

Title: Statistical mechanical models for quantum codes with correlated noise

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Abstract: <p>In this talk I will discuss generalising the mapping from quantum error correcting codes to stat. mech. models (originally due to Dennis, Kitaev, Landahl and Preskill) to the case of correlated Pauli noise. This mapping connects the error correcting threshold to a phase transition, allowing us to use methods for studying stat. mech. models to estimate code thresholds. Using this, I will present numerical estimates of toric code thresholds under correlated noise. Time permitting, I will also discuss how this mapping can be used to give a tensor network-based algorithm for approximate optimal decoding of 2d topological codes, generalising the decoder of Bravyi, Suchara and Vargo.</p>

Statistical mechanical models for quantum codes subject to correlated noise

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Perimeter Institute
2018-11



THE UNIVERSITY OF
SYDNEY



Quantum codes

Two important questions about quantum codes:

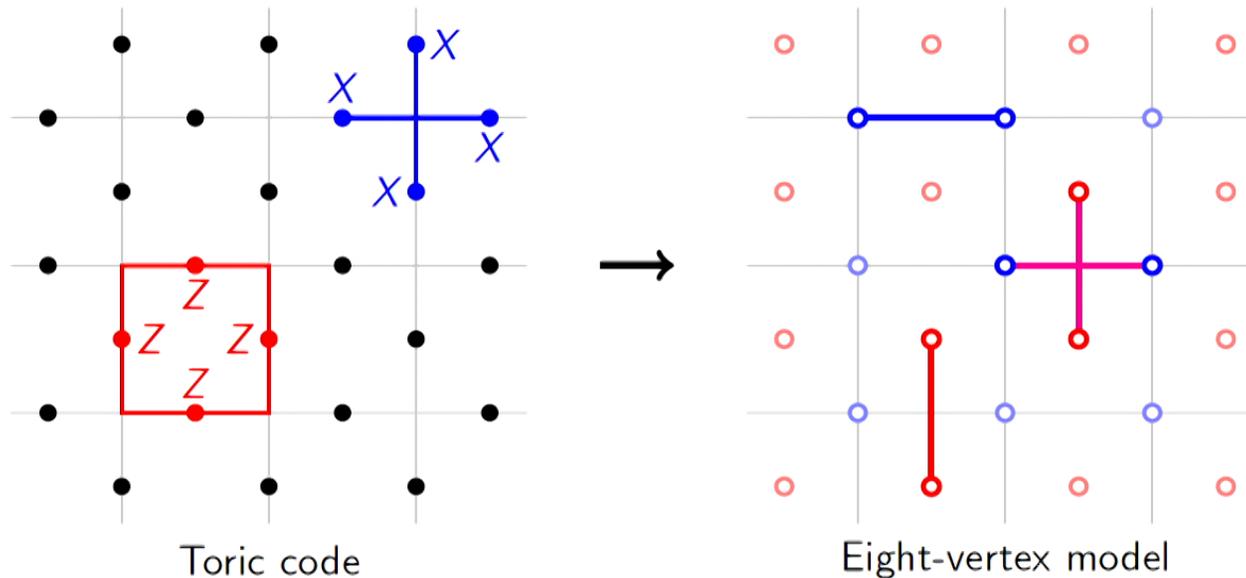
- How do I decode optimally?
- What is the fault-tolerant threshold?

The statistical mechanical mapping¹ allows us to address both questions codes subject to Pauli noise.

¹Dennis, Kitaev, Landahl, Preskill, JMP 2002, doi:10/cs2mtf, arXiv:quant-ph/0110143

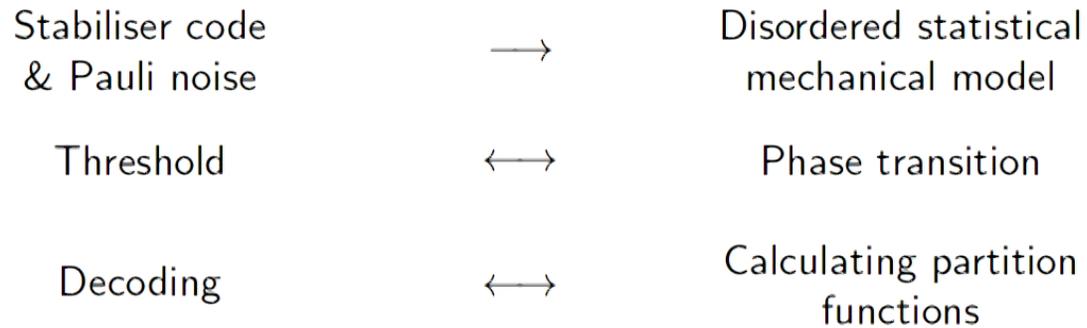
Statistical mechanical mapping

The idea here is to construct a family of statistical mechanical models, whose thermodynamic properties reflect the error correction properties of the code.

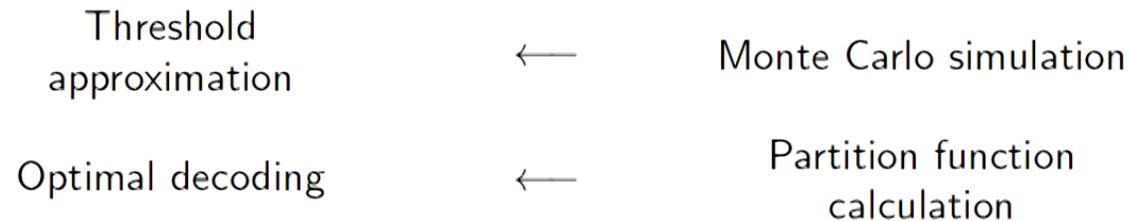


This will allow us to use the analytic and numerical tools developed to study stat mech systems to study quantum codes.

Statistical mechanical mapping



Allows us to reappropriate techniques for studying stat. mech. systems to study quantum codes:



Our results

- Generalise mapping to correlated noise for arbitrary subsystem codes
- Numerically show that mild correlations can lower the threshold of the toric code considerably
- Show how to apply our mapping to circuit noise via the history code, allowing us to approximate fault tolerant thresholds
- Show that the stat. mech. mapping gives tensor network maximum likelihood decoders which generalises the MPS decoder of Bravyi, Suchara and Vargo

Stabiliser codes and Pauli noise

For qubits, the Paulis $\mathcal{P} := \{I, X, Y, Z\}$ are defined

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will be considering stabiliser codes, which are specified by an Abelian subgroup of the Paulis \mathcal{S} , and whose code space \mathcal{C} is the joint +1 eigenspace,

$$\mathcal{C} = \left\{ |\psi\rangle \mid \mathcal{S} |\psi\rangle = |\psi\rangle, \forall \mathcal{S} \in \mathcal{S} \right\}.$$

Any two errors which differ by a stabiliser are logically equivalent, so the logical classes of errors are

$$\bar{E} := \{ES \mid S \in \mathcal{S}\}$$

Independent case: Hamiltonian

Let $\llbracket A, B \rrbracket$ be the scalar commutator of two Paulis, such that $AB =: \llbracket A, B \rrbracket BA$.

For a stabiliser code generated by $\{S_k\}_k$, and an error Pauli E , the (disordered) Hamiltonian H_E is defined

$$H_E(\vec{s}) := - \sum_i \sum_{\sigma \in \mathcal{P}_i} \overbrace{J_i(\sigma)}^{\text{Coupling Disorder}} \overbrace{\llbracket \sigma, E \rrbracket}^{\text{DoF}} \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

for $s_k = \pm 1$, and coupling strengths $J_i(\sigma) \in \mathbb{R}$.

Take-aways:

- Ising-type, with interactions corresponding to single-site Paulis σ
- Disorder E flips some interactions (Ferro \leftrightarrow Anti-ferro)
- Local code \implies local stat. mech. model

Independent case: Gauge symmetry

$$H_E(\vec{s}) = - \sum_i \sum_{\sigma \in \mathcal{P}_i} J_i(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

Using $\llbracket A, B \rrbracket \llbracket A, C \rrbracket = \llbracket A, BC \rrbracket$, we see this system has a gauge symmetry

$$s_k \rightarrow -s_k \quad \text{and} \quad E \rightarrow ES_k.$$

This gauge symmetry will capture the logical equivalence of errors, $Z_E = Z_{ES_k}$.

Independent case: Nishimori condition

Suppose we have an independent error model

$$\Pr(E) = \prod_i p_i(E_i),$$

we now want $Z_E = \Pr(\bar{E})$.

Using the gauge symmetry we have that the partition function can be written as a sum stabiliser-equivalent errors

$$Z_E = \sum_{\vec{s}} e^{-\beta H_E(\vec{s})} = \sum_S e^{-\beta H_{ES}(\vec{1})} = \sum_{F \in \bar{E}} e^{-\beta H_F(\vec{1})}.$$

If we select the coupling strength such that $e^{-\beta H_E(\vec{1})} = \Pr(E)$, then $Z_E = \Pr(\bar{E})$ will follow.

Independent case: Nishimori condition

We now want to pick our couplings such that $e^{-\beta H_E(\vec{I})} = \Pr(E)$. Expanding this out, we get

$$\sum_i \log p_i(E) = - \sum_i \sum_{\sigma} \beta J_i(\sigma) [[\sigma, E]].$$

Using the Fourier-like orthogonality relation $\frac{1}{4} \sum_{\sigma} [[\sigma, \tau]] = \delta_{\tau, I}$, this becomes

$$\text{Nishimori condition: } \beta J_i(\sigma) = \frac{1}{4} \sum_{\tau \in \mathcal{P}} \log p_i(\tau) [[\sigma, \tau]],$$

which implies $e^{-\beta H_E(\vec{I})} = \Pr(E)$, and therefore $Z_E = \Pr(\bar{E})$.

This intrinsically links the error correcting behaviour of the code to the thermodynamic behaviour of the model (along the Nishimori line).

Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

Step 1: Degrees of freedom

$s_v = \pm 1$ on each vertex v

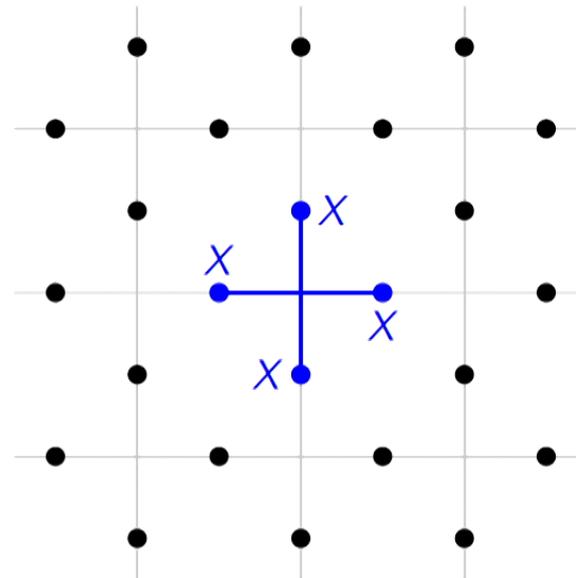
Step 2: Interactions

$$H_I = \sum_{v \sim v'} J s_v s_{v'}$$

Step 3: Disorder

$$H_E = - \sum_{v \sim v'} J e_{vv'} s_v s_{v'}$$

$$\text{where } e_{vv'} = \begin{cases} +1 & E_{vv'} = I, \\ -1 & E_{vv'} = X. \end{cases}$$



Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(X_e) = p, \quad \Pr(I_e) = 1 - p.$$

Step 1: Degrees of freedom

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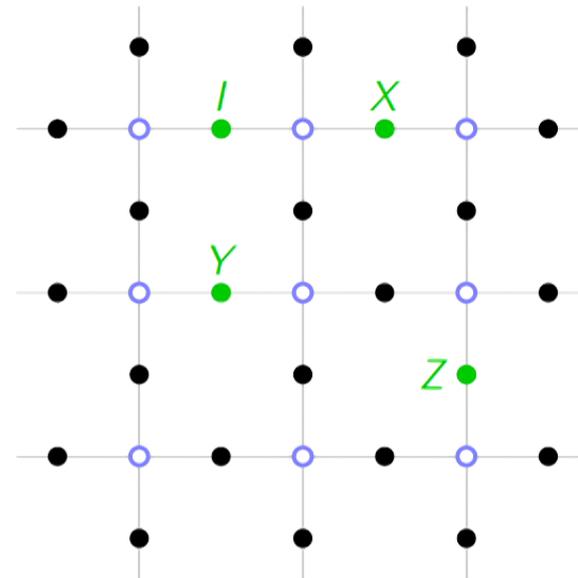
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$$[[\sigma_e, S_v]] = \begin{cases} -1 & \text{if } \sigma = X, Y, v \in e, \\ +1 & \text{else.} \end{cases}$$

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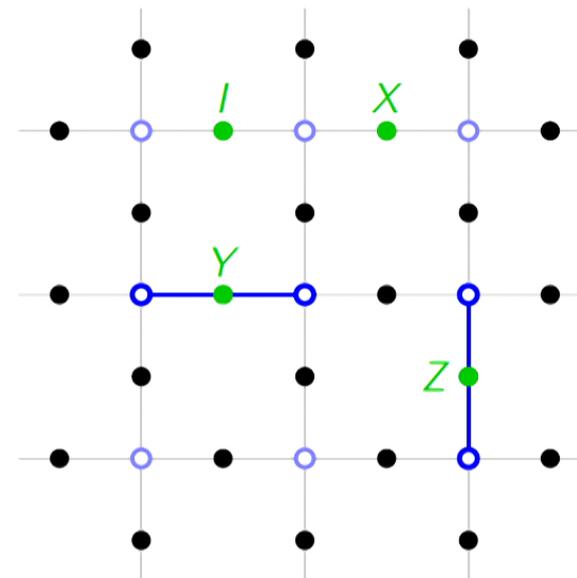
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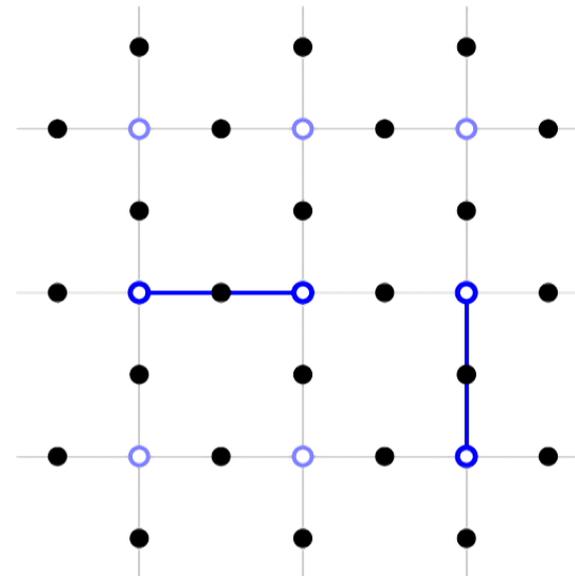
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$$[[\sigma, S_k]] \rightarrow [[\sigma, ES_k]]$$

Toric code and the random-bond Ising model

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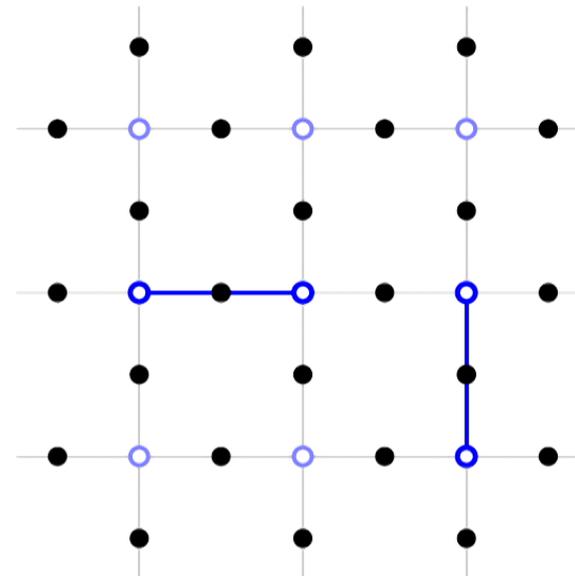
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$$[[\sigma, S_k]] \rightarrow [[\sigma, ES_k]]$$

Toric code and the random-bond Ising model

Step 0: Code and noise model

Toric code with iid bit-flips

$$\Pr(+J) = p, \quad \Pr(-J) = 1 - p.$$

Step 1: Degrees of freedom

$$s_v = \pm 1 \text{ on each vertex } v$$

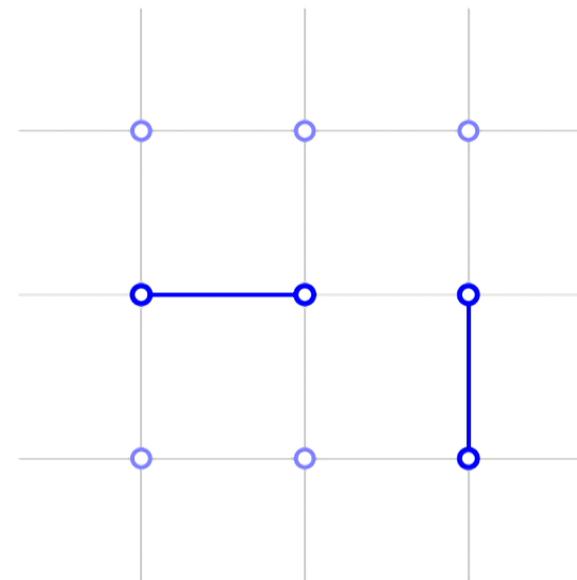
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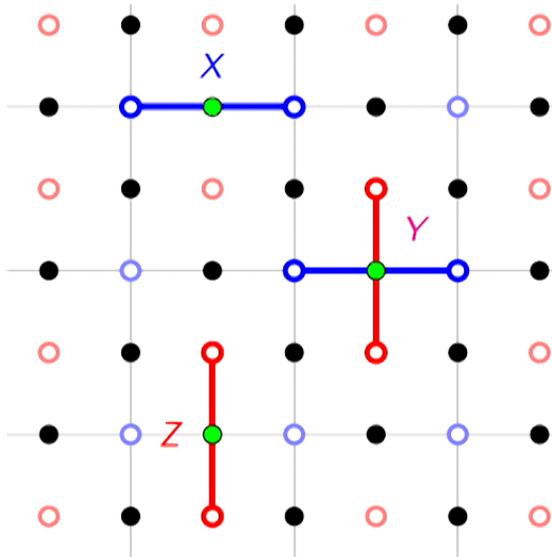


$\pm J$ Random-bond Ising Model

Other independent examples

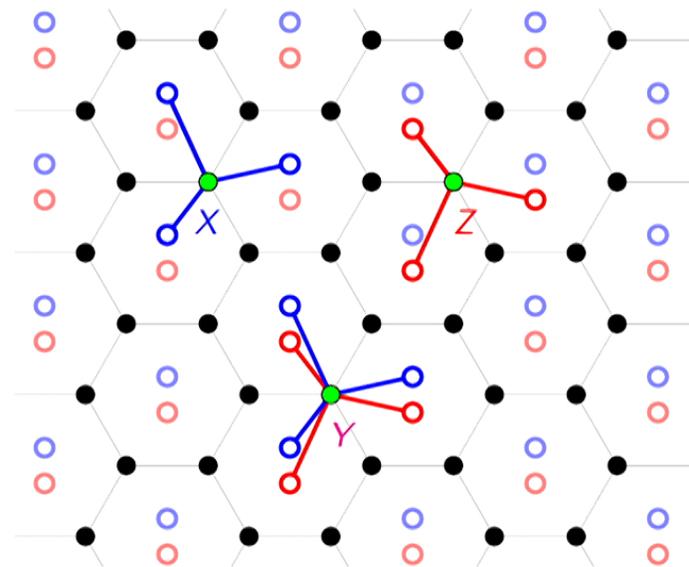
Toric code

Bit-flip \rightarrow Random-bond Ising¹
Indep. X&Z \rightarrow 2 \times Random-bond Ising
Depolarising \rightarrow Random 8-vertex model²



Colour code

Bit-flip \rightarrow Random 3-spin Ising
Indep. X&Z \rightarrow 2 \times Random 3-spin Ising
Depolarising \rightarrow Random interacting 8-vertex²



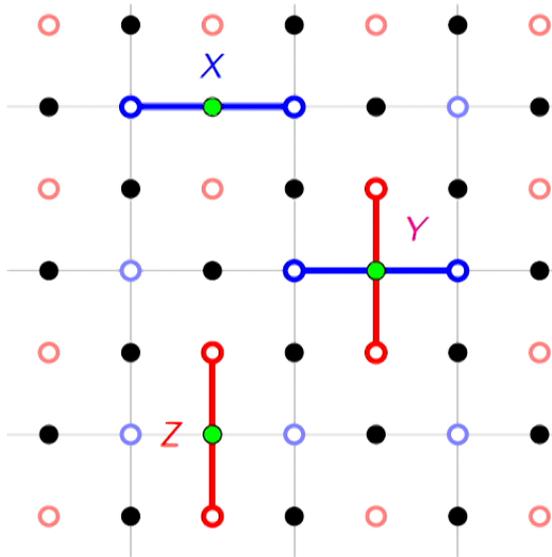
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²Bombin et.al., PRX 2012, doi:10/crz5, arXiv:1202.1852

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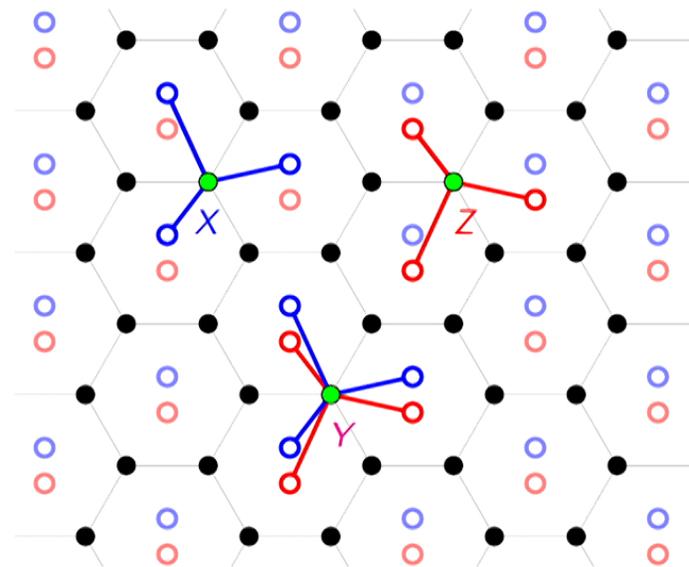
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Error correction threshold as a quenched phase transition

Consider the free energy cost of a logical error L ,

$$\Delta_E(L) = -\frac{1}{\beta} \log Z_{EL} + \frac{1}{\beta} \log Z_E.$$

Along the Nishimori line

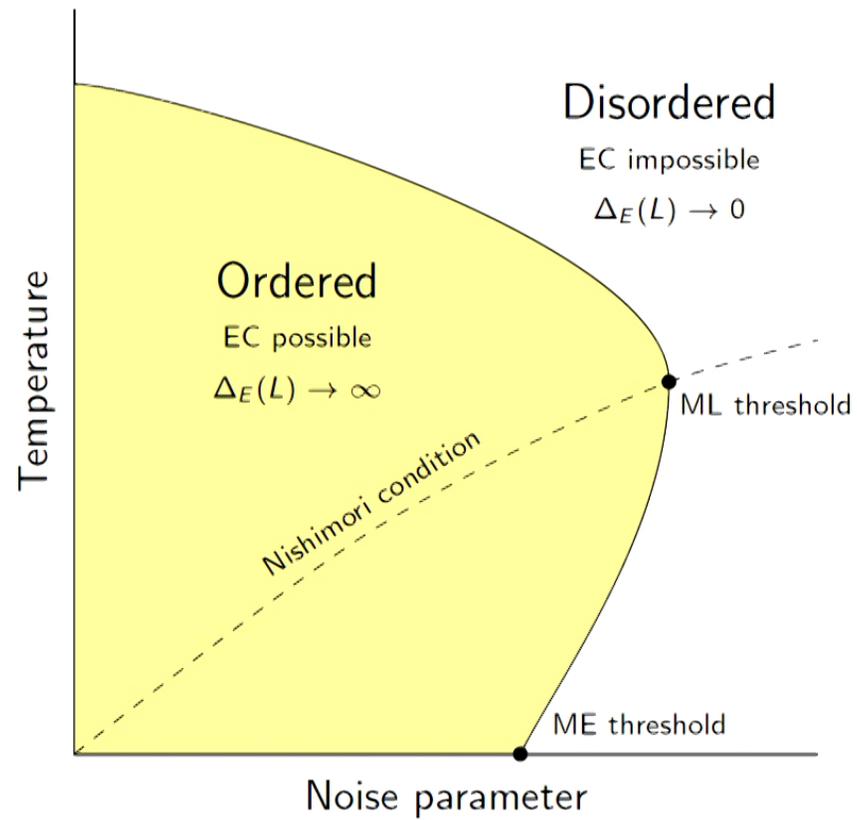
$$\Delta_E(L) = \frac{1}{\beta} \log \frac{\Pr(\bar{E})}{\Pr(\bar{EL})},$$

which implies

Below threshold : $\Delta_E(L) \rightarrow \infty$ (in mean)

Above threshold : $\Delta_E(L) \rightarrow 0$ (in prob.)

Phase diagram sketch



Correlated case

The key point independence gave us was the ability to factor our noise model

$$\Pr(E) = \prod_i p_i(E_i).$$

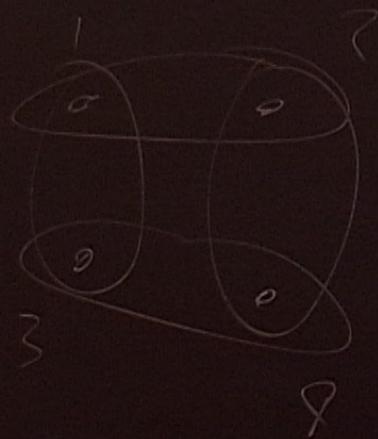
We can generalise this to correlated models:

Factored distribution

An error model factors over regions $\{R_j\}_j$ if there exist $\phi_j : \mathcal{P}_{R_j} \rightarrow \mathbb{R}$ such that

$$\Pr(E) = \prod_j \phi_j(E_{R_j})$$

This model includes many probabilistic graphical models, such as Bayesian Networks and Markov/Gibbs Random Fields.



Correlated case

By construction, we can extend to the correlated case by changing $\sigma \in \mathcal{P}_i$ to $\sigma \in \mathcal{P}_{R_j}$:

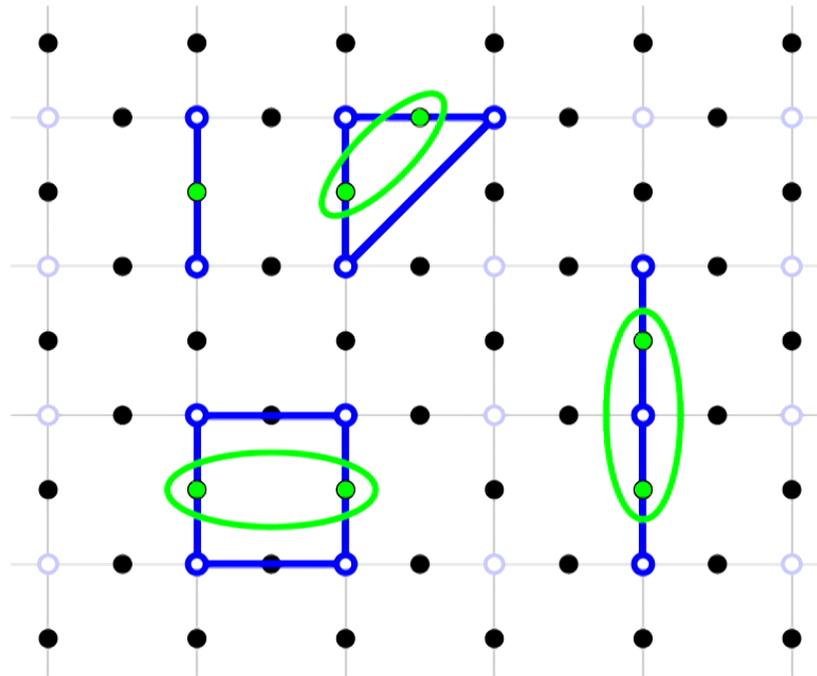
$$H_E(\vec{s}) := - \sum_j \sum_{\sigma \in \mathcal{P}_{R_j}} J_j(\sigma) \llbracket \sigma, E \rrbracket \prod_{k: \llbracket \sigma, S_k \rrbracket = -1} s_k$$

Nishimori condition:
$$\beta J_j(\sigma) = \frac{1}{|\mathcal{P}_{R_j}|} \sum_{\tau \in \mathcal{P}_{R_j}} \log \phi_j(\tau) \llbracket \sigma, \tau \rrbracket,$$

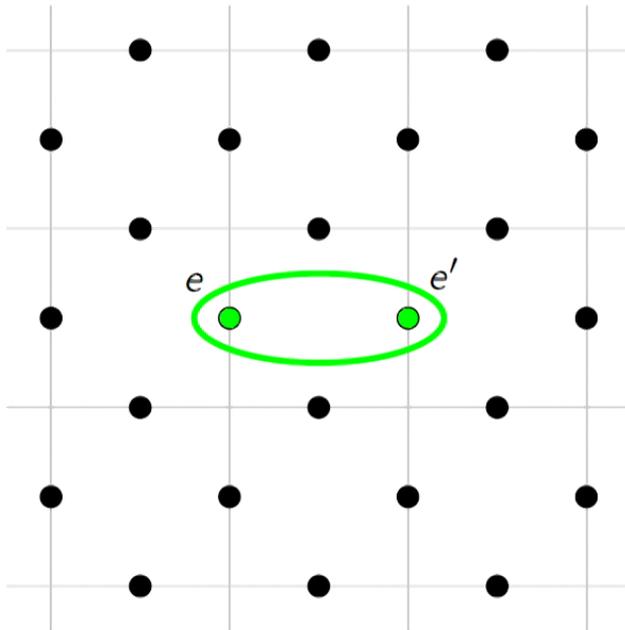
As before we get that $Z_E = \Pr(\bar{E})$, and so the threshold manifests as a phase transition.

Correlated example

Toric code with correlated bit-flips
Correlations induce longer-range interactions



'Across plaquette' correlated bit-flips



This error model is entirely specified by the conditional error probabilities

$$\begin{array}{ll} \Pr(I_e|I_{e'}) & \Pr(I_e|X_{e'}) \\ \Pr(X_e|I_{e'}) & \Pr(X_e|X_{e'}) \end{array}$$

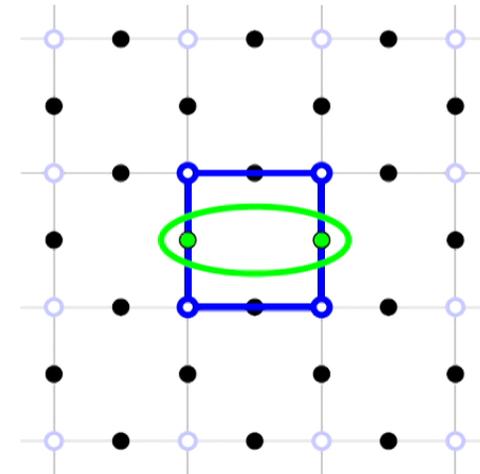
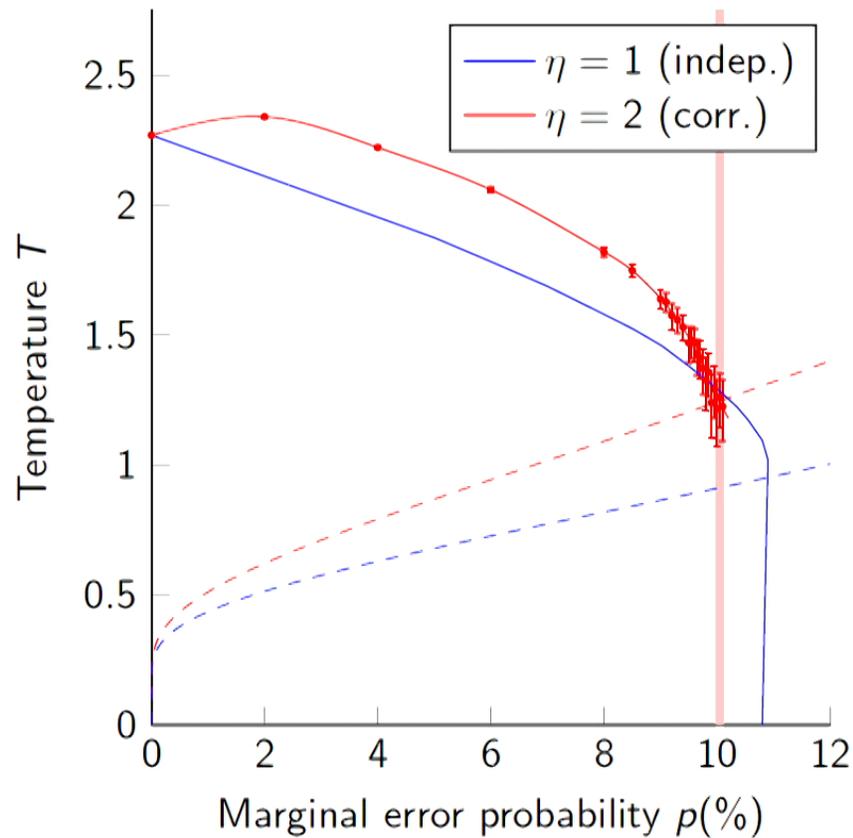
for all neighbouring edges e and e' .

For our purposes, it will be convenient to parameterise things by

$$p := \Pr(X_e), \quad \eta := \frac{\Pr(X_e|X_{e'})}{\Pr(X_e|I_{e'})}.$$

Here p is the marginal error rate, and η is a measure of the correlations.

Monte Carlo simulations



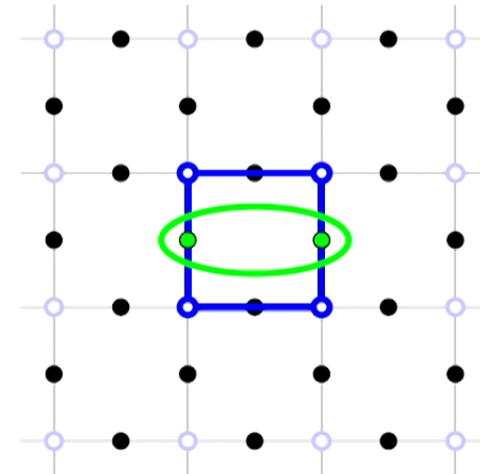
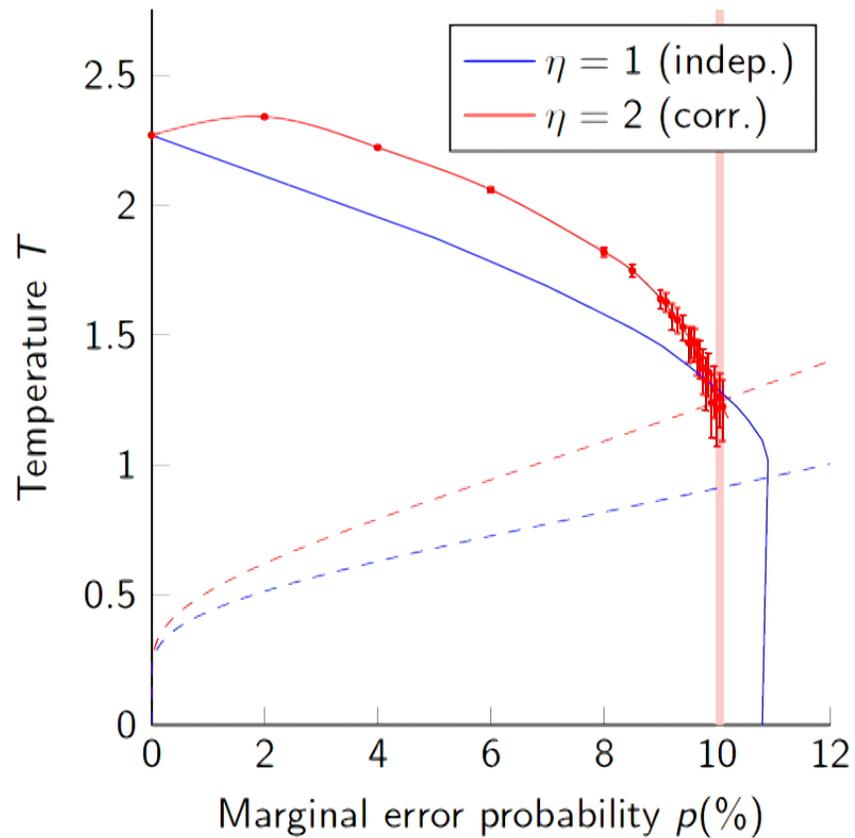
Thresholds

Indep.: $p_{th} = 10.917(2)\%^{1,2}$
 Corr.: $p_{th} = 10.04(6)\%$

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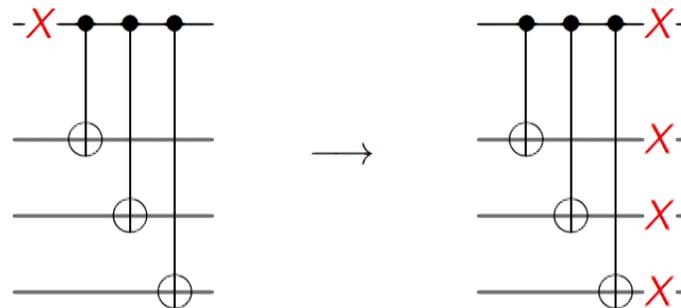
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Circuit noise

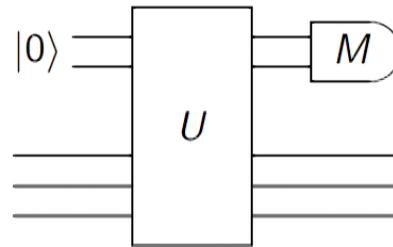
Noise followed by ideal measurements is unrealistic. In reality, circuits will be faulty.

Noisy circuits give faulty outcomes, and will spread around and correlate noise:



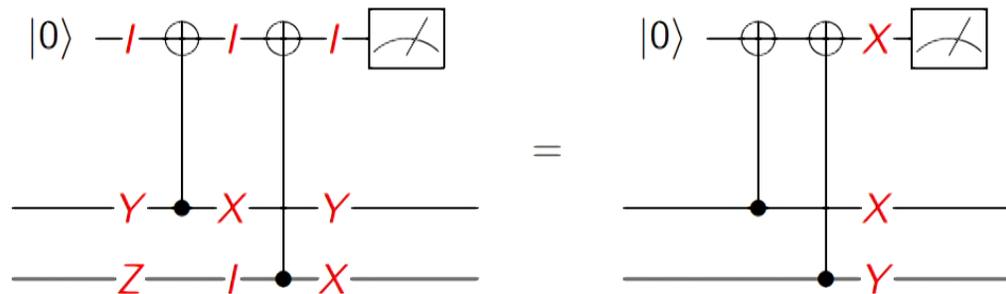
Circuit noise

We will consider measurement circuits of the form

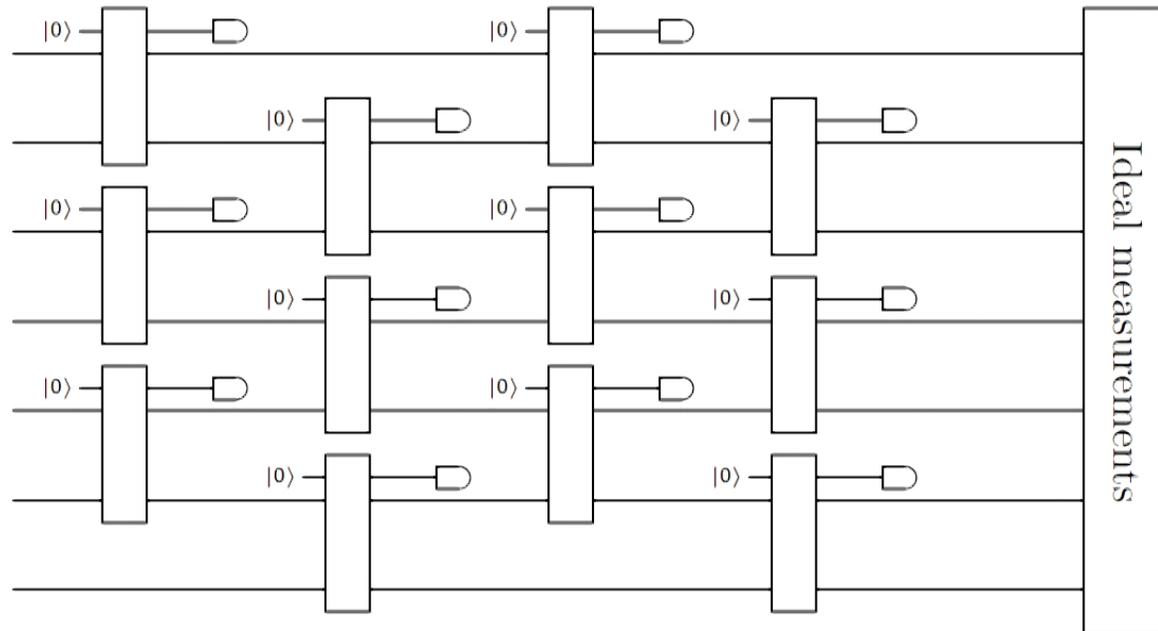


where U is a Clifford and M is a Pauli.

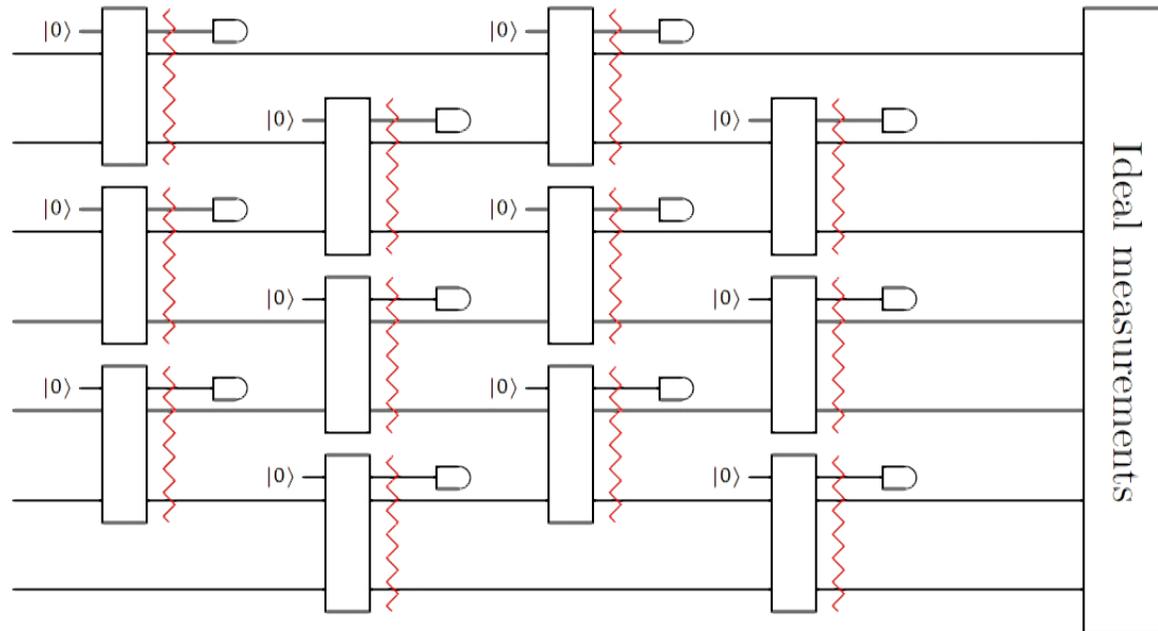
For convenience we only consider independent noise on each circuit. We also will push noise through until after the unitary:



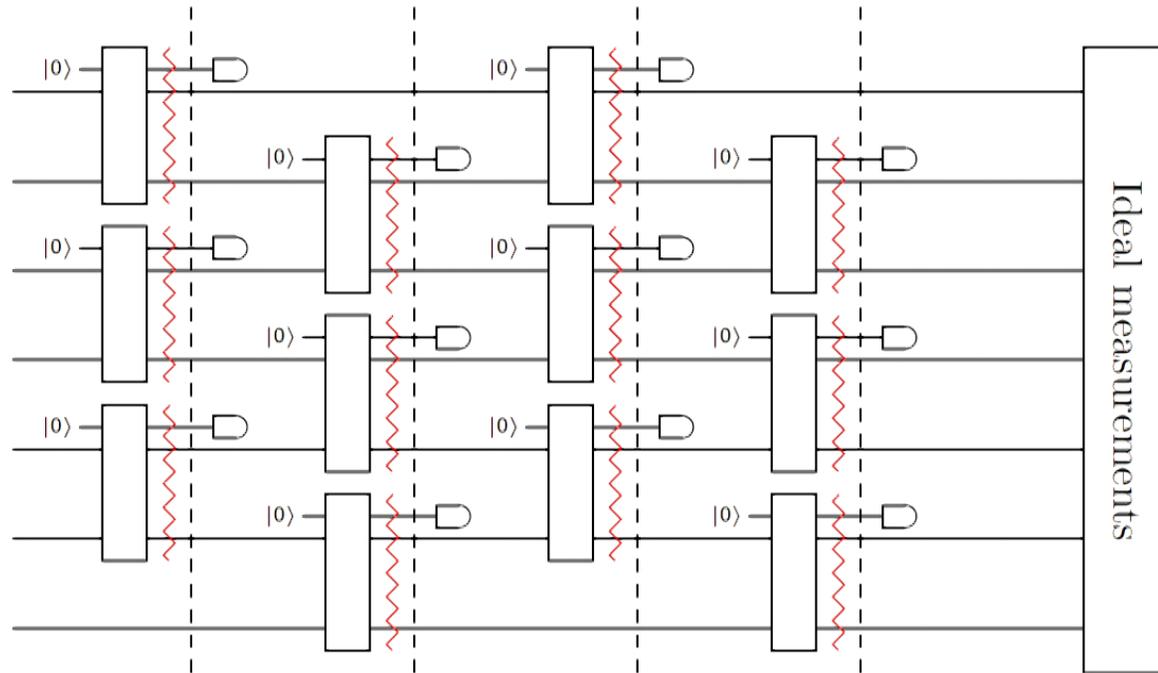
Circuit noise



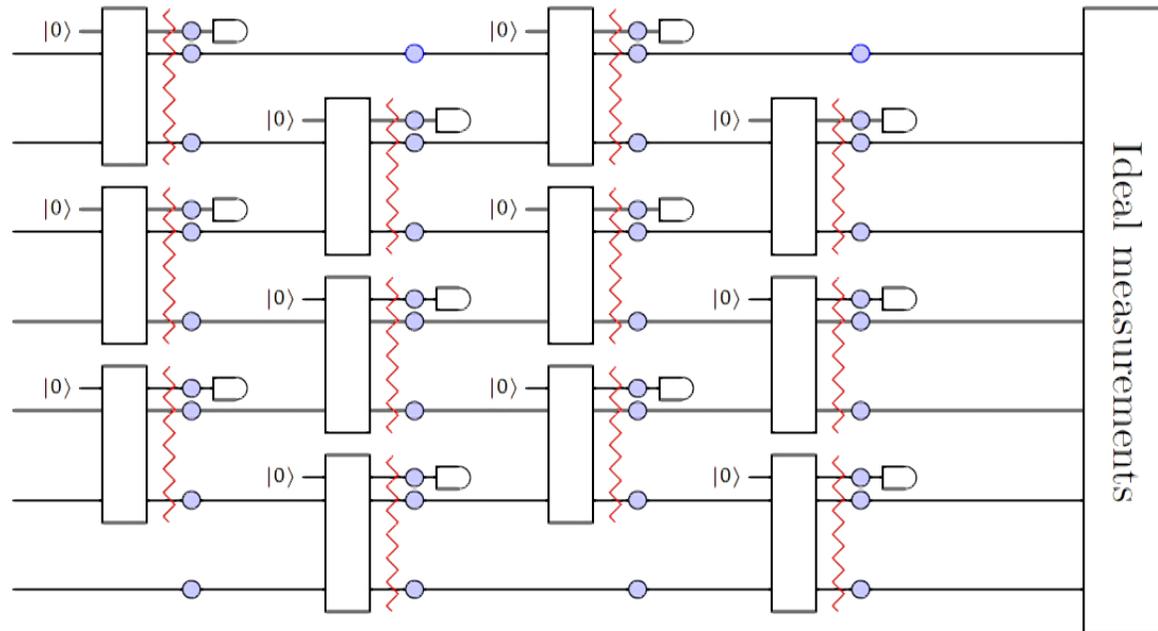
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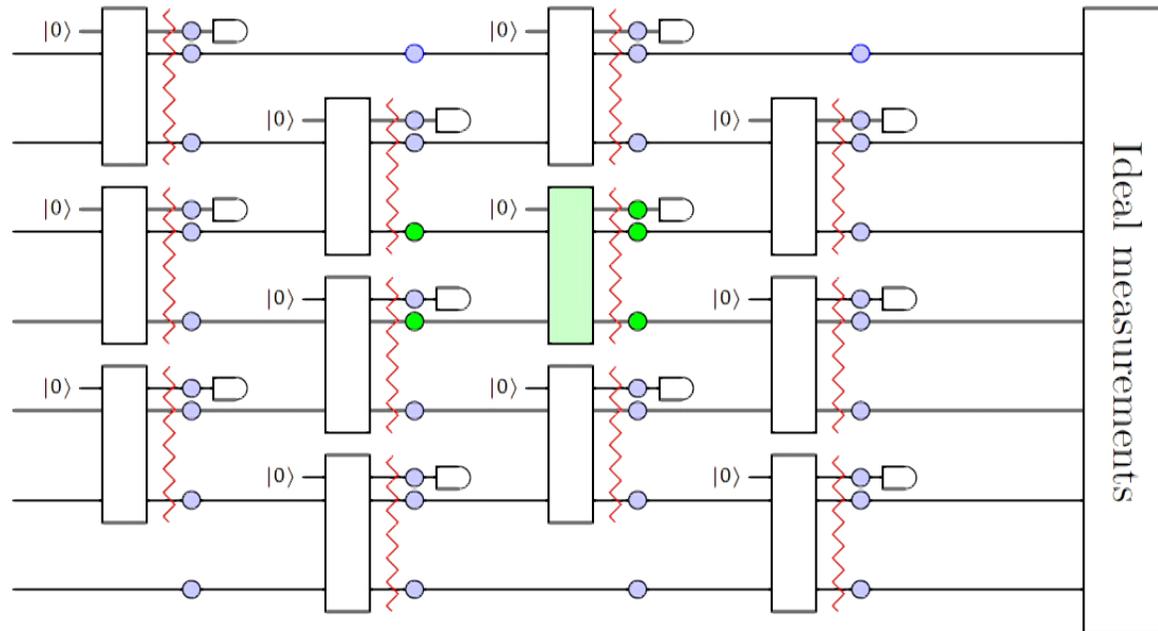
Circuit noise



Circuit noise

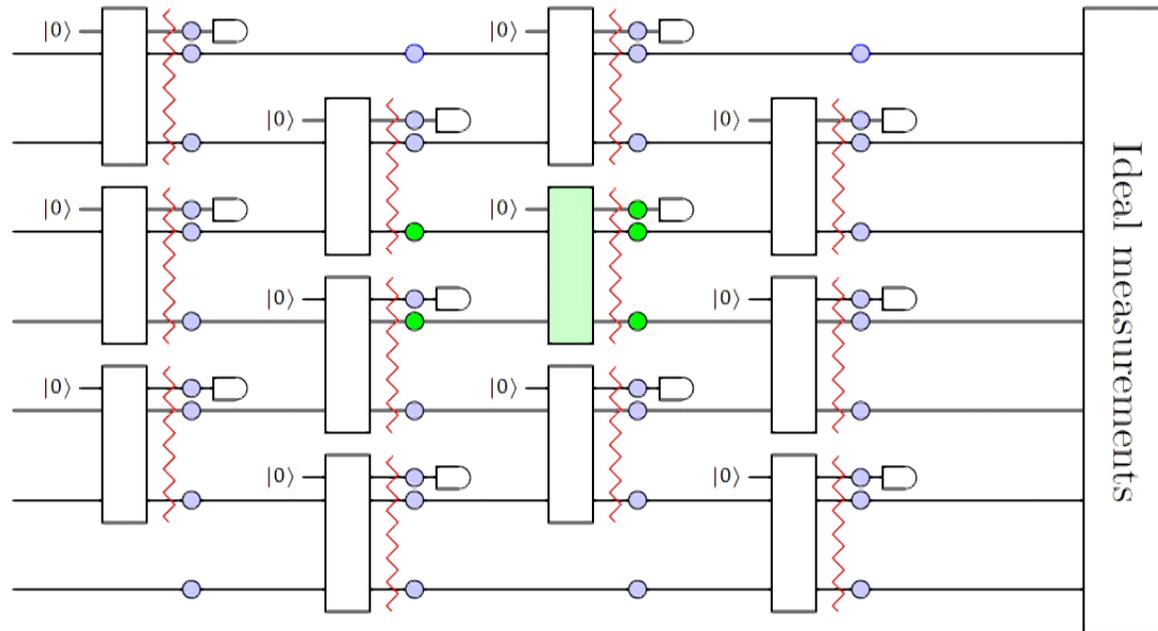


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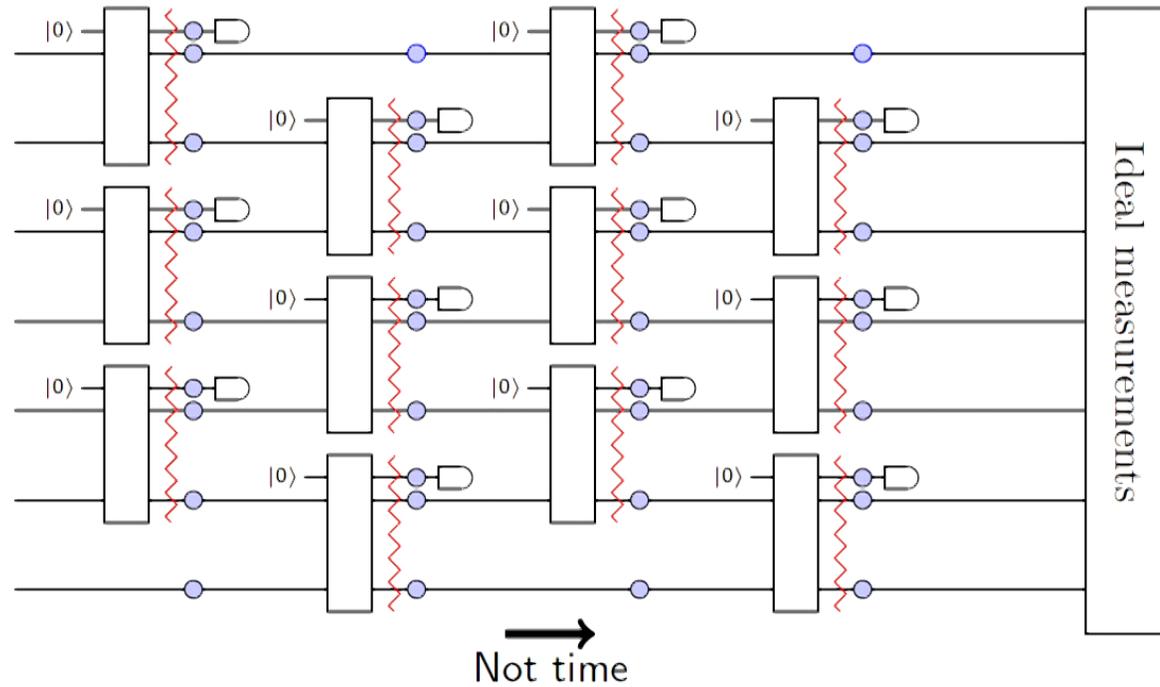
$$\Pr(\text{Error history}) = \prod_{j,t} \Pr(\text{Error after meas. } (x, t) \mid \text{Error before meas. } (x, t))$$

Circuit noise



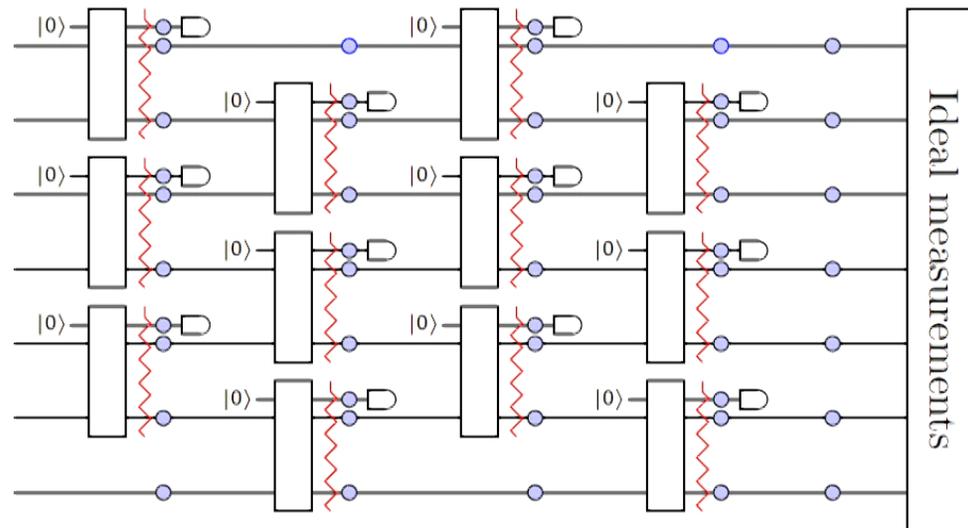
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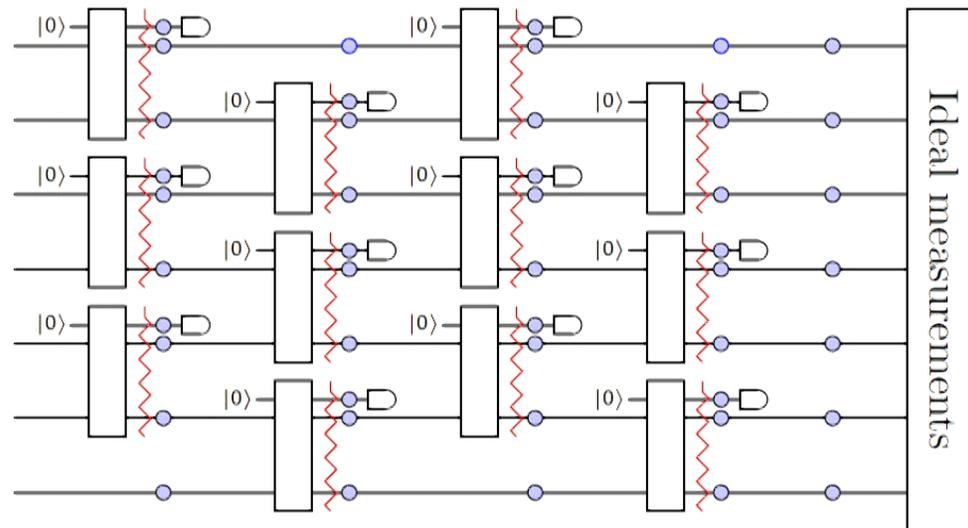
History code



History code:

- Qubits placed at points in space-time (including ancillae)
- Stabilisers correspond to measurements and stabilisers at final time
- Logicals correspond to logicals at final time
- Circuit noise on original code Spatially correlated noise on history code
- FT decoding of original code Decoding the history code
- FT threshold of original code = Threshold of history code

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- FT decoding of original code = Decoding the history code
- FT threshold of original code = Threshold of history code

Decoding

Can the stat. mech. model give us a decoder?

If an error E occurs, a decoder needs to select one of the degenerate logical error classes

$$\overline{E} \quad \overline{EL_1} \quad \overline{EL_2} \quad \overline{EL_3} \quad \dots$$

The optimal (maximum likelihood) decoder selects the most likely class

$$D_{\text{ML}} = \overline{EL_I} \quad \text{where} \quad I = \arg \max_I \Pr(\overline{EL_I}).$$

Decoding from partition functions

Along the Nishimori line, the maximum likelihood condition corresponds to maximising the partition function

$$l = \arg \max_l Z_{EL_l}.$$

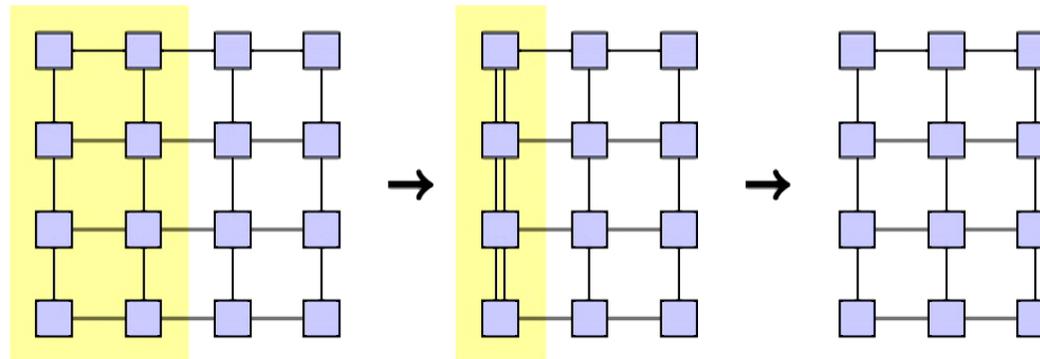
Approximating Z_{EL_l} therefore allows us to approximate the ML decoder.

- Step 1: Measure the syndrome s
- Step 2: Construct an arbitrary error C_s which has syndrome s
- Step 3: Approximate $Z_{C_s L_l} = \Pr(\overline{C_s L_l})$ for each logical l
- Step 4: Find the l such that $Z_{C_s L_l}$ is maximised
- Step 5: Apply $(C_s L_l)^{-1}$

Decoding from (approximate) tensor network contraction

Partition functions can be expressed as tensor networks^{2,3}, allowing us to use approximate tensor network contraction schemes.

For 2D codes and locally correlated noise, this tensor network is also 2D. Here we can use the MPS-MPO approximation contraction scheme considered by Bravyi, Suchara and Vargo⁴:



⁴Verstraete et. al., PRL 2006, doi:10/dfgcz8, arXiv:quant-ph/0601075

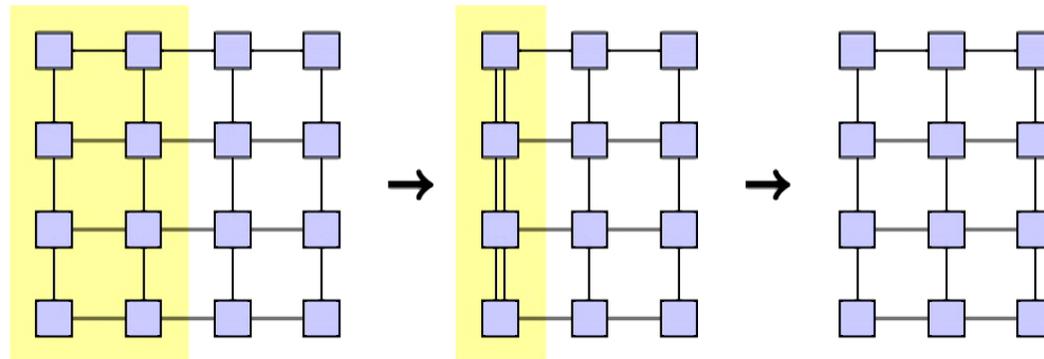
³Bridgeman and Chubb, JPA 2017, doi:10/cv7m, arXiv:1603.03039

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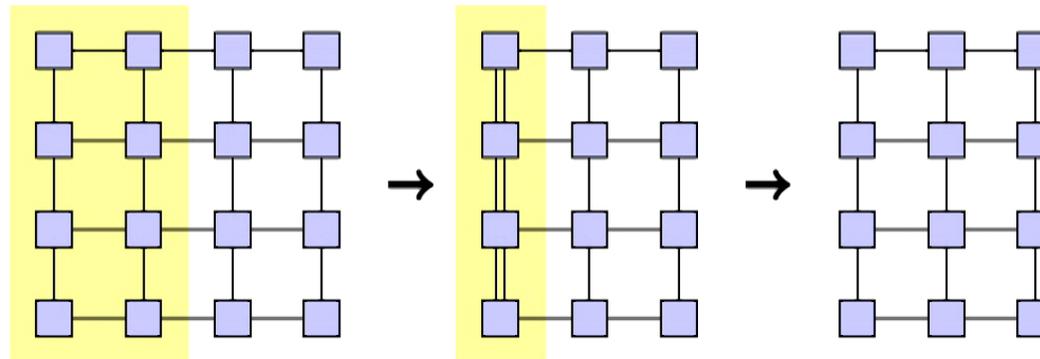
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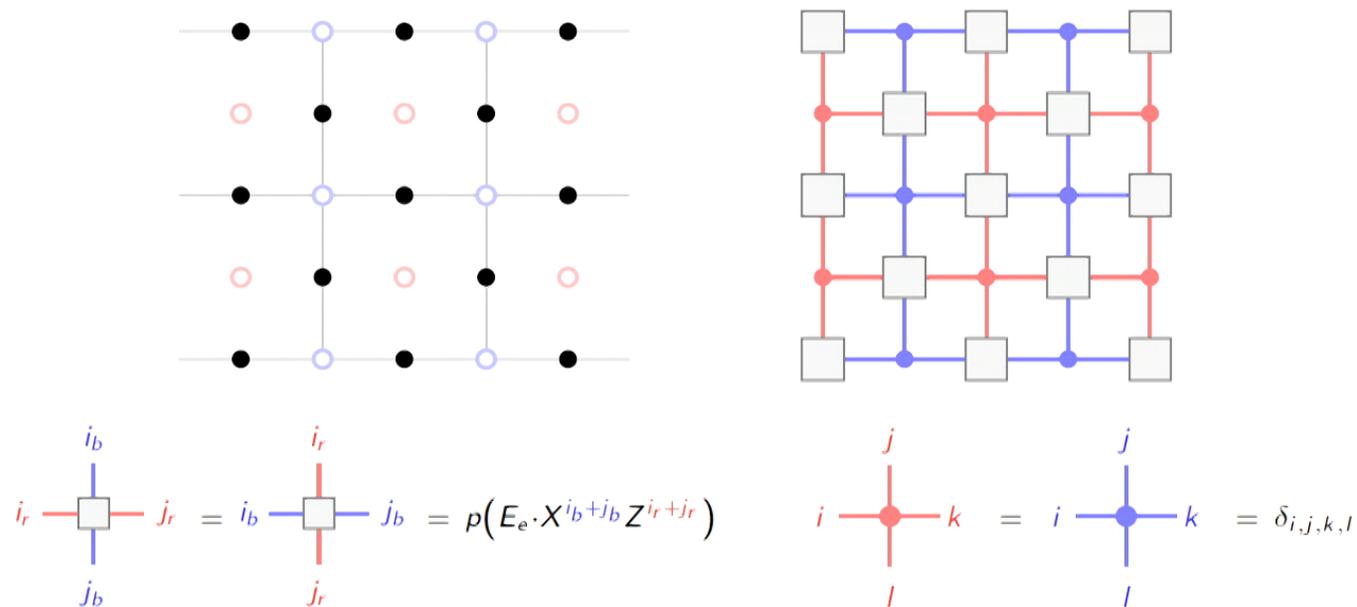
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Decoding from (approximate) tensor network contraction

This gives an algorithm for (approximate) maximum likelihood decoding for any 2D code, subject to any locally correlated noise, generalising BSV.

Indeed, applying this to iid noise in the surface code reproduces BSV:



Conclusions and further work

- Extended the stat. mech. mapping to correlated models
 - Can apply stat. mech. mapping to circuit noise via the history code
 - Numerically evaluated the threshold of correlated bit-flips in the toric code
 - Stat. mech. mapping gives tensor network maximum likelihood decoders
-
- Can we apply this to experimentally relevant correlated models?
 - Construction extends to all Abelian quantum doubles. Can we extend non-Abelian models?
 - Non-stochastic noise? Non-Pauli noise? Coherent noise?
 - Can we use the decoders to understand to better understand the connection between correlation and the threshold (ongoing work with David Tuckett and Benjamin Brown).

Thank you!

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