$$KO_0 = \mathbb{Z}, \qquad \mathrm{Hom}(\mathrm{Tors}(\Omega_2^{\mathrm{pin}^-}), U(1)) = \mathbb{Z}/8$$

Free	0	1	2	3	4	5	6	7
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$

$\longrightarrow$

Int.	0	1	2	3	4	5	6	7
BDI	$\mathbb{Z}_2$	$\mathbb{Z}_8$				$\mathbb{Z}_{16}$		$\mathbb{Z}_2^2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}^2$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$				$\mathbb{Z}_2 \times \mathbb{Z}_{32}$

- Note: consider only strong invariants; translation-invariance is not protected.
- Intrinsically interacting phases. For example, class D systems in 6D.

$$\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

In the past, examples of intrinsically interacting crystalline SPT phases.[3]
Non-invertible (topologically ordered) examples are well-known (eg FQHE).

[2] Kitaev, 2008; Schnyder-Ryu-Furusaki-Ludwig, 2009

[3] Lapa, Teo, Hughes 2014

Two Goals

1. Understand classification of free fermionic SRE phases with on-site symmetry.
   - Representation Theory + Periodic Table = Symmetric Free Classification
2. Study the map into the (already known) interacting classification of SRE phases.

$$\{ \text{ Free Phases } \} \longrightarrow \{ \text{ Interacting Phases } \}$$

   - Surprising finding: examples of unstable free phases and intrinsically interacting invertible phases are common and exist in dimension as low as zero.
   - Understand interacting invariants in the language of band theory.

Band Hamiltonians

- Momentum space variables $(\Gamma_k^A)^\dagger = \Gamma_{-k}^A, \quad A = 1, \ldots, 2n, \quad k \in \mathrm{BZ}$
- Free fermionic $\hat{H} \longrightarrow$ Class D band Hamiltonian $X(k)$

$$\hat{H} = \frac{i}{2}\Gamma_k X(k) \Gamma_{-k}$$

  - $X(k)$ is a $2n \times 2n$ skew-Hermitian matrix
  - PHS condition (of BdG Hamiltonian) becomes a reality condition in the Γ basis:

$$X(k)^* = \mathcal{K} X(k) \mathcal{K}^{-1} = X(-k),$$

    in particular, $X(k)$ is real skew-symmetric at PHS fixed points

- Other relevant symmetry classes:
  - Class A. No PHS constraint. Typically written in Dirac basis:

$$\hat{H} = a_k^\dagger H(k) a_k$$

  - Class C. Has PHS constraint with $C^2 = -1$. $C^{-1} H(k) C = -H(-k)$

On-site Symmetry

- Work in 0D (no k index) for simplicity. Easy to generalize.
- The total symmetry group is a central extension of the bosonic symmetries by fermion parity:

$$\mathbb{Z}_2^f \to \hat{G} \to G_b$$

- A symmetry $\hat{R}$ of a free Hamiltonian is represented on the fermions as R:

$$\hat{R}\Gamma^A\hat{R}^{-1} = \sum_B R_{AB}\Gamma^B.$$

  - R preserves fermionic commutation relations iff $R \in O(2n)$, ie R is real.
  - R is said to be allowed if fermion parity actions by $R(p) = -\mathbb{1}$.
- R decomposes into real irreps r_α with multiplicity n_α.
  - The band Hamiltonian X decomposes into blocks X_α acting on $r_\alpha \otimes \mathbb{R}^{n_\alpha}$.

$\mathbb{R}$-type irreducibles

- Schur's Lemma: the commutant of an irrep r is a division algebra.
  - Complex representations: must be $\mathbb{C}$
  - Real representations: may be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$
- $\mathbb{R}$-type: commutant is spanned by $\mathbb{1}$ (equivalently, $r^{\mathbb{C}} := r \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible)

$$\Longrightarrow \qquad X_\alpha = \mathbb{1} \otimes \mathcal{A} \qquad (\text{acting on } r_\alpha \otimes \mathbb{R}^{n_\alpha})$$

  for some $n_\alpha \times n_\alpha$ real skew-symmetric matrix $\mathcal{A}$

- The block X_α is $\dim r_\alpha$ copies of the class D (no other symmetry) system $\mathcal{A}$:

$$\hat{H} = \frac{i}{2}\Gamma^A X_{AB} \Gamma^B = \frac{i}{2} \sum_{\mu}^{\dim r_\alpha} \Gamma^a_\mu \mathcal{A}_{ab} \Gamma^b_\mu$$

- Contribution to classification:
  - Each allowed $\mathbb{R}$-type irrep of $\hat{G}$ contributes a class D invariant:

0	1	2	3	4	5	6	7
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	

$\mathbb{H}$-type irreducibles

- $\mathbb{H}$-type: $\mathbb{1}, \mathcal{I}, \mathcal{J}, \mathcal{K} \in O(\dim r_\alpha)$ with $\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -\mathbb{1}$ and $\mathcal{I}\mathcal{J} = \mathcal{K}$

$$\implies \qquad X_\alpha = \mathbb{1} \otimes \mathcal{A} + \mathcal{I} \otimes \mathcal{B} + \mathcal{J} \otimes \mathcal{C} + \mathcal{K} \otimes \mathcal{D}$$

  for a real skew-symmetric matrix $\mathcal{A}$ and real symmetric matrices $\mathcal{B}, \mathcal{C}, \mathcal{D}$.

- Similarly, in the basis of Nambu-Dirac spinors $\Upsilon = \begin{pmatrix} \Psi & \bar{\Psi}^T \end{pmatrix}^T$,

$$\hat{H} = \frac{1}{2} \bar{\Upsilon} \mathcal{Z} \Upsilon \qquad \text{where} \qquad \mathcal{Z} = \begin{pmatrix} i\mathcal{A} - \mathcal{C} & \mathcal{B} + i\mathcal{D} \\ \mathcal{B} - i\mathcal{D} & i\mathcal{A} + \mathcal{C} \end{pmatrix}.$$

- The matrix $\mathcal{Z}$ is equivalent to a class C system, as is satisfies a PHS constraint

$$\sigma^\dagger \mathcal{Z} \sigma = -\mathcal{Z}, \quad \sigma^* \sigma = -\mathbb{1} \qquad \text{with} \qquad \sigma := i\sigma_2 \otimes \mathbb{1}.$$

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		$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

Summary: The Free $\hat{G}$-Symmetric Classification

- In each dimension, the free invariants are read off of the periodic table

$d=0$	$d=1$	$d=2$	$d=3$
$\oplus_{r\in\mathbb{R}}\mathbb{Z}_2 \times \oplus_{r\in\mathbb{C}}\mathbb{Z}$	$\oplus_{r\in\mathbb{R}}\mathbb{Z}_2$	$\oplus_r\mathbb{Z}$	trivial

- Examples:
  - Superconductors with spin parity symmetry: $\hat{G} = \mathbb{Z}_2^f \times \mathbb{Z}_2$.
    Two allowed irreps: $R(p) = -1, R(u) = \pm 1$. Both are $\mathbb{R}$-type.
    $\implies$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification in 0d and 1d
    $\mathbb{Z} \times \mathbb{Z}$ classification in 2d
  - Charge-4e superconductors: $\hat{G} = \mathbb{Z}_4^f$. Single allowed irrep is $\mathbb{C}$-type.
    $\implies$ $\mathbb{Z}$ classification in 0d and 2d
    trivial classification in 1d
  - Class A insulators: $\hat{G} = U(1)^f$. Allowed $\mathbb{C}$-type irrep for each charge $n \in \mathbb{N}$.
    $\implies$ $\mathbb{Z}^{\mathbb{N}}$ classification in 0d and 2d
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- 3D SPT phases protected by a unitary on-site symmetry (for example, the group supercohomology phases) never have free fermion realizations.

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Sidenote: Twisted Equivariant K-Theory

- When $\hat{G} = G_b \times \mathbb{Z}_2^f$, the allowed irreps of $\hat{G}$ are simply the irreps of G_b.
- In this case, our classification is given by real equivariant K-theory:[4]

$$KO^d(G) = KO^d \otimes R_{\mathbb{R}}(G) + KU^d \otimes R_{\mathbb{C}}(G) + KH^d \otimes R_{\mathbb{H}}(G)$$

- For more general extensions $\hat{G}$, the invariants live in twisted K-theory.
- When $\hat{G} = G_b \times \mathbb{Z}_2^f$, some information about the interacting classification may be extracted from the Atiyah-Hirzebruch Spectral Sequence (AHSS).
  - Related to the layered / decorated domain wall construction.
  - Not as useful for the free classification due to non-connective spectrum. Returns infinite tower of free invariants, the "completion" of which is $KO^d(G)$.

[4] Segal, 1968

Relative Charge of Ground States

- Focus on a $\mathbb{C}$-type block $r_\alpha \otimes \mathbb{R}^{n_\alpha}$. The class A invariant $m_\alpha \in \mathbb{Z}$ counts the negative eigenvalues of $\mathcal{H}$.

$$\hat{H} = \sum_{\mu}^{\dim q_\alpha} \bar{\Psi}^a_\mu \mathcal{H}_{ab} \Psi^b_\mu$$

  - The relative class A invariant of a pair $\mathcal{H}, \mathcal{H}'$ is $\rho_\alpha = m'_\alpha - m_\alpha$.

- Deform $\mathcal{H}$ to $\mathcal{H}'$ along a path that flips an eigenvalue from positive to negative ρ_α-many times. Each time this happens, the ground state changes by

$$|\text{g.s.}'\rangle = \prod_{\mu} (\bar{\Psi}^a_\mu v^a) |\text{g.s.}\rangle,$$

which is an operator of charge $\det \bar{q}$.

- Therefore the irrep r_α contributes a ground state charge

$$\rho_\alpha \mapsto (\det \bar{q}_\alpha)^{\rho_\alpha}.$$

Relative Charge of Ground States

- The charge $(\det \bar{q}_\alpha)^{\rho_\alpha} \in H^1(\hat{G}; U(1))$ may be regarded as the first Chern class

$$c_1(\rho_\alpha \bar{q}_\alpha) \in H^2(\hat{G}, \mathbb{Z})$$

of $\rho_\alpha \bar{q}_\alpha$ as a complex vector bundle over $B\hat{G}$.

- Similarly, $\mathbb{R}$-type blocks contribute

$$\rho_\alpha \mapsto (\det r_\alpha)^{\rho_\alpha},$$

which is the first Stiefel-Whitney class

$$w_1(\rho_\alpha r_\alpha) \in H^1(\hat{G}, \mathbb{Z}/2)$$

of $\rho_\alpha r_\alpha$ as a real vector bundle over $B\hat{G}$.

- In higher dimensions, the map from free to interacting phases will involve other representation theoretic invariants w_2, p_1, etc.

Turning on Interactions in One Dimension

- Old news: An interacting 1d SPT protected by an on-site unitary $\hat{G}$ is completely characterized by the following invariants:
  - $\gamma \in \mathbb{Z}/2$ – the number of fermionic boundary zero modes mod 2
  - if $\gamma = 0$, an invariant $\omega \in H^2(\hat{G}, U(1))$ – the projectivity of $\hat{G}$ at the boundary
  - if $\gamma = 1$, a map $\mu : \hat{G} \to \mathbb{Z}/2$ with $\mu(p) \neq 0$ and an $\alpha \in H^2(G_b, U(1))$
- Let $\mathcal{R} : \hat{G} \to O(M)$ be the symmetry action on the M boundary modes.

$$\mathcal{R} = \oplus_\alpha \nu_\alpha r_\alpha$$

  - $\nu_\alpha \in \mathbb{Z}$ records the number of boundary modes transforming as r_α
- We just learned: A free 1d SPT is characterized by, for each allowed $\mathbb{R}$-type irrep, a class D invariant

$$\rho_\alpha = \nu_\alpha \text{ mod } 2 \in \mathbb{Z}_2$$

- The first interacting invariant γ is simply...

$$\gamma = M \text{ mod } 2 = \dim \mathcal{R} \text{ mod } 2 = \sum_{\alpha \in \mathbb{R}\text{-type}} \rho_\alpha \dim r_\alpha \quad \text{mod } 2$$

Turning on Interactions in One Dimension

- How are other other 1d interacting invariants determined by $\{\rho_\alpha\}$ (via $\mathcal{R}$)?
- When $\gamma = 0$ (M even), the interacting invariant $\omega \in H^2(\hat{G}, U(1))$ records the projectivity of the $\hat{G}$ action on the $2^{M/2}$-dimensional boundary Fock space.
- In other words, ω is the obstruction to lifting $\mathcal{R} : \hat{G} \to O(M)$ to $Pin^c(M)$, i.e. the image in $H^2(\hat{G}; U(1))$ of the second Stiefel-Whitney class

$$w_2(\mathcal{R}) = \sum_\alpha \rho_\alpha w_2(r_\alpha) + \sum_{\alpha<\beta} \rho_\alpha \rho_\beta w_1(r_\alpha) \cup w_1(r_\beta) \mod 2.$$

- Similarly, one may express the $\gamma = 1$ (M odd) invariants μ, α in terms of $\mathcal{R}$.

Turning on Interactions in One Dimension

The finding that ω is given by w_2 has two interesting consequences:

1. The known stacking law for interacting phases[5] is recovered:

$$\omega_A \circ \omega_B = \omega_A + \omega_B + \beta_A \cup \beta_B \text{ where } \beta(g) = \omega(g,p) - \omega(p,g)$$

   is explained by

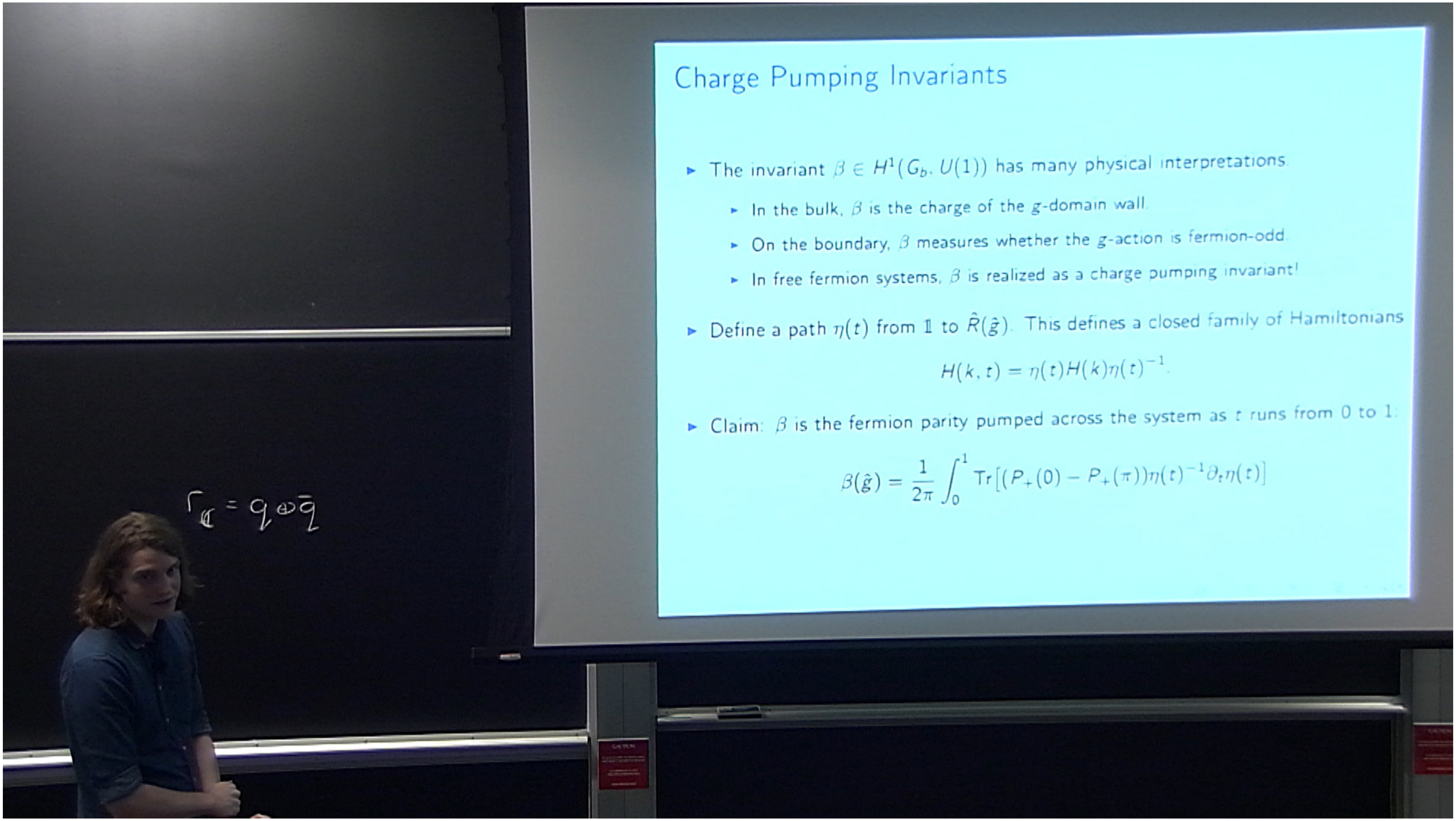
$$w_2(\mathcal{R} \oplus \mathcal{R}') = w_2(\mathcal{R}) + w_2(\mathcal{R}') + w_1(\mathcal{R}) \cup w_1(\mathcal{R}').$$

2. There are intrinsically interacting phases of order 2, as w_2 satisfies (an infinite number of) special relations such as

$$\mathrm{Bock}(w_2 \cup w_2) = 0 \in H^5(\hat{G}; \mathbb{Z}),$$

   which a generic $\omega \in H^2(\hat{G}, U(1))$ does not.

[5] Gaiotto-Kapustin 2015


Charge Pumping Invariants
The invariant $\beta \in H^1(G_b, U(1))$ has many physical interpretations.
In the bulk, β is the charge of the g-domain wall.
On the boundary, β measures whether the g-action is fermion-odd.
In free fermion systems, β is realized as a charge pumping invariant!
Define a path $\eta(t)$ from $\mathbb{1}$ to $\hat{R}(\hat{g})$. This defines a closed family of Hamiltonians
$H(k,t) = \eta(t)H(k)\eta(t)^{-1}$.
Claim: β is the fermion parity pumped across the system as t runs from 0 to 1:
$\beta(\hat{g}) = \frac{1}{2\pi}\int_0^1 \mathrm{Tr}\left[(P_+(0) - P_+(\pi))\eta(t)^{-1}\partial_t\eta(t)\right]$

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Two Dimensions

- Assume $\hat{G} = G_b \times \mathbb{Z}_2^f$ for finite G_b. The interacting invariants are
  - $\kappa \in \frac{1}{2}\mathbb{Z}$ – chiral central charge of the boundary theory
  - $(\alpha, \beta, \gamma) \in C^3(G_b, U(1)) \times H^2(G_b, \mathbb{Z}_2) \times H^1(G_b, \mathbb{Z}_2)$ satisfying[6] $\delta\alpha = \frac{1}{2}\beta \cup \beta$.
- The free class D, A, C invariants $\rho_\alpha \in \mathbb{Z}$ count the chiral boundary modes.

$$\implies \qquad \kappa = \frac{1}{2}\sum_\alpha \rho_\alpha \dim r_\alpha .$$

- Understanding (α, β, γ) requires understanding their physical meanings in terms of boundary physics. However, we can make a stacking-compatible guess
  - $\gamma = w_1(\mathcal{R})$ where $\mathcal{R}$ is the virtual representation $\oplus_\alpha \rho_\alpha r_\alpha$
  - $\beta = w_2(\mathcal{R})$ or $w_2(\mathcal{R}) + w_1^2(\mathcal{R})$, the obstructions to $Pin^\pm$-lifts of $\mathcal{R}$
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6 Gu-Wen, 2012

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Conclusions and Takeaways

1. We've given a classification of free SPT's protected by on-site unitary $\hat{G}$.
   - Each type of irrep contributes a symmetry protected invariant $\rho_\alpha \in 0, \mathbb{Z}_2, \mathbb{Z}$, depending on the spatial dimension and the type of irrep ($\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$).
   - The SPT invariants arise from the class D, A, and C rows of the periodic table.
2. We've constructed a map to interacting invariants in 0d, 1d, and (partially) 2d.
   - Check whether a free phase $\{\rho_\alpha\}$ is stable to interactions by looking at its image.
   - Check whether an interacting phase is intrinsically interacting by asking whether it lives outside the image. Examples (incl. of order 2) exist in all dimensions.
   - Interacting invariants are realized in free fermion systems as characteristic classes of a representation $\mathcal{R}$ on the fermion operators.