Title: PSI 2018/2019 - Condensed Matter - Lecture 2

Date: Nov 13, 2018 10:45 AM

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Abstract:

a) Feynmann diag
b) Noether
c) Klein-Gordon
h) Wick
i) Feynman prop.
j) LSZ assumptions → 2nd order
k) Kallen-Lehmann
5) Conjugate
6) Fermion Lags
7) Qtz Dirac
8) Anti particles
9) Causality
12) $\mathcal{L}_{int}$ → a vertex rule
65) QED lag

II) Quantization of Free scalar Field by path integrals (Functional integral)

2.1 Classical K.G field

Minkowski spacetime d dimensional ↳ 1 time, $d-1$ space

$ds^2 = -dt^2 + d\vec{x}^2 \quad X = (t, \vec{x}) \quad c = 1$

Lagrangian density

Field $\phi(x)$ real spin = 0, no charge

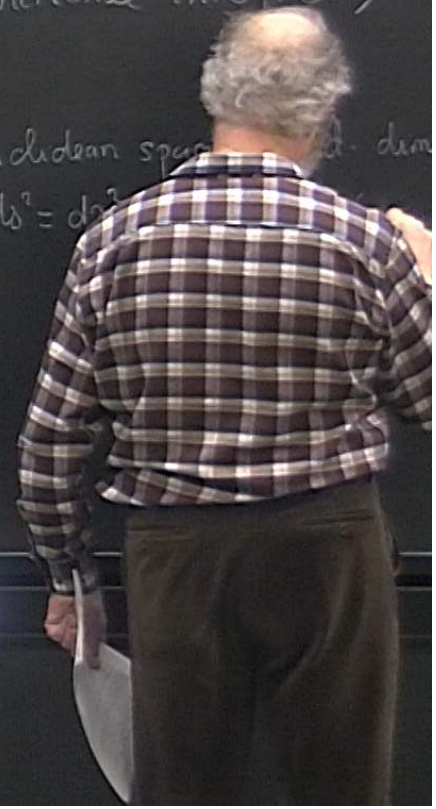
Klein-Gordon Equation

Action $S[\phi] = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}_{\vec{x}} \phi)^2 - \frac{m^2}{2}\phi^2 \right] \Rightarrow (-\Delta + m^2)\phi = 0$

$= \int d^d x \left[-\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right]$

Euclidean spa... d. dim

$ds^2 = d...$



b) Noether
c) Klein-Gordon
d) Casimir

i) LSZ assumptions → 2nd order
k) Kallen-Lehmann

III MAT
III SAM

8) Anti particles
9) Causality

III

$$b^\dagger_{k_1} b^\dagger_{k_2} \cdots b^\dagger_{k_N} |0,0,\ 0\rangle = \hat{S}_\mp \prod_{j=1}^{N} |\psi_{k_j}\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{k_1}(\vec{r}_1) & \psi_{k_1}(\vec{r}_2) & \cdots & \psi_{k_1}(\vec{r}_N) \\ \psi_{k_2}(\vec{r}_1) & \psi_{k_2}(\vec{r}_2) & & \psi_{k_2}(\vec{r}_N) \\ \vdots & & & \\ \psi_{k_N}(\vec{r}_1) & \psi_{k_N}(\vec{r}_2) & \cdots & \psi_{k_N}(\vec{r}_N) \end{vmatrix}_\mp$$

TWO Fermions

Vacuum $= |0_{k_1}, 0_{k_2}\rangle$

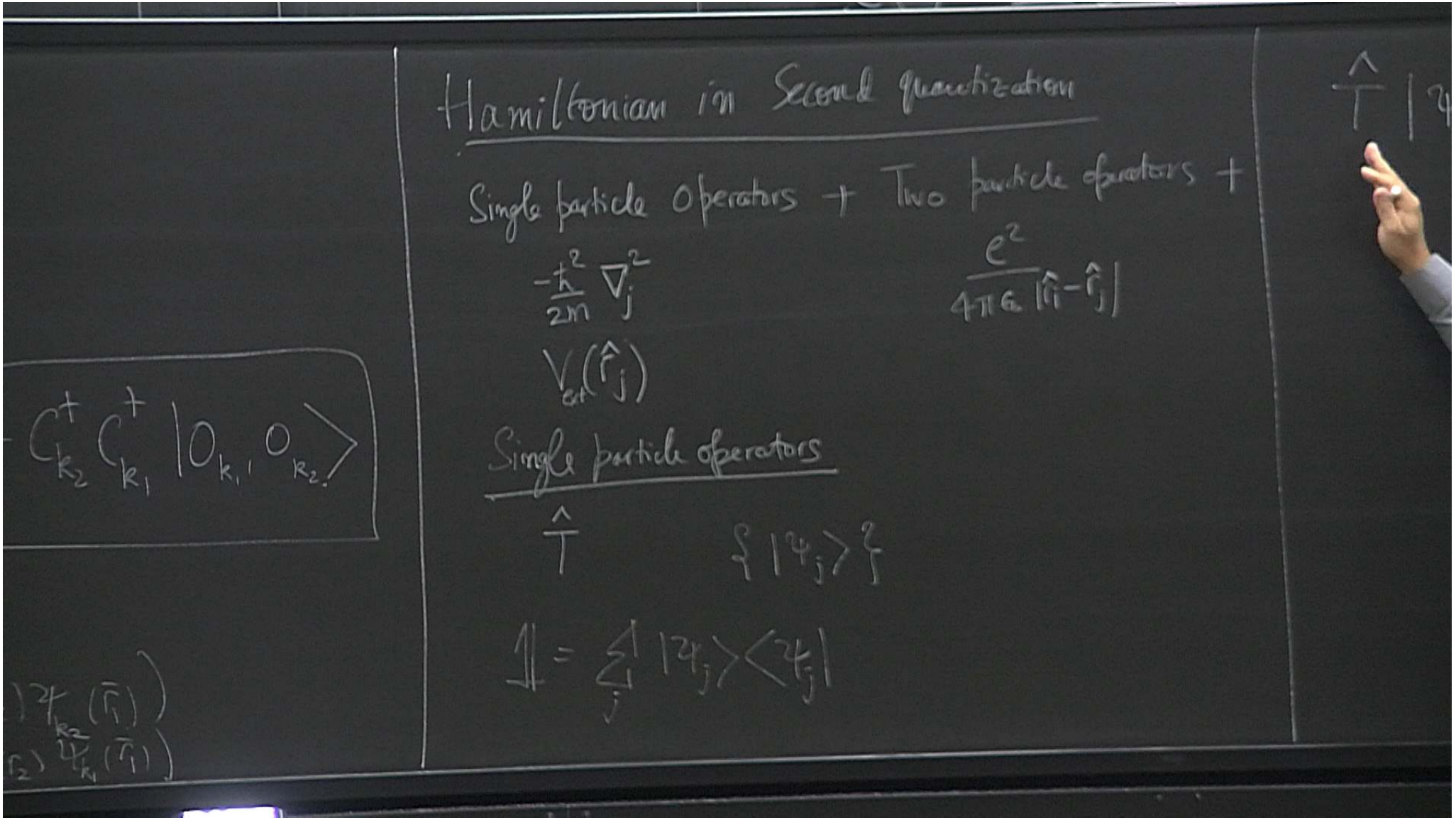
$$c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 0_{k_2}\rangle$$

$$c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = |0_{k_1}, 1_{k_2}\rangle$$

$$c^\dagger_{k_2} c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 1_{k_2}\rangle = \hat{S}_- \psi_{k_1}(\vec{r}_1)\psi_{k_2}(\vec{r}_2) = \frac{1}{\sqrt{2!}} \left(\psi_{k_1}(\vec{r}_1)\psi_{k_2}(\vec{r}_2) - \psi_{k_1}(\vec{r}_2)\psi_{k_2}(\vec{r}_1) \right)$$

$$c^\dagger_{k_1} c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 1_{k_2}\rangle = \hat{S}_- \psi_{k_2}(\vec{r}_1)\psi_{k_1}(\vec{r}_2) = \frac{1}{\sqrt{2!}} \left(\psi_{k_2}(\vec{r}_1)\psi_{k_1}(\vec{r}_2) - \psi_{k_2}(\vec{r}_2)\psi_{k_1}(\vec{r}_1) \right)$$

$$c^\dagger_{k_1} c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = -c^\dagger_{k_2} c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle$$


Hamiltonian in Second quantization
Single particle Operators + Two particle operators +
$-\frac{\hbar^2}{2m}\nabla_j^2$
$\frac{e^2}{4\pi\epsilon_0|\hat{r}_i-\hat{r}_j|}$
$V_{ext}(\hat{r}_j)$
Single particle operators
$\hat{T}$
$\{|\psi_j\rangle\}$
$\mathbb{1} = \sum_j |\psi_j\rangle\langle\psi_j|$
$\hat{T}|\psi$
$c^\dagger_{k_2} c^\dagger_{k_1} |0_{k_1} 0_{k_2}\rangle$

$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{n} = \pi s$

Hamiltonian in Second quantization

Single particle Operators + Two particle operators +

$-\frac{\hbar^2}{2m}\nabla_j^2$ $\qquad$ $\frac{e^2}{4\pi\epsilon_0|\hat{r}_i - \hat{r}_j|}$

$V_{ext}(\hat{r}_j)$

$|0_{k_2}\rangle$

Single particle operators

$\hat{T}$ $\qquad$ $\{|\psi_j\rangle\}$

$\mathbb{1} = \sum_j |\psi_j\rangle\langle\psi_j|$

$\hat{T}|\psi_k\rangle = \sum_j |\psi_j\rangle\langle\psi_j|\hat{T}|\psi_k\rangle$

$= \sum_j T_{jk}|\psi_j\rangle$

$\hat{T}_{tot} = \sum_{n=1}^{N} \hat{T}_n$

$\hat{T}_1 = \hat{T} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$

$\hat{T}_2 = \mathbb{1} \otimes \hat{T} \otimes \cdots \otimes \mathbb{1}$

$\hat{T}_n = \mathbb{1} \otimes \cdots \otimes \hat{T} \otimes \cdots \otimes \mathbb{1}$

position "n"

$\hat{T}_{tot} |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots |\psi_{k_N}\rangle = \sum_{n=1}^{N} \hat{T}_n |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots |\psi_{k_N}\rangle = \sum_{n=1}^{N} \sum_j T_{jk_n} |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots \underbrace{|\psi_j\rangle}_{\text{position "n"}} \cdots |\psi_{k_N}\rangle$

$\hat{T}_{tot}$ is invariant under Exchange

$[\hat{T}_{tot}, \hat{S}_\pm] = 0$

$\hat{S}_\pm \hat{T}_{tot} |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots |\psi_{k_N}\rangle = \hat{T}_{tot} \hat{S}_\pm |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots |\psi_{k_N}\rangle = \hat{T}_{tot}\, b^\dagger_{k_1} b^\dagger_{k_2} \cdots b^\dagger_{k_N} |0,0,\cdots,0\rangle$

$\sum_{n=1}^{N} \sum_j T_{jk_N} \hat{S}_\pm |\psi_{k_1}\rangle |\psi_{k_2}\rangle \cdots \underbrace{|\psi_j\rangle}_{\text{position "n"}} \cdots |\psi_{k_N}\rangle = \sum_{n=1}^{N} \sum_j T_{jk_n}\, b^\dagger_{k_1} b^\dagger_{k_2} \cdots \underbrace{b^\dagger_j}_{\text{position "n"}} \cdots b^\dagger_{k_N} |0,0,\cdots,0\rangle$

① $\left(b^\dagger_{k_n}\right)^p$

② $b^\dagger_j \left(b^\dagger_{k_n}\right)^{p-1}$

Bosons

(p particles with quantum # k_n)

$$\hat{T}_{tot}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\hat{T}_n\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_{j} T_{jk_n}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots\underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle$$

$\hat{T}_{tot}$ is invariant under Exchange

$$[\hat{T}_{tot},\hat{S}_{\pm}] = 0$$

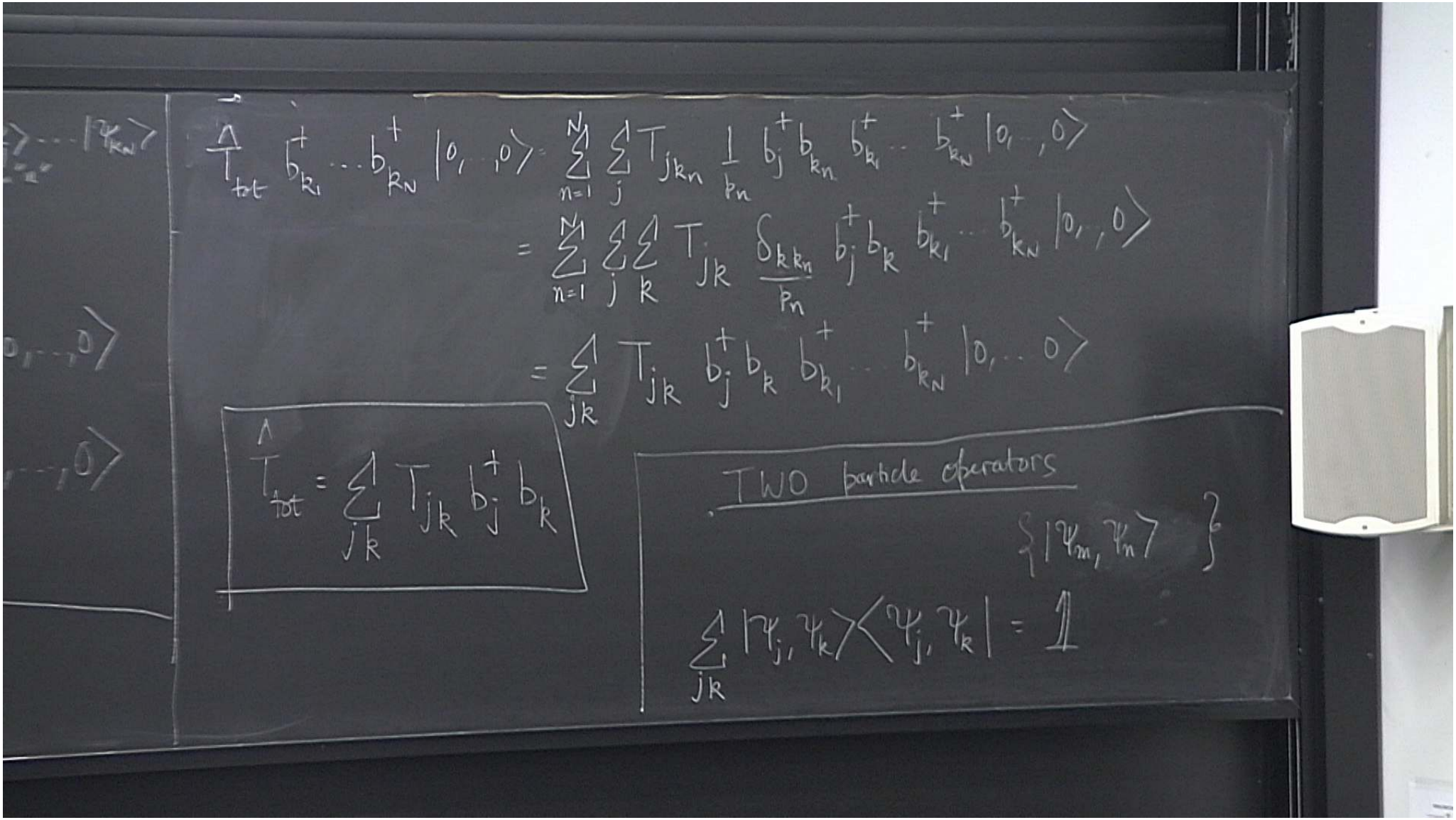
$$\hat{T}_{tot}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat{T}_{tot}\,\hat{S}_{\pm}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat{T}_{tot}\,b^{\dagger}_{k_1}b^{\dagger}_{k_2}\cdots b^{\dagger}_{k_N}\,|0,0,\ldots,0\rangle$$

$$\hat{S}_{\pm}\hat{T}_{tot}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_{j} T_{jk_N}\,\hat{S}_{\pm}\,|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots\underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_{j} T_{jk_n}\,b^{\dagger}_{k_1}b^{\dagger}_{k_2}\cdots\underbrace{b^{\dagger}_{j}}_{\text{position "n"}}\cdots b^{\dagger}_{k_N}\,|0,0,\ldots,0\rangle$$

① $\left(b^{\dagger}_{k_n}\right)^{p_n}$

② $b^{\dagger}_{j}\left(b^{\dagger}_{k_n}\right)^{p_n-1}$

$$\left(b^{\dagger}_{k_n}\right)^{p_n-1}|0\rangle = \frac{1}{p_n}\,b_{k_n}\left(b^{\dagger}_{k_n}\right)^{p_n}|0\rangle$$


$\hat{T}_{tot}\, b^{+}_{k_1} \cdots b^{+}_{k_N} |0, \ldots, 0\rangle = \sum_{n=1}^{N} \sum_{j} \sum_{k_n} T_{j k_n} \frac{1}{p_n} b^{+}_{j} b_{k_n} b^{+}_{k_1} \cdots b^{+}_{k_N} |0, \ldots, 0\rangle$
$= \sum_{n=1}^{N} \sum_{j} \sum_{k} T_{jk} \frac{\delta_{k k_n}}{p_n} b^{+}_{j} b_{k} b^{+}_{k_1} \cdots b^{+}_{k_N} |0, \ldots, 0\rangle$
$= \sum_{jk} T_{jk} b^{+}_{j} b_{k} b^{+}_{k_1} \cdots b^{+}_{k_N} |0, \ldots 0\rangle$
$\hat{T}_{tot} = \sum_{jk} T_{jk} b^{+}_{j} b_{k}$
TWO particle operators
$\{ |\psi_m, \psi_n\rangle \}$
$\sum_{jk} |\psi_j, \psi_k\rangle\langle\psi_j, \psi_k| = \mathbb{1}$

$$\hat{V}\,|\psi_m,\psi_n\rangle = \sum_{jk} |\psi_j,\psi_k\rangle\langle\psi_j,\psi_k|\hat{V}|\psi_m,\psi_n\rangle = \sum_{j,k} V_{jk}^{mn}\,|\psi_j,\psi_k\rangle$$

$$V_{jk}^{mn} = \langle\psi_j,\psi_k|\hat{V}|\psi_m,\psi_n\rangle$$

↓ position representations

$$V_{jk}^{mn} = \int d^3r_1\, d^3r_2\, \langle\psi_j,\psi_k|\vec{r}_1,\vec{r}_2\rangle\langle\vec{r}_1,\vec{r}_2|\hat{V}|\vec{r}_1,\vec{r}_2\rangle\langle\vec{r}_1,\vec{r}_2|\psi_m,\psi_n\rangle$$

$$= \int d^3r_1 d^3r_2\, V(\vec{r}_1,\vec{r}_2)\,\psi_j^*(\vec{r}_1)\,\psi_k^*(\vec{r}_2)\,\psi_m(\vec{r}_1)\,\psi_n(\vec{r}_2)$$

$$\hat{V}|\psi_m, \psi_n\rangle = \sum_{j,k} V_{jk}^{mn} |\psi_j, \psi_k\rangle$$

$$\hat{V}_{tot} = \sum_{m,n,j,k} V_{jk}^{mn} b_j^\dagger b_k^\dagger b_m b_n$$

$$T_{jk} = \langle \varphi_j | \hat{T} | \varphi_k$$

$$\langle \vec{r}_1, \vec{r}_2 | \hat{V} | \vec{r}_1, \vec{r}_2 \rangle \langle \vec{r}_1, \vec{r}_2 | \psi_m, \psi_n \rangle$$

$$(\vec{r}_2)\, \psi_m(\vec{r}_1)\, \psi_n(\vec{r}_2)$$

Example:

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

Use momentum eigenstates $|\varphi_{\vec{k}}\rangle$

$$\varphi_{\vec{k}}(r) = \langle \vec{r} | \varphi_{\vec{k}} \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{r}}$$

$$V_{tot} = \sum_{m,n,j,k} V^{mn}_{jk}\, b^\dagger_j\, b^\dagger_k\, b_m\, b_n$$

$$T_{jk} = \langle \varphi_j | \hat{T} | \varphi_k \rangle = \frac{-\hbar^2}{2m} \int d^3r\, \varphi_j^*(\vec{r})\, \nabla^2\, \varphi_k(\vec{r})$$

$$= \frac{-\hbar^2}{2m} \frac{i\vec{k}\cdot i\vec{k}}{\Omega} \int d^3r\, e^{i(\vec{k}-\vec{j})\cdot r}$$

Example:

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i \neq j} V(|\vec{r}_i - \vec{r}_j|)$$

Use momentum eigenstates $|\varphi_{\vec{k}}\rangle$

$$\varphi_{\vec{k}}(\vec{r}) = \langle \vec{r} | \varphi_{\vec{k}} \rangle = \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{r}}$$

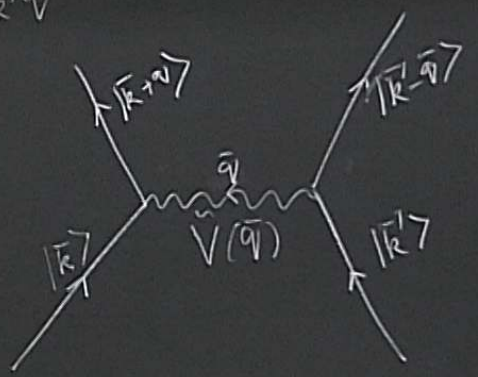
$$\frac{1}{\Omega} \int d^3r\, e^{i(\vec{k}-\vec{j})\cdot r} = \delta_{j\vec{k}}$$

$$\Rightarrow T_{j\vec{k}} = \frac{\hbar^2 |\vec{k}|^2}{2m} \delta_{j\vec{k}}$$

$$\Rightarrow \hat{T}_{tot} = \sum_{\vec{k}} \frac{\hbar^2 |\vec{k}|^2}{2m} C^\dagger_{\vec{k}} C_{\vec{k}}$$

$$\hat{V}_{int} = \sum_{\bar{k}_1,\bar{k}_2,\bar{q}_1,\bar{q}_2} V^{\bar{k}_1\bar{k}_2}_{\bar{q}_1\bar{q}_2}\, c^\dagger_{q_1} c^\dagger_{q_2} c_{k_1} c_{k_2} = \sum_{\bar{k}\bar{k}'\bar{q}} \tilde{V}(\bar{q})\, c^\dagger_{\bar{k}'-\bar{q}}\, c^\dagger_{\bar{k}+\bar{q}}\, c_{\bar{k}'}\, c_{\bar{k}}$$ [Tutorial]

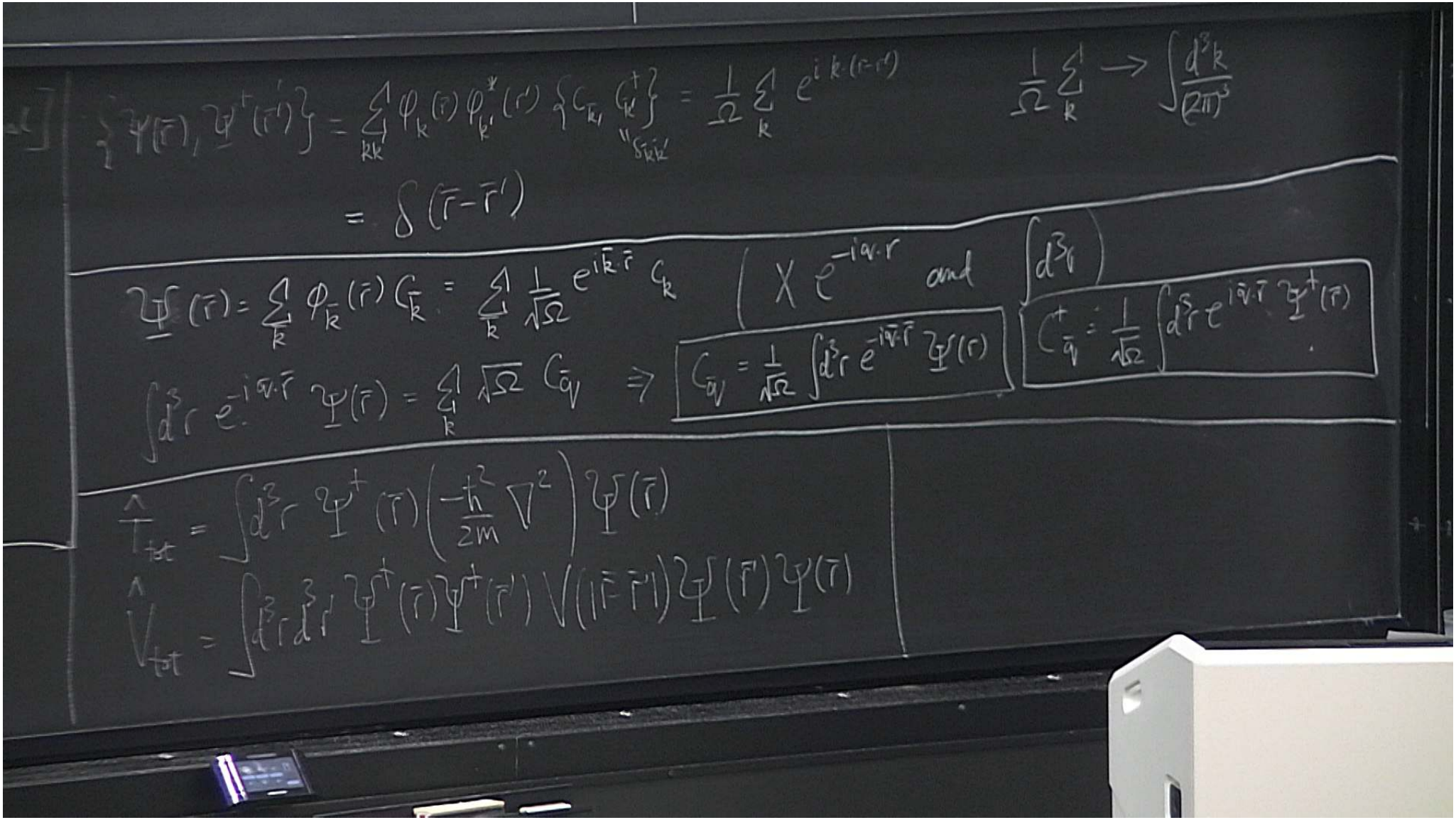
$$\tilde{V}(\bar{q}) = \frac{1}{\Omega}\int d^3r\, V(\bar{r})\, e^{i\bar{r}\cdot\bar{q}}$$



$$\hat{\Psi}(\bar{r}) = \sum_{\bar{k}} \phi_{\bar{k}}(\bar{r})\, c_{\bar{k}}$$

$$\hat{\Psi}^\dagger(\bar{r}) = \sum_{\bar{k}} \phi^*_{\bar{k}}(\bar{r})\, c^\dagger_{\bar{k}}$$

$$\hat{\Psi}^\dagger(\bar{r})|0\rangle = \sum_{\bar{k}} \phi^*_{\bar{k}}(\bar{r})\, c^\dagger_{\bar{k}}|0\rangle$$


$\{\Psi(\vec{r}), \Psi^\dagger(\vec{r}')\} = \sum_{kk'} \phi_k(\vec{r})\phi^*_{k'}(\vec{r}')\{c_k, c_{k'}^\dagger\} = \frac{1}{\Omega}\sum_k e^{ik\cdot(\vec{r}-\vec{r}')}$
$\frac{1}{\Omega}\sum_k \to \int \frac{d^3k}{(2\pi)^3}$
$= \delta(\vec{r}-\vec{r}')$
$\Psi(\vec{r}) = \sum_k \phi_k(\vec{r})\, c_k = \sum_k \frac{1}{\sqrt{\Omega}} e^{i\vec{k}\cdot\vec{r}}\, c_k$
$\int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, \Psi(\vec{r}) = \sum_k \sqrt{\Omega}\, c_{\vec{q}} \Rightarrow$
$c_{\vec{q}} = \frac{1}{\sqrt{\Omega}} \int d^3r\, e^{-i\vec{q}\cdot\vec{r}}\, \Psi(\vec{r})$
$c^\dagger_{\vec{q}} = \frac{1}{\sqrt{\Omega}} \int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, \Psi^\dagger(\vec{r})$
$\hat{T}_{tot} = \int d^3r\, \Psi^\dagger(\vec{r}) \left(\frac{-\hbar^2}{2m}\nabla^2\right) \Psi(\vec{r})$
$\hat{V}_{tot} = \int d^3r\, d^3r'\, \Psi^\dagger(\vec{r})\Psi^\dagger(\vec{r}')\, V(|\vec{r}-\vec{r}'|)\, \Psi(\vec{r}')\Psi(\vec{r})$