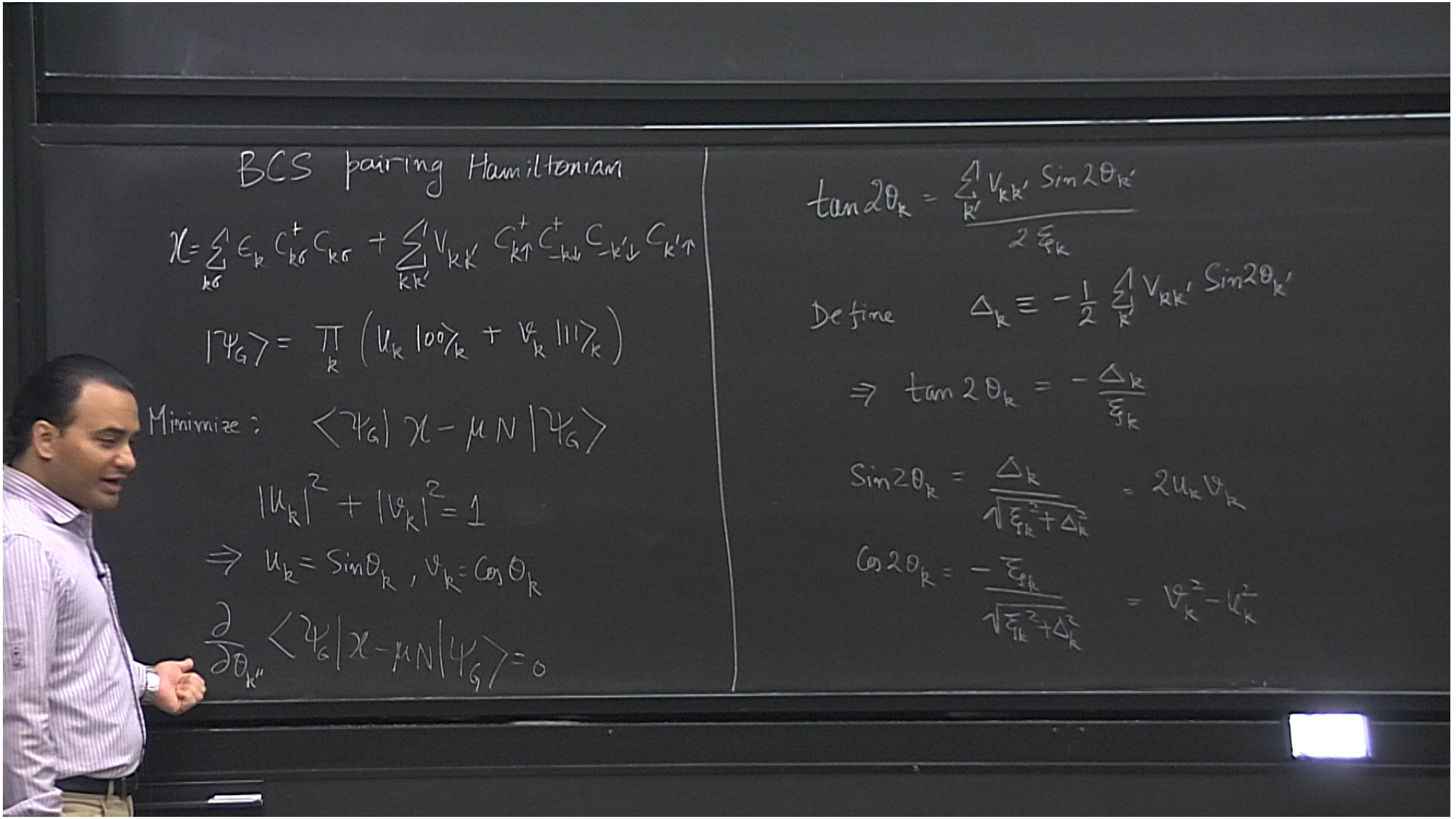
Title: PSI 2018/2019 - Condensed Matter - Lecture 11

Date: Nov 26, 2018 10:45 AM

URL: http://pirsa.org/18110029

Abstract:


BCS pairing Hamiltonian
$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum'_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$
$|\Psi_G\rangle = \prod_k \left(u_k |0\rangle_k + v_k |1\rangle_k \right)$
Minimize: $\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle$
$|u_k|^2 + |v_k|^2 = 1$
$\Rightarrow u_k = \sin\theta_k, \; v_k = \cos\theta_k$
$\frac{\partial}{\partial \theta_{k''}} \langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle = 0$
$\tan 2\theta_k = \frac{\sum'_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$
Define $\Delta_k \equiv -\frac{1}{2} \sum'_{k'} V_{kk'} \sin 2\theta_{k'}$
$\Rightarrow \tan 2\theta_k = -\frac{\Delta_k}{\xi_k}$
$\sin 2\theta_k = \frac{\Delta_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = 2u_k v_k$
$\cos 2\theta_k = -\frac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = v_k^2 - u_k^2$

BCS pairing Hamiltonian

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum'_{kk'} V_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

$$|\Psi_G\rangle = \prod_k \left(u_k |0\rangle_k + v_k |1\rangle_k \right)$$

Minimize: $\langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle$

$$|u_k|^2 + |v_k|^2 = 1$$

$$\Rightarrow u_k = \sin\theta_k, \; v_k = \cos\theta_k$$

$$\frac{\partial}{\partial \theta_{k''}} \langle \Psi_G | \mathcal{H} - \mu N | \Psi_G \rangle = 0$$

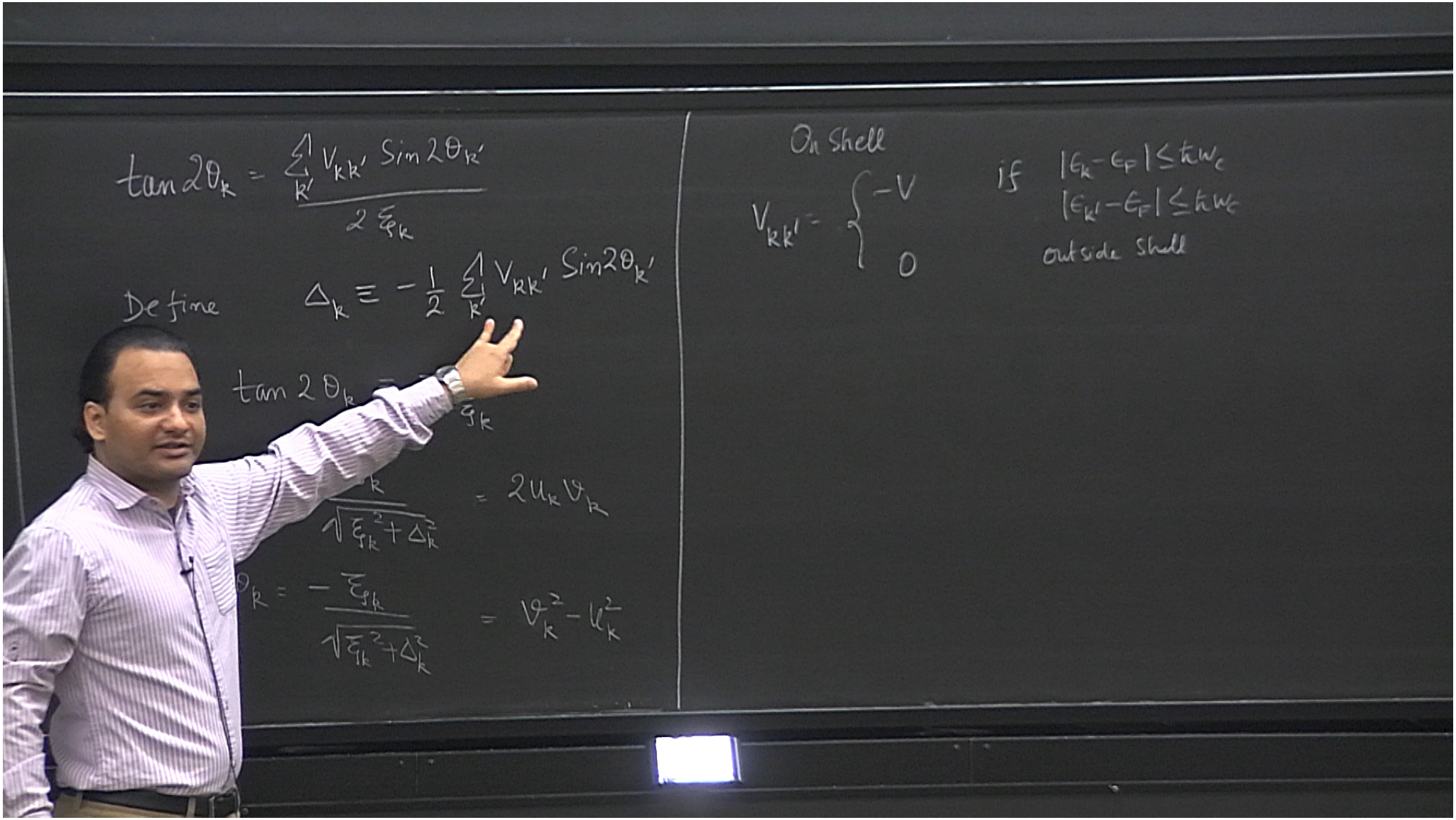
$$\tan 2\theta_k = \frac{\sum'_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$$

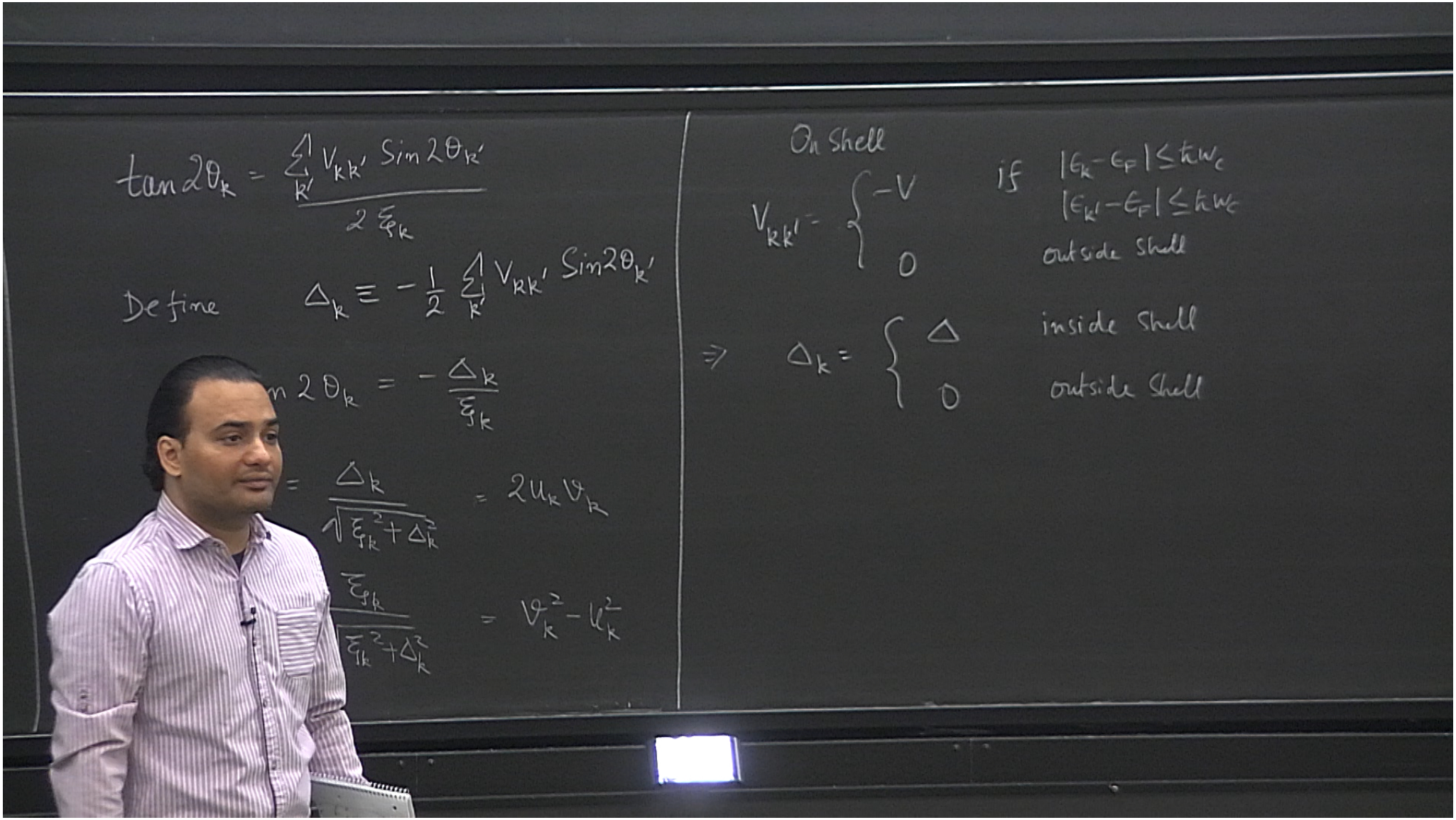
Define $\Delta_k \equiv -\frac{1}{2} \sum'_{k'} V_{kk'} \sin 2\theta_{k'}$

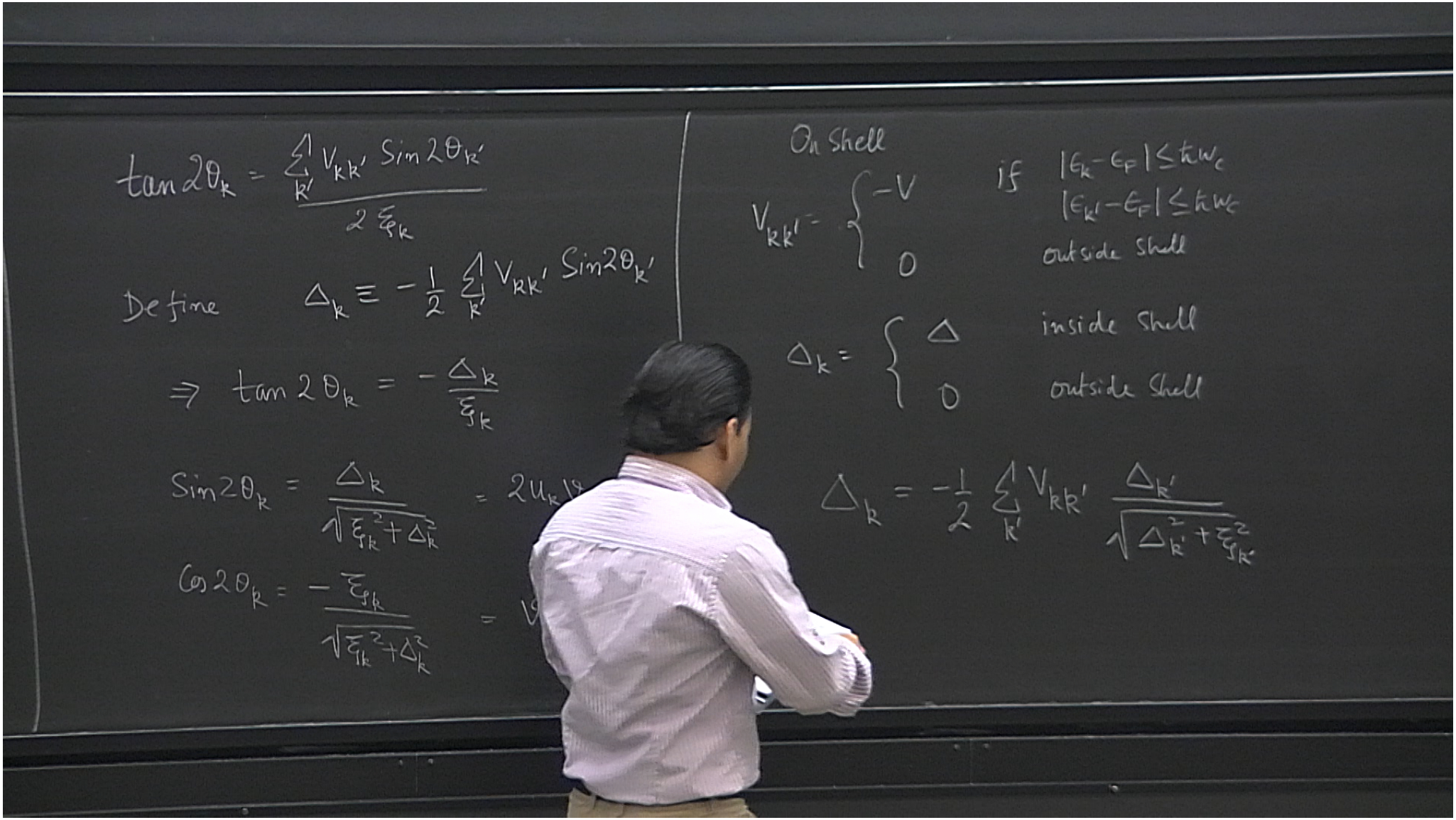
$$\Rightarrow \tan 2\theta_k = -\frac{\Delta_k}{\xi_k}$$

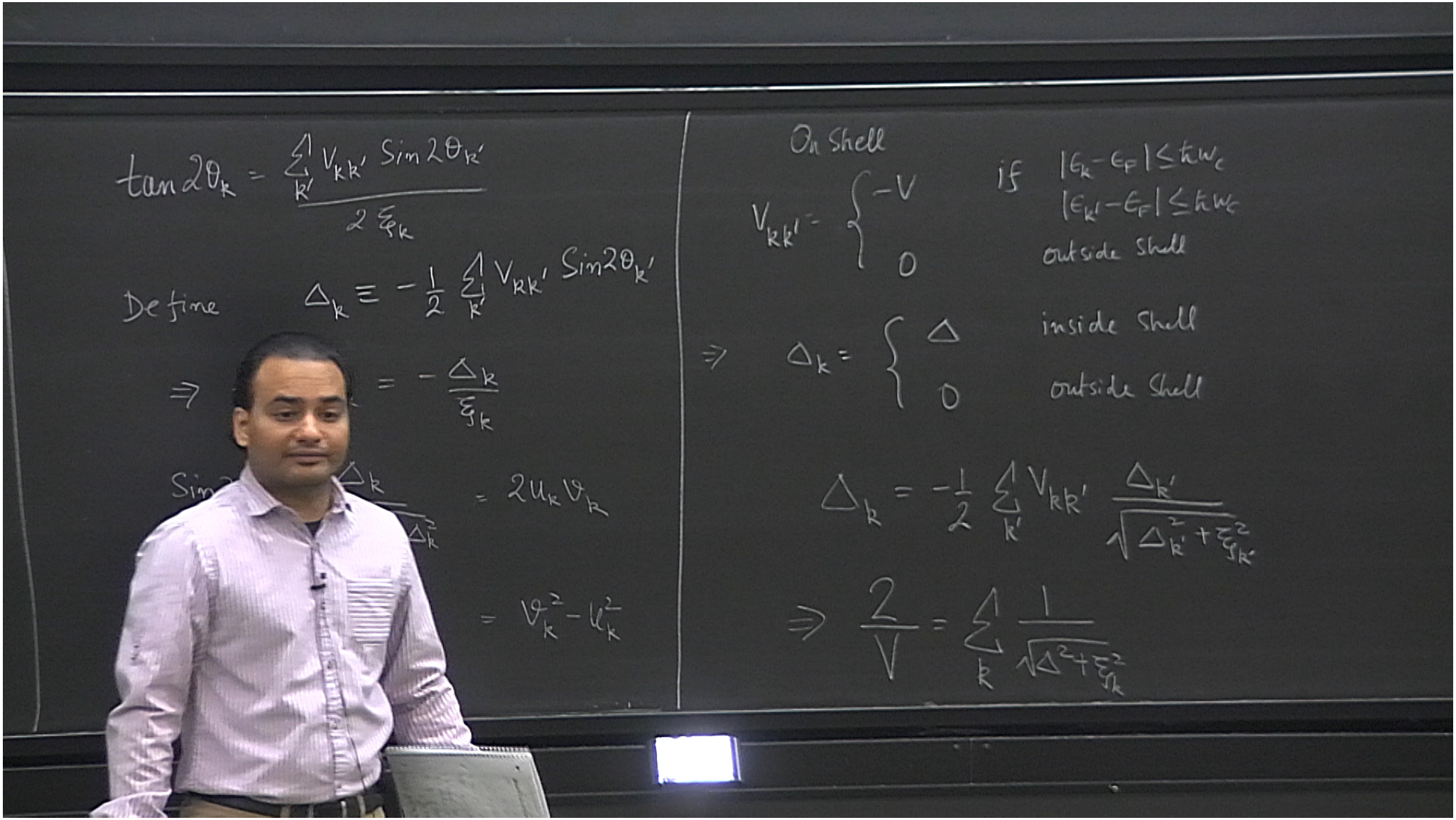
$$\sin 2\theta_k = \frac{\Delta_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = 2u_k v_k$$

$$\cos 2\theta_k = -\frac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = v_k^2 - u_k^2$$


$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$
Define $\Delta_k \equiv -\frac{1}{2}\sum_{k'} V_{kk'} \sin 2\theta_{k'}$
$\tan 2\theta_k$
$= 2u_k v_k$
$\theta_k = -\frac{\xi_{1k}}{\sqrt{\xi_{1k}^2+\Delta_k^2}}$
$= v_k^2 - u_k^2$
On Shell
$V_{kk'} = \begin{cases} -V \\ 0 \end{cases}$
if $|\epsilon_k - \epsilon_F| \le \hbar\omega_c$
$|\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c$
outside Shell


$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$
Define $\Delta_k \equiv -\frac{1}{2}\sum_{k'} V_{kk'} \sin 2\theta_{k'}$
$\sin 2\theta_k = -\frac{\Delta_k}{\xi_k}$
$= \frac{\Delta_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = 2u_k v_k$
$\frac{\xi_k}{\sqrt{\xi_k^2 + \Delta_k^2}} = v_k^2 - u_k^2$
On Shell
$V_{kk'} = \begin{cases} -V \\ 0 \end{cases}$ if $|\epsilon_k - \epsilon_F| \le \hbar\omega_c$, $|\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c$
outside Shell
$\Rightarrow \Delta_k = \begin{cases} \Delta \\ 0 \end{cases}$ inside Shell
outside Shell


$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$
Define $\Delta_k \equiv -\frac{1}{2}\sum_{k'} V_{kk'} \sin 2\theta_{k'}$
$\Rightarrow \tan 2\theta_k = -\frac{\Delta_k}{\xi_k}$
$\sin 2\theta_k = \frac{\Delta_k}{\sqrt{\xi_k^2+\Delta_k^2}}$
$\cos 2\theta_k = -\frac{\xi_k}{\sqrt{\xi_k^2+\Delta_k^2}}$
On Shell
$V_{kk'} = \begin{cases} -V \\ 0 \end{cases}$ if $|\epsilon_k - \epsilon_F| \le \hbar\omega_c$, $|\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c$
outside Shell
$\Delta_k = \begin{cases} \Delta \\ 0 \end{cases}$ inside Shell
outside Shell
$\Delta_k = -\frac{1}{2}\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2+\xi_{k'}^2}}$


$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\xi_k}$
Define $\Delta_k \equiv -\frac{1}{2}\sum_{k'} V_{kk'} \sin 2\theta_{k'}$
$\Rightarrow$ $= -\frac{\Delta_k}{\xi_k}$
$= 2u_k v_k$
$= v_k^2 - u_k^2$
On Shell
$V_{kk'} = \begin{cases} -V \\ 0 \end{cases}$ if $|\epsilon_k - \epsilon_F| \le \hbar\omega_c$ $|\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c$
outside Shell
$\Rightarrow$ $\Delta_k = \begin{cases} \Delta \\ 0 \end{cases}$ inside Shell
outside Shell
$\Delta_k = -\frac{1}{2}\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \xi_{k'}^2}}$
$\Rightarrow \frac{2}{V} = \sum_k \frac{1}{\sqrt{\Delta^2 + \xi_k^2}}$

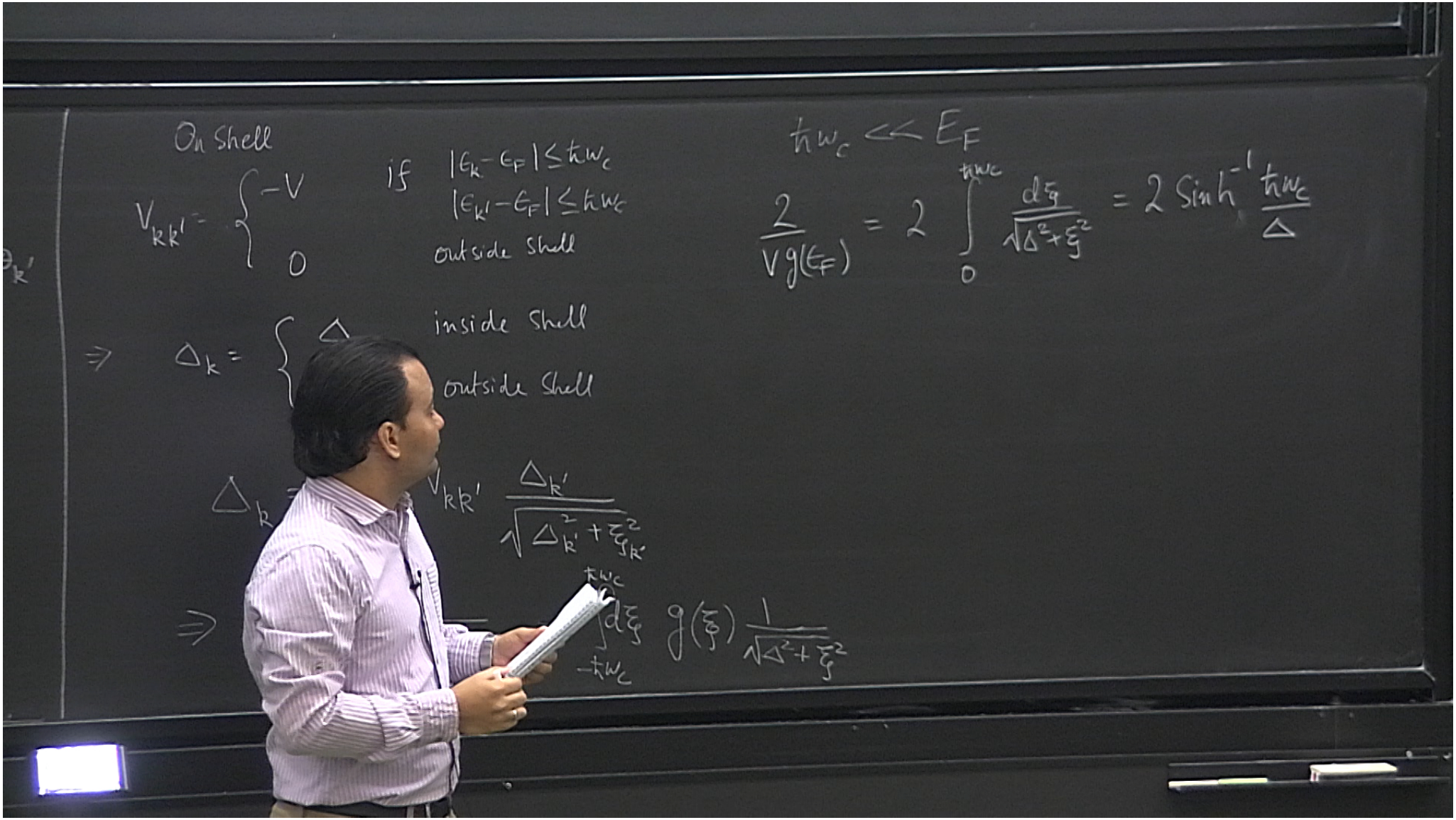
On Shell

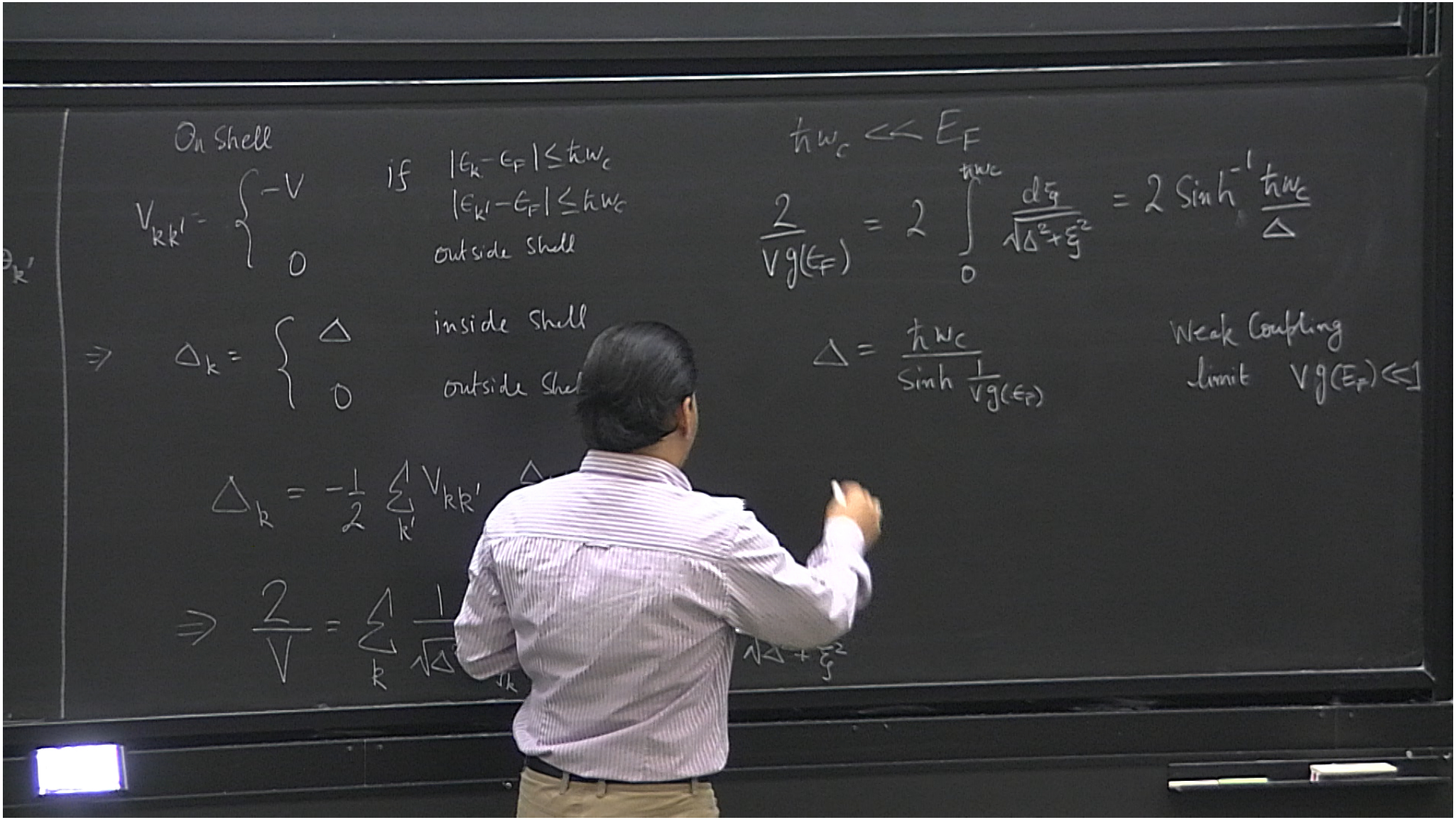
$$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \\ & |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \\ 0 & \text{outside shell} \end{cases}$$

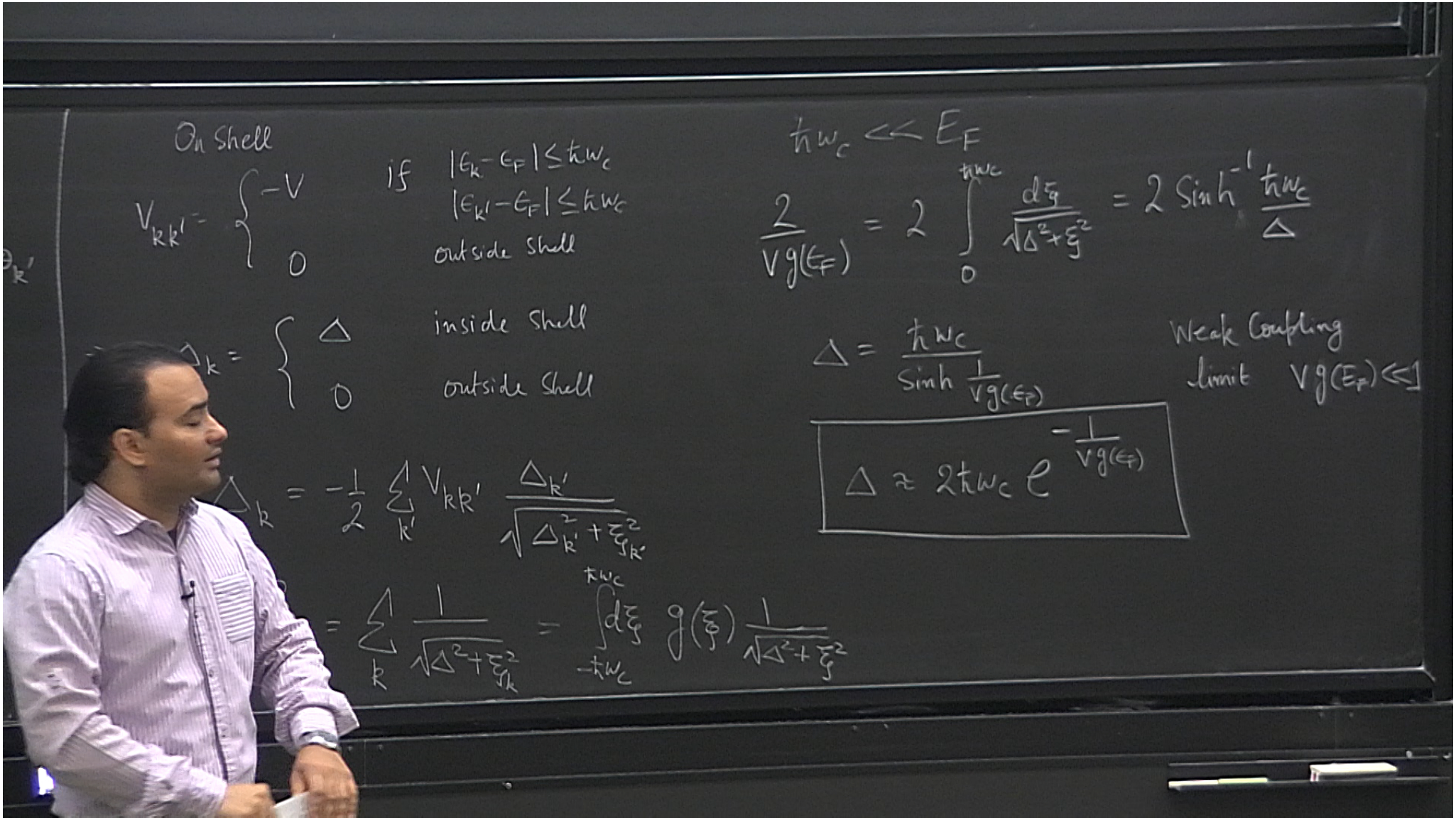
$$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{inside shell} \\ 0 & \text{outside shell} \end{cases}$$

$$\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \xi_{k'}^2}}$$

$$\Rightarrow \frac{2}{V} = \sum_k \frac{1}{\sqrt{\Delta^2 + \xi_k^2}} = \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \; g(\xi) \frac{1}{\sqrt{\Delta^2 + \xi^2}}$$


On Shell
$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \\ 0 & |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \end{cases}$
outside Shell
$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{inside Shell} \\ & \text{outside Shell} \end{cases}$
$\Delta_k = V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \xi_{k'}^2}}$
$\Rightarrow \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \; g(\xi) \frac{1}{\sqrt{\Delta^2 + \xi^2}}$
$\hbar\omega_c \ll E_F$
$\frac{2}{V g(\epsilon_F)} = 2 \int_0^{\hbar\omega_c} \frac{d\xi}{\sqrt{\Delta^2 + \xi^2}} = 2 \sinh^{-1} \frac{\hbar\omega_c}{\Delta}$


On Shell
$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \\ 0 & |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \end{cases}$
outside Shell
$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{inside Shell} \\ 0 & \text{outside Shell} \end{cases}$
$\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'}$
$\Rightarrow \frac{2}{V} = \sum_k$
$\hbar\omega_c \ll E_F$
$\frac{2}{V g(\epsilon_F)} = 2 \int_0^{\hbar\omega_c} \frac{d\xi}{\sqrt{\Delta^2 + \xi^2}} = 2 \sinh^{-1} \frac{\hbar\omega_c}{\Delta}$
$\Delta = \frac{\hbar\omega_c}{\sinh \frac{1}{V g(\epsilon_F)}}$
Weak Coupling
limit $V g(E_F) \ll 1$


On Shell
$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \text{ and } |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \\ 0 & \text{outside shell} \end{cases}$
$\Delta_k = \begin{cases} \Delta & \text{inside shell} \\ 0 & \text{outside shell} \end{cases}$
$\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \xi_{k'}^2}}$
$= \sum_k \frac{1}{\sqrt{\Delta^2 + \xi_k^2}} = \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi\, g(\xi) \frac{1}{\sqrt{\Delta^2 + \xi^2}}$
$\hbar\omega_c \ll E_F$
$\frac{2}{V g(\epsilon_F)} = 2 \int_0^{\hbar\omega_c} \frac{d\xi}{\sqrt{\Delta^2 + \xi^2}} = 2 \sinh^{-1} \frac{\hbar\omega_c}{\Delta}$
$\Delta = \frac{\hbar\omega_c}{\sinh \frac{1}{V g(\epsilon_F)}}$
Weak Coupling limit $V g(E_F) \ll 1$
$\Delta \approx 2\hbar\omega_c e^{-\frac{1}{V g(\epsilon_F)}}$

On Shell

$$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \le \hbar\omega_c \\ & |\epsilon_{k'} - \epsilon_F| \le \hbar\omega_c \\ 0 & \text{outside shell} \end{cases}$$

$$\Rightarrow \Delta_k = \begin{cases} \Delta & \text{inside shell} \\ 0 & \text{outside shell} \end{cases}$$

$$\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \xi_{k'}^2}}$$

$$\Rightarrow \frac{2}{V} = \sum_k \frac{1}{\sqrt{\Delta^2 + \xi_k^2}} = \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \, g(\xi) \frac{1}{\sqrt{\Delta^2 + \xi^2}}$$

$\hbar\omega_c \ll \epsilon_F$

$$\frac{2}{V g(\epsilon_F)} = 2 \int_0^{\hbar\omega_c} \frac{d\xi}{\sqrt{\Delta^2 + \xi^2}} = 2 \sinh^{-1} \frac{\hbar\omega_c}{\Delta}$$

$$\Delta = \frac{\hbar\omega_c}{\sinh \frac{1}{V g(\epsilon_F)}}$$

Weak Coupling limit $V g(E_F) \ll 1$

$$\boxed{\Delta \approx 2\hbar\omega_c \, e^{-\frac{1}{V g(\epsilon_F)}}}$$

$\Delta \ll \hbar\omega_c \ll \epsilon_F$

$$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = 0$$

$$\sqrt{\xi_k^2 + \Delta_k^2}$$

$$v_k^2 - u_k^2 = \frac{-\xi_{pk}}{E_k} \quad \text{and} \quad v_k^2 + u_k^2 = 1$$

$$E_k = \sqrt{\Delta^2 + \xi_{pk}^2}$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$$

$$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = 0$$

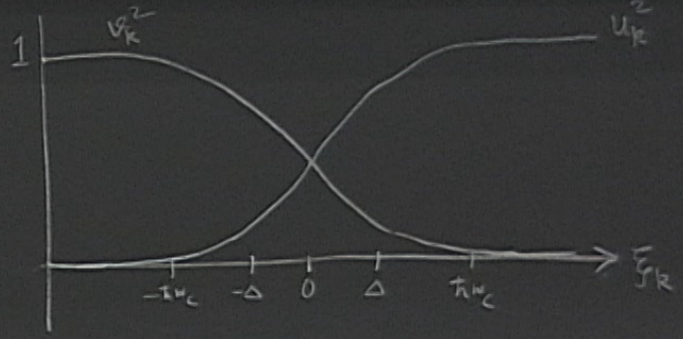
$$\sqrt{\xi_k^2 + \Delta_k^2}$$

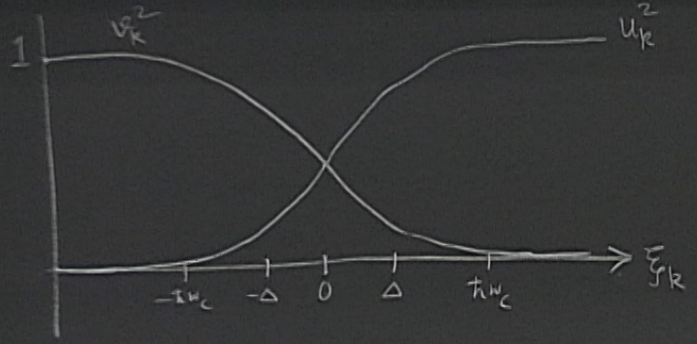
$$v_k^2 - u_k^2 = \frac{-\xi_{pk}}{E_k} \quad \text{and} \quad v_k^2 + u_k^2 = 1$$

$$E_k = \sqrt{\Delta^2 + \xi_{pk}^2}$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$$

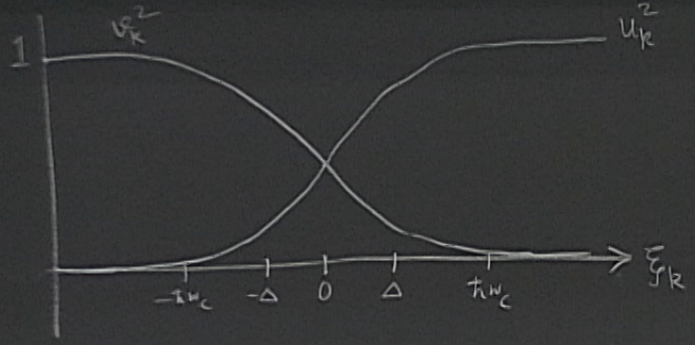
$$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$$





$$\langle c^\dagger_{k\uparrow} c_{k\uparrow} \rangle = v_k^2$$

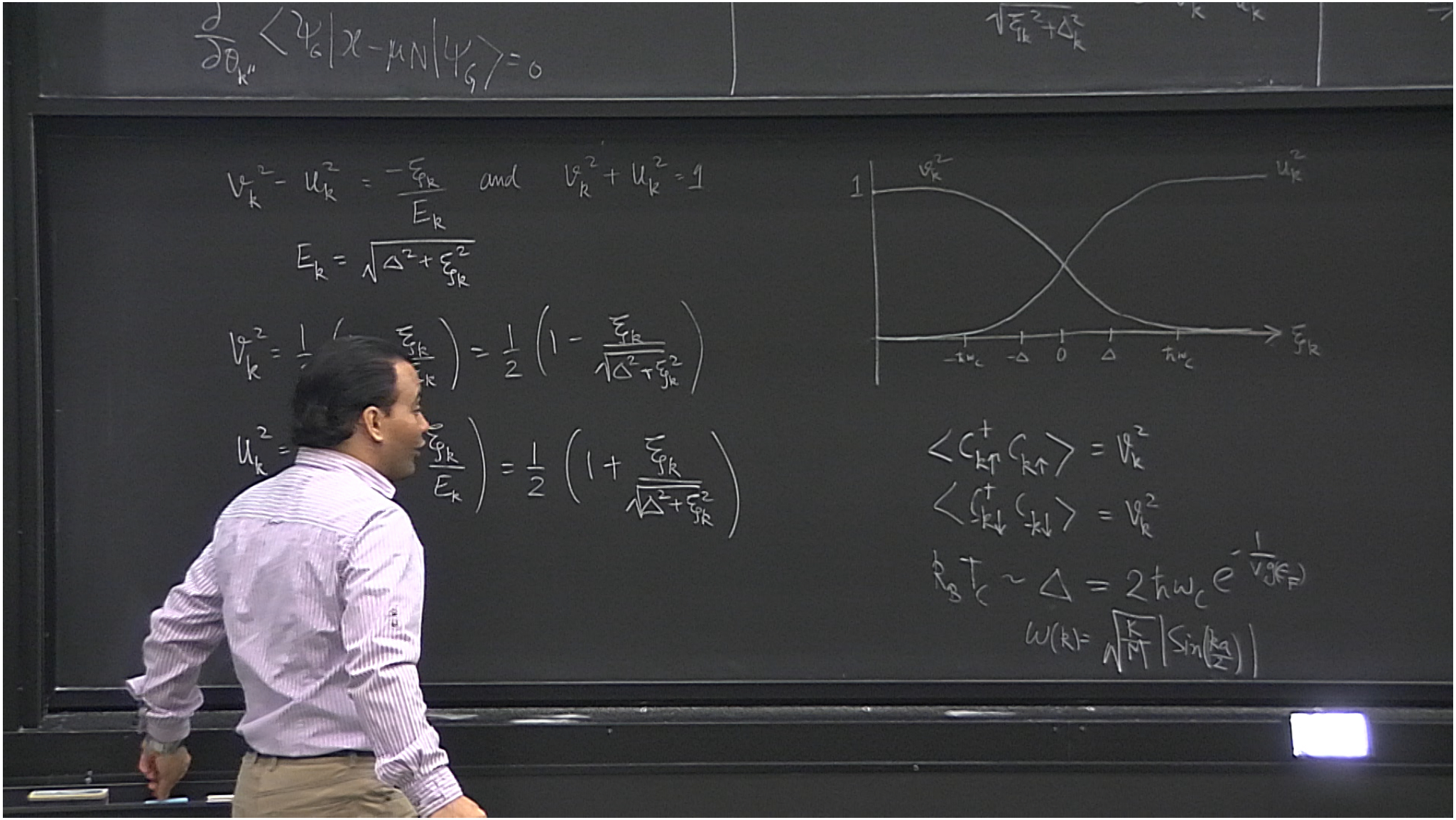
$$\langle c^\dagger_{-k\downarrow} c_{-k\downarrow} \rangle = v_k^2$$

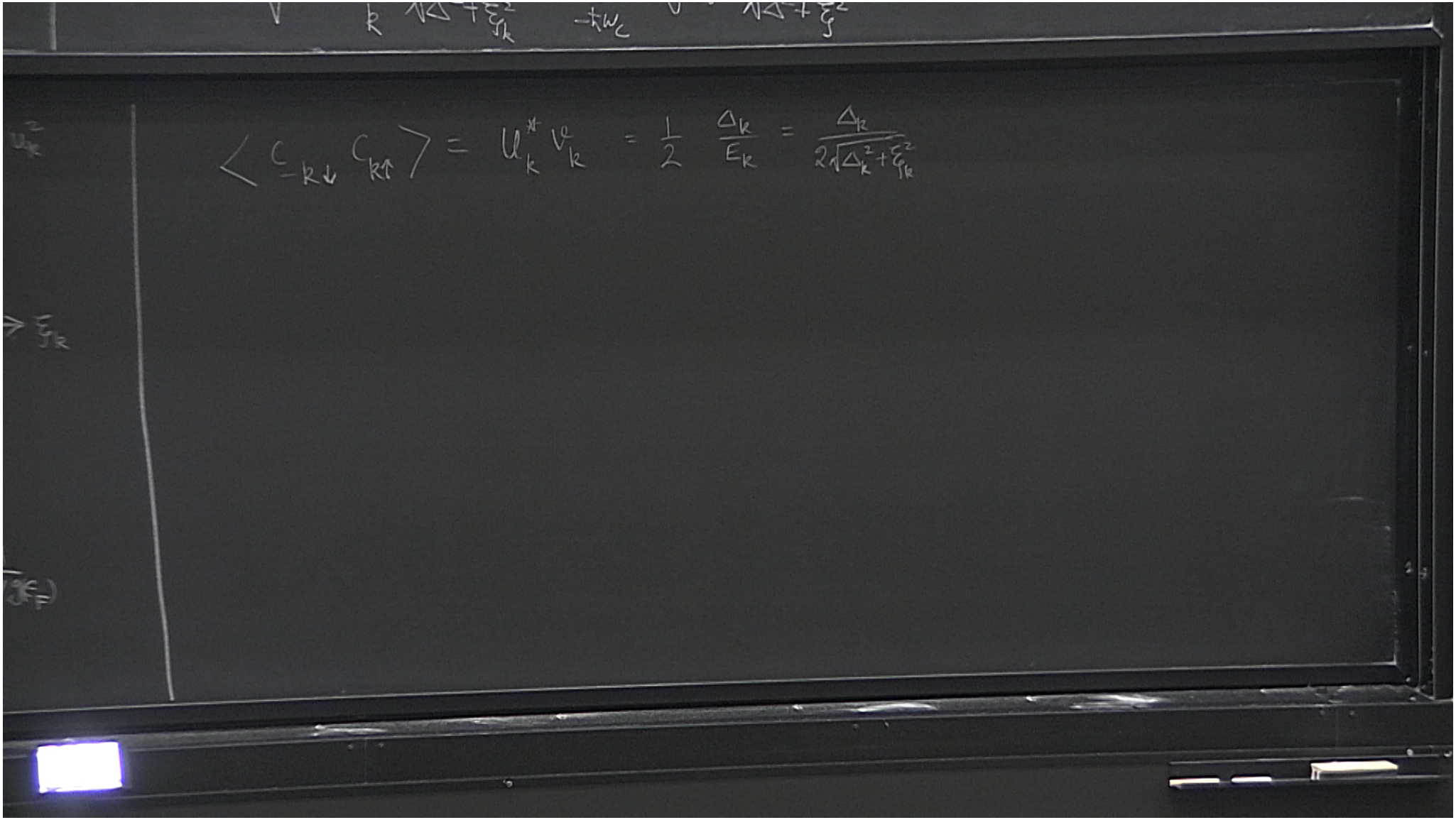


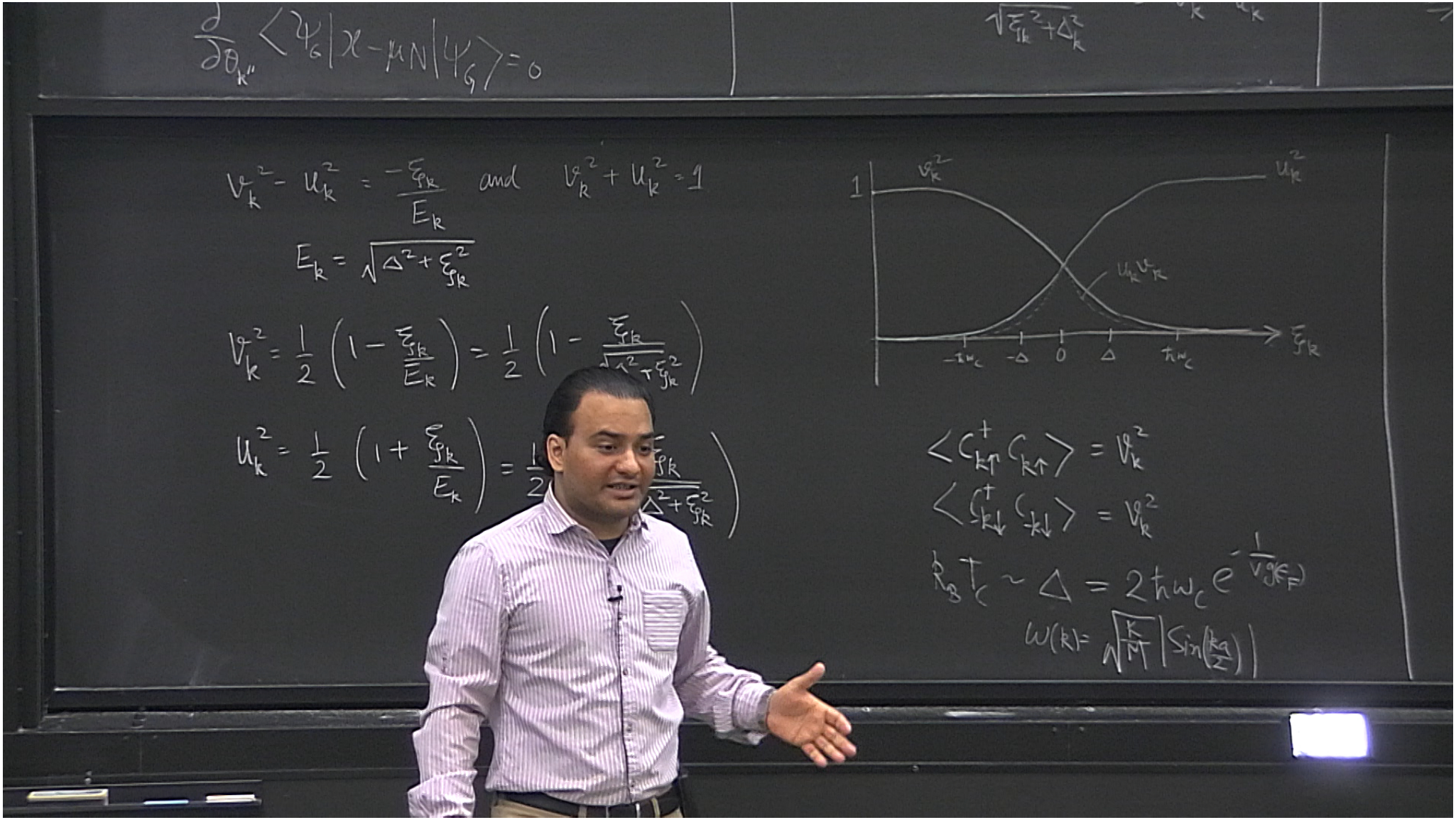
$$\langle c^\dagger_{k\uparrow} c_{k\uparrow} \rangle = v_k^2$$

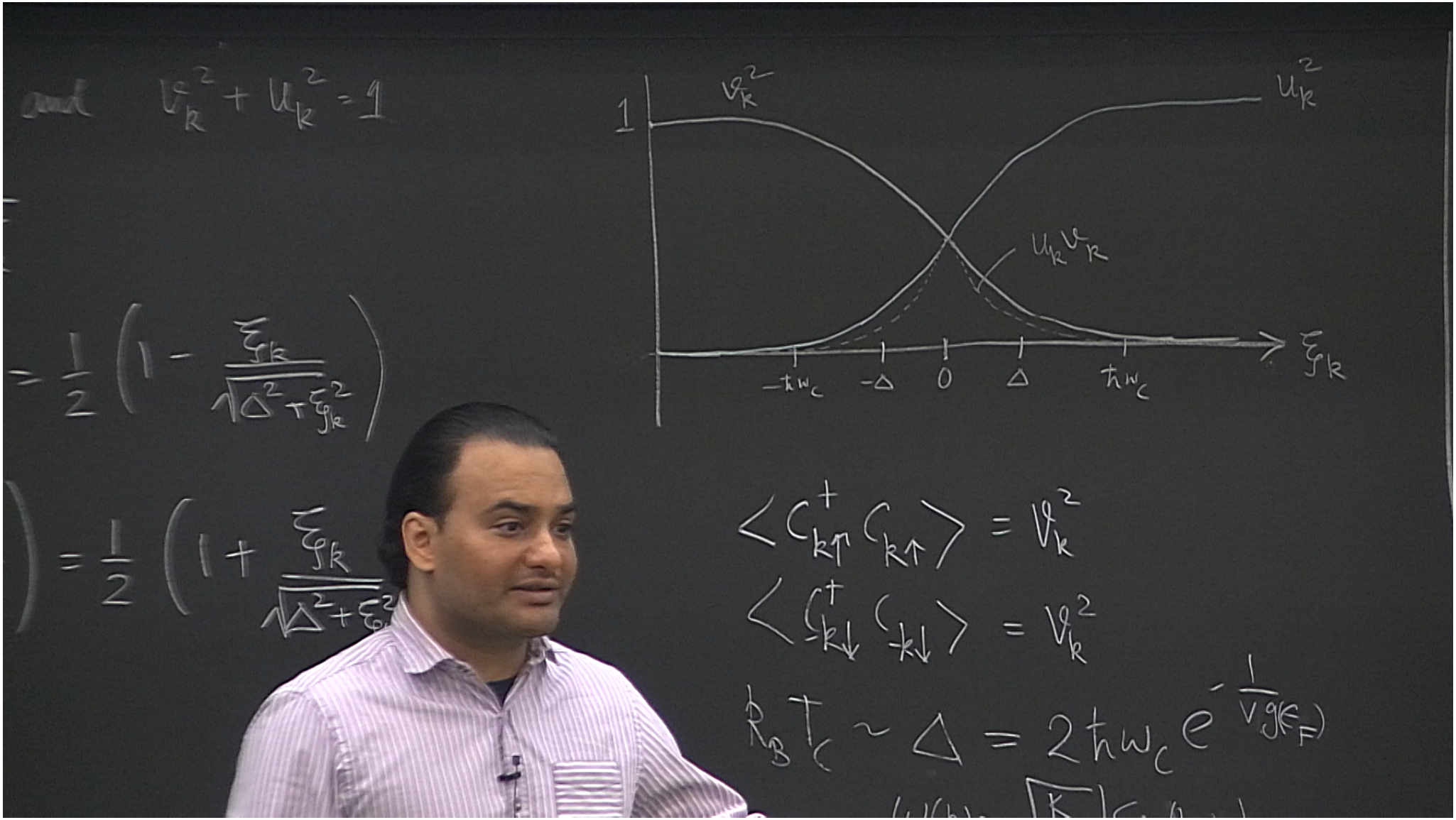
$$\langle c^\dagger_{-k\downarrow} c_{-k\downarrow} \rangle = v_k^2$$

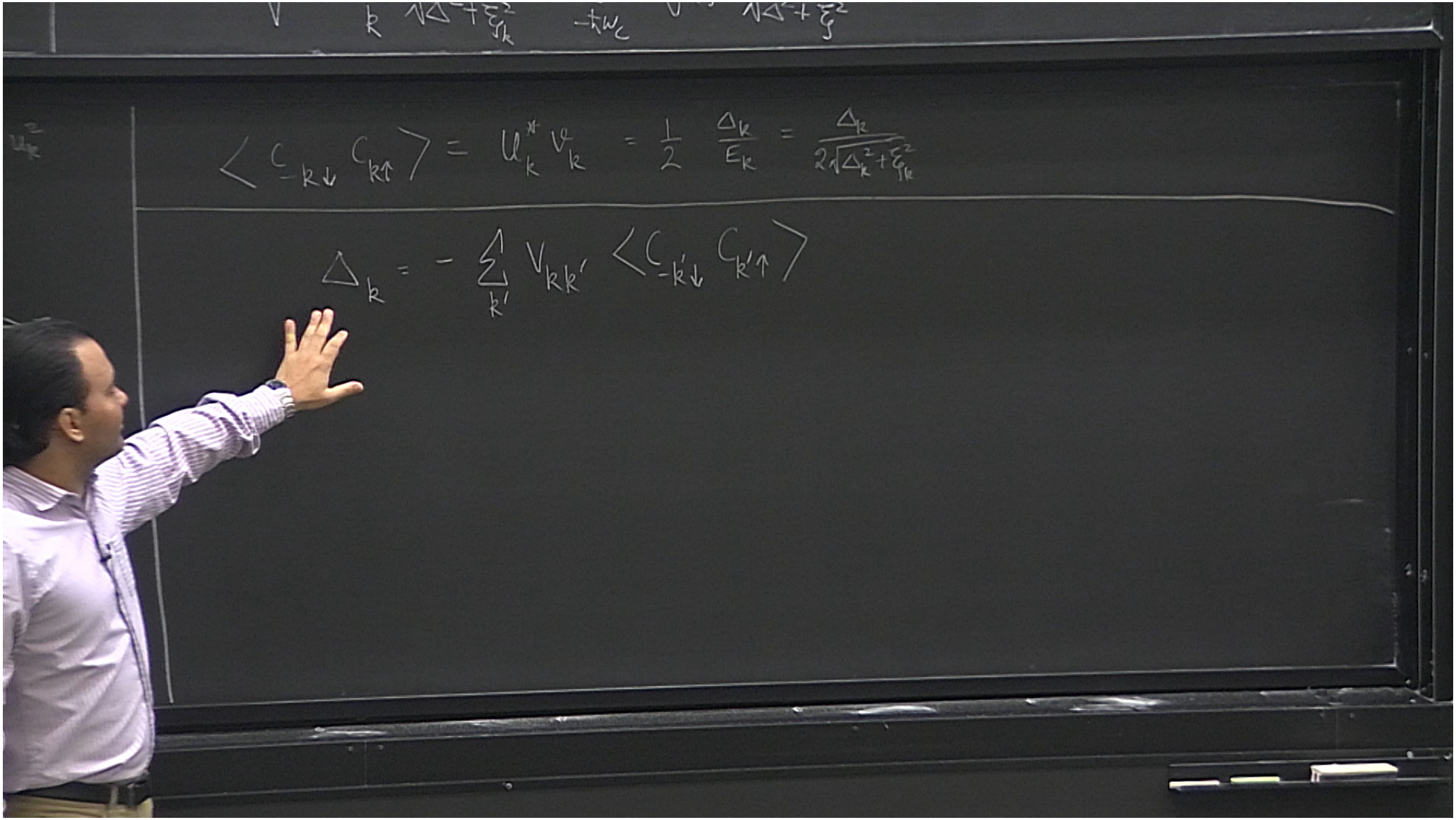
$$k_B T_c \sim \Delta = 2\hbar\omega_c e^{-\frac{1}{V g(E_F)}}$$

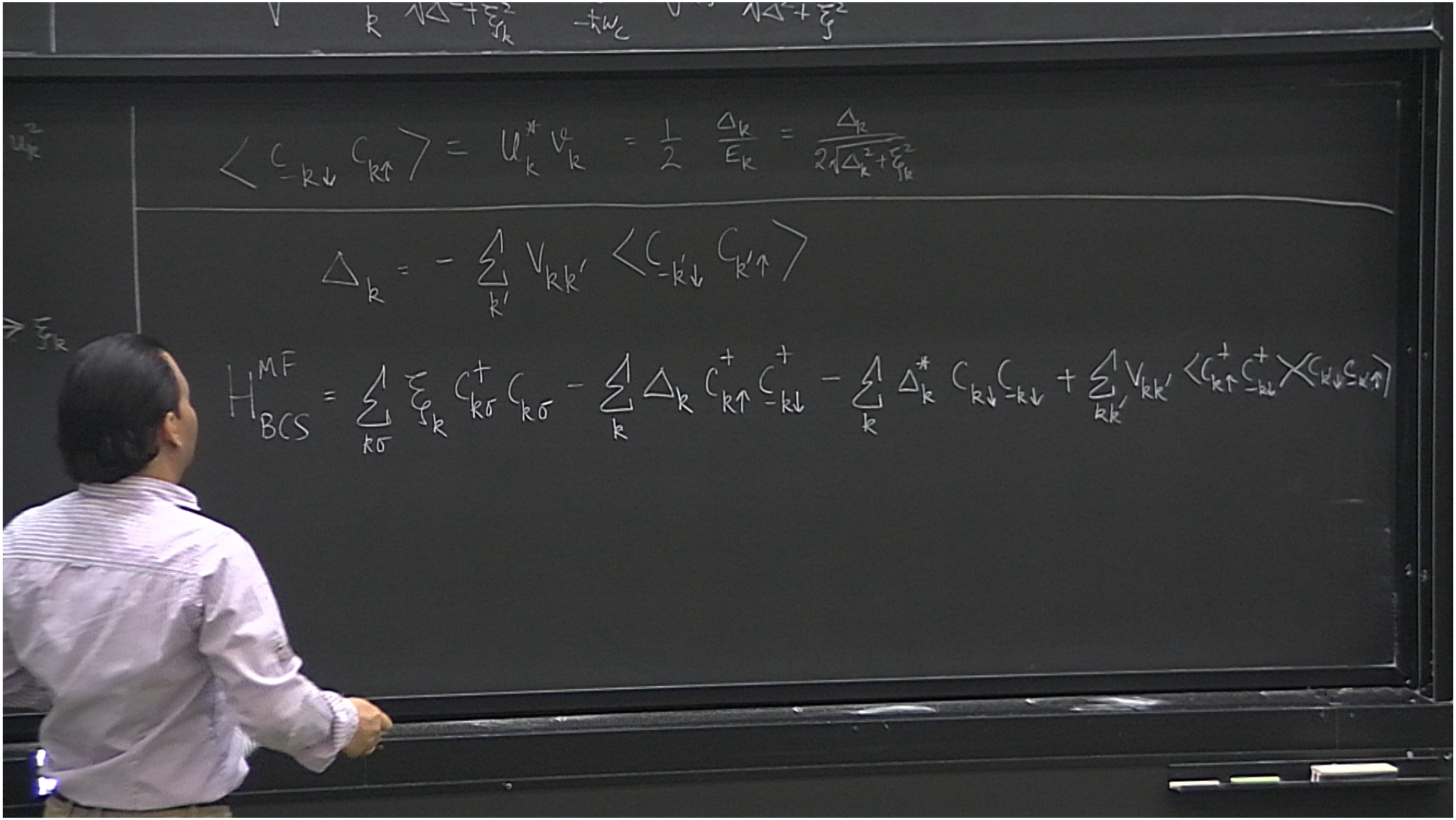

$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = 0$
$\sqrt{\xi_k^2 + \Delta_k^2}$
$v_k^2 - u_k^2 = \frac{-\xi_{pk}}{E_k}$ and $v_k^2 + u_k^2 = 1$
$E_k = \sqrt{\Delta^2 + \xi_{pk}^2}$
$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$
$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$
1
v_k^2
u_k^2
$-\hbar\omega_c$
$-\Delta$
0
Δ
$\hbar\omega_c$
ξ_{pk}
$\langle c_{k\uparrow}^\dagger c_{k\uparrow} \rangle = v_k^2$
$\langle c_{-k\downarrow}^\dagger c_{-k\downarrow} \rangle = v_k^2$
$k_B T_c \sim \Delta = 2\hbar\omega_c e^{-\frac{1}{V_0 g(\epsilon_F)}}$
$\omega(k) = \sqrt{\frac{k}{M}} \left| \sin\left(\frac{ka}{2}\right) \right|$

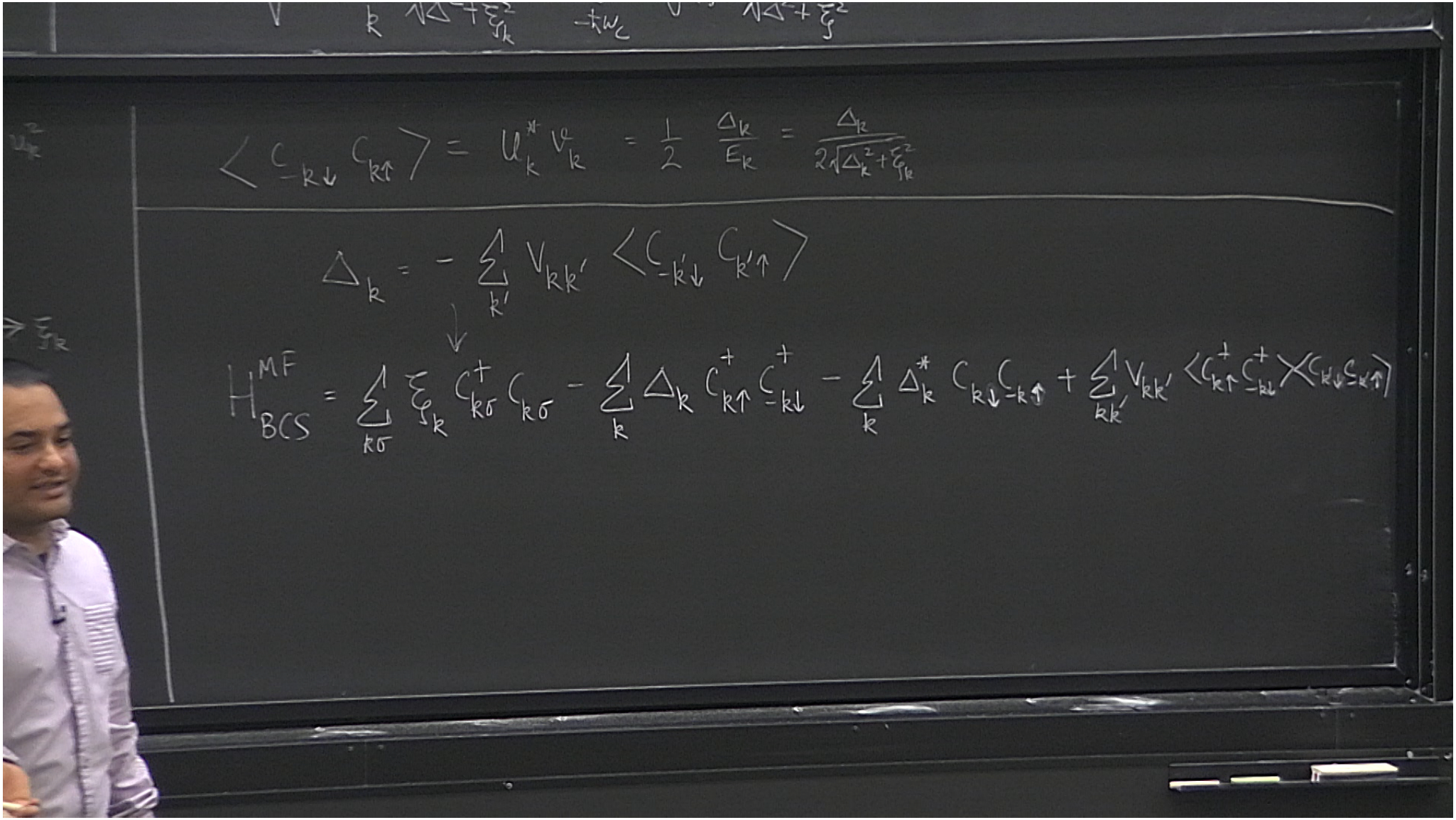

$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$


$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \mathcal{H} - \mu N | \psi_G \rangle = 0$
$\sqrt{\xi_k^2 + \Delta_k^2}$
$v_k^2 - u_k^2 = \frac{-\xi_{pk}}{E_k}$ and $v_k^2 + u_k^2 = 1$
$E_k = \sqrt{\Delta^2 + \xi_{pk}^2}$
$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$
$u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{E_k}\right) = \frac{1}{2}\left(1 + \frac{\xi_{pk}}{\sqrt{\Delta^2 + \xi_{pk}^2}}\right)$
1
v_k^2
u_k^2
$u_k v_k$
$-\hbar\omega_c$
$-\Delta$
0
Δ
$\hbar\omega_c$
ξ_{pk}
$\langle c_{k\uparrow}^\dagger c_{k\uparrow} \rangle = v_k^2$
$\langle c_{-k\downarrow}^\dagger c_{-k\downarrow} \rangle = v_k^2$
$k_B T_c \sim \Delta = 2\hbar\omega_c e^{-\frac{1}{V g(\epsilon_F)}}$
$\omega(k) = \sqrt{\frac{k}{M}} \left| \sin\left(\frac{ka}{2}\right) \right|$


and $v_k^2 + u_k^2 = 1$
$= \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\Delta^2 + \xi_k^2}}\right)$
$= \frac{1}{2}\left(1 + \frac{\xi_k}{\sqrt{\Delta^2 + \xi_k^2}}\right)$
1
v_k^2
u_k^2
$u_k v_k$
$-\hbar\omega_c$
$-\Delta$
0
Δ
$\hbar\omega_c$
ξ_k
$\langle c_{k\uparrow}^\dagger c_{k\uparrow} \rangle = v_k^2$
$\langle c_{-k\downarrow}^\dagger c_{-k\downarrow} \rangle = v_k^2$
$k_B T_c \sim \Delta = 2\hbar\omega_c e^{-\frac{1}{V g(\epsilon_F)}}$


$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$
$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$


$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$
$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$
$H_{BCS}^{MF} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - \sum_k \Delta_k^* c_{k\downarrow} c_{-k\downarrow} + \sum_{kk'} V_{kk'} \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \times c_{k\downarrow} c_{-k'\uparrow} \rangle$


$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$
$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$
$H_{BCS}^{MF} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - \sum_k \Delta_k^* c_{k\downarrow} c_{-k\uparrow} + \sum_{kk'} V_{kk'} \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \times c_{k\downarrow} c_{-k\uparrow} \rangle$

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$$

$$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$$

$$\downarrow$$

$$H_{BCS}^{MF} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - \sum_k \Delta_k^* c_{k\downarrow} c_{-k\uparrow} + \sum_{kk'} V_{kk'} \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \times c_{k\downarrow} c_{-k\uparrow} \rangle$$

$$\downarrow$$ Bogoliubov - Valatin transformation

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle = u_k^* v_k = \frac{1}{2} \frac{\Delta_k}{E_k} = \frac{\Delta_k}{2\sqrt{\Delta_k^2 + \xi_k^2}}$$

$$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$$

$$\downarrow$$

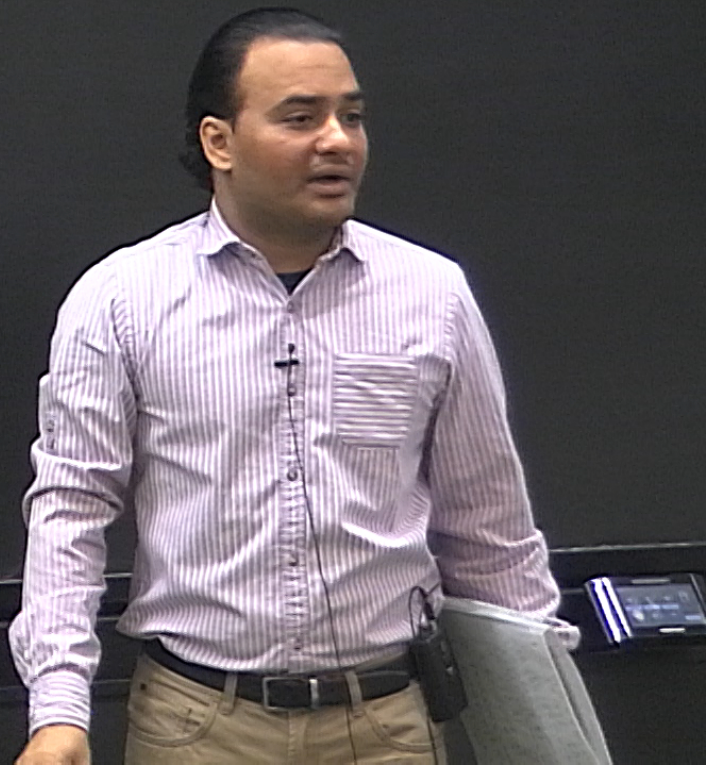
$$H_{BCS}^{MF} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger - \sum_k \Delta_k^* c_{k\downarrow} c_{-k\uparrow} + \sum_{kk'} V_{kk'} \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \times c_{k'\downarrow} c_{-k'\uparrow} \rangle$$

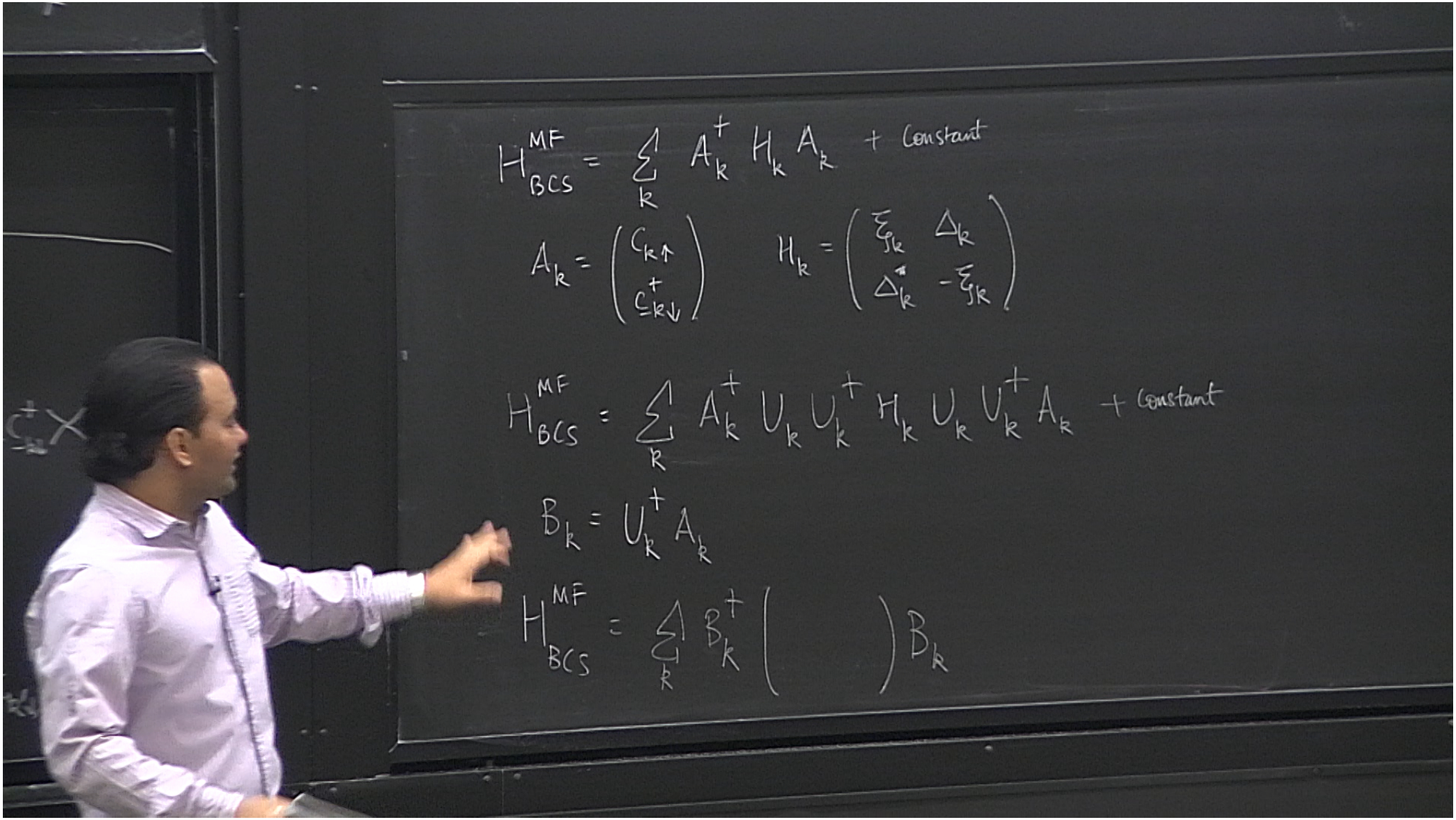
$$\downarrow \text{ Bogoliubov - Valatin transformation}$$

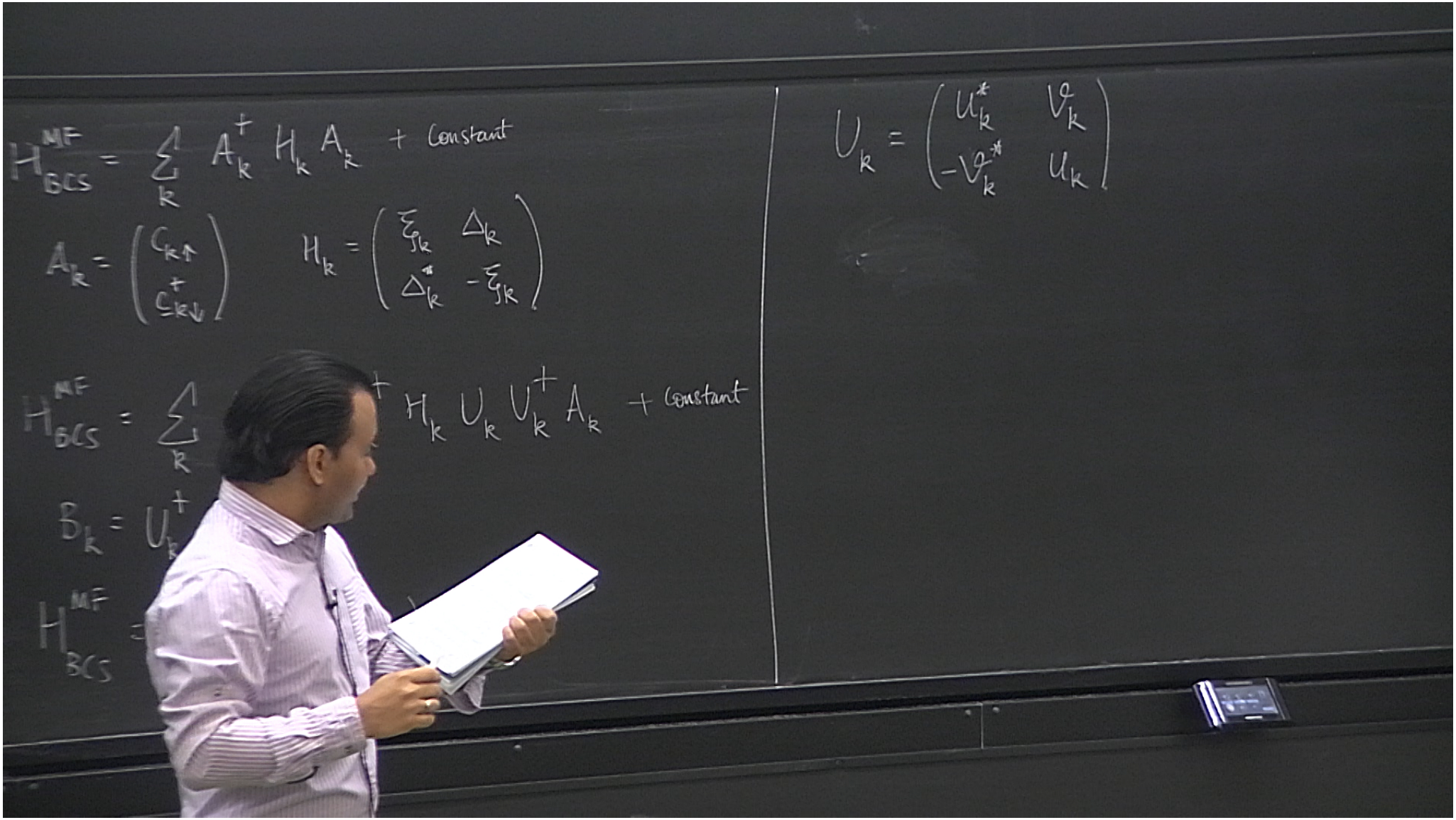
$$H_{BCS}^{MF} = \sum_k \begin{pmatrix} c_{k\uparrow}^\dagger & c_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} + \sum_k \xi_k + \sum_{kk'} V_{kk'} \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle \langle c_{k'\downarrow} c_{-k'\uparrow} \rangle$$

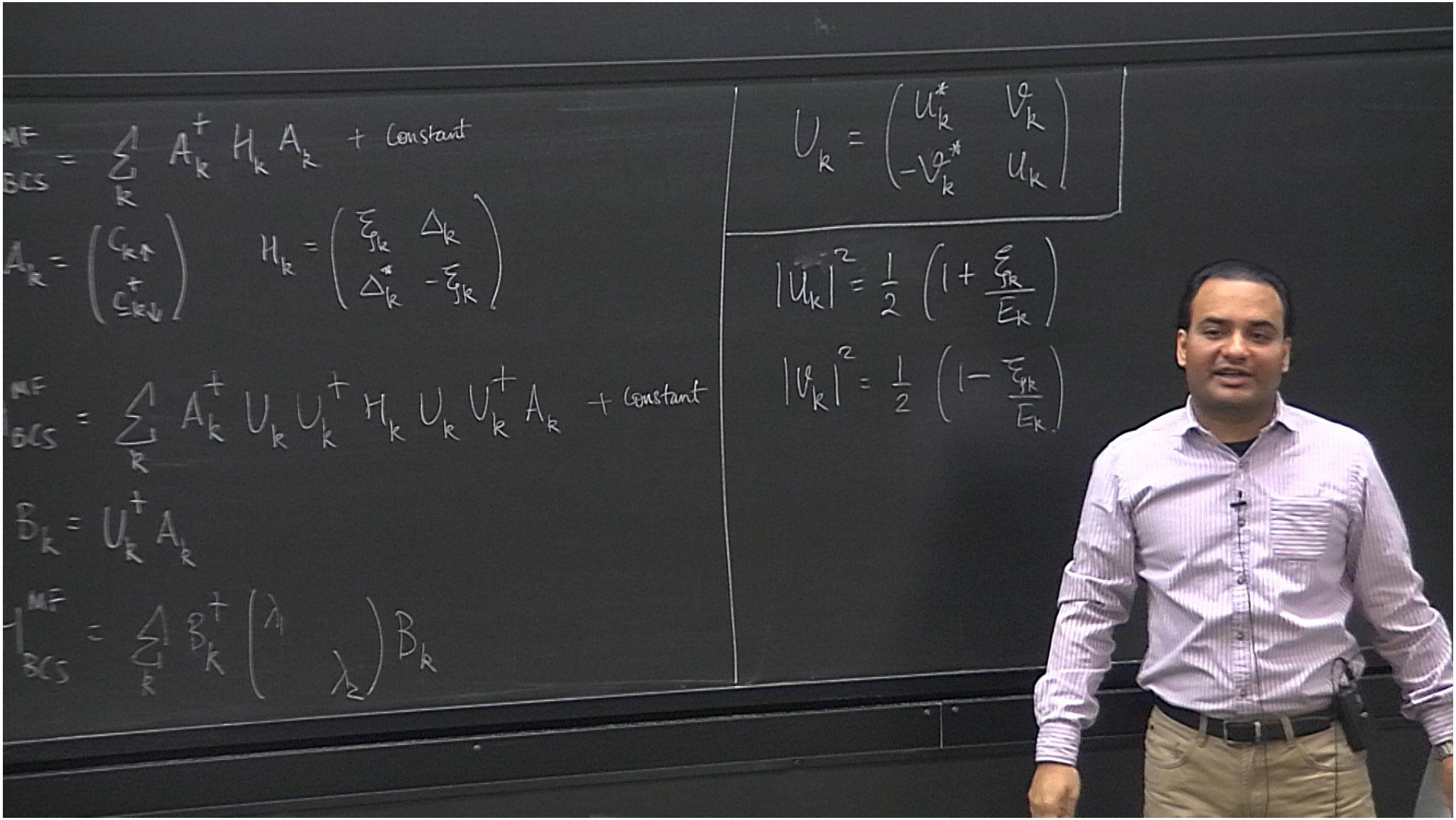
$$H_{BCS}^{MF} = \sum_k A_k^\dagger H_k A_k + \text{Constant}$$

$$A_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} \qquad H_k = \begin{pmatrix} \xi_{jk} & \Delta_k \\ \Delta_k^* & -\xi_{jk} \end{pmatrix}$$




$H_{BCS}^{MF} = \sum_k A_k^\dagger H_k A_k$ + Constant
$A_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$
$H_k = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}$
$H_{BCS}^{MF} = \sum_k A_k^\dagger U_k U_k^\dagger H_k U_k U_k^\dagger A_k$ + Constant
$B_k = U_k^\dagger A_k$
$H_{BCS}^{MF} = \sum_k B_k^\dagger \left(\quad \right) B_k$


$H_{BCS}^{MF} = \sum_k A_k^\dagger H_k A_k + \text{Constant}$
$A_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix}$
$H_k = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}$
$U_k = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix}$
$H_{BCS}^{MF} = \sum_k \; + H_k U_k U_k^\dagger A_k + \text{Constant}$
$B_k = U_k^\dagger$
$H_{BCS}^{MF} =$


$H^{MF}_{BCS} = \sum_k A_k^\dagger H_k A_k + \text{Constant}$
$A_k = \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}$
$H_k = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}$
$H^{MF}_{BCS} = \sum_k A_k^\dagger U_k U_k^\dagger H_k U_k U_k^\dagger A_k + \text{Constant}$
$B_k = U_k^\dagger A_k$
$H^{MF}_{BCS} = \sum_k B_k^\dagger \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} B_k$
$U_k = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix}$
$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$
$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$

$$H_{BCS}^{MF} = \sum_k A_k^\dagger H_k A_k + \text{Constant}$$

$$A_k = \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} \qquad H_k = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix}$$

$$H_{BCS}^{MF} = \sum_k A_k^\dagger U_k U_k^\dagger H_k U_k U_k^\dagger A_k + \text{Constant}$$

$$B_k = U_k^\dagger A_k$$

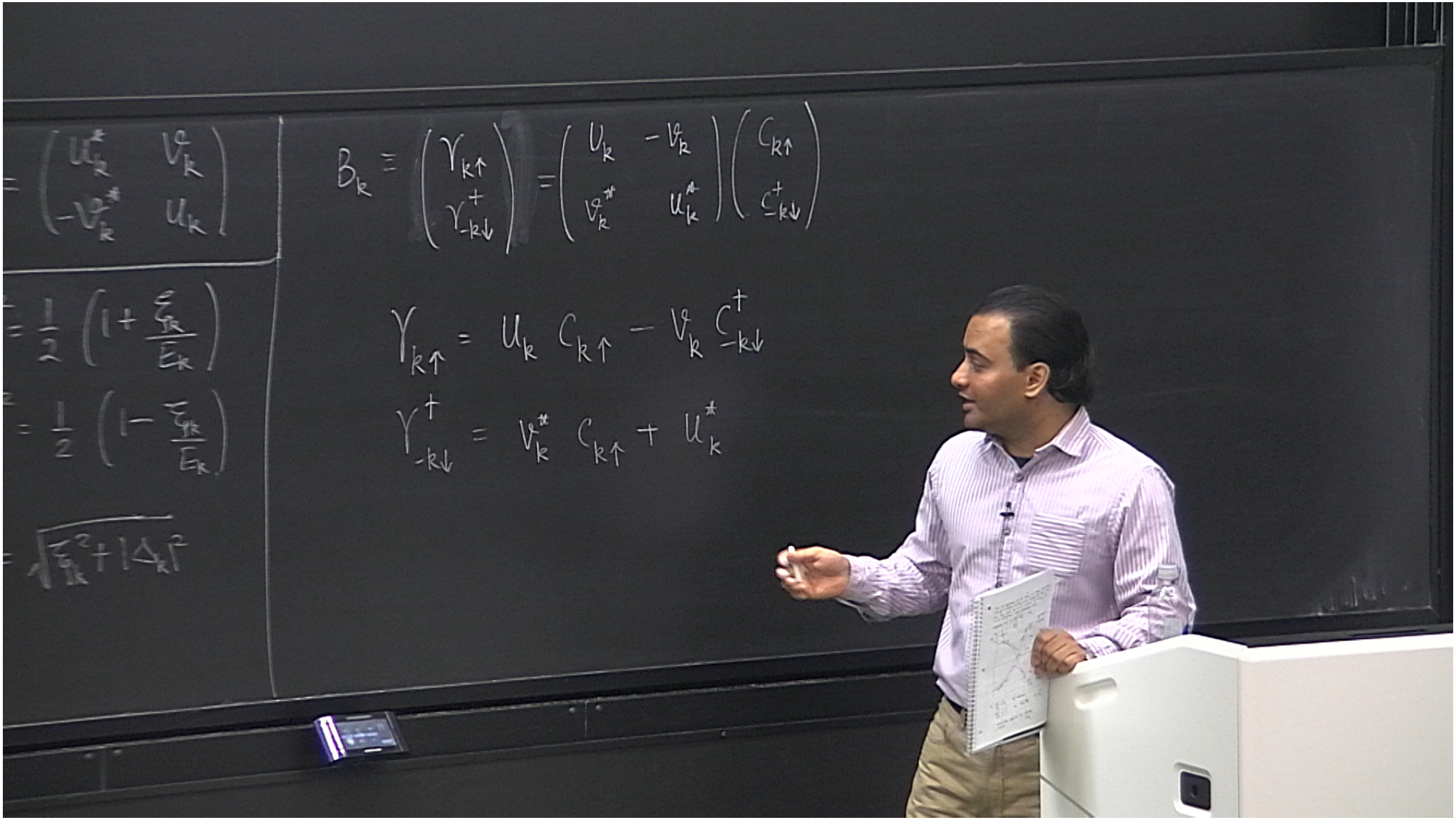
$$H_{BCS}^{MF} = \sum_k B_k^\dagger \begin{pmatrix} E_k & 0 \\ 0 & -E_k \end{pmatrix} B_k$$

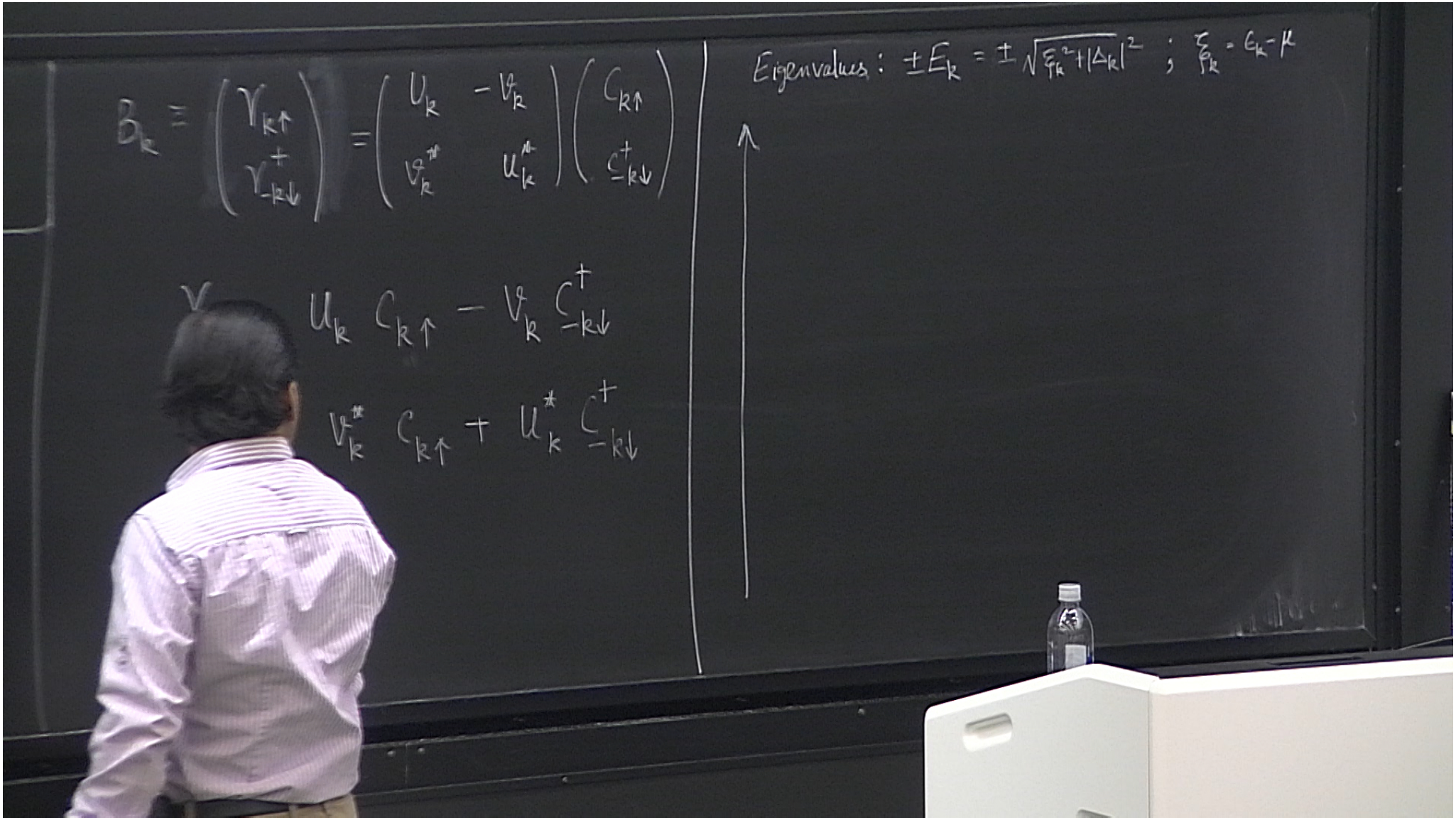
$$U_k = \begin{pmatrix} u_k^* & v_k \\ -v_k^* & u_k \end{pmatrix}$$

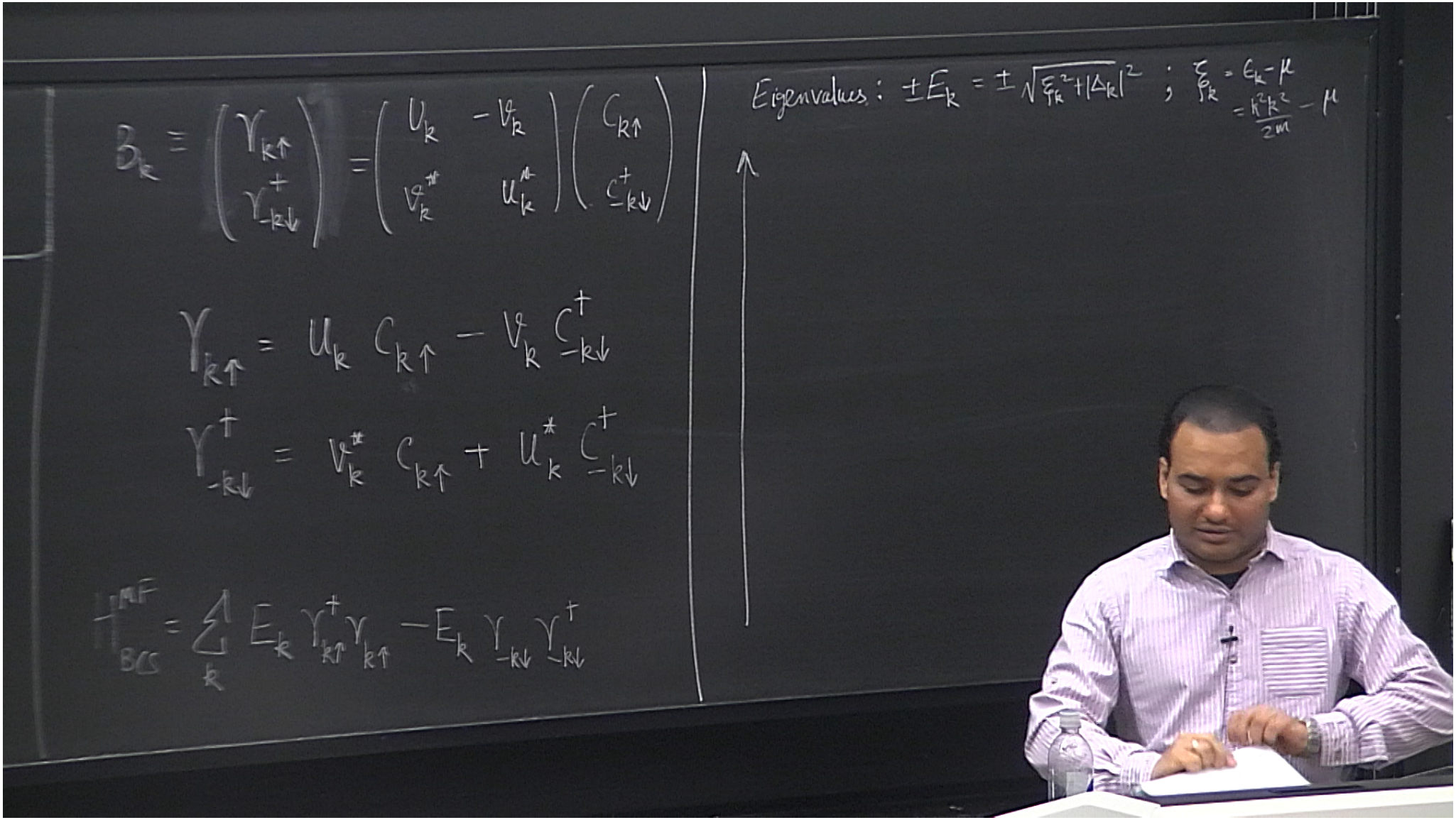
$$|u_k|^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$$

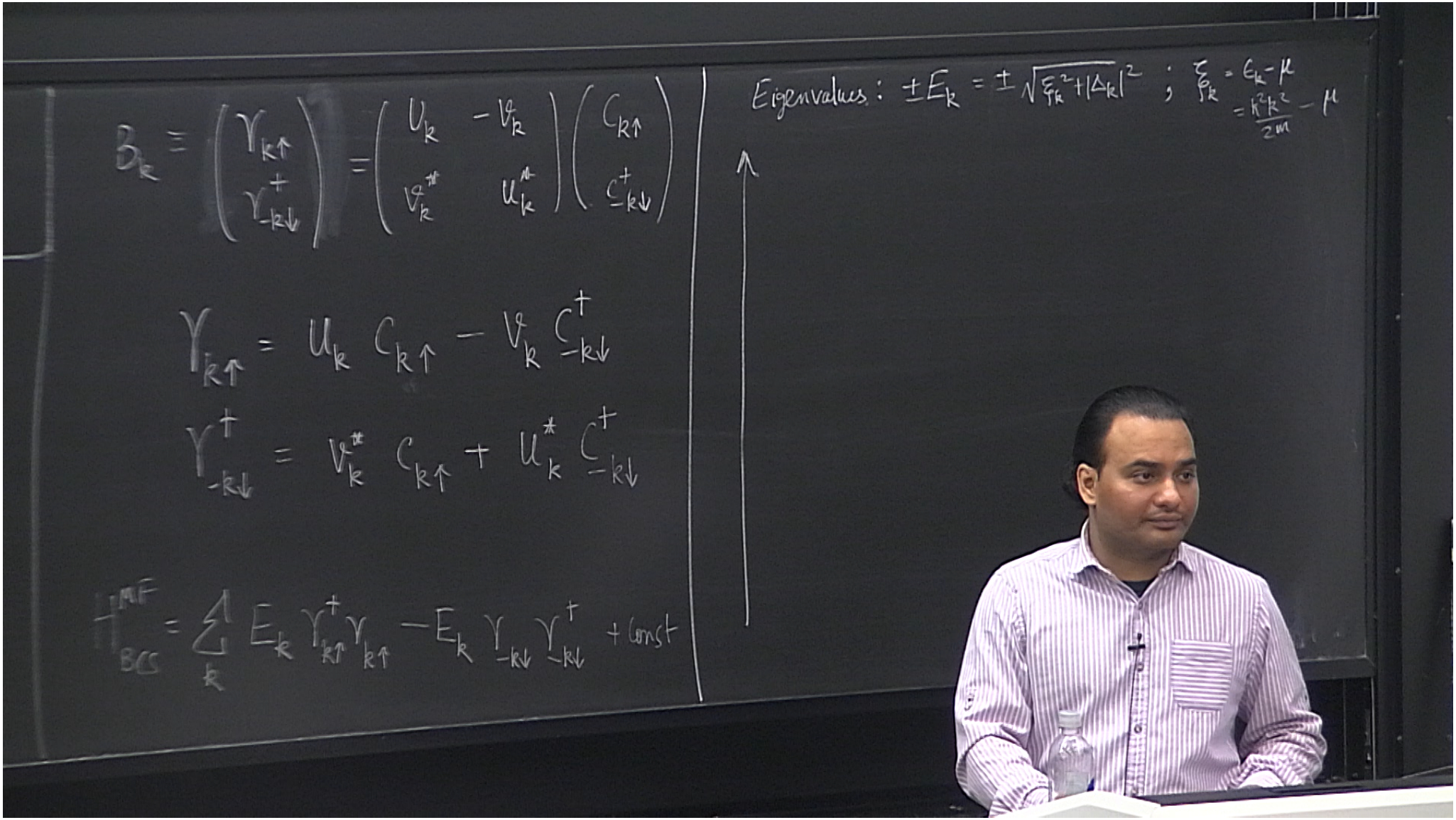
$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

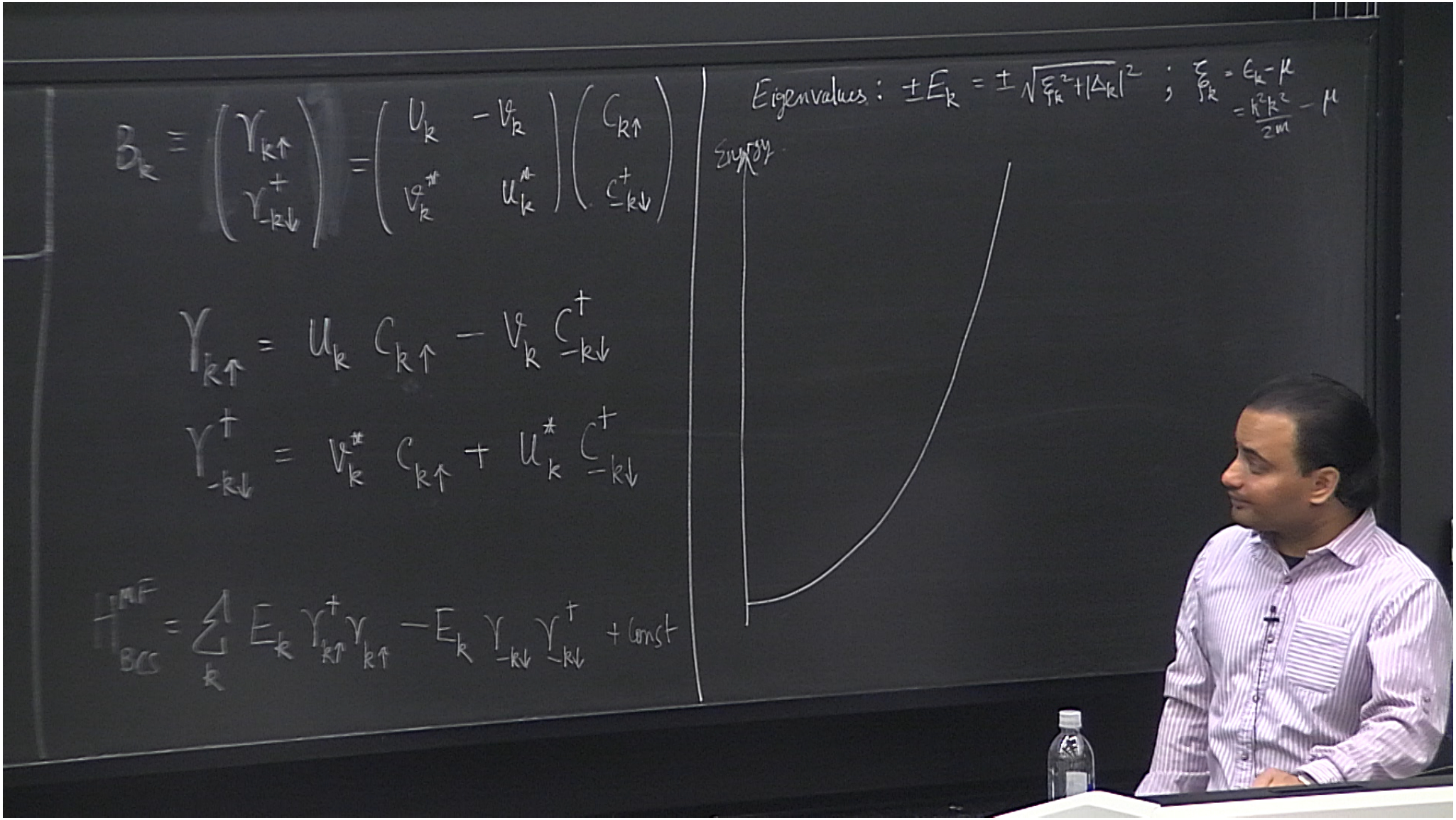
$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$


$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^* & u_k^* \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}$
$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}$
$\gamma^\dagger_{-k\downarrow} = v_k^* c_{k\uparrow} + u_k^*$


$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$
$u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$
$v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$
Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu$


$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$
$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^{+}_{-k\downarrow}$
$\gamma^{+}_{-k\downarrow} = v_k^{*} c_{k\uparrow} + u_k^{*} c^{+}_{-k\downarrow}$
$H^{MF}_{BCS} = \sum_k E_k \gamma^{+}_{k\uparrow}\gamma_{k\uparrow} - E_k \gamma_{-k\downarrow}\gamma^{+}_{-k\downarrow}$
Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$


$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$
$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^{+}_{-k\downarrow}$
$\gamma^{+}_{-k\downarrow} = v_k^{*} c_{k\uparrow} + u_k^{*} c^{+}_{-k\downarrow}$
$H^{MF}_{BCS} = \sum_k E_k \gamma^{+}_{k\uparrow}\gamma_{k\uparrow} - E_k \gamma_{-k\downarrow}\gamma^{+}_{-k\downarrow} + \text{Const}$
Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$


$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$
$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^{+}_{-k\downarrow}$
$\gamma^{+}_{-k\downarrow} = v_k^{*} c_{k\uparrow} + u_k^{*} c^{+}_{-k\downarrow}$
$H^{MF}_{BCS} = \sum_k E_k \gamma^{+}_{k\uparrow}\gamma_{k\uparrow} - E_k \gamma_{-k\downarrow}\gamma^{+}_{-k\downarrow} + \text{Const}$
Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$
Energy

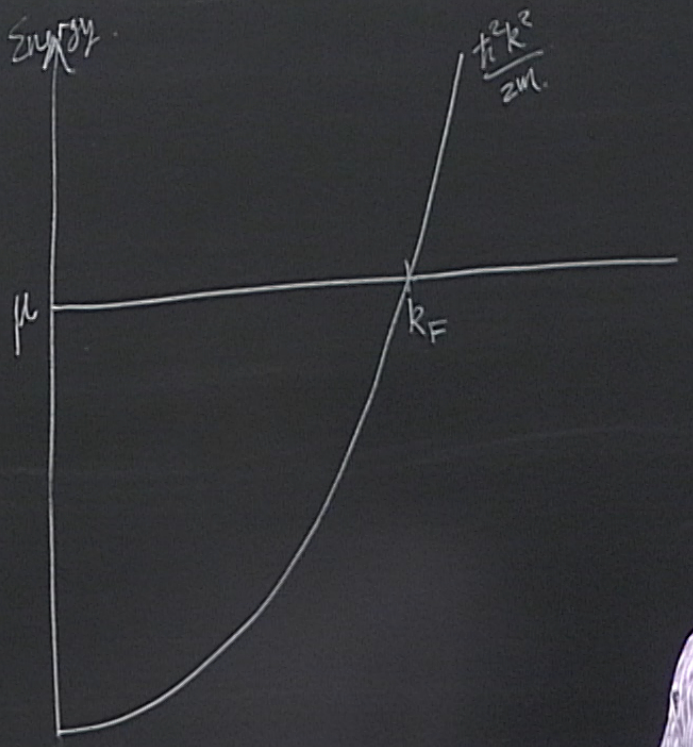
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow} + \text{Const}$$

Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow} + \text{Const}$$

Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



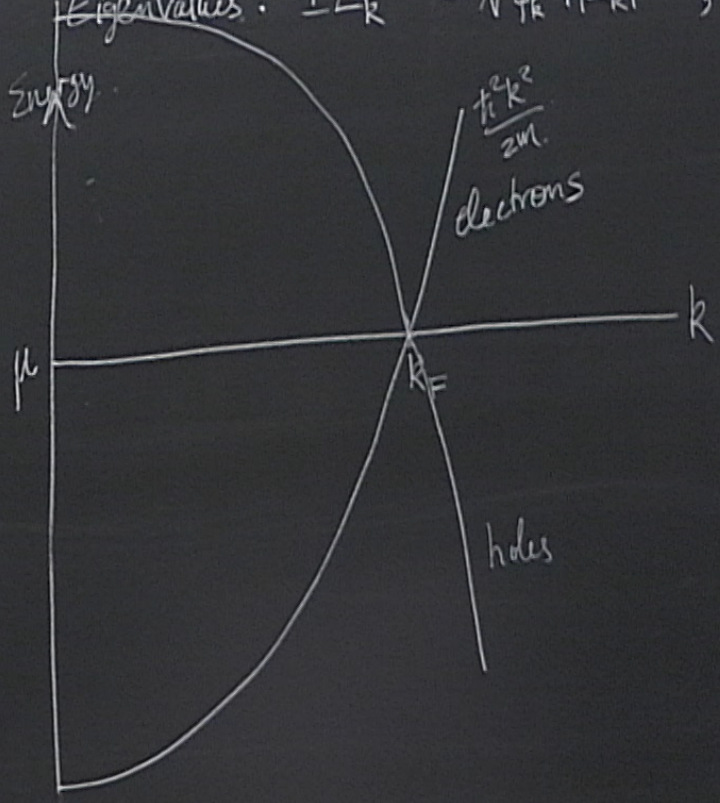
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow} + \text{Const}$$

Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



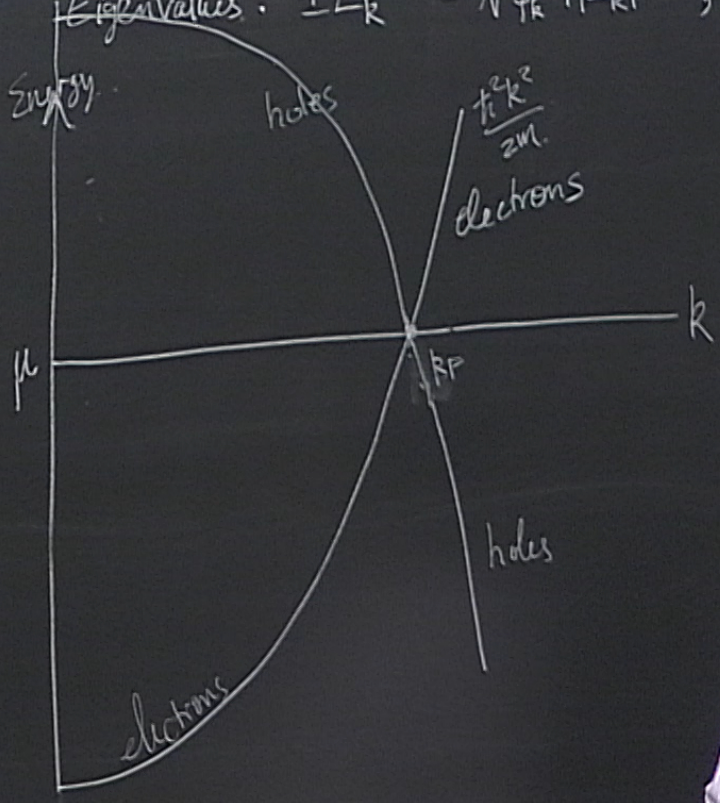
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} U_k & -V_k \\ V_k^{*} & U_k^{*} \end{pmatrix} \begin{pmatrix} C_{k\uparrow} \\ C^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = U_k C_{k\uparrow} - V_k C^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = V_k^{*} C_{k\uparrow} + U_k^{*} C^{\dagger}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k \gamma^{\dagger}_{k\uparrow} \gamma_{k\uparrow} - E_k \gamma_{-k\downarrow} \gamma^{\dagger}_{-k\downarrow} + \text{Const}$$

Eigenvalues: $\pm E_k = \pm \sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



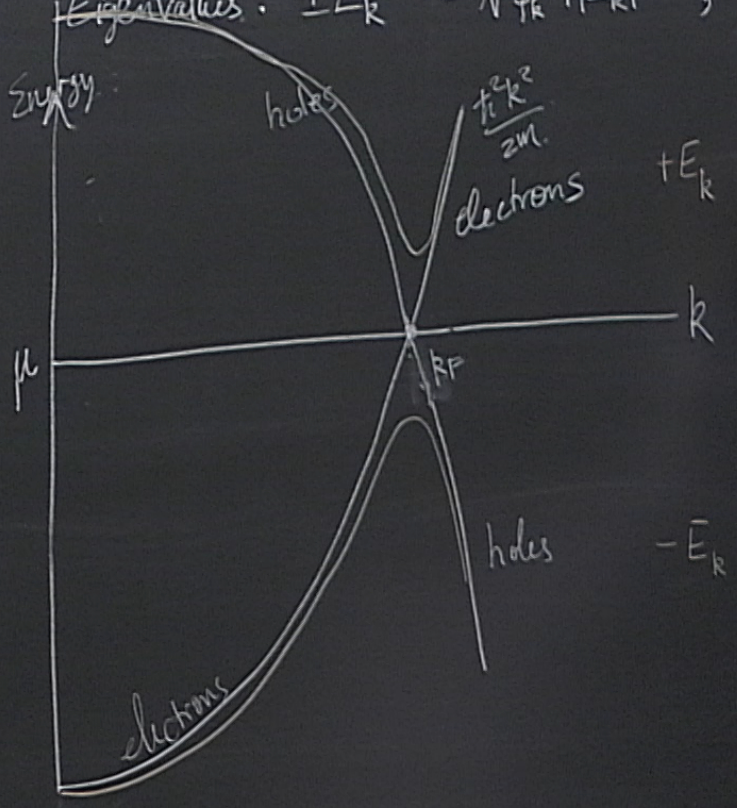
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow} + \text{const}$$

Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



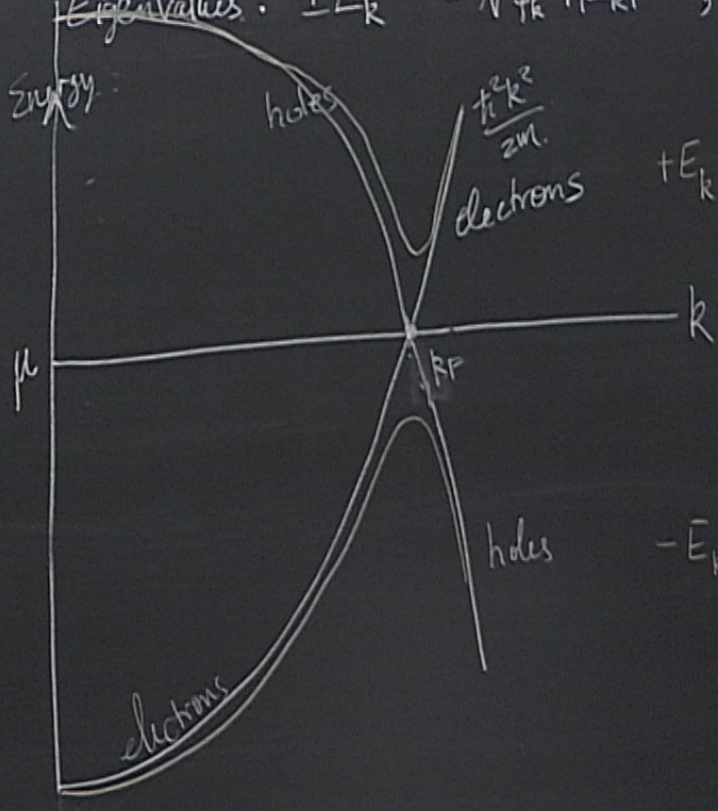
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{+}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{+}_{-k\downarrow}$$

$$\gamma^{+}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{+}_{-k\downarrow}$$

$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{+}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{+}_{-k\downarrow} + \text{Const}$$

Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$



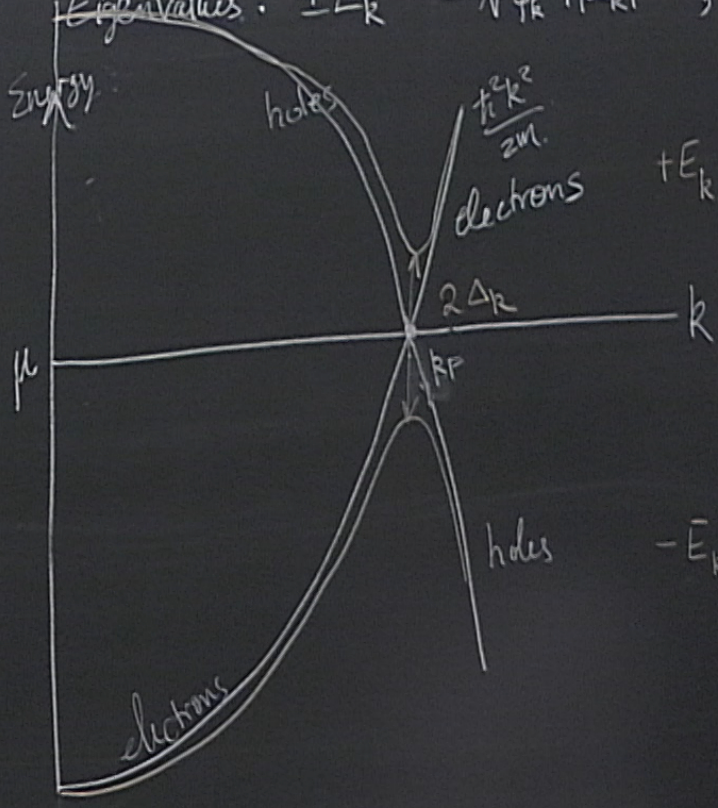
$$B_k \equiv \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^{\dagger}_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ v_k^{*} & u_k^{*} \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^{\dagger}_{-k\downarrow} \end{pmatrix}$$

$$\gamma_{k\uparrow} = u_k\, c_{k\uparrow} - v_k\, c^{\dagger}_{-k\downarrow}$$

$$\gamma^{\dagger}_{-k\downarrow} = v_k^{*}\, c_{k\uparrow} + u_k^{*}\, c^{\dagger}_{-k\downarrow}$$

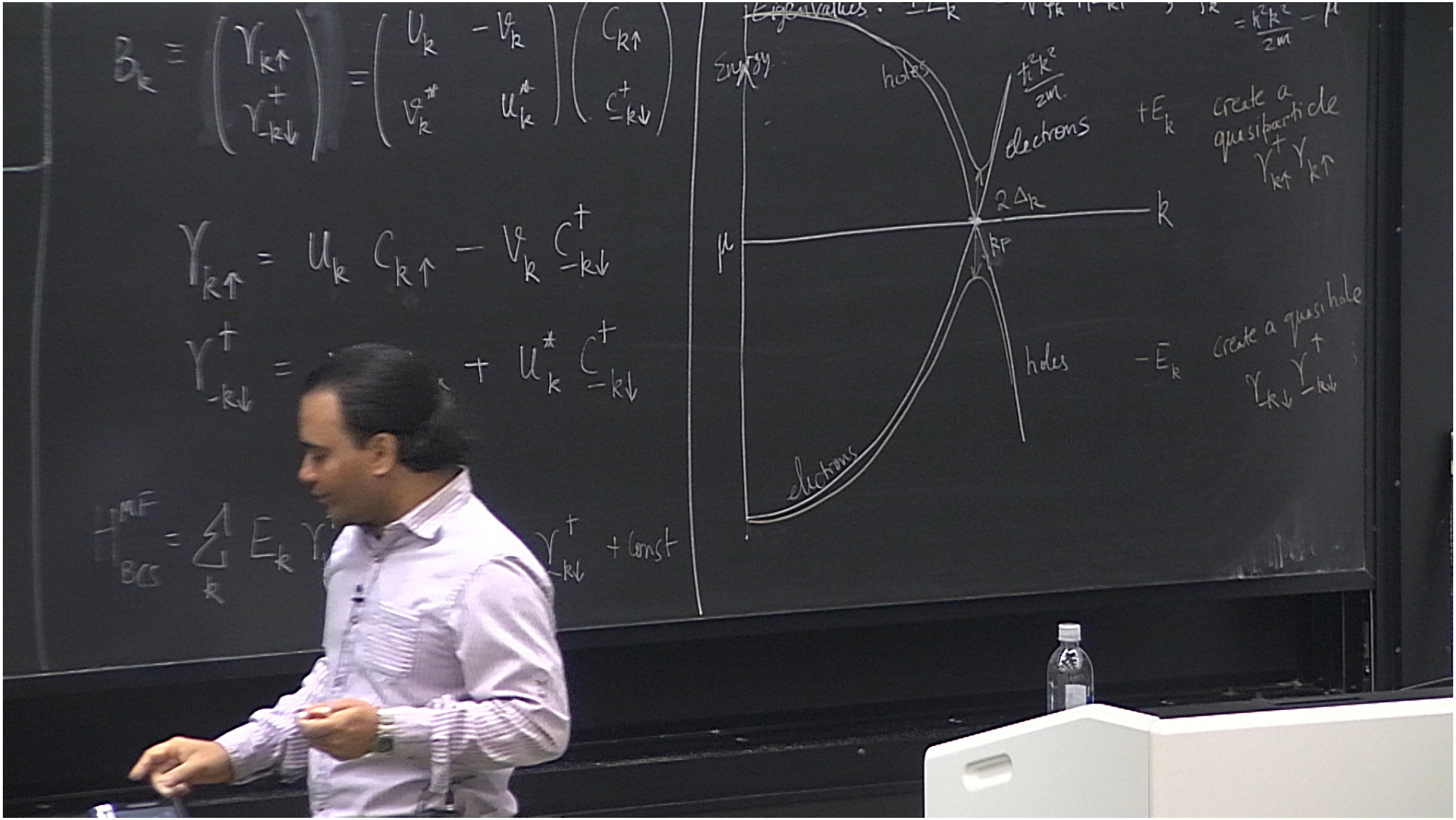
$$H^{MF}_{BCS} = \sum_k E_k\, \gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow} - E_k\, \gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow} + \text{const}$$

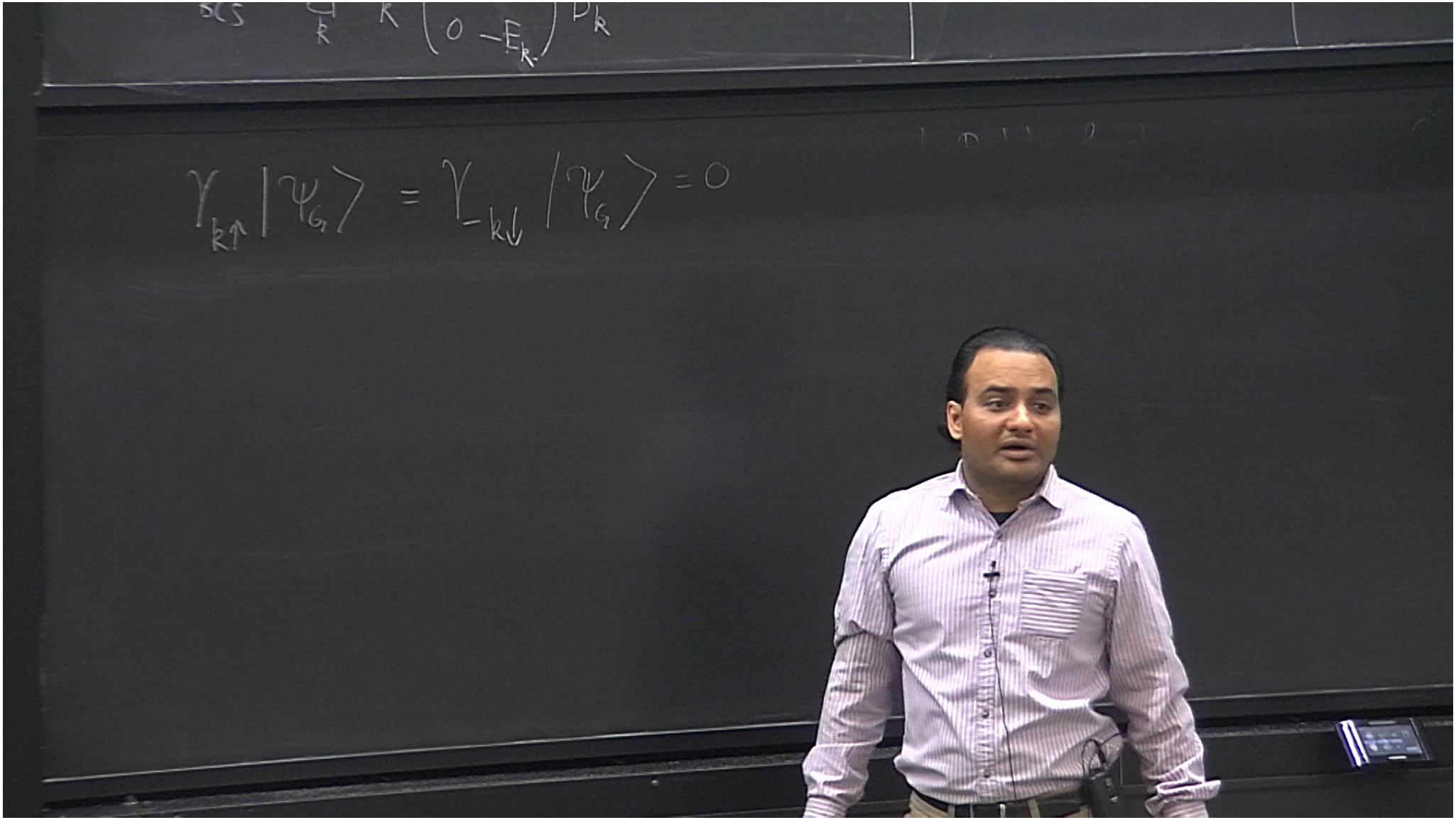
Eigenvalues: $\pm E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}$; $\xi_k = \epsilon_k - \mu = \frac{\hbar^2 k^2}{2m} - \mu$

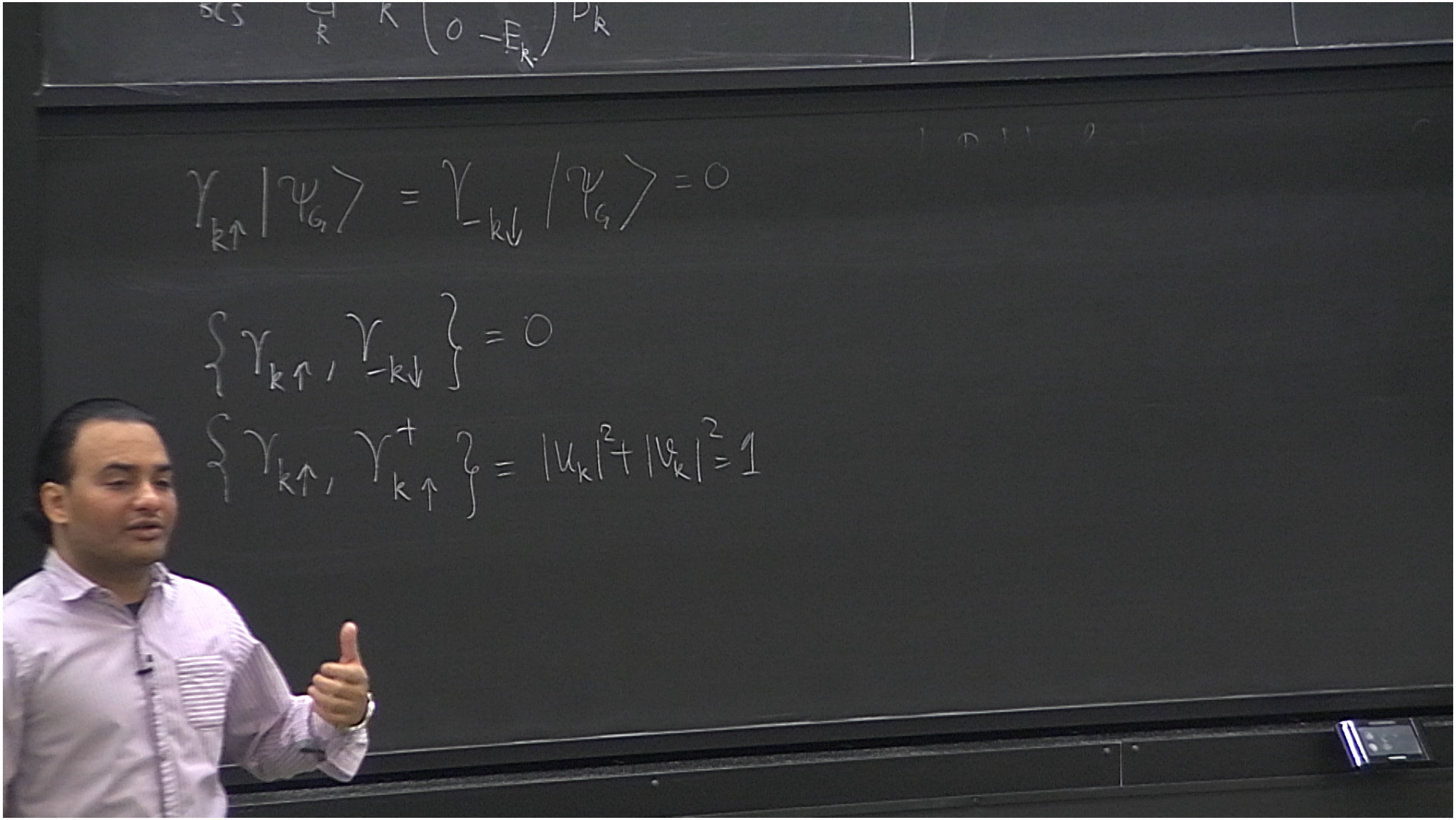


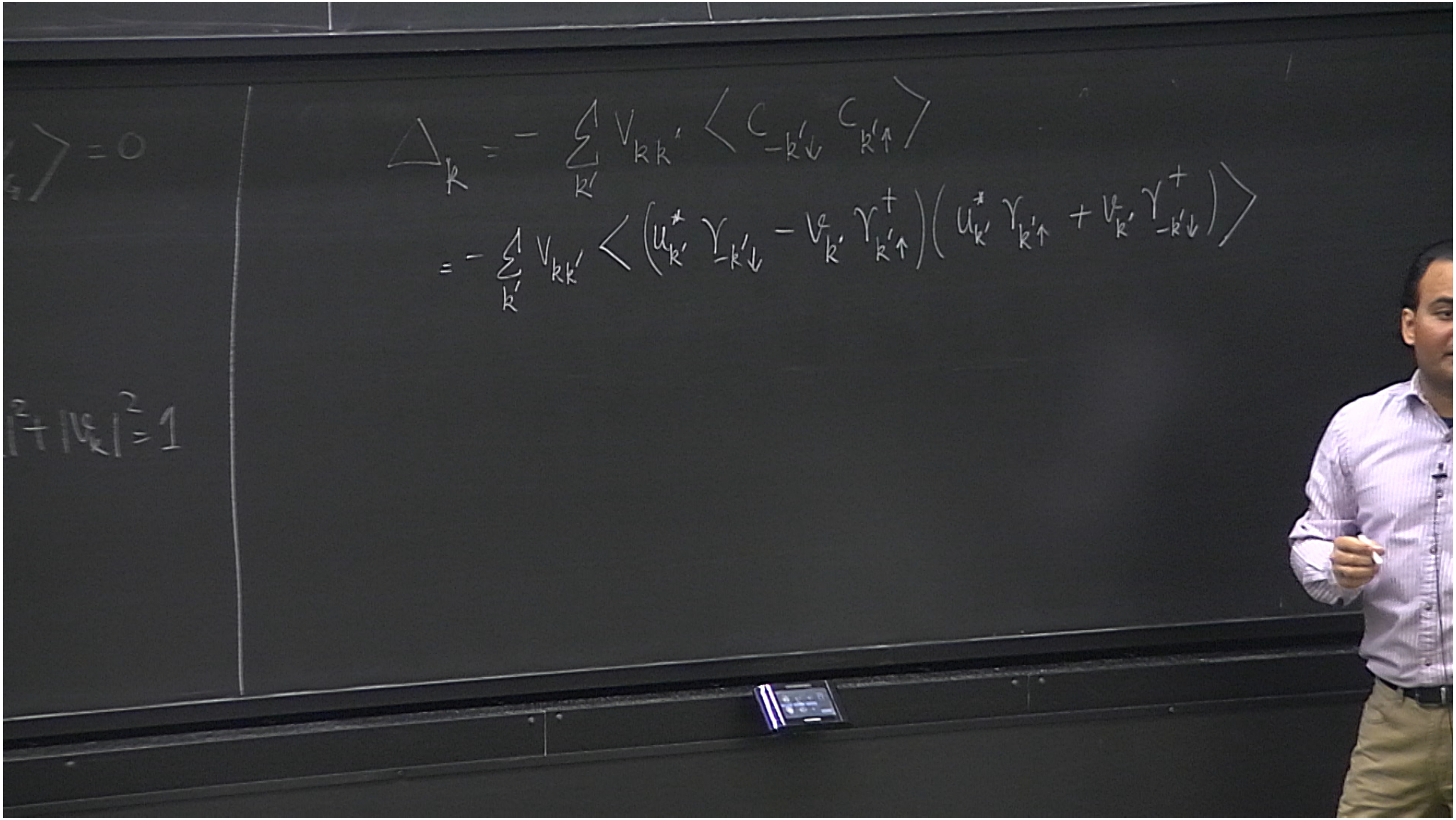
create a quasiparticle $\gamma^{\dagger}_{k\uparrow}\gamma_{k\uparrow}$

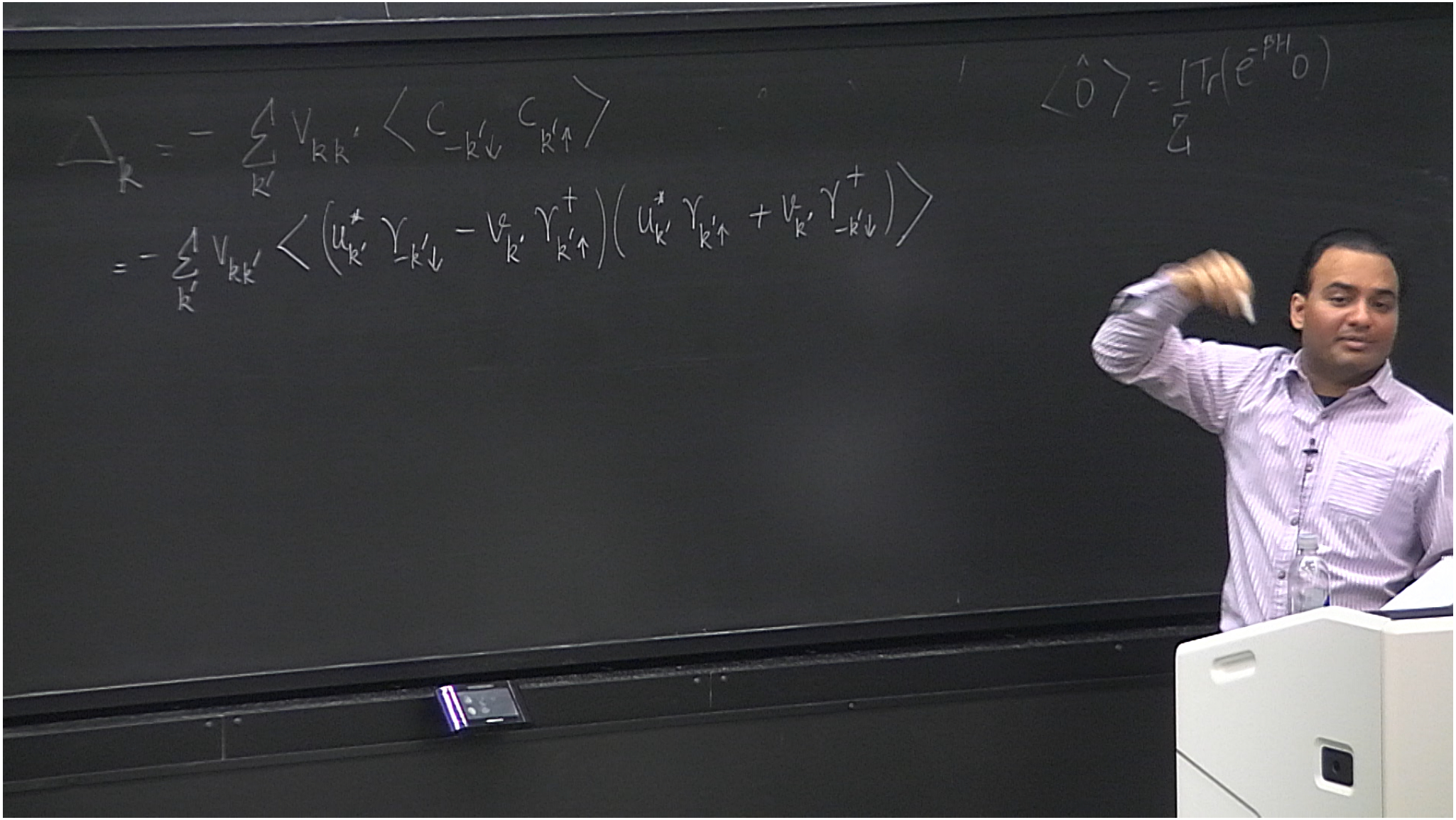
create a qua… $\gamma_{-k\downarrow}\gamma^{\dagger}_{-k\downarrow}$

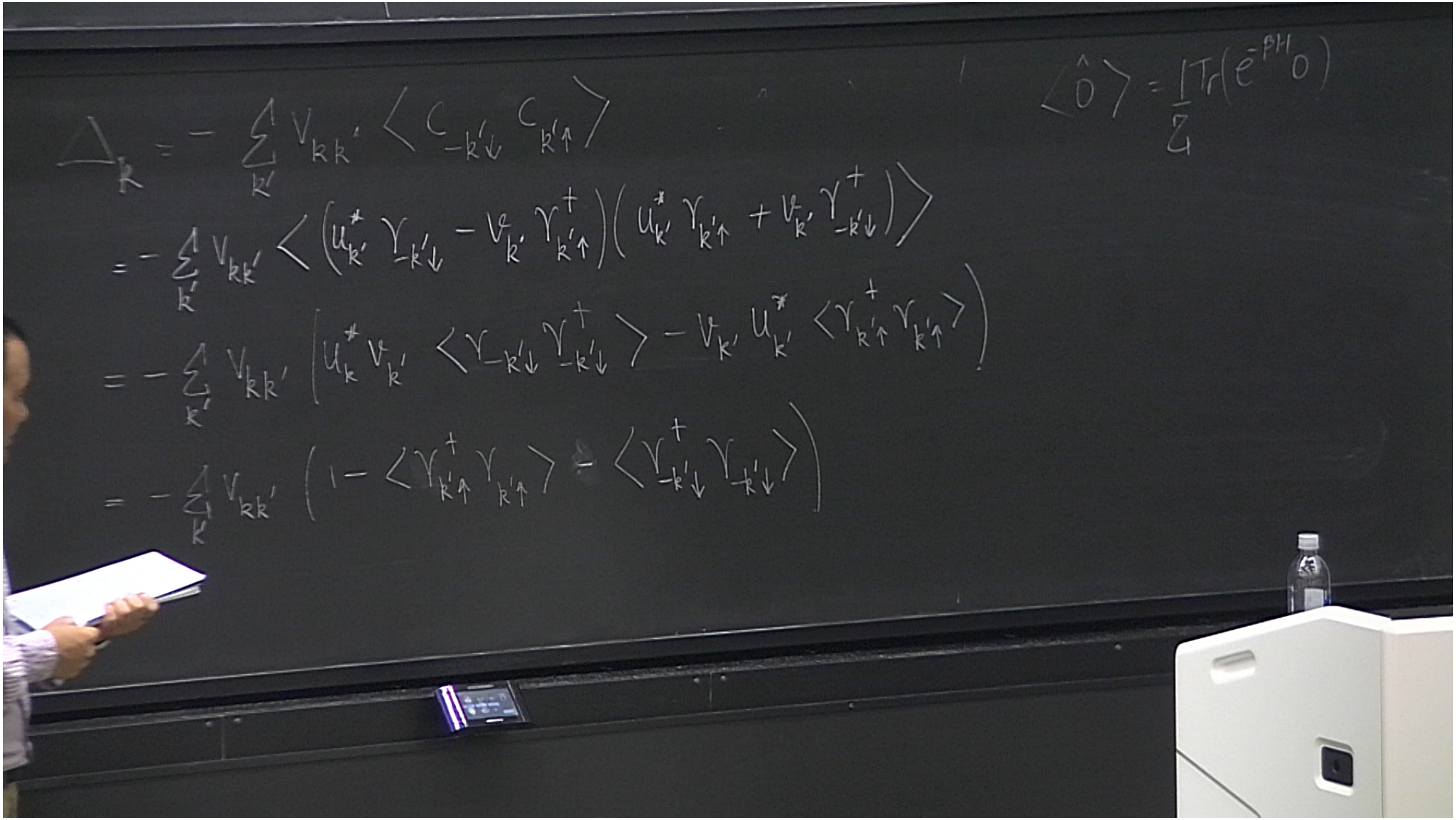

holes
electrons
$+E_k$
$2\Delta_k$
k
μ
k_F
holes
$-E_k$
electrons
create a quasiparticle
create a quasi hole
+ Const


$\gamma_{k\uparrow}|\psi_G\rangle = \gamma_{-k\downarrow}|\psi_G\rangle = 0$


$\gamma_{k\uparrow}|\psi_G\rangle = \gamma_{-k\downarrow}|\psi_G\rangle = 0$
$\{\gamma_{k\uparrow}, \gamma_{-k\downarrow}\} = 0$
$\{\gamma_{k\uparrow}, \gamma^{+}_{k\uparrow}\} = |u_k|^2 + |v_k|^2 = 1$


$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$
$= -\sum_{k'} V_{kk'} \langle \left(u_{k'}^* \gamma_{-k'\downarrow} - v_{k'}^* \gamma_{k'\uparrow}^\dagger \right) \left(u_{k'} \gamma_{k'\uparrow} + v_{k'} \gamma_{-k'\downarrow}^\dagger \right) \rangle$
$= 0$
$|^2 + |v_k|^2 = 1$


$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$
$= -\sum_{k'} V_{kk'} \langle (u^*_{k'} \gamma_{-k'\downarrow} - v_{k'} \gamma^\dagger_{k'\uparrow})(u^*_{k'} \gamma_{k'\uparrow} + v_{k'} \gamma^\dagger_{-k'\downarrow}) \rangle$
$\langle \hat{O} \rangle = \frac{1}{Z} \mathrm{Tr}(e^{-\beta H} O)$


$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$
$= -\sum_{k'} V_{kk'} \langle (u^*_{k'} \gamma_{-k'\downarrow} - v_{k'} \gamma^\dagger_{k'\uparrow})(u^*_{k'} \gamma_{k'\uparrow} + v_{k'} \gamma^\dagger_{-k'\downarrow}) \rangle$
$= -\sum_{k'} V_{kk'} (u^*_k v_{k'} \langle \gamma_{-k'\downarrow} \gamma^\dagger_{-k'\downarrow} \rangle - v_{k'} u^*_{k'} \langle \gamma^\dagger_{k'\uparrow} \gamma_{k'\uparrow} \rangle)$
$= -\sum_{k'} V_{kk'} (1 - \langle \gamma^\dagger_{k'\uparrow} \gamma_{k'\uparrow} \rangle - \langle \gamma^\dagger_{-k'\downarrow} \gamma_{-k'\downarrow} \rangle)$
$\langle \hat{O} \rangle = \frac{1}{Z} Tr(e^{-\beta H} O)$

$$\Delta_k = -\sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$$

$$\langle \hat{O} \rangle = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} O\right)$$

$$= -\sum_{k'} V_{kk'} \left\langle \left(u_{k'}^{*} \gamma_{-k'\downarrow} - v_{k'} \gamma_{k'\uparrow}^{\dagger}\right)\left(u_{k'}^{*} \gamma_{k'\uparrow} + v_{k'} \gamma_{-k'\downarrow}^{\dagger}\right) \right\rangle$$

$$= -\sum_{k'} V_{kk'} \left(u_k^{*} v_{k'} \langle \gamma_{-k'\downarrow} \gamma_{-k'\downarrow}^{\dagger} \rangle - v_{k'} u_{k'}^{*} \langle \gamma_{k'\uparrow}^{\dagger} \gamma_{k'\uparrow} \rangle \right)$$

$$= -\sum_{k} V_{kk'} \left(1 - \langle \gamma_{k'\uparrow}^{\dagger} \gamma_{k'\uparrow} \rangle - \langle \gamma_{-k'\downarrow}^{\dagger} \gamma_{-k'\downarrow} \rangle \right)$$

$$= -\sum_{k'} V_{kk'} u_{k'}^{*} v_{k'} \left(1 - 2 n_F(E_{k'})\right) \quad ; \; n_F \equiv \text{Fermi-Dirac}$$

$$= -\sum_{k'} V_{kk'}\, u_{k'}^{*}\, v_{k'} \left(1 - 2\right.$$

On shell approx.

$$\Delta = V\left(g(E_F)\right) \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \; \frac{\Delta}{2\sqrt{\xi^2+\Delta^2}} \left(1 - \frac{2}{e^{\beta\sqrt{\xi^2+\Delta^2}}+1}\right)$$



$$= -\sum_{k'} V_{kk'} u_{k'}^* v_{k'} (1-2$$

On shell approx.

$$1 = V(g(E_F)) \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \, \frac{1}{2\sqrt{\xi^2+\Delta^2}} \left(1 - \frac{2}{e^{\beta\sqrt{\xi^2+\Delta^2}}+1}\right)$$

$$= -\sum_{k'} V_{kk'} u_{k'}^{*} v_{k'} (1-2$$

Onshell approx.

$$1 = V g(E_F) \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \, \frac{1}{2\sqrt{\xi^2+\Delta^2}} \left(1 - \frac{2}{e^{\beta\sqrt{\xi^2+\Delta^2}}+1}\right)$$

$$\frac{1}{V g(E_F)} = \int_0^{\hbar\omega_c} d\xi \, \frac{\tanh\left(\frac{\beta\sqrt{\xi^2+\Delta^2}}{2}\right)}{\sqrt{\xi^2+\Delta^2}}$$

Gap Equation

