Title: Path Integral Optimization and Complexity in 2d Conformal Field Theories

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Abstract: <p>I will review recent and ongoing developments in path integral optimization and path integral complexity in 2d CFTs. After, I will discuss the connection with geometric approach to circuit complexity and point out several open problems.</p>

Path Integral Optimization and Complexity in 2d CFTs

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Perimeter Institute, 28.11.2018

Based on collaborations with:

"AdS from Optimization of Path Integrals in Conformal Field Theories" **PRL 119 (2017) 7, 071602**

"Liouville Action as Path-Integral Complexity: From cTN to AdS/CFT" **JHEP11 (2017) 097**

with Nilay Kundu (IIT Kanpur), Masamichi Miyaji, Tadashi Takayanagi (YITP), Kento Watanabe (U. of Tokyo)

"Path Integral Complexity for Perturbed CFTs" **JHEP 1807 (2018) 086**
with Arpan Bhattacharyya, Nilay Kundu (IIT Kanpur), Masamichi Miyaji, Tadashi Takayanagi (YITP), Sumit R. Das (Kentucky U)

"Quantum Computation as Gravity" with Javier M. Magan (C.A. Bariloche) **arXiv: 1807.04422 [hep-th]**

Some work/ideas in progress to discuss during my visit …

Plan

- Motivation/Background
- Optimization of Path Integrals (old and new)
- Liouville Action as “Path Integral Complexity”
- “CFT gates” and “Quantum Computation as Gravity”
- Open Questions

Motivation/Goals:

What is the basic mechanism behind AdS/CFT?

How to "extract" geometry from CFT states?

Geometry from QI: entanglement? Is EE sufficient (HRT)? No…

Use distance measures? Information Metric? Complexity? (Independent?)….

How to define "complexity" in CFT?

Motivation/Goals:

This talk

What is the basic mechanism behind AdS/CFT?

How to “extract” geometry from CFT states? ✓

Geometry from entanglement? Is EE sufficient (HRT)? No…

Use distance measures? Information Metric? Complexity? (Independent?)….

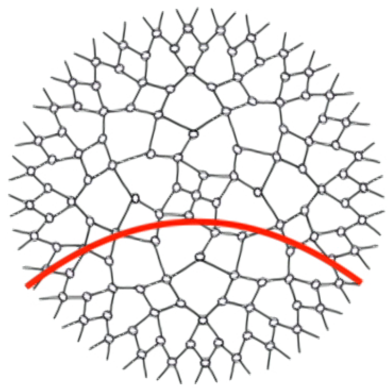
How to define “complexity” in 2d CFT? ✓

Background: Geometry and TN (AdS/TN)

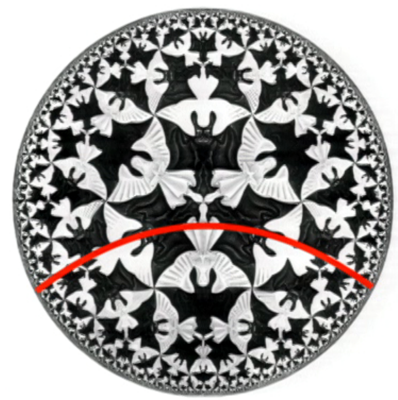
[Vidal '05] [Swingle '12] [HAPPY '15]
[Evenbly,Vidal '14,'15] [Evenbly'17]

CFT wave functions and time slice of AdS?

MERA



AdS



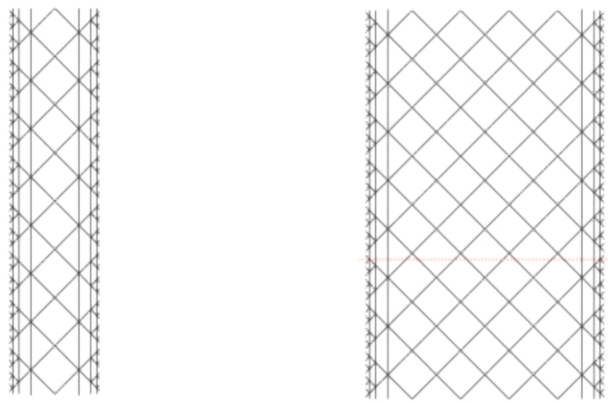
Geometry can be thought of as TN approximation to the CFT wave function (at large N)

Promising but for AdS/CFT we need this for strongly interacting CFTs.

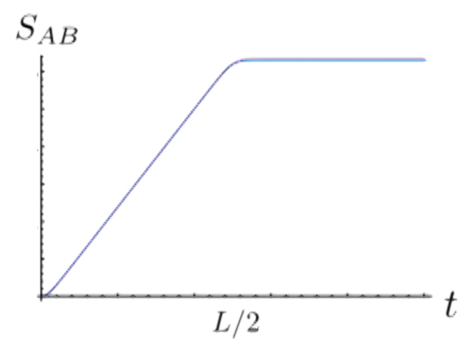
Gravity dual of a global quench (TN)

[Hartman&Maldacena '13]

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n\rangle_L|n\rangle_R \qquad |\psi_\beta(t)\rangle = e^{-i(H_L+H_R)t}|\psi_\beta\rangle$$



TN grows ~ t



TN grows but EE saturates after time ~ size of the interval

"Entanglement is not enough"…

[Susskind,'14]

What is the "CFT dual" of this (ERB) growth? "Complexity" of the TFD state ~ # of Tensors

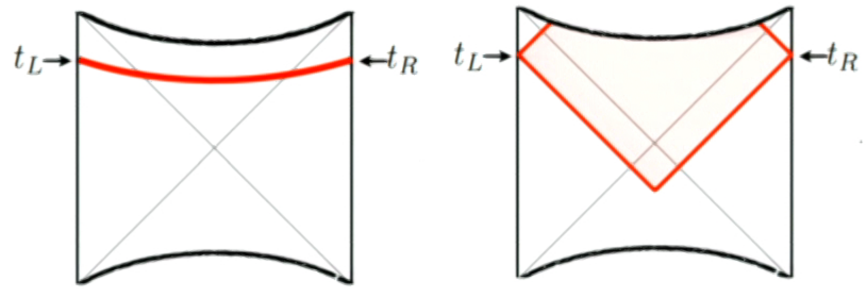
Holographic complexity “proposals”

[Brown,Roberts,Susskind,Swingle,Zhao’15;
Lehner,Myers,Poisson,Sorkin;
Chapman,Marrochio,Myers’16….]

C=Vol

C=Action

C~ GR



Field theory: Circuit complexity, Nielsen approach….

[Susskind et al..’15; Jefferson&Myers’17; Chapman,Heller,Marrochio,Pastawski,’17; Abt et al.’17;
Myers et al..’17’18; Balasubramanian et al. ’18….]

New exploration (similar to entanglement developments: GR->free->CFTs…?)

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]

Path Integral Optimization

1. AdS/TN in the language of 2d CFT (continuum, incl. interactions)?

2. How to measure the size of the TN ~ "Complexity" in 2d CFTs ?

Optimization of Path Integrals

The basic tool to "define/compute" wave functions in QFT is the Euclidean PI

$$\Psi[\varphi_0(x)] = \int_{\varphi(0,x)=\varphi_0(x)} D\varphi \, e^{-S_E}$$

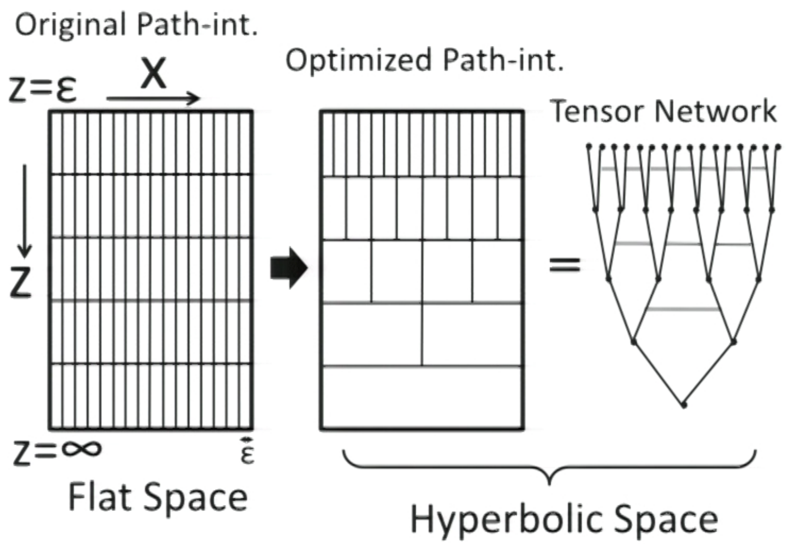
How can we optimize it?

How can we extract a geometry from PI for a given quantum state?

How can we quantify its "complexity"?

IDEA (PI optimization):

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]



Optimization:

$$\frac{\Psi_g}{\Psi_{flat}} = e^{I_\Psi[g]}$$

Minimize "Path Integral Complexity"

2D CFTs and Liouville

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]

Background metric for path integral $z = -\tau$

$$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$$

Once we introduce the background metric [Polyakov'81]

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}$$

The wave functional is

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x)) = e^{S_L[\phi]-S_L[0]} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x))$$

Path Integral Complexity given by the Liouville action

$$S_L[\phi] = \frac{c}{24\pi}\int dxdz\left[(\partial_z\phi)^2 + (\partial_x\phi)^2 + e^{2\phi}\right]$$

c - central charge

Optimization <=> Minimizing PI complexity

Optimized metric satisfies Liouville equation with the appropriate b.c.

$$4\partial_w \partial_{\bar{w}} \phi = e^{2\phi} \qquad\qquad e^{2\phi(z=\epsilon,x)} = 1/\epsilon^2 \quad \text{cut-off}$$

$w = z + ix$

$$e^{2\phi} = \frac{4f'(w)g'(\bar{w})}{(1 - f(w)g(\bar{w}))^2}$$

Equivalently

$$\left(\partial_w^2 + \frac{1}{2}T(w)\right) e^{-\phi(w,\bar{w})} = 0,$$
$$\left(\partial_{\bar{w}}^2 + \frac{1}{2}\bar{T}(\bar{w})\right) e^{-\phi(w,\bar{w})} = 0$$

Hill's eq (Virasoro co-adjoint orbits)

$T(w) = 2\left(\partial_w^2 \phi - (\partial_w \phi)^2\right) = \{f(w), w\}$

$$\partial_{\bar{w}} T(w) = 0 \qquad \text{Liouville eq}$$

Optimized metrics and AdS/CFT solutions!

[PC,Kundu,Miyaji,Takayanagi,Watanabe '17]
[PC,Bhattacharyya,Das,Kundu,Miyaji,Takayanagi,'18]

1. Vacuum: PI on u.h.p

$$e^{2\phi} = \frac{4}{(w+\bar{w})^2} = z^{-2}$$

H_2 !

2. TFD PI on a strip

$-\frac{\beta}{4}(\equiv z_1) < z < \frac{\beta}{4}(\equiv z_2)$

$$e^{2\phi} = \frac{4\pi^2}{\beta^2}\sec^2\left(\frac{2\pi z}{\beta}\right)$$

Time slice of eternal BH

3. Primary PI on a disc with insertion

$$e^{2\phi} = \frac{4a^2}{|w|^{2(1-a)}(1-|w|^{2a})^2}$$

Time slice of con. sing.

$$a = 1 - \frac{12h}{c}$$

Perturbations of CFTs with position dep. coupling => Time slice of AdS3 + scalar

[Hung,Myers,Smolkin'11]

Entanglement Entropy and Liouville Field

[PC,Kundu,Miyaji,Takayanagi,Watanabe '17]

$$S_A = \frac{c}{6}\int_{\partial\Sigma_+} e^{\phi} ds \qquad\qquad ds^2 = e^{2\phi(w,\bar{w})} dw d\bar{w}$$

After the integration we can get (single interval entropy in 2d CFT)

$$S_l = \frac{c}{12}\log\left(\frac{(f(w_1)-f(w_2))^2}{f'(w_1)f'(w_2)\epsilon^2}\right) + \frac{\bar{c}}{12}\log\left(\frac{(g(\bar{w}_1)-g(\bar{w}_2))^2}{g'(\bar{w}_1)g'(\bar{w}_2)\epsilon^2}\right)$$

This itself can be written as a Liouville field

$$e^{2\phi(w,\bar{w})} = \frac{\epsilon^2 f'(w)g'(\bar{w})}{(f(w)-g(\bar{w}))^2}$$

After setting w=iu, w=iv, f=g and EE satisfies Liouville equation

[de Boer,Haehl,Heller,Myers'16]

[Czech '17]

$$\partial_u\partial_v\left(-\frac{6}{c}S_A\right) = -\frac{1}{\epsilon^2}e^{2\left(-\frac{6}{c}S_A\right)}$$

Summary PI:

- We propose how to extract geometry from states of 2d CFT (cTN)
- It reproduces "time slices" of holographic geometries
- Universal: Valid for arbitrary 2d CFT at large c (also very strongly interacting)
- Key role played by the Liouville action
- "RT formula" on cTN reproduces EE in CFT states
- Deformations under control
- (c)TN perspective? Relation with [Milsted&Vidal '17'18]

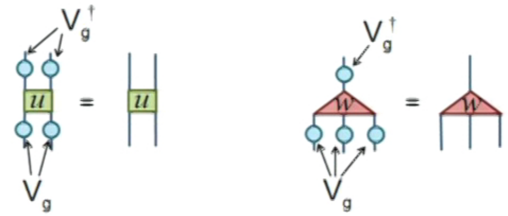
New 1: Extended Symmetries

How to generalize our prescription to CFTs with extended symmetries (Potts model, W algebras, etc.) ?

Hint 1) MERA with symmetries [Singh,Pfeifer,Vidal'10]

$$\Gamma_g \; H \; \Gamma_g^\dagger = H, \qquad \forall g \in \mathcal{G}$$

Tensors satisfy extra constraints



Hint 2) EE in CFT with W3 and SL(3,R) Toda [de Boer,Heller,Hael,Myers'16]

$$\epsilon_{\rm fun} \frac{\partial S^{(2)}_{\rm EE}}{\partial u \partial v} = 2\, c_{\rm fun}\, e^{-S^{(2)}_{\rm EE}/(2c_{\rm fun})} \cosh\left(3S^{(3)}_{\rm EE}/c_{\rm fun}\right)$$
$$\epsilon_{\rm fun} \frac{\partial S^{(3)}_{\rm EE}}{\partial u \partial v} = -c_{\rm fun}\, e^{-S^{(2)}_{\rm EE}/(2c_{\rm fun})} \sinh\left(3S^{(3)}_{\rm EE}/c_{\rm fun}\right)$$

PI optimization with Liouville->Toda. Interpretation? W-geometry? Complexity?

New 2: Optimization of Transition Amplitudes? Time dependent TN?

Lorentzian PI in QFT

$$\langle\varphi_f(x),t_f|\varphi_i(x),t_i\rangle = \int_{\varphi(t_i,x)=\varphi_i(x)}^{\varphi(t_f,x)=\varphi_f(x)}[D\varphi]e^{i\int_{t_i}^{t_f}dtdx\mathcal{L}}$$

Again in 2d CFTs

$$ds^2 = e^{2\phi(x,t)}\eta_{\mu\nu}dx^\mu dx^\nu = e^{2\phi(x,t)}\left(-dt^2+dx^2\right)$$

Optimize the ratio (PI measure argument=>Lorentzian Liouville)

$$\frac{\langle\varphi_f(x),t_f|\varphi_i(x),t_i\rangle_{e^{2\phi}\eta}}{\langle\varphi_f(x),t_f|\varphi_i(x),t_i\rangle_\eta} \equiv e^{C[\phi]}$$

Tensor Network Counterpart? TNR of Lorenzian PI?

Complexity? See more general construction by [Takayanagi '18]

Path Integral "Complexity"

Continuous TN "interpretation" (for free theories)

$$S_L[\phi] = \frac{c}{24\pi} \int dxdz \left[(\partial_z \phi)^2 + (\partial_x \phi)^2 + e^{2\phi} \right]$$

Curvature
(~Number of Isometries [Czech'17])

Volume
(~Number of tensors)

PI complexity = 2d Gravity ! (Eucl.)

$$S_P[g] = \frac{c}{24\pi} \int d^2x \sqrt{g} \left(-\frac{1}{4} R \frac{1}{\Box} R + \Lambda \right)$$

[Polyakov'87]

Minimization of complexity <=> full eom of 2d gravity

Great: Based on "universal" features of the CFT (arbitrary c!) and computable

Q: Relation to conventional "Complexity":

Non-Unitarity… $e^{-\beta H}$ What kind of Complexity, Gates,Costs? Time dep.?

Geometric approach to circuit complexity

[Nielsen + et al. 05]
[Jefferson-Myers'17]
[Chapman,Heller,Marrochio,Pastawski'17]

Quantum circuit

$$|\Psi_T\rangle = U(t)\,|\Psi_R\rangle$$

Where the unitary operator is

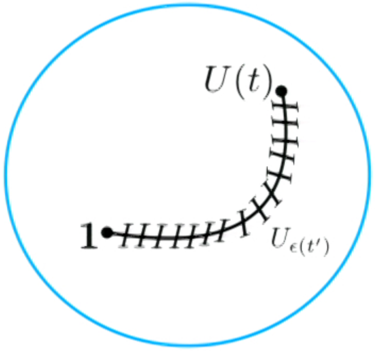
$$U(t) = \mathcal{P}\exp\left(\int_0^t d\tau \mathcal{H}(\tau)\right)$$

Decompose it into infinitesimal (instantaneous) gates (Key!)

$$U(t) = U_{\epsilon(t)}U_{\epsilon(t-dt)}\cdots U_{\epsilon(dt)}\mathbb{1}$$

where

$$U(t+dt) = U_{\epsilon(t)}U(t)$$



Cost functions chosen such that they define a geometry on the space of U

$$C(t) = d[U(t)] = \int_0^t dt' F\left(U_{\epsilon(t')}, \dot{U}_{\epsilon(t')}\right)$$

Complexity of implementing U <=> geodesic distance on this manifold.

Optimal circuit <=> Free fall between 1 and U.

In general we can think of Nielsen's approach as "particle on a group"

Geometric approach to circuit complexity

Generally we expand the instantaneous gate operator in algebra generators

$$\mathcal{H}(t) = \sum_I Y^I(t) M_I$$

and then define cost(s) (a lot of freedom)

$$F_1(U,Y) = \sum_I |Y^I| \qquad F_2(U,Y) = \sqrt{\sum_I (Y^I)^2} \qquad F_q(U,Y) = \sqrt{\sum_I q_I \, (Y^I)^2}$$

These (local!) costs functions can be expressed as expectation values of the infinitesimal gate operator (MC and with some projectors) and also penalty factors.

Target: Geodesics for metric

$$\partial_t U U^{-1} = Y^I(t) M_I \qquad ds^2 = G_{IJ} dY^I dY^J$$

Penalty factors etc.

e.g. Free scalar: (Killing form) $Tr(M_I M_J^T) = \delta_{IJ}$

Purely classical problem! (Nielsen: SU(2^n), Ro&Rob: GL(2,R)…)

Main Idea:

[PC, J.Magan '18]

Since Nielsen's approach for the Virasoro group (Diff(S^1)xR).
CFT=two copies on the LC coords.

Is there a natural/universal way to define "gates" and "cost functions"?

Could we derive Liouville action that way?

Results:

We can consider a subset of "symmetry gates" that implement Diffs.

One natural generalization of Nielsen's costs leads to the Alexeev-Shatashvili geometric action on the coadjoint orbits = Complexity functional for diffeos f(t,z)

Complexity action becomes the Polyakov action of induced gravity in 2d

"Quantum Computation as Gravity"

CFT "symmetry setup":

Consider reparam. of the unit circle (z = $e^{i\sigma}$) Diff(S1)xR $\quad (v(\sigma)\partial_\sigma, c)$

Group action is a composition $\quad f \cdot g = f \circ g$

$$f(z) = z + \epsilon(z) \qquad \epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{-n}$$

Gates

$$\boxed{U_\epsilon = \exp\left(-\int_0^{2\pi} \frac{d\sigma}{2\pi}\epsilon(z)T(z)\right) = \exp\left(-Q_\epsilon\right)} \qquad U_f U_g = U_{f\circ g}$$

where

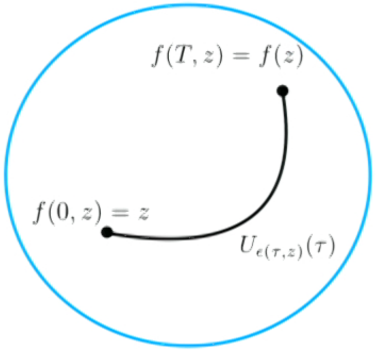
$$T(z) = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) z^{-n} \qquad Q_\epsilon = -\sum_{n\in\mathbb{Z}} \epsilon_n L_{-n} \qquad \epsilon_n^* = -\epsilon_{-n}$$

with generators of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

(No phases,1 is free)

"CFT circuit"



$$|\Psi_T\rangle = U_f(t)\,|\Psi_R\rangle$$

where

$$f_\tau(z) \equiv f(\tau, z) \qquad f(0,z) = z \qquad f(T,z) = f(z)$$

Because we are dealing with the "symmetry" gate (representation) we have

$$U = U_{g_N} \cdots U_{g_1} \mathbb{1} = U_{g_N \cdots g_1}$$

Protocol = Path in the group!

The instantaneous infinitesimal gate is

$$U_\epsilon(\tau) = \exp\left(-\int_0^{2\pi} \frac{d\sigma}{2\pi} \epsilon(\tau, z) T(z)\right) = \exp\left(-Q_{\epsilon(\tau)}\right)$$

The instantaneous gate parameter (from the composition of the symmetry gates)

$$\epsilon(\tau, z) = \dot{f}(\tau, f^{-1}(\tau, z)) = \frac{-\dot{f}^{-1}(\tau, z)}{(f^{-1})'(\tau, z)} \qquad f(\tau, f^{-1}(\tau, z)) = z$$

Problem: Complexity (cost) of these reparametrizations?

Cost functions

Pick a point on the (sub)manifold of U and define cost as expectation value of the instantaneous gate operator(s). [J.M.Magan '18]

Liouville action suggests that a way to do that is by

$$F_1(\tau) = |\langle\Delta|U_f^\dagger Q_{\epsilon(\tau)} U_f|\Delta\rangle|$$

$$F_2(\tau) = \sqrt{\langle\Delta|U_f^\dagger Q_{\epsilon(\tau)} Q_{\epsilon(\tau)}^\dagger U_f|\Delta\rangle}$$

The Nielsen-type complexity is then

$$C(t) = \int_0^t d\tau F(\tau)$$

Complexity

Our complexity becomes the Schwarzian action <=> geometric action Vir.

$$C(t) = \frac{c}{24\pi}\int d\tau d\sigma \frac{\dot{f}}{f'}\{\tan(af), z\} + O(1/c)$$

[Alexeev-Shatashvili'89]
[Witten'88]

(Schwarzian action f^{-1})

Polyakov 2d GR action in a particular gauge!

Combining two (L-R) transformations as an infinitesimal gate

$$Q_{\epsilon,\bar{\epsilon}}(\tau) = \int_0^{2\pi} \frac{d\sigma_1}{2\pi}\epsilon(\tau, z_1)T(\tau, z_1) + \int_0^{2\pi} \frac{d\sigma_1}{2\pi}\bar{\epsilon}(\tau, \bar{z}_1)\bar{T}(\tau, \bar{z}_1)$$

The large c cost function becomes the sum of two geometric actions: for f and g and is equivalent to Liouville

[Henneaux,et al.'99]

Recently: [SYK], [Mandal et al.] and closely Berry phases for Virasoro group [Oblak]

Comment 2: metrics

$$U(t) = P\exp\left(\int_0^t d\tau \epsilon_n(\tau) L_n\right) \qquad \partial_t U(t) U^{-1}(t) = \epsilon_n(t) L_n$$

1) Use the correlation function in some primary state

$$\langle h|.....|h\rangle \qquad ds^2 = G_{ij} d\epsilon_i d\epsilon_j$$

2) Right invariant metrics and Euler-Arnold equations: Virasoro $L_n = z^{n+1}\partial_z$

for the product

$$\langle (v(\sigma)\partial_\sigma, c), (w(\sigma)\partial_\sigma, a)\rangle = \int_{S^1} v(\sigma) w(\sigma) d\sigma + c \cdot a$$

geodesic equation (EA equation) becomes KdV equation [Ovsienko,Khesin'87]

or Burger's equation without central extension

3) Other choices and "complexity" interpretation…

Conclusions

- A new proposal for AdS/(c)TN and “PI Complexity” at any c!
- Classical geometries from Minimization of PI Complexity.
- Open: Extended symmetries and Toda, Transition amplitudes, Purifications?
- Applications to TN (hyperbolic TN: [A. Jahn, M. Gluza, F. Pastawski, J. Eisert])?
- Complexity <=> Dynamics of Geometry (Gravity)
- Universal gates in CFT implement conformal transformations
- Choice of the metric?
- QFT complexity deserves its own life !