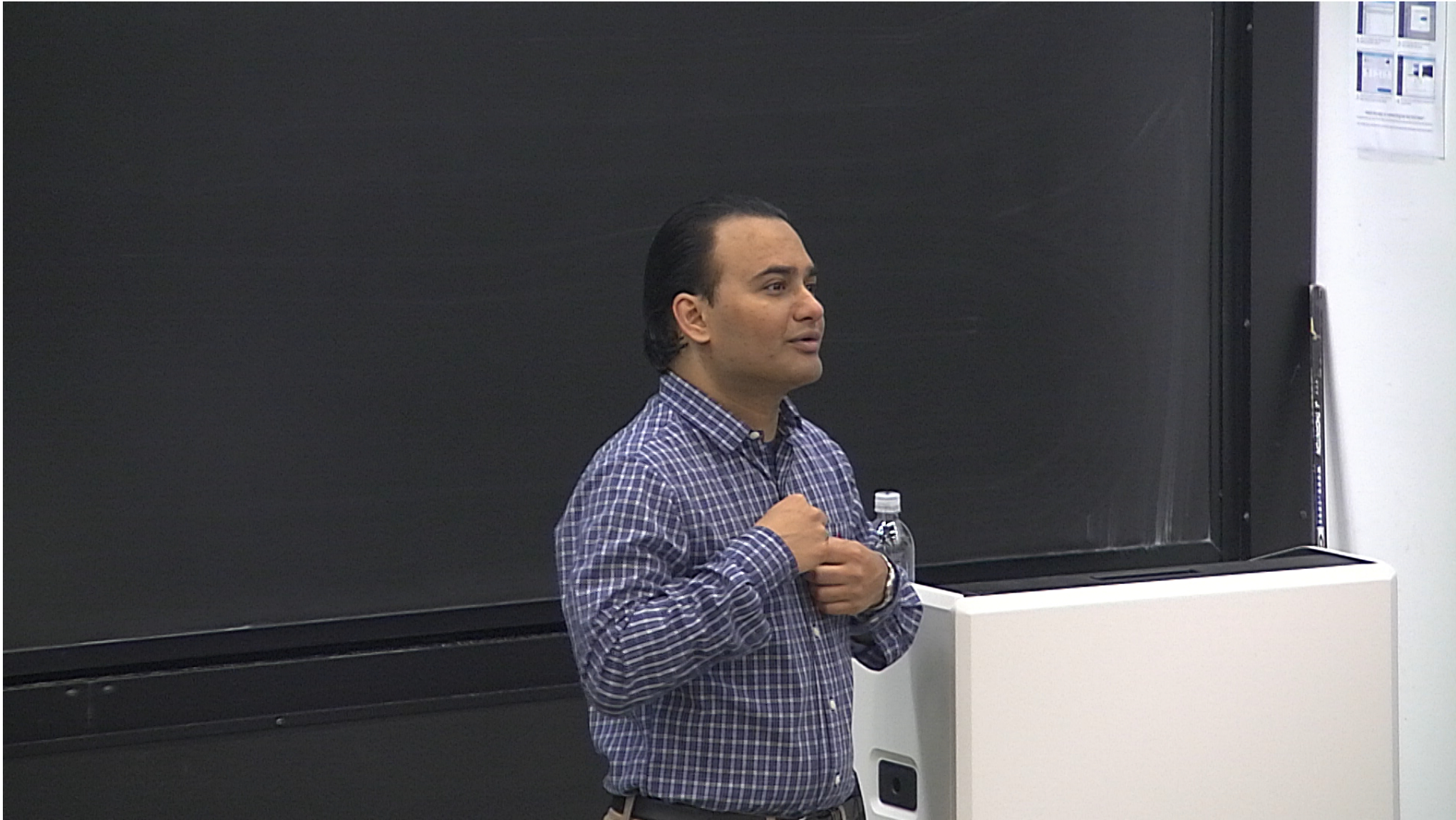


Title: PSI 2018/2019 - Condensed Matter - Lecture 10

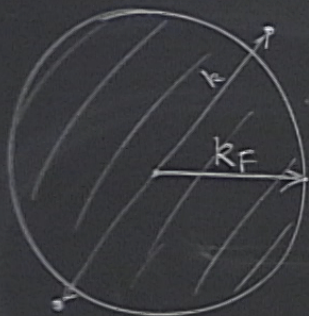
Date: Nov 23, 2018 10:45 AM

URL: <http://pirsa.org/18110028>

Abstract:



Cooper problem



$$\psi(\vec{r}_1, \vec{r}_2) = \psi_{cm}(\vec{R}_{cm}) \psi(\vec{r})$$

$$H = H_{cm} + H_{rel}$$

$$H_{cm} = \frac{p_{cm}^2}{2m} ; \quad \vec{p}_{cm} = 0 ; \quad (\vec{R}, -\vec{R})$$

$$H_{rel} \psi(r) = (2\epsilon_F + E) \psi(r)$$

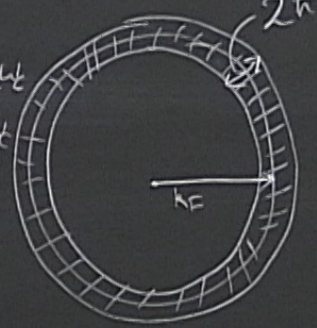
$$(E - 2\epsilon_{Fk}) \psi_k = \sum_{k'} V_{kk'} \psi_{k'} ; \quad \epsilon_{Fk} = \epsilon_k - \epsilon_F = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

Onshell approximation

$$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \leq \hbar\omega \\ & \& |\epsilon_{k'} - \epsilon_F| \leq \hbar\omega \\ 0 & \text{Outside shell} \end{cases}$$

$$\Rightarrow \psi_k =$$

Onshell approximation

$$V_{kk'} = \begin{cases} -V & \text{if } |\epsilon_k - \epsilon_F| \leq \hbar\omega_c \\ & \& |\epsilon_{k'} - \epsilon_F| \leq \hbar\omega_c \\ 0 & \text{Outside shell} \end{cases}$$


The diagram shows a circular Fermi disk in the \$k\$-plane. A horizontal line from the center to the right edge is labeled \$\epsilon_F\$. A shaded annular region (shell) is drawn around the center, with its thickness indicated by a double-headed arrow and labeled \$2\hbar\omega_c\$.

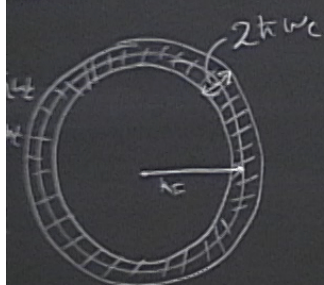
$$\Rightarrow \frac{1}{V} = \sum_k^{\#} \frac{1}{2\epsilon_k - E}$$

$$\Rightarrow \psi_k = \frac{V}{2\epsilon_k - E} \sum_{k'}^* \psi_{k'}$$

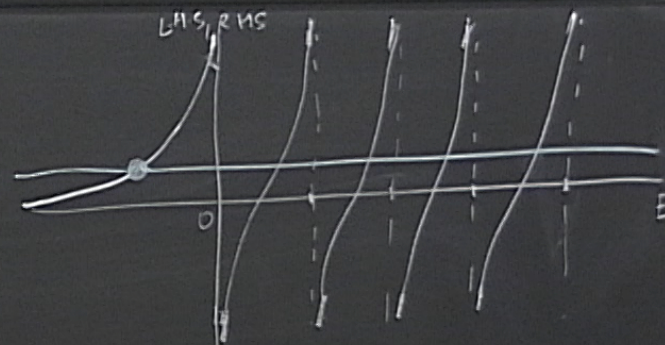
$$\epsilon_k = \epsilon_k - \epsilon_F = \frac{\hbar^2 k^2}{2m} - \epsilon_F$$

$$\sum_k^* \psi_k = \sum_k^* \frac{V}{2\epsilon_k - E} \sum_{k'}^* \psi_{k'}$$

tion



$$\Rightarrow \frac{1}{V} = \sum_k^{\#} \frac{1}{2\epsilon_k - E}$$



$$\frac{V}{2\epsilon_k - E} \sum_{k'}^* \psi_{k'}$$

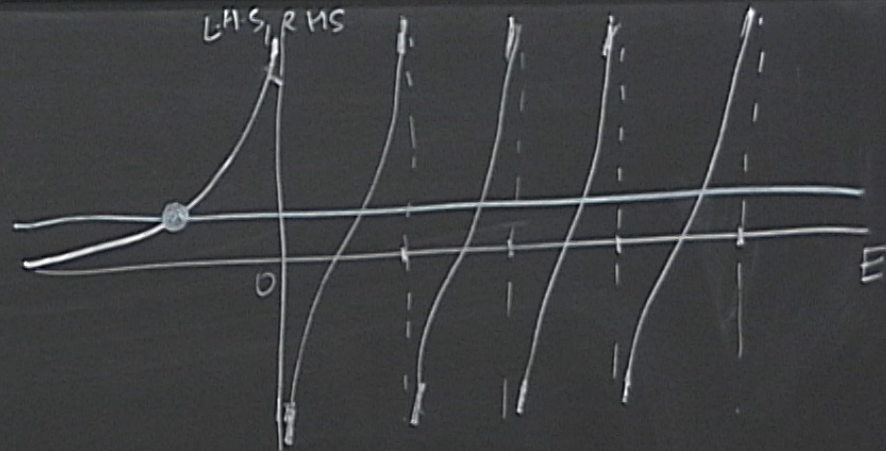
$$\sum_k^* \psi_k = \sum_k^* \frac{V}{2\epsilon_k - E} \sum_{k'}^* \psi_{k'}$$

$$\Rightarrow \frac{1}{V} = \sum_k^{\#} \frac{1}{2\xi_{sk} - E}$$

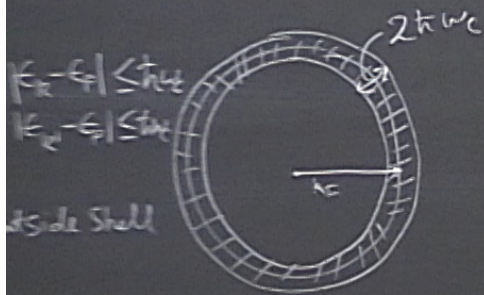
$$= \frac{1}{\Omega} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\xi_{sk} - E}$$

$$= \frac{1}{\Omega} \int \frac{d^3k}{(2\pi)^3} \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \delta(\xi - \xi_{sk}) \frac{1}{2\xi - E}$$

$$= \int_{-\hbar\omega_c}^{\hbar\omega_c} d\xi \frac{g(\xi)}{2\xi - E}$$



approximation

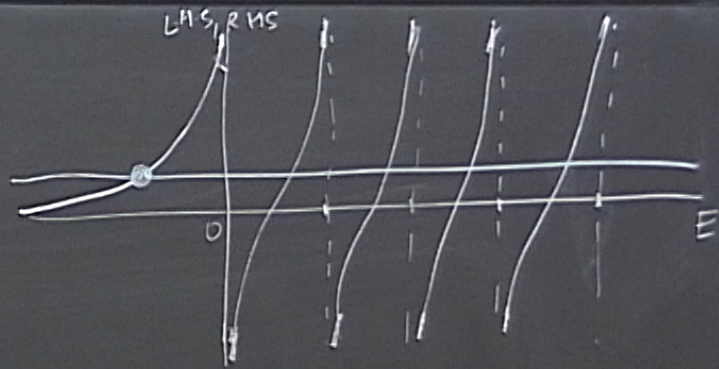


$$\psi_k = \frac{V}{2\xi_{jk} - E} \sum_{k'}^* \psi_{k'}$$

$$\sum_k^* \psi_k = \sum_k^* \frac{V}{2\xi_{jk} - E} \sum_{k'}^* \psi_{k'}$$

$$\Rightarrow \frac{1}{V} = \sum_k^* \frac{1}{2\xi_{jk} - E}$$

$$= \Omega \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\xi_{jk} - E}$$



$$= \Omega \int \frac{d^3k}{(2\pi)^3} \int d\xi \delta(\xi - \xi_{jk}) \frac{1}{2\xi_{jk} - E}$$

$$= \int_0^{\hbar\omega_c} d\xi \frac{g(\xi)}{2\xi - E} = g(E_F) \int_0^{\hbar\omega_c} d\xi \frac{1}{2\xi - E} = \frac{g(E_F)}{2} \ln \frac{E - 2\hbar\omega_c}{E}$$

$$E = 2\hbar\omega_c \left(1 - e^{\frac{2}{g(E_F)V}}\right)^{-1}$$

Weak coupling limit: $g(E_F)V \ll 1$

$$E = -2\hbar\omega_c e^{-\frac{2}{g(E_F)V}}$$

$M = \# \text{ of } k \text{ values}$

$$\frac{M!}{\left(\frac{N}{2}\right)! \left(\frac{M-N}{2}\right)!} \sim 10^{10^2}$$

$$|\psi_0\rangle = \sum_{|k| > k_F} \psi_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger |F\rangle$$

↑ Fermi sea

$$|\psi^{(N)}\rangle = \left(\sum_k \psi_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right)^{N/2} |0\rangle$$

$$M = \# \text{ of } k \text{ values}$$

$$\frac{M!}{(\frac{N}{2})! (\frac{M-N}{2})!} \sim 10^{10^{23}}$$

$$|\psi_G\rangle = \prod_k (u_k + v_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger) |0\rangle$$

where, $|u_k|^2 + |v_k|^2 = 1$

$$|\psi_G\rangle = \exp\left(\sum_k \psi_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger\right) |0\rangle$$

$$= \prod_k \exp(\psi_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger) |0\rangle$$

$$= \prod_k (1 + \psi_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger) |0\rangle$$

$$\sum_k \psi_k = \sum_k \frac{V}{2\xi_k - E} \sum_{k'} \psi_{k'}$$

$$\int_0^{\infty} g \frac{g(\epsilon)}{2\xi_g - E} = g(\epsilon_F) \int_0^{\infty} d\xi_g \frac{1}{2\xi_g - E} = \frac{g(\epsilon_F)}{2} \int_0^{\infty} d\xi_g \frac{E - \epsilon_F}{E}$$

$$| \psi \rangle = \prod_k (u_k + v_k c_{k\uparrow}^+ c_{-k\downarrow}^+) | 0 \rangle$$

$$e, \quad |u_k|^2 + |v_k|^2 = 1$$

$$c_{k\uparrow}^+ c_{-k\downarrow}^+ | 0 \rangle$$

$$c_{k\uparrow}^+ c_{-k\downarrow}^+ | 0 \rangle$$

$$c_{k\uparrow}^+ c_{-k\downarrow}^+ | 0 \rangle$$

$$|00\rangle_k = k\uparrow \& -k\downarrow \text{ unoccupied}$$

$$|11\rangle_k = k\uparrow \& -k\downarrow \text{ occupied}$$

$$|0\rangle = \prod_k |00\rangle_k$$

$$| \psi_G \rangle = \prod_k (u_k |00\rangle_k + v_k |11\rangle_k)$$

$$\langle \psi_G | \psi_G \rangle = \prod_k (|u_k|^2 + |v_k|^2) = 1$$

$$\begin{aligned} \bar{N} &= \langle \psi_G | \hat{N} | \psi_G \rangle = \langle \psi_G | \sum_k (c_{k\uparrow}^+ c_{k\uparrow} + c_{-k\downarrow}^+ c_{-k\downarrow}) | \psi_G \rangle \\ &= 2 \sum_k |v_k|^2 \end{aligned}$$

$$\langle N^2 \rangle = 4 \sum_k |v_k|^2 + 4 \sum_{k, k' \neq k} |v_k|^2 |v_{k'}|^2$$

$$\begin{aligned} \langle (\delta N)^2 \rangle &= \langle (N - \bar{N})^2 \rangle \\ &= 4 \sum_k |v_k|^2 |u_k|^2 \end{aligned}$$

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'} V_{kk'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow}$$

$$\Omega = E - TS - \mu N$$

↓ T=0

minimize: $\langle \psi_g | H - \mu N | \psi_g \rangle$

$$\delta \left(\langle \psi_g | \sum_{k\sigma} \tilde{\epsilon}_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'} V_{kk'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow} | \psi_g \rangle \right) = 0$$

$$\tilde{\epsilon}_k = \epsilon_k - \mu$$

$$c_{k\sigma}^\dagger c_{k\sigma} (u_{k_1} |00\rangle_{k_1} + v_{k_1} |11\rangle_{k_1}) \dots (u_{k_M} |00\rangle_{k_M} + v_{k_M} |11\rangle_{k_M})$$

$$(u_k |00\rangle_k + v_k |11\rangle_k)$$

$$\langle \psi_G | \sum_{k\sigma} \sum_{\sigma'} c_{k\sigma}^\dagger c_{k\sigma'} | \psi_G \rangle = 2 \sum_k \sum_{\sigma} |v_k|^2$$

$$\langle \psi_G | \sum_{kk'} v_{kk'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow} | \psi_G \rangle = \sum_{kk'} v_k^* u_k u_{k'}^* v_{k'}$$

Using $|u_k|^2 + |v_k|^2 = 1$; $u_k = \sin \theta_k$ & $v_k = \cos \theta_k$

$$\left(\langle \psi_G | \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'} V_{kk'} c_{k\uparrow}^\dagger c_{k'\downarrow} c_{k'\downarrow}^\dagger c_{k\uparrow} | \psi_G \rangle \right) = 0$$

$$\epsilon_k = \epsilon_k - \mu$$

$$\langle \psi_G | \epsilon_k - \mu | \psi_G \rangle = 2 \sum_k \epsilon_k \cos^2 \theta_k + \sum_{kk'} V_{kk'} \sin \theta_k \cos \theta_k \sin \theta_{k'} \cos \theta_{k'}$$

$$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | \epsilon_k - \mu | \psi_G \rangle = 0$$

$$-2 \epsilon_{k''} \sin 2\theta_{k''} + \frac{1}{4} \sum_{k'} V_{k''k'} (2 \cos 2\theta_{k''}) \sin 2\theta_{k'} + \frac{1}{4} \sum_k V_{kk''} (\sin 2\theta_k) (2 \cos 2\theta_{k''}) = 0$$

$$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2 \epsilon_k}$$

$$\langle \psi_G | H - \mu N | \psi_G \rangle = 2 \sum_k \sum_{j_k} \cos^2 \theta_k + \sum_{kk'} V_{kk'} \sin \theta_k \cos \theta_k \sin \theta_{k'} \cos \theta_{k'}$$

$$\frac{\partial}{\partial \theta_{k''}} \langle \psi_G | H - \mu N | \psi_G \rangle = 0$$

$$-2 \sum_{j_{k''}} \sin 2\theta_{k''} + \frac{1}{4} \sum_{k'} V_{k''k'} (2 \cos 2\theta_{k''}) \sin 2\theta_{k'} + \frac{1}{4} \sum_k V_{kk''} (\sin 2\theta_k) (2 \cos 2\theta_{k''}) = 0$$

$$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2 \sum_{j_k} \cos 2\theta_k}$$

Define $\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \sin 2\theta_{k'} = -\sum_{k'} V_{kk'} u_{k'} v_{k'}$

$$\tan 2\theta_k = -\frac{\Delta_k}{\sum_{j_k} \cos 2\theta_k}$$

$$\sin 2\theta_k = 2 u_k v_k = \frac{\Delta_k}{\sqrt{\sum_{j_k} \cos^2 2\theta_k + \Delta_k^2}}$$

$$\tan 2\theta_k = \frac{\sum_{k'} V_{kk'} \sin 2\theta_{k'}}{2\bar{\epsilon}_{jk}}$$

Define $\Delta_k = -\frac{1}{2} \sum_{k'} V_{kk'} \sin 2\theta_{k'}$

$$\tan 2\theta_k = -\frac{\Delta_k}{\bar{\epsilon}_{jk}}$$

$$\sin 2\theta_k = 2u_k v_k = \frac{\Delta_k}{\sqrt{\bar{\epsilon}_{jk}^2 + \Delta_k^2}} = \frac{\Delta_k}{E_k}$$

$$\cos 2\theta_k = v_k^2 - u_k^2 = -\frac{\bar{\epsilon}_{jk}}{\sqrt{\bar{\epsilon}_{jk}^2 + \Delta_k^2}} = -\frac{\bar{\epsilon}_{jk}}{E_k}$$

$$E_k = \sqrt{\bar{\epsilon}_{jk}^2 + \Delta_k^2}$$

$$\bar{\epsilon}_{jk} \rightarrow \infty$$

