Title: Operator dynamics and quantum chaos: an approach from Brownian circuit

Date: Nov 01, 2018 02:00 PM

URL: http://pirsa.org/18110070

Abstract: <p>Operator scrambling is a crucial ingredient of quantum chaos. Specifically, in the quantum chaotic system, a simple operator can become increasingly complicated under unitary time evolution. This can be diagnosed by various measures such as square of the commutator (out-of-time-ordered correlator), operator entanglement entropy etc. In this talk, we discuss operator dynamics in three representative models: a 2-local spin model with all-to-all interaction, a chaotic spin chain with long-range interactions, and the quantum linear map. In the first two examples, we explore the operator dynamics by using the quantum Brownian circuit approach and transform the operator spreading into a classical stochastic problem. Although the speeds of scrambling are quite different, a simple operator can eventually approach a "highly entangled" operator with operator entanglement entropy taking a volume law value (close to the Page value). Meanwhile, the spectrum of the operator reduced density matrix develops a universal spectral correlation which can be characterized by the Wishart random matrix ensemble. In contrast, in the third example (the quantum linear map), although the square of commutator can increase exponentially with time, a simple operator does not scramble but performs chaotic motion in the operator basis space determined by the classical linear map. We show that once we modify the quantum linear map such that operator can mix in the operator basis, the operator entanglement entropy can grow and eventually saturate to its Page value, thus making it a truly quantum chaotic model.</p>

Kavli Institute for Theoretical Physics
University of California, Santa Barbara

GORDON AND BETTY MOORE FOUNDATION

Operator dynamics and quantum chaos: An approach from Brownian circuit

Xiao Chen
KITP

arXiv: 1804.08655
arXiv: 1805.09307
arXiv: 1808.09812
with Tianci Zhou

+ unpublished work with Andreas Ludwig

Nov. 1, PI

Outline

- Definition of operator dynamics
- Operator dynamics in systems with all-to-all interaction
- Operator dynamics in systems with local interaction
- Operator dynamics in systems with long-range interaction
- Operator entanglement entropy and random matrix physics

Quantum chaos

- Classical chaos

$$|\delta \mathbf{d}(t)| \sim e^{\lambda_c t} |\delta \mathbf{d}(0)|$$

- Quantum chaos

(1) Square of commutator (out-of-time-ordered correlator)

$$C(t) = -\langle [V(t), W]^2 \rangle_\beta$$

It can grow exponentially in time. The growth rate is called quantum Lyapunov exponent. Larkin, Ovchinnikov 1969

(2) Level repulsion and random matrix physics Bohigas, Giannoni, Schmit 1984

Operator dynamics

- For a quantum wave function, it can be expanded into a linear combination of basis in the Hilbert space,

$$|\psi\rangle = \sum_n c_n |n\rangle$$

- For a quantum operator, it can be treated as a wave function living in an operator Hilbert space.

$$\hat{O} = \sum_{m,n} C_{m,n} |n\rangle\langle m| \longleftrightarrow |O\rangle = \sum_{m,n} C_{m,n} |n\rangle |m\rangle$$

- Under unitary time evolution, it can be expanded as

$$\hat{O}(t) = \sum_j \alpha_j(t) \hat{\mathcal{B}}_j \qquad \sum_j |\alpha_j(t)|^2 = 1$$

$\{\hat{\mathcal{B}}_j\}$ is a set of operator basis satisfying $\mathrm{Tr}\hat{\mathcal{B}}_i^\dagger \hat{\mathcal{B}}_j = \delta_{ij}$

- The evolution of $\alpha_j(t)$ determines the operator dynamics.

Operator dynamics

- For a quantum wave function, it can be expanded into a linear combination of basis in the Hilbert space,

$$|\psi\rangle = \sum_n c_n |n\rangle$$

- For a quantum operator, it can be treated as a wave function living in an operator Hilbert space.

$$\hat{O} = \sum_{m,n} C_{m,n} |n\rangle\langle m| \quad \longleftrightarrow \quad |O\rangle = \sum_{m,n} C_{m,n} |n\rangle|m\rangle$$

- Under unitary time evolution, it can be expanded as

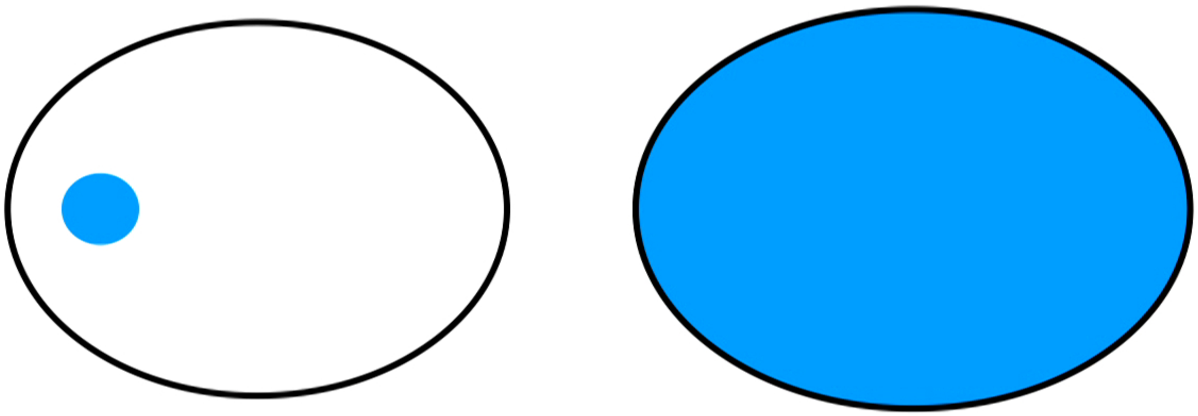
$$\hat{O}(t) = \sum_j \alpha_j(t)\hat{\mathcal{B}}_j \qquad \sum_j |\alpha_j(t)|^2 = 1$$

$\{\hat{\mathcal{B}}_j\}$ is a set of operator basis satisfying $\mathrm{Tr}\hat{\mathcal{B}}_i^\dagger \hat{\mathcal{B}}_j = \delta_{ij}$

- The evolution of $\alpha_j(t)$ determines the operator dynamics.

Chaotic operator dynamics

- In a chaotic system, under unitary time evolution, a simple operator becomes increasingly complicated.
- The quantum information encoded in this operator is delocalized and this phenomenon is also called scrambling.



- After sufficient time evolution, an initially local operator will approach an entirely non-local random operator.

The general features of operator growth

In chaotic system, it is impossible to keep track of each $\alpha_j(t)$

The emergent hydrodynamics

Nahum, Vijay and Haah, 2017, Keyserlingk, Rakovszky, Pollmann and Sondhi, 2017
Khemani, Vishwanath and Huse, 2017, Rakovszky, Pollmann and Keyserlingk, 2017
Roberts, Stanford and Streicher, 2018, Xu and Swingle, 2018, XC and Zhou, 2018

(1) All-to-all interaction
(2) Local interaction
(3) Power-law interaction

The development of random matrix physics

XC and Ludwig, 2017

Fast scrambler

- Sachdev-Ye-Kitaev (SYK) model

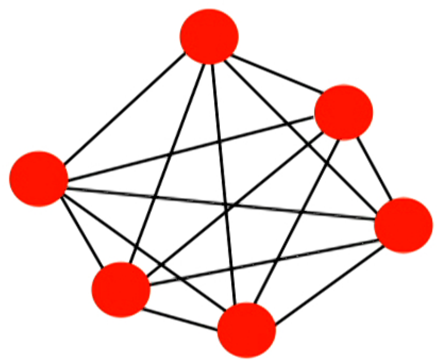
$$H_{\mathrm{SYK}_4} = \sum_{i,j,k,l=1}^{N} \frac{J_{ijkl}}{4!} \chi_i \chi_j \chi_k \chi_l \qquad \overline{J_{ijkl}} = 0, \quad \overline{J_{ijkl}^2} = \frac{3! J^2}{N^3}$$

(1) SYK model can be analytically solved in the large N limit
(2) Extensive T=0 residual entropy
(3) Maximally chaotic & saturate the chaos bound at low temperature

Sachdev and Ye, 1993, Kitaev, 2015, Maldacena, Shenker and Stanford 2016
Sekino and Susskind, 2008, Maldacena, Shenker and Stanford 2016

- Two-local qubit model

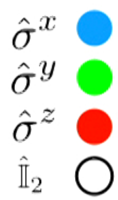
$$H = \frac{1}{\sqrt{9N}} \sum_{1 \le i < j \le N} \sum_{a,b=1}^{3} \alpha_{a,b,(i,j)} \sigma_i^a \sigma_j^b$$

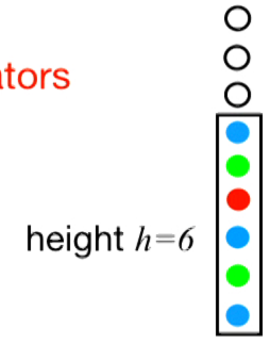


The height of the operator

- For systems composed of 1/2 spins, a natural basis operator is the Pauli string operator

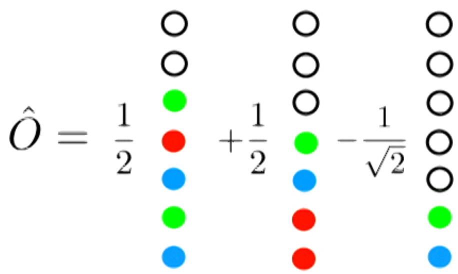
- The height is the number of the non-identity operators

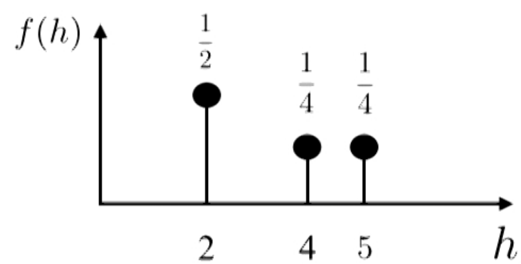




- The height distribution is a weighted sum $f(h,t) = \sum_{\text{height}(B_j)=h} |\alpha_j(t)|^2$

Ignore the explicit structure of Pauli string operator and the phase information





The Brownian two-local circuit

The complete unitary time evolution is generated in the continuum limit of

$$e^{-iH_s\Delta t}e^{-iH_{s-1}\Delta t}\dots$$

Lashkari, Stanford, Hastings, Osborne and Patrick Hayden, 2011
Xu and Swingle, 2018, Zhou and XC, 2018

where $$H_s = J\sum_{i<j}\sum_{\mu_i,\mu_j=0}^{q^2-1}\sigma_i^{\mu_i}\otimes\sigma_j^{\mu_j}\Delta B^s_{i,j,\mu_i,\mu_j}$$

The height distribution $\mathbf{f}(h, t)$ is determined by the master equation:

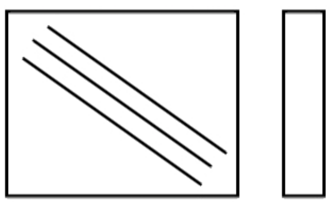
$$\frac{d\mathbf{f}(t)}{dt} = A_f\mathbf{f}(t)$$

$$(A_f)_{k,k} = \frac{4}{n}k\big[-(n-k)+\frac{1}{q^2}(n-2k+1)\big]$$

$$(A_f)_{k-1,k} = \frac{4}{n}\frac{k(k-1)}{q^2}$$

$$(A_f)_{k+1,k} = \frac{4}{n}k(n-k)\big[1-\frac{1}{q^2}\big].$$

Zhou and XC, 2018



A is tri-diagonal matrix ***f***

The above master equation has two static solutions which satisfy A**f**=0

Early time behavior (large N limit)

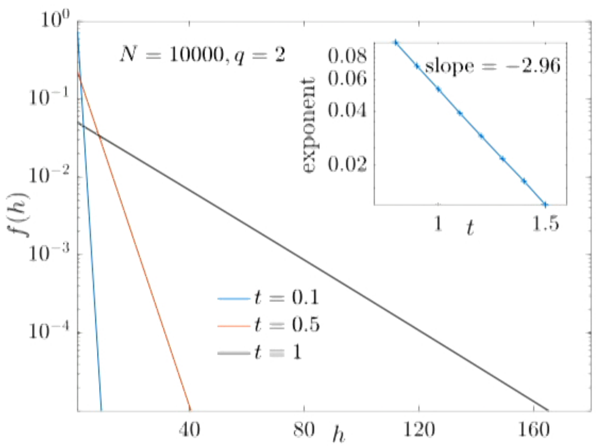
$$\frac{df_k(t)}{dt} = -\lambda_q k f_k + \lambda_q (k-1) f_{k-1}$$

The above result is similar to that found in SYK model in some limits

Roberts, Stanford and Streicher, 2018

$$f_k(t) = e^{-\lambda_q t}[1 - e^{-\lambda_q t}]^{(k-1)}$$

$$\langle h(t) \rangle = e^{\lambda_q t} \langle h(t=0) \rangle$$

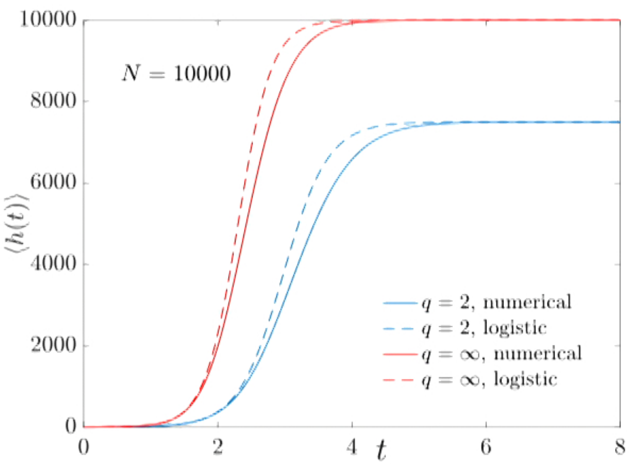


In the continuum limit

$$\partial_t f(h,t) = -\frac{4}{N}\partial_h[h(h_{\rm sat} - h)f]$$

$$\partial_t \langle h \rangle = \frac{4}{N}(h_{\rm sat} - \langle h \rangle)\langle h \rangle - \frac{4}{N}(\langle h^2 \rangle - \langle h \rangle^2)$$

The logistic differential equation

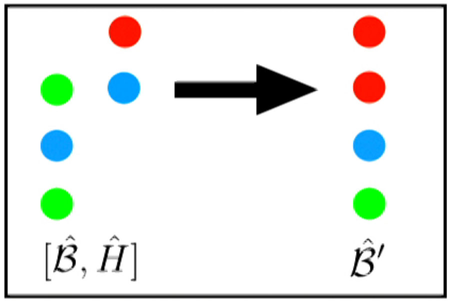


The operator growth in two-local model

The short time evolution h to $h+1$:

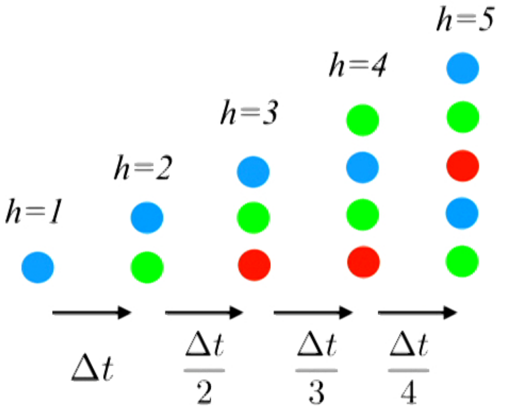
$$\sigma_i^a \sigma_j^b \quad \rightarrow \quad 2h, \quad 3(N-h)$$

$$\hat{\mathcal{B}}(t) \sim \hat{\mathcal{B}}(0) + i[\hat{H}, \hat{\mathcal{B}}(0)]t$$



(1) The number of terms which changes the height (does not commute with B) from h to $h+1$ is $6h(N-h)$

(2) The transition time from h to $h+1$ is inversely proportional to the number of terms and scales as $1/h$ when $h << N$



$$t = \sum_{l=1}^{h} \Delta t \frac{N-1}{l(N-l)} \approx \Delta t \log \frac{h(N-1)}{N-h} \quad \Rightarrow \quad h(t) = \frac{N e^{\frac{t}{\Delta t}}}{N + e^{\frac{t}{\Delta t}} - 1}$$

The mean height satisfies the logistic differential equation and is linearly proportional to OTOC

$$\frac{dh}{d(t/\Delta t)} = h\left(1 - \frac{h}{N}\right)$$

Assume that the operator has a typical height with no fluctuation

XC, Zhou, 2018

Early time behavior (large *N* limit)

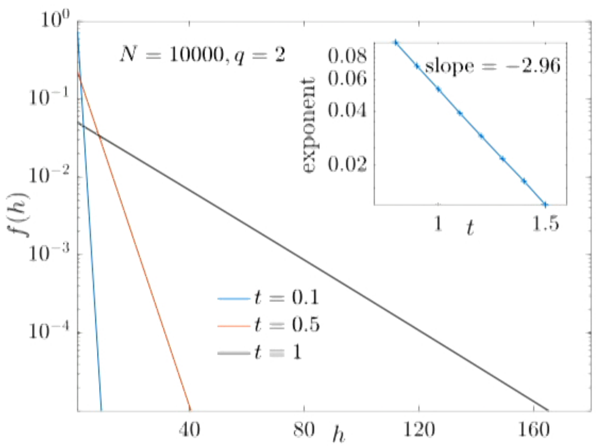
$$\frac{df_k(t)}{dt} = -\lambda_q k f_k + \lambda_q (k-1) f_{k-1}$$

The above result is similar to that found in SYK model in some limits

Roberts, Stanford and Streicher, 2018

$$f_k(t) = e^{-\lambda_q t}[1 - e^{-\lambda_q t}]^{(k-1)}$$

$$\langle h(t) \rangle = e^{\lambda_q t} \langle h(t=0) \rangle$$

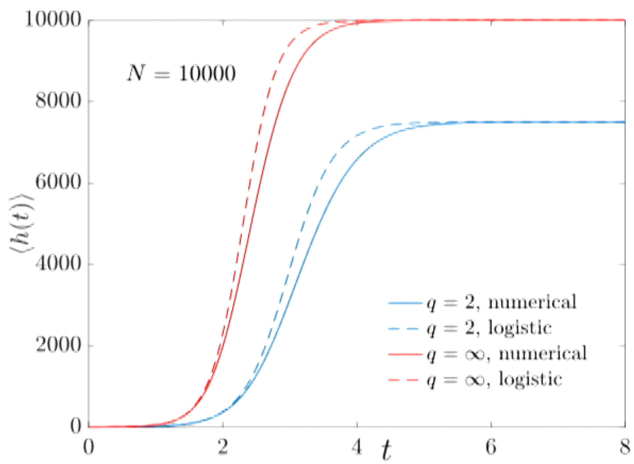


In the continuum limit

$$\partial_t f(h,t) = -\frac{4}{N} \partial_h [h(h_{\text{sat}} - h) f]$$

$$\partial_t \langle h \rangle = \frac{4}{N} (h_{\text{sat}} - \langle h \rangle) \langle h \rangle - \frac{4}{N} (\langle h^2 \rangle - \langle h \rangle^2)$$

The logistic differential equation



Operator dynamics in spin-1/2 chain with local interaction

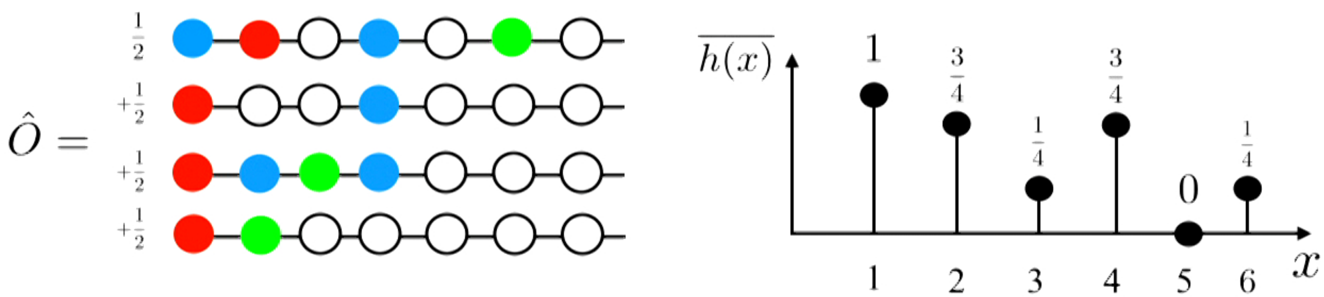
- Similar to the previous model with all-to-all interaction, a local operator will become increasingly non-local and approach a random superposition of the basis operators at late time

The operator height distribution

$$f(\mathbf{h}, t) = \sum_{\text{height}(B_\mu)=\mathbf{h}} |\alpha_\mu(t)|^2$$

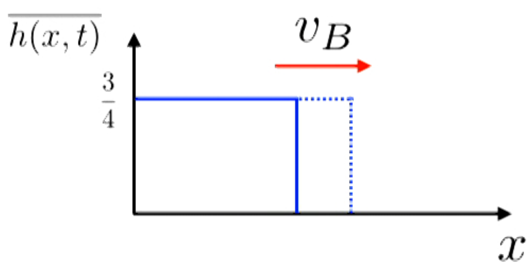
where **h** is a L-component vector

The mean height $\overline{h(x)}$



Operator dynamics in spin-1/2 chain with local interaction

- It is reasonable to assume the coefficients $\alpha_j(t)$ are uniformly distributed among the Pauli string operators with the same length
- The zeroth order solution neglects the possible dispersion as it moves with group velocity



- At infinite high temperature, the OTOC is the same as $\overline{h(x,t)}$

The Brownian circuit

The joint distribution f(**h**) is governed by this master equation:

$$\frac{\partial f(\mathbf{h},t)}{\partial t} = \sum_{j\neq i} 3D_{ij}h_{i+j} f(\mathbf{h}-\mathbf{e}_i,t) + \sum_{j\neq i} D_{ij}h_{i+j} f(\mathbf{h}+\mathbf{e}_i,t) - \{3D_{ij}h_{i+j}(1-h_i) + D_{ij}h_{i+j}h_i\}\, f(\mathbf{h},t)$$

before	after	rate($q=2$)	rate($q=\infty$)
↑	↓	D_{ij}	0
↓	↑	$3D_{ij}$	$4D_{ij}$

(1) The endpoint is performing biased random walk
(2) The length distribution of h=1 domain is a moving Gaussian packet
(2) The wavefront interpolates between the left h=1 and the right h=0 domains
(3) The physics is the same as Haar random circuit

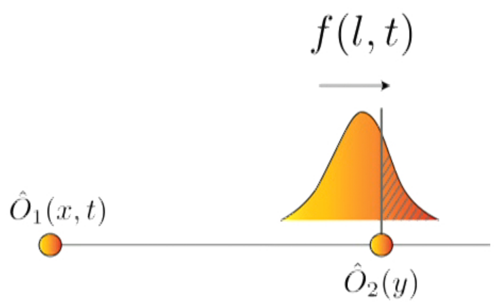
q here is onsite Hilbert space

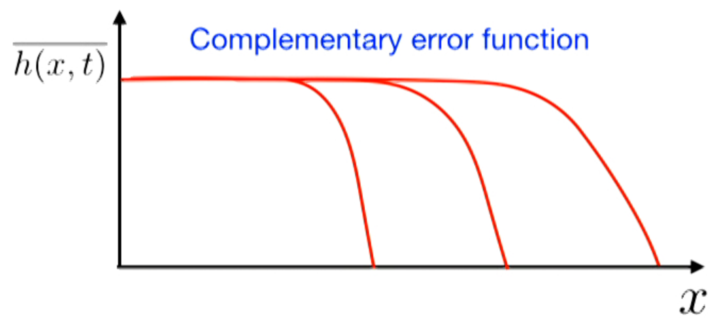
Nahum, Vijay and Haah, 2017
Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017

Initial condition ① ⓪ ⓪ ⓪ ⓪ ⓪

Final steady state ① ① ① ① ① ①

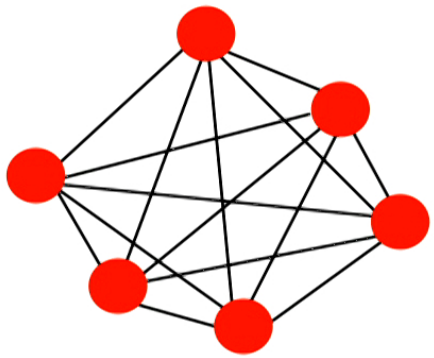
$(q=\infty)$



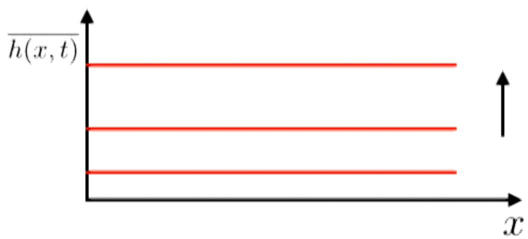


Two pictures of operator dynamics

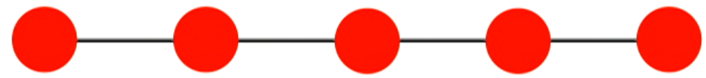
All-to-all interaction



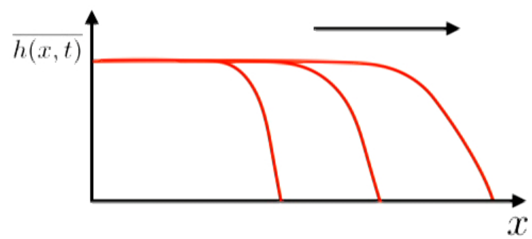
(1) Scrambling time ~ log N



Local interaction

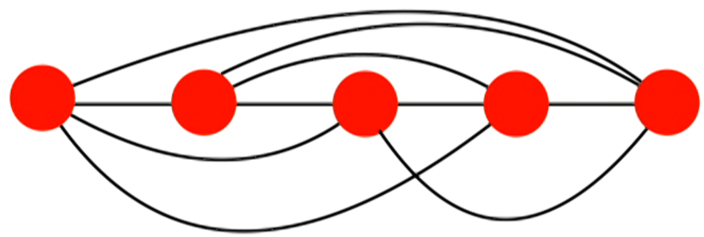


(1) Scrambling time ~ N
(2) The linear light cone structure

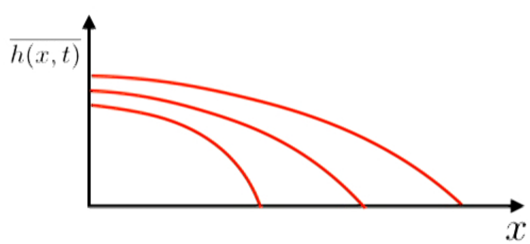


Operator dynamics → **Brownian circuit** → Classical stochastic problem

Operator dynamics in long range interaction systems



- The emergent light cone behavior
- The OTOC (mean height) dynamics



?

Kinetic Ising model with non-local power-law interaction

$$\frac{\partial f(\mathbf{h},t)}{\partial t} = \sum_{j\neq i} 3D_{ij}h_{i+j}f(\mathbf{h}-\mathbf{e}_i,t) + \sum_{j\neq i} D_{ij}h_{i+j}f(\mathbf{h}+\mathbf{e}_i,t) - \{3D_{ij}h_{i+j}(1-h_i) + D_{ij}h_{i+j}h_i\}\, f(\mathbf{h},t)$$

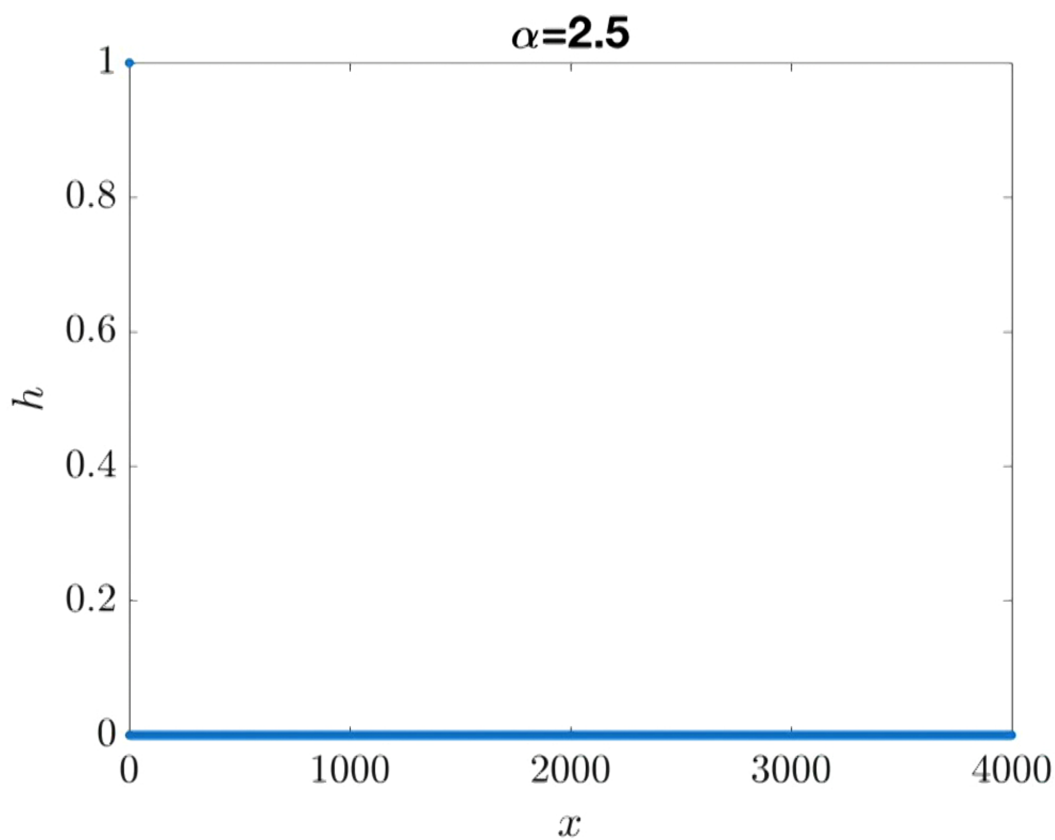
One spin facilitated Fredrickson-Andersen model

Non-local power-law interaction $D_{ij} \sim \frac{1}{|i-j|^{\alpha}}$

Perform classical simulation for this process

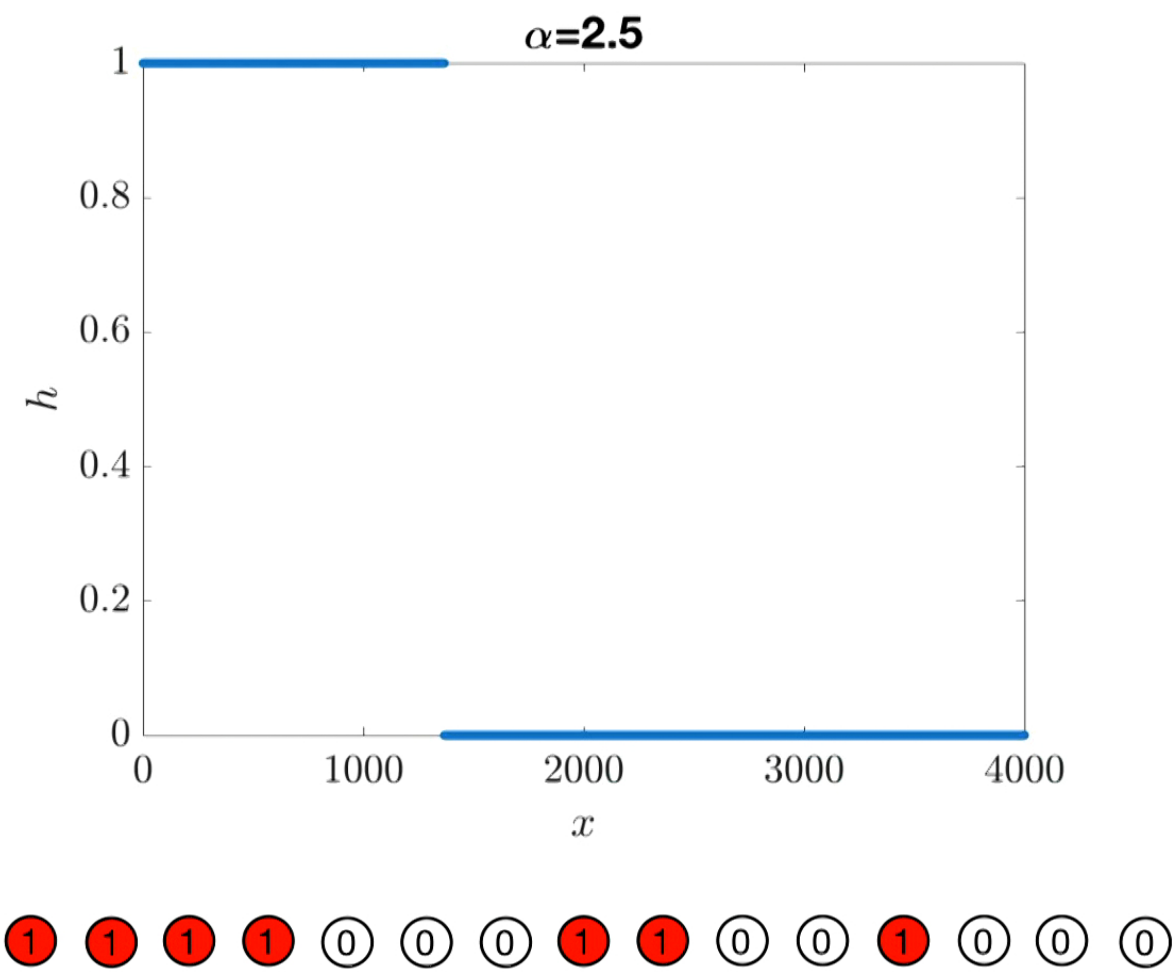
(1) The growth of the h=1 domain

(2) The non-local flipping process



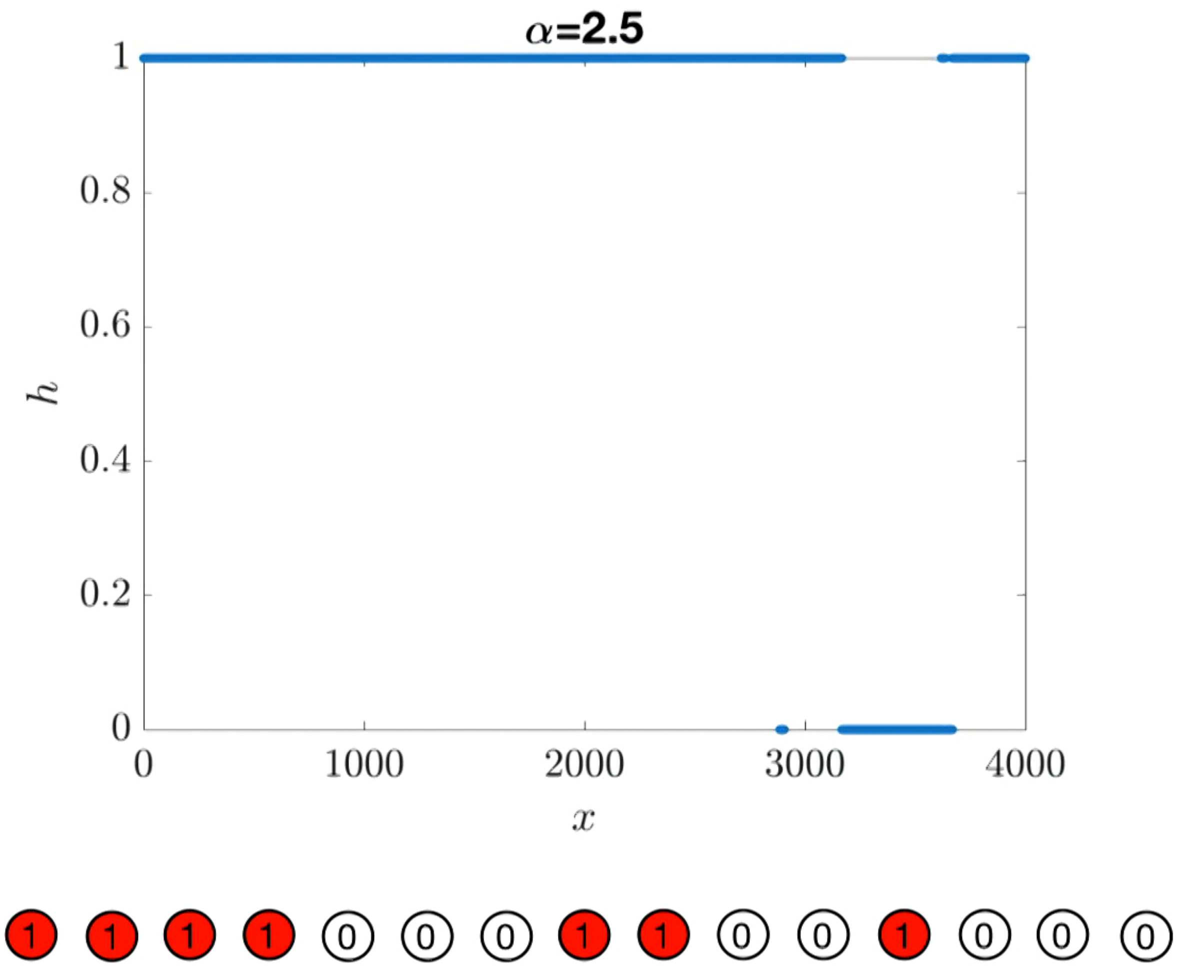
(1) The growth of the h=1 domain

(2) The non-local flipping process



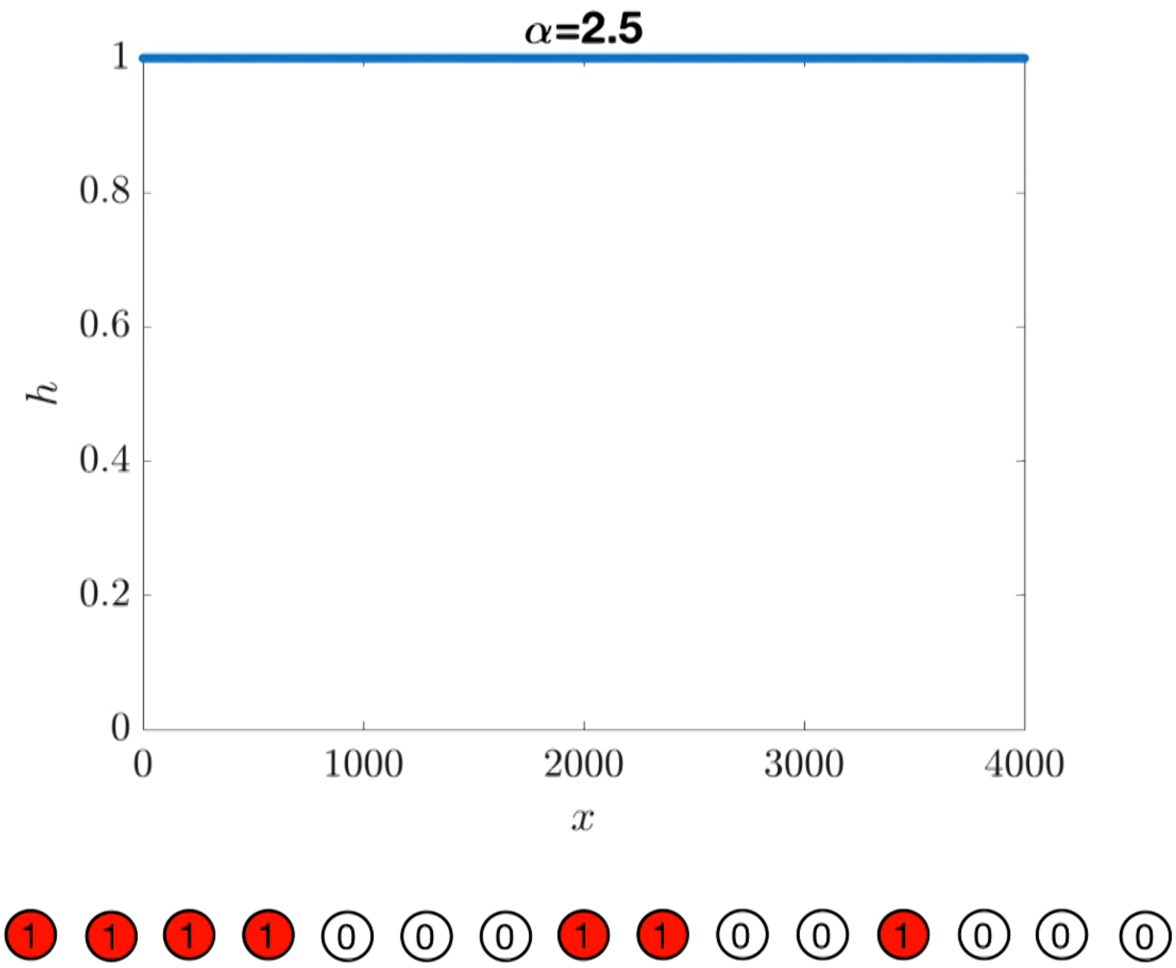
(1) The growth of the h=1 domain

(2) The non-local flipping process



(1) The growth of the h=1 domain

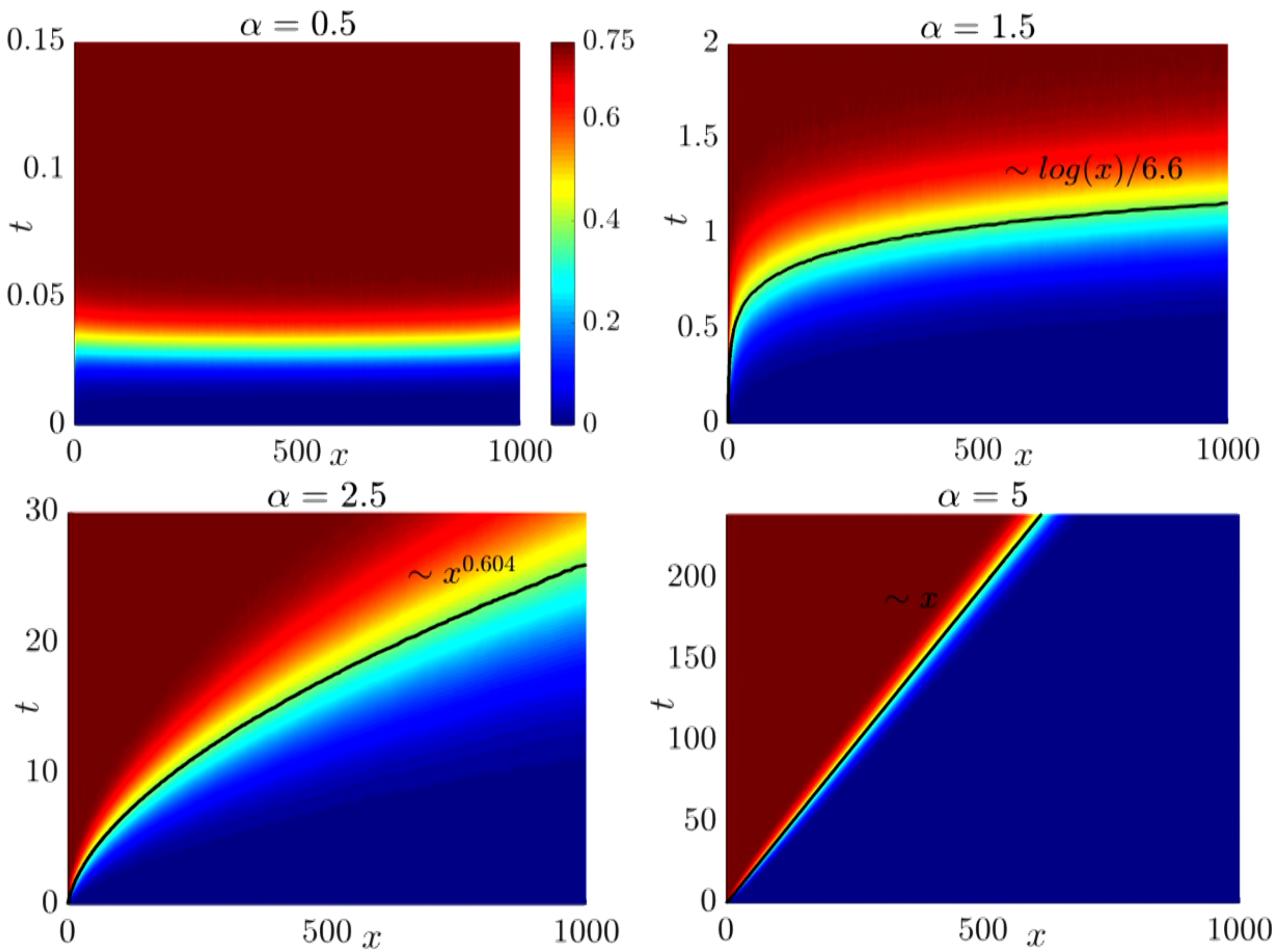
(2) The non-local flipping process



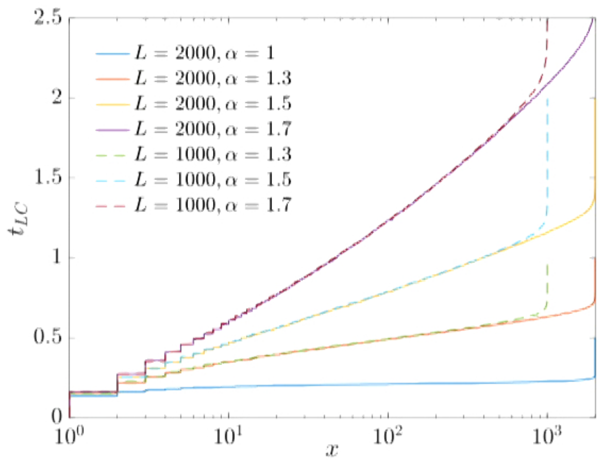
(1) The growth of the h=1 domain

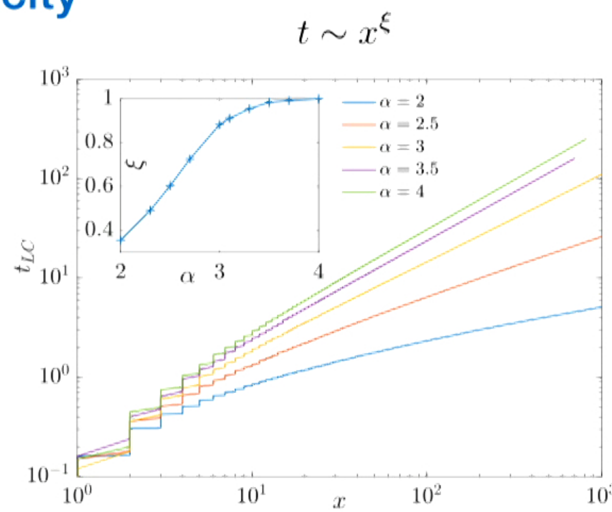
(2) The non-local flipping process

The formation of effective light cone



Light cone and butterfly velocity





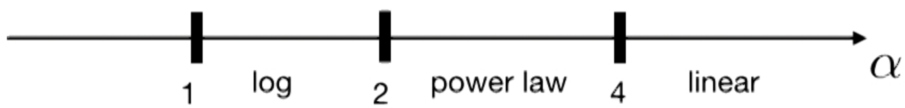
Local interaction

v_B is a constant and is bounded by Lieb-Robinson velocity

$$v_B = x/t_{LC}$$

Power law interaction

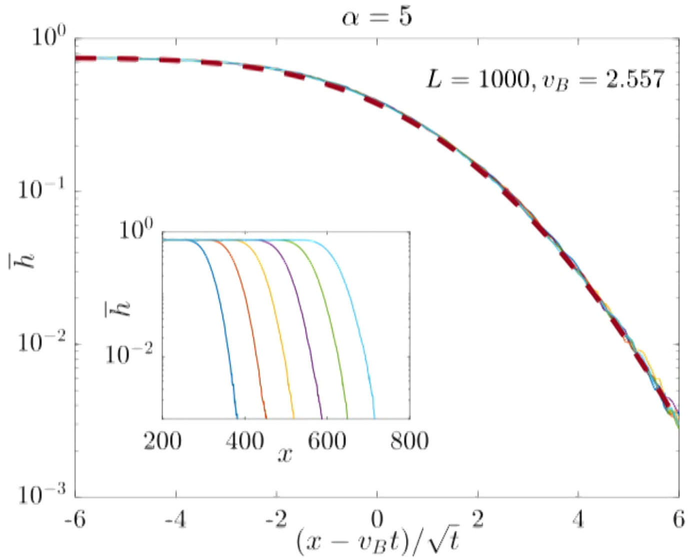
v_B can be (1) a constant, (2) grows algebraically or (3) grows exponentially in time



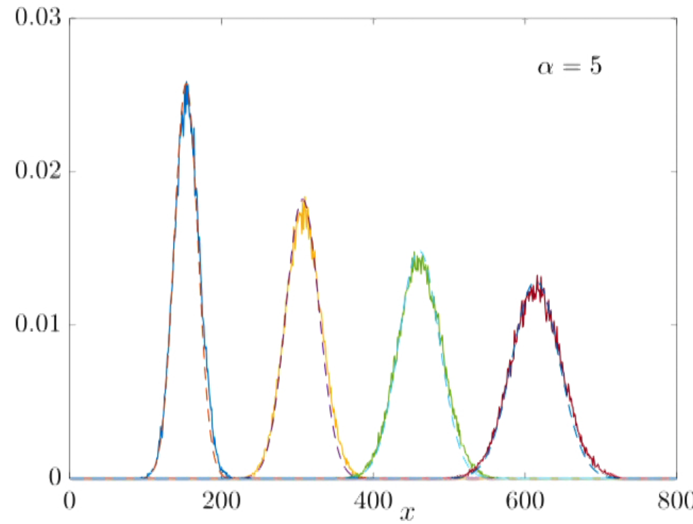
Hastings and Koma, 2006, Foss-Feig, Gong, Clark and Gorshkov, 2015

Emergent linear light cone and locality

$$h(x,t) \to h\left(\frac{x - v_B t}{g(t)}\right) \longrightarrow \text{scaling argument}$$

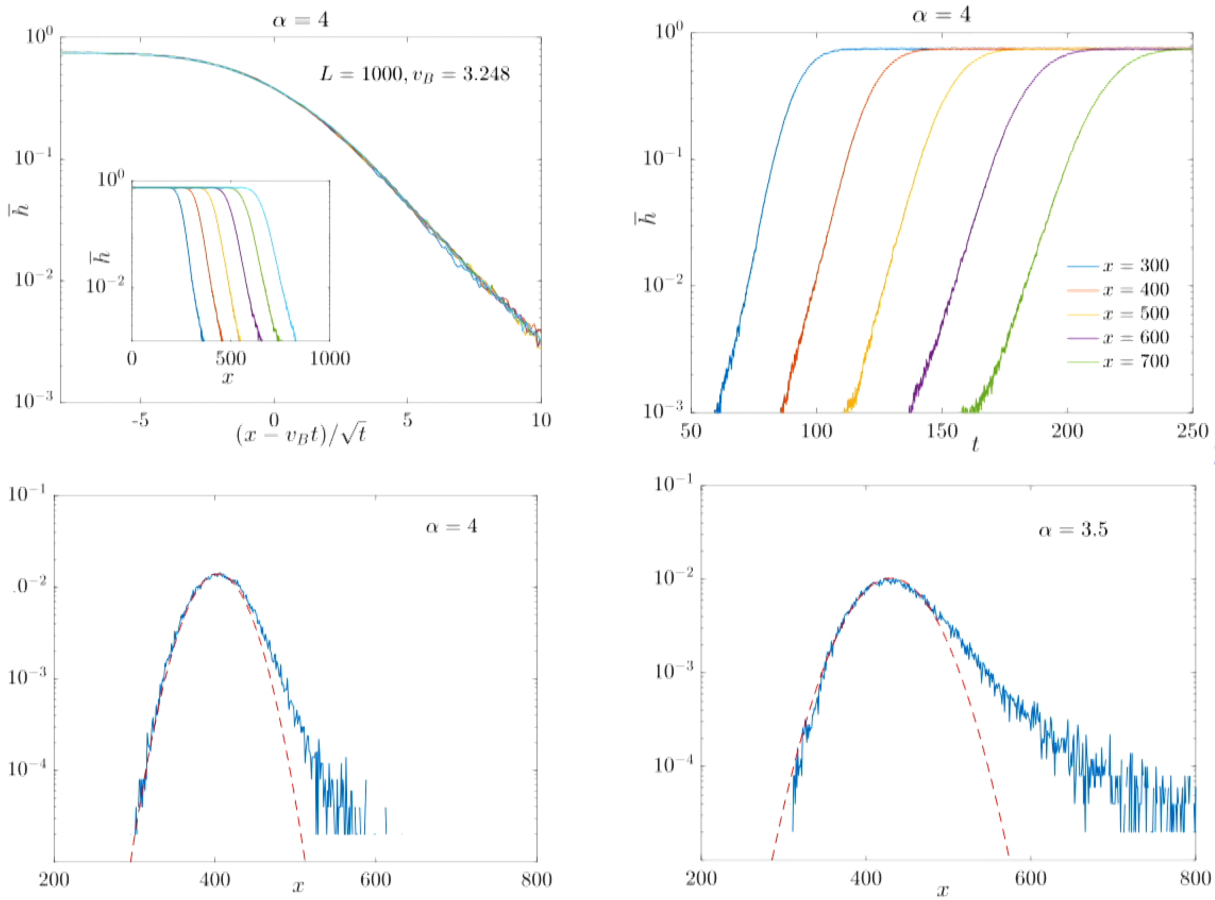


Data collapse of mean height



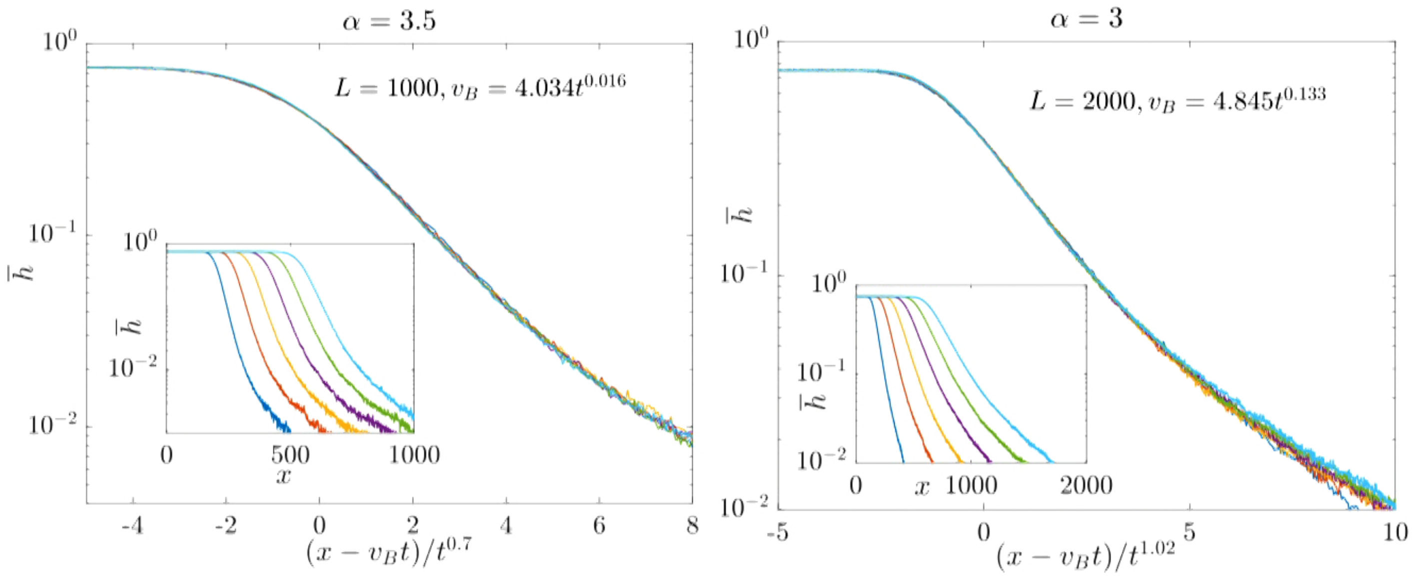
Endpoint distribution

Non-Gaussian tail



XC, Zhou and Xu, 2017

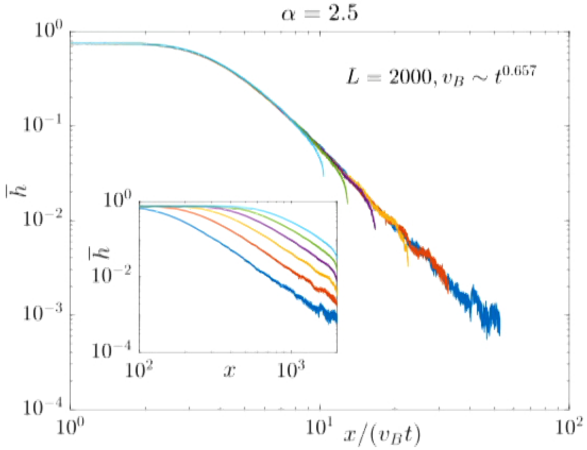
The endpoint distribution cannot be directly connected to mean height when α is small

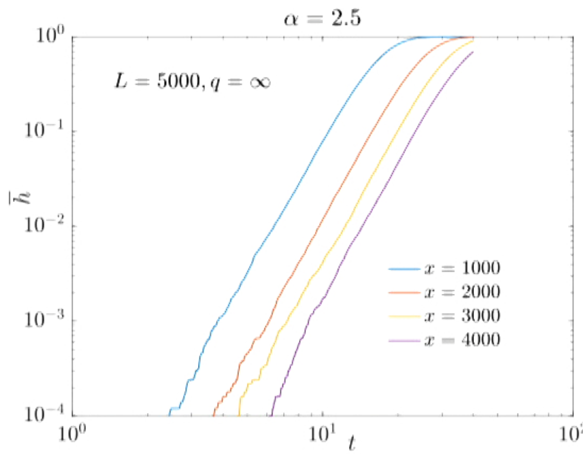


$$g(t) \sim t^b \text{ with } b > 0.5$$

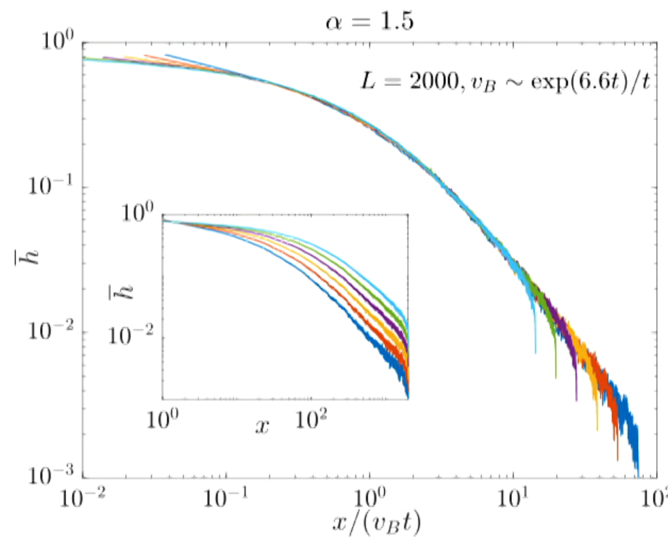
$$\overline{h(x,t)} \sim \left(\frac{t^b}{x}\right)^c$$

Power law in both directions

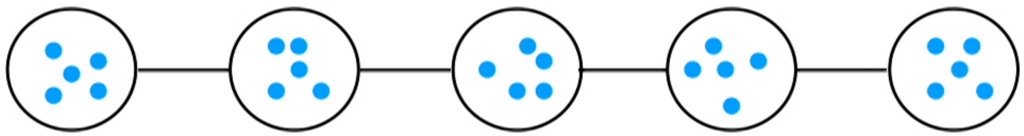




$$\overline{h(x,t)} \sim \frac{e^{\lambda t}}{x^c}$$



Large N limit with local interaction



Similar to SYK chain

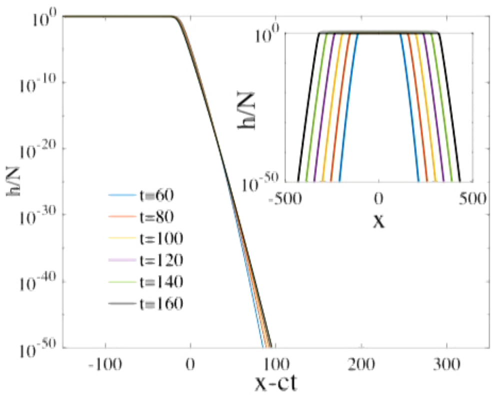
Gu, Qi and Stanford, 2016

Operator dynamics in two directions:
(1) the local onsite Hilbert space
(2) the spatial direction

Logistic differential equation + Spatial diffusion → Fisher-KPP equation

$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2} + \lambda h(1 - \frac{h}{N})$$

XC and Zhou, 2018

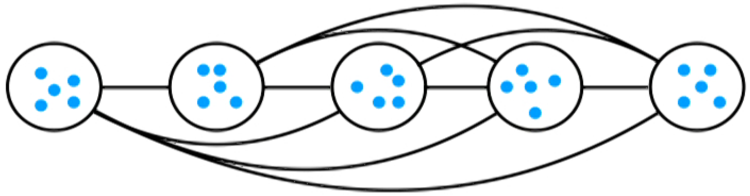


(1) Stable traveling wave solution & no dispersion
(2) Crossover to the diffusive wavefront picture at finite N.

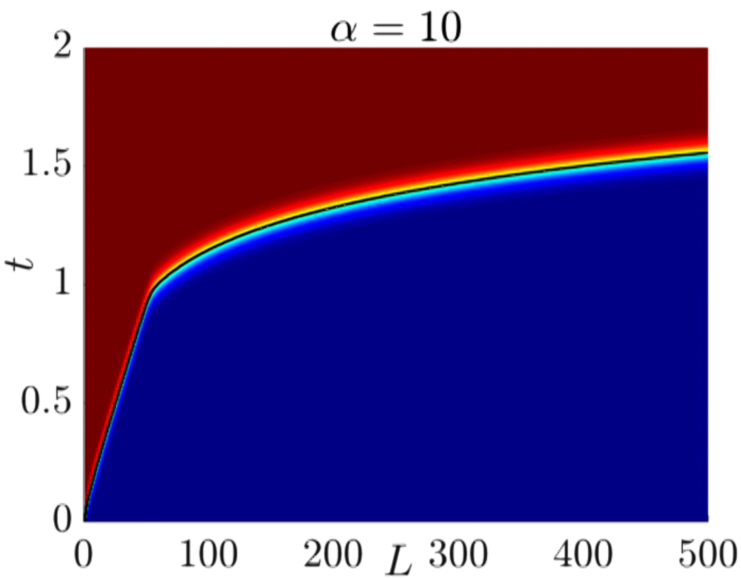
Xu and Swingle, 2018

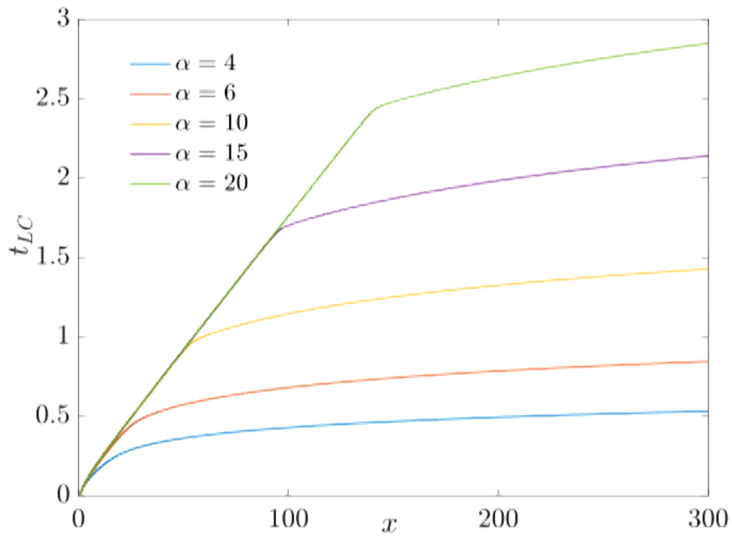
Large N limit with power-law interaction

$$\frac{\partial \underline{h}(x,t)}{\partial t} = \int dy \underline{h}(y,t) D(y,x)(1 - \frac{1}{h_{sat}} \underline{h}(x,t))$$



The power law kernel $D(x,y) = \frac{1}{|x-y|^{\alpha}}$





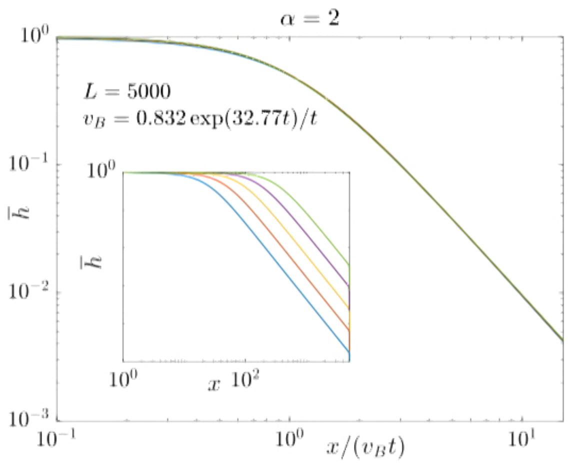
Three observations:

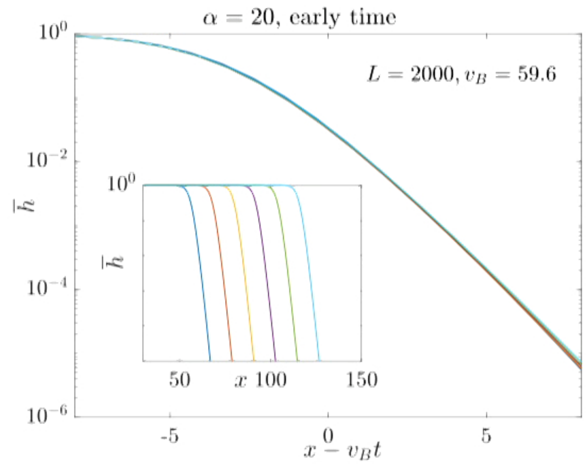
(1) The log light cone scales as $t_{LC} \sim \alpha \log x$

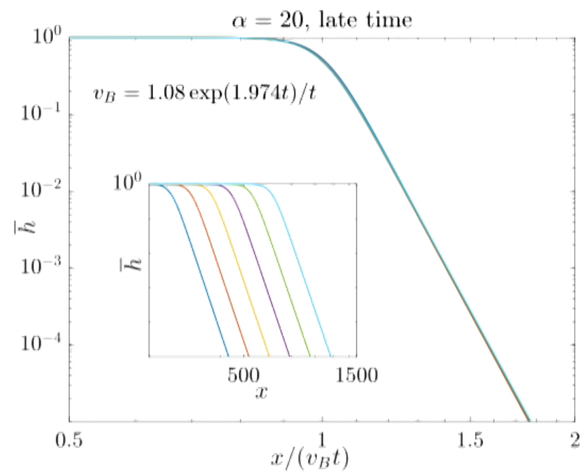
(2) The butterfly velocity in the linear light cone regime is independent of α

(3) The transition from linear to log light cone occurs at the intersection $x \sim \alpha \log x$

Data collapse







$$\overline{h(x,t)} \sim \frac{e^{\lambda t}}{x^c}$$

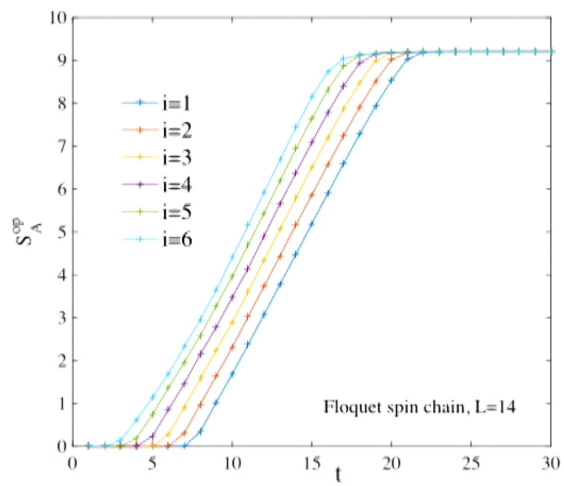
The complexity of the operator

Operator entanglement entropy

Pizorn and Prosen, 2007, Dubail, 2017

The reduced density matrix for subsystem A

$$\hat{\rho}_A = \mathrm{Tr}_B\left[|\hat{O}\rangle\langle\hat{O}|\right]$$



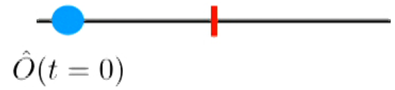
It grows linearly in time in chaotic systems with local interaction

XC and Ludwig, 2018

The operator entanglement entropy

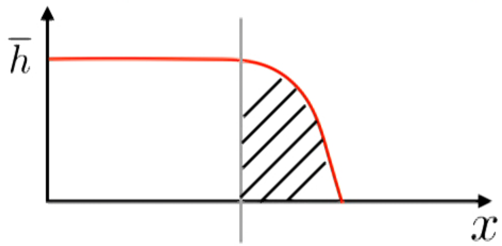
The reduced density matrix for subsystem A

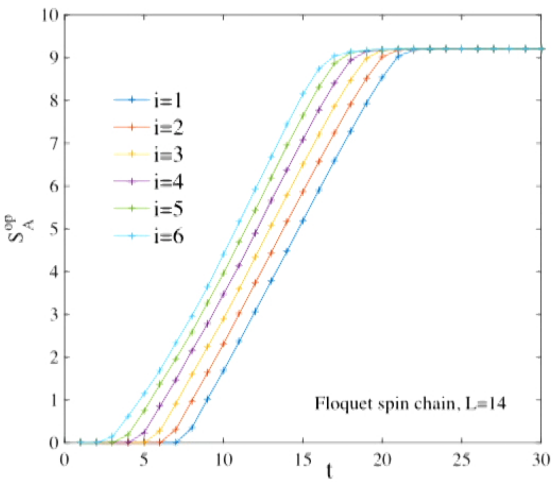
$$\hat{\rho}_A = \mathrm{Tr}_B \left[|\hat{O}\rangle\langle\hat{O}| \right]$$



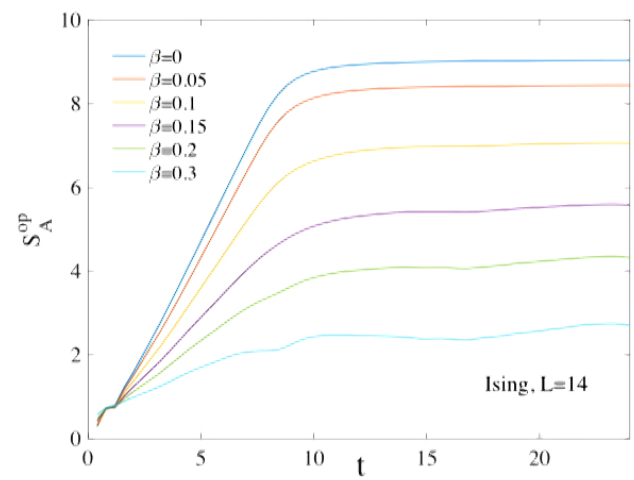
Pizorn and Prosen, 2007, Dubail, 2017

Measuring the area of mean height in subsystem A





It grows linearly in time in systems with local interaction

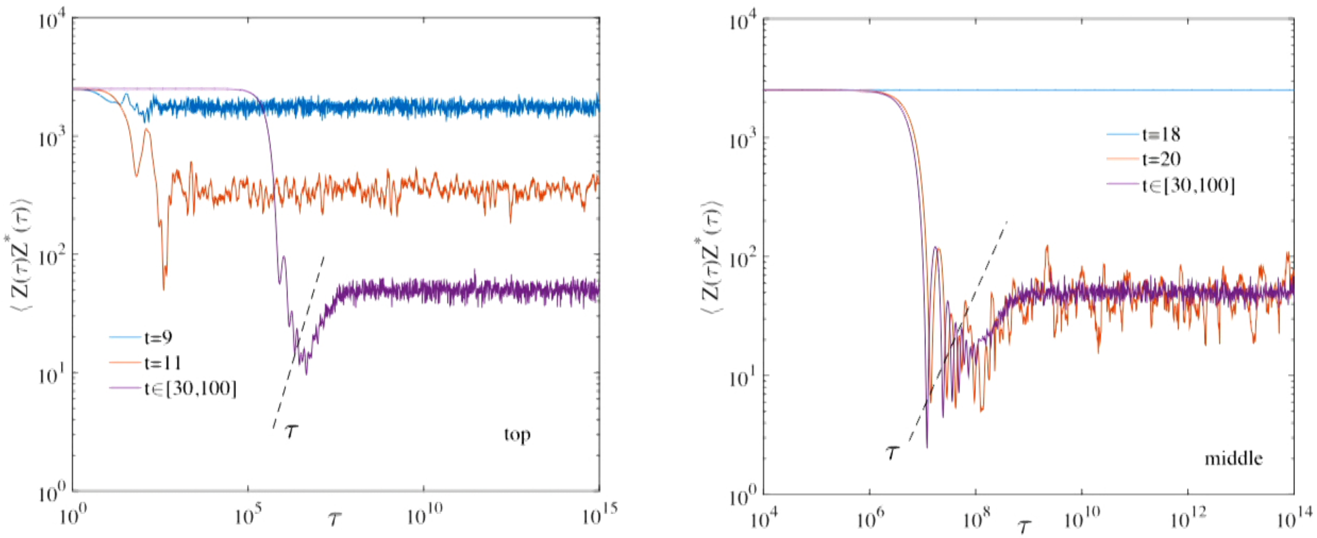


This can be generalized to finite temperature

$$e^{-\beta H} O e^{-\beta H}$$

The spectral correlation in the operator reduced density matrix

$$g(\tau) = \langle Z(\tau) Z^*(\tau) \rangle = \langle e^{-i\tau(\lambda_i - \lambda_j)} \rangle$$



The development of (Wishart) random matrix physics in a subsystem under unitary time evolution

XC and Ludwig, 2018

Quantum linear map

- Classical linear map (Arnold's cat map)

$$\begin{pmatrix} q \\ p \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \bmod 1 \qquad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}(2, \mathbb{Z})$$

- Quantum linear map (defined on the torus)

$$M = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 2 \end{smallmatrix} \right) \qquad \hat{U}(q', q) = \left(\frac{i}{K} \right)^{1/2} \exp \left[\frac{i\pi}{K} (2q^2 - 2qq' + 2(q')^2) \right]$$

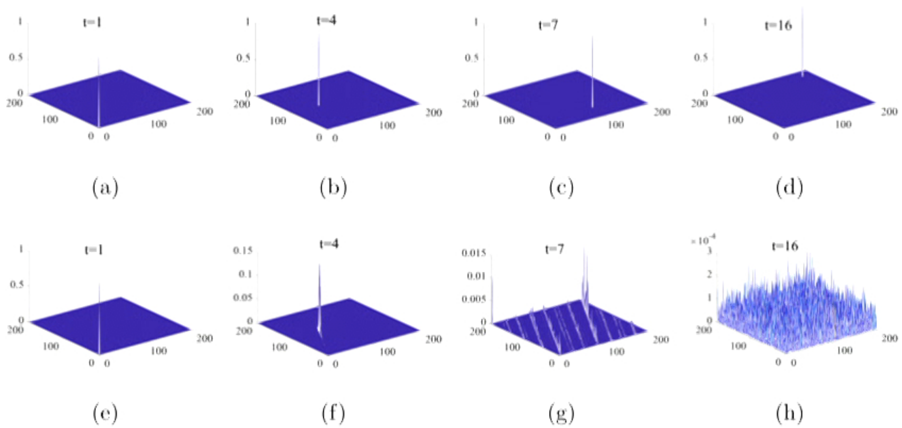
Under U operator

$$\hat{U}^\dagger \hat{\sigma} \hat{U} \sim \hat{\sigma}^2 \hat{\tau} \qquad \hat{U}^\dagger \hat{\tau} \hat{U} \sim \hat{\sigma}^3 \hat{\tau}^2$$

The motion of the operator is determined by the classical dynamics

Operator dynamics

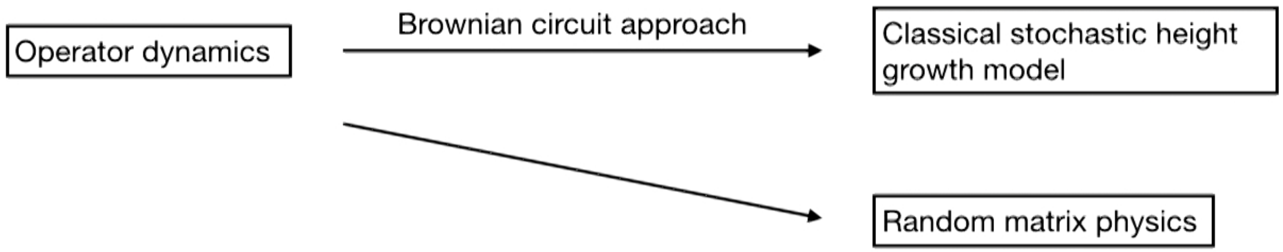
- The operator performs "chaotic" motion in operator Hilbert space
- The operator spreads out in the operator space



$$\hat{U} = \hat{U}_1 \hat{U}_2$$

$$\langle q'|\hat{U}_1|q\rangle = \exp\left[i\frac{\kappa K}{2\pi}\left(\sin(\frac{2\pi q}{K}) - \frac{1}{2}\sin(\frac{4\pi q}{K})\right)\right]\delta_{q,q'}$$

Summary and Outlook

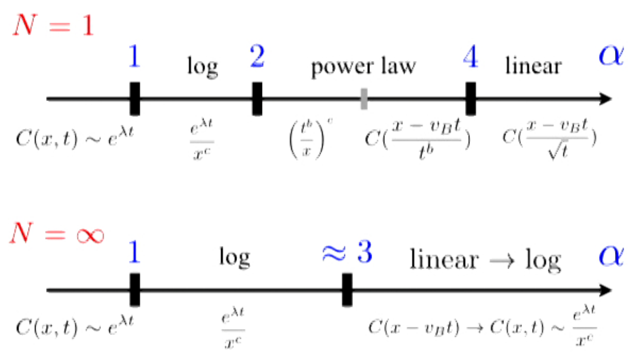


Operator height growth in:

(1) All-to-all interaction

(2) Local interaction

(3) Power law interaction



Possible future directions:
(1) Power law regime $2 < \alpha < 3$
(2) Entanglement dynamics after quench
(3) Operator dynamics at finite temperature