Title: PSI 2018/2019 - Condensed Matter - Lecture 2

Date: Nov 13, 2018 10:45 AM

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Abstract:

a) Feynmann diag.
b) Noether
c) Klein-Gordon

h) Wick
i) Feynman prop.
j) LSZ assumptions → 2nd order
k) Kallen-Lehmann

6) Fermion Lags
7) Qtz Dirac
8) Anti particles
9) Causality

17) $\mathcal{L}_{int}$ → vertex rule
65) QED lag

II) Quantization of Free scalar Field by path integrals (Functional integral)

2.1 Classical K.G field.

Minkowski space time d dimensional $\left\langle \begin{array}{l} 1 \text{ time} \\ d-1 \text{ space} \end{array} \right.$

$ds^2 = -dt^2 + d\vec{x}^2 \qquad X = (t, \vec{x}) \qquad c = 1$

Lagrangian density

Field $\phi(x)$ real spin = 0, no charge

Klein-Gordon Equation

$$\text{Action } S[\phi] = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 - \frac{m^2}{2}\phi^2 \right] \Rightarrow (-\Delta + m^2)\phi = 0$$

$$= \int d^d x \left[-\frac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right]$$

Euclidean spa[ce] d. dim

$ds^2 = d$

b) Noether
c) Klein-Gordon
d) Casimir

j) LSZ assumptions $\to$ 2nd order
k) Kallen-Lehmann

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8) Anti particles
9) Causality

$$b^\dagger_{k_1} b^\dagger_{k_2} \cdots b^\dagger_{k_N} |0,0,\dots 0\rangle = \hat{S}_\pm \prod_{j=1}^{N} |\psi_{k_j}\rangle = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{k_1}(\bar r_1) & \psi_{k_1}(\bar r_2) & \cdots & \psi_{k_1}(\bar r_N) \\ \psi_{k_2}(\bar r_1) & \psi_{k_2}(\bar r_2) & \cdots & \psi_{k_2}(\bar r_N) \\ \vdots & & & \\ \psi_{k_N}(\bar r_1) & \psi_{k_N}(\bar r_2) & \cdots & \psi_{k_N}(\bar r_N) \end{vmatrix}_\pm$$

TWO Fermions

Vacuum $= |0_{k_1}, 0_{k_2}\rangle$

$$c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 0_{k_2}\rangle$$

$$c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = |0_{k_1}, 1_{k_2}\rangle$$

$$c^\dagger_{k_2} c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 1_{k_2}\rangle = \hat{S}_- \psi_{k_1}(\bar r_1)\psi_{k_2}(\bar r_2) = \frac{1}{\sqrt{2!}}\left(\psi_{k_1}(\bar r_1)\psi_{k_2}(\bar r_2) - \psi_{k_1}(\bar r_2)\psi_{k_2}(\bar r_1)\right)$$

$$c^\dagger_{k_1} c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = |1_{k_1}, 1_{k_2}\rangle = \hat{S}_- \psi_{k_2}(\bar r_1)\psi_{k_1}(\bar r_2) = \frac{1}{\sqrt{2!}}\left(\psi_{k_2}(\bar r_1)\psi_{k_1}(\bar r_2) - \psi_{k_2}(\bar r_2)\psi_{k_1}(\bar r_1)\right)$$

$$\boxed{c^\dagger_{k_1} c^\dagger_{k_2} |0_{k_1}, 0_{k_2}\rangle = - c^\dagger_{k_2} c^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle}$$

Hamiltonian in Second quantization

Single particle Operators + Two particle operators +

$$-\frac{\hbar^2}{2m}\nabla_j^2 \qquad\qquad \frac{e^2}{4\pi\epsilon_0|\hat{r}_i-\hat{r}_j|}$$

$$V_{ext}(\hat{r}_j)$$

Single particle operators

$$\hat{T} \qquad \{|\psi_j\rangle\}$$

$$\mathbb{1} = \sum_j |\psi_j\rangle\langle\psi_j|$$

$$\boxed{C^\dagger_{k_2} C^\dagger_{k_1} |0_{k_1}, 0_{k_2}\rangle}$$

$$\psi_{k_2}(\bar{r}_1)$$

$$(\bar{r}_2)\ \psi_{k_1}(\bar{r}_1)$$

$$\hat{T}\,|\psi$$

$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{\ldots} = \frac{\pi}{\ldots}\sum$

$\ldots, 0_{k_2} \rangle$

Hamiltonian in Second quantization

Single particle Operators + Two particle operators +

$-\frac{\hbar^2}{2m}\nabla_j^2 \qquad \frac{e^2}{4\pi\epsilon_0 |\hat{r}_i - \hat{r}_j|}$

$V_{ext}(\hat{r}_j)$

Single particle operators

$\hat{T} \qquad \{|\psi_j\rangle\}_j$

$\mathbb{1} = \sum_j |\psi_j\rangle\langle\psi_j|$

$$\hat{T}|\psi_k\rangle = \sum_j |\psi_j\rangle\langle\psi_j|\hat{T}|\psi_k\rangle = \sum_j T_{jk}|\psi_j\rangle$$

$$\hat{T}_{tot} = \sum_{n=1}^{N} \hat{T}_n$$

$$\hat{T}_1 = \hat{T}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$$

$$\hat{T}_2 = \mathbb{1}\otimes\hat{T}\otimes\cdots\otimes\mathbb{1}$$

$$\hat{T}_n = \mathbb{1}\otimes\cdots\otimes\underset{\text{position "}n\text{"}}{\hat{T}}\otimes\cdots\otimes\mathbb{1}$$

$$\hat{T}_{tot}|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\hat{T}_n|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_j T_{jk_n}|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots\underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle$$

$\hat{T}_{tot}$ is invariant under Exchange

$$[\hat{T}_{tot}, \hat{S}_\pm] = 0$$

① $$\hat{S}_\pm\hat{T}_{tot}|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat{T}_{tot}\hat{S}_\pm|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat{T}_{tot}\, b^\dagger_{k_1} b^\dagger_{k_2}\cdots b^\dagger_{k_N}|0,0,\dots,0\rangle$$

② $$\sum_{n=1}^{N}\sum_j T_{jk_N}\hat{S}_\pm|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots\underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_j T_{jk_n}\, b^\dagger_{k_1} b^\dagger_{k_2}\cdots\underbrace{b^\dagger_j}_{\text{position "n"}}\cdots b^\dagger_{k_N}|0,0,\dots,0\rangle$$

Bosons

(p particles with quantum # k_n)

① $(b^\dagger_{k_n})^p$

② $b^\dagger_j (b^\dagger_{k_n})^{p-1}$

$$\hat T_{tot} |\psi_{k_1}\rangle|\psi_{k_2}\rangle \cdots |\psi_{k_N}\rangle = \sum_{n=1}^{N} \hat T_n |\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \sum_{n=1}^{N}\sum_j T_{jk_n} |\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots \underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle$$

$\hat T_{tot}$ is invariant under Exchange

$$[\hat T_{tot}, \hat S_\pm] = 0$$

① $$\hat S_\pm \hat T_{tot}|\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat T_{tot}\hat S_\pm |\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots|\psi_{k_N}\rangle = \hat T_{tot}\, b^\dagger_{k_1} b^\dagger_{k_2}\cdots b^\dagger_{k_N}|0,0,\dots,0\rangle$$

② $$\sum_{n=1}^N\sum_j T_{jk_N}\hat S_\pm |\psi_{k_1}\rangle|\psi_{k_2}\rangle\cdots \underbrace{|\psi_j\rangle}_{\text{position "n"}}\cdots|\psi_{k_N}\rangle = \sum_{n=1}^N\sum_j T_{jk_n}\, b^\dagger_{k_1}b^\dagger_{k_2}\cdots \underbrace{b^\dagger_j}_{\text{position "n"}}\cdots b^\dagger_{k_N}|0,0,\dots,0\rangle$$

① $(b^\dagger_{k_n})^{p_n}$

② $b^\dagger_j (b^\dagger_{k_n})^{p_n-1}$

$$(b^\dagger_{k_n})^{p_n-1}|0\rangle = \frac{1}{p_n}\, b_{k_n}(b^\dagger_{k_n})^{p_n}|0\rangle$$

$\dots|\psi_{k_N}\rangle$

$|0,\dots,0\rangle$

$\dots,0\rangle$

$$\hat{T}_{tot}\, b^{\dagger}_{k_1}\cdots b^{\dagger}_{k_N}|0,\dots,0\rangle = \sum_{n=1}^{N}\sum_{j} T_{jk_n}\frac{1}{p_n}\, b_j^{\dagger} b_{k_n}\, b^{\dagger}_{k_1}\cdots b^{\dagger}_{k_N}|0,\dots,0\rangle$$

$$= \sum_{n=1}^{N}\sum_{j}\sum_{k} T_{jk}\frac{\delta_{k k_n}}{p_n}\, b_j^{\dagger} b_k\, b^{\dagger}_{k_1}\cdots b^{\dagger}_{k_N}|0,\dots,0\rangle$$

$$= \sum_{jk} T_{jk}\, b_j^{\dagger} b_k\, b^{\dagger}_{k_1}\cdots b^{\dagger}_{k_N}|0,\dots,0\rangle$$

$$\boxed{\hat{T}_{tot} = \sum_{jk} T_{jk}\, b_j^{\dagger} b_k}$$

TWO particle operators

$$\{|\psi_m,\psi_n\rangle\}$$

$$\sum_{jk}|\psi_j,\psi_k\rangle\langle\psi_j,\psi_k| = \mathbb{1}$$

$$\hat{V}|\psi_m,\psi_n\rangle = \sum_{jk}|\psi_j,\psi_k\rangle\langle\psi_j,\psi_k|\hat{V}|\psi_m,\psi_n\rangle = \sum_{j,k} V_{jk}^{mn}|\psi_j,\psi_k\rangle$$

$$V_{jk}^{mn} = \langle\psi_j,\psi_k|\hat{V}|\psi_m,\psi_n\rangle$$

$\downarrow$ position representations

$$V_{jk}^{mn} = \int d^3r_1\, d^3r_2 \,\langle\psi_j,\psi_k|\bar{r}_1\bar{r}_2\rangle\langle\bar{r}_1,\bar{r}_2|\hat{V}|\bar{r}_1,\bar{r}_2\rangle\langle\bar{r}_1,\bar{r}_2|\psi_m,\psi_n\rangle$$

$$= \int d^3r_1\, d^3r_2\; V(\bar{r}_1,\bar{r}_2)\,\psi_j^*(\bar{r}_1)\,\psi_k^*(\bar{r}_2)\,\psi_m(\bar{r}_1)\,\psi_n(\bar{r}_2)$$

$$\hat{V} | \psi_m, \psi_n \rangle = \sum_{j,k} V^{mn}_{jk} | \psi_j, \psi_k \rangle$$

$$\hat{V}_{tot} = \sum_{m,n,j,k} V^{mn}_{jk} \, b_j^{\dagger} b_k^{\dagger} b_m b_n$$

$$T_{\bar{j}\bar{k}} = \langle \varphi_j | \hat{T} | \varphi_{\bar{k}}$$

$$\bar{r}_2 | \hat{V} | \bar{r}_1, \bar{r}_2 \rangle \langle \bar{r}_1, \bar{r}_2 | \psi_m, \psi_n \rangle$$

$$(\bar{r}_2)\, \psi_m(\bar{r}_1)\, \psi_n(\bar{r}_2)$$

Example:

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \nabla_i^2 + \sum_{i \neq j} V(|\bar{r}_i - \bar{r}_j|)$$

Use momentum eigenstates $|\varphi_{\bar{k}}\rangle$

$$\varphi_{\bar{k}}(r) = \langle \bar{r} | \varphi_{\bar{k}} \rangle = \frac{1}{\sqrt{\Omega}} e^{i\bar{k}\cdot\bar{r}}$$

$$V_{tot} = \sum_{m,n,j,k} V_{jk}^{mn}\, b_j^\dagger b_k^\dagger b_m b_n$$

$$T_{\bar{j}\bar{k}} = \langle \varphi_j | \hat{T} | \varphi_{\bar{k}} \rangle = \frac{-\hbar^2}{2m} \int d^3r\, \varphi_j^*(r)\, \nabla^2 \varphi_{\bar{k}}(r)$$

$$= \frac{-\hbar^2}{2m} \frac{i\bar{k}\cdot i\bar{k}}{\Omega} \int d^3r\, e^{i(\bar{k}-\bar{j})\cdot r}$$

$$\boxed{\frac{1}{\Omega}\int d^3r\, e^{i(\bar{k}-\bar{j})\cdot r} = \delta_{\bar{j}\bar{k}}}$$

Example:

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N} \nabla_i^2 + \sum_{i\neq j} V(|\bar{r}_i - \bar{r}_j|)$$

Use momentum eigenstates $|\varphi_{\bar{k}}\rangle$

$$\varphi_{\bar{k}}(r) = \langle \bar{r} | \varphi_{\bar{k}} \rangle = \frac{1}{\sqrt{\Omega}} e^{i\bar{k}\cdot\bar{r}}$$

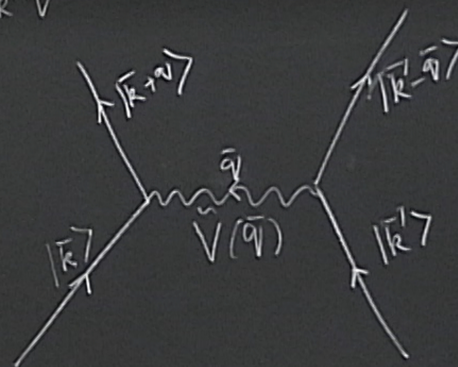
$$\Rightarrow T_{\bar{j}\bar{k}} = \frac{\hbar^2 |\bar{k}|^2}{2m} \delta_{\bar{j}\bar{k}}$$

$$\Rightarrow \hat{T}_{tot} = \sum_{\bar{k}} \frac{\hbar^2 |\bar{k}|^2}{2m} c_{\bar{k}}^\dagger c_{\bar{k}}$$

$$\hat{V}_{tot} = \sum_{\bar{k}_1,\bar{k}_2,\bar{q}_1,\bar{q}_2} V^{\bar{k}_1 \bar{k}_2}_{\bar{q}_1 \bar{q}_2}\, c^\dagger_{q_1} c^\dagger_{q_2} c_{k_1} c_{k_2} = \sum_{\bar{k}\,\bar{k}'\,\bar{q}} \tilde{V}(\bar{q})\, c^\dagger_{\bar{k}'-\bar{q}}\, c^\dagger_{\bar{k}+\bar{q}}\, c_{\bar{k}'}\, c_{\bar{k}}$$

[Tutorial]

$$\tilde{V}(\bar{q}) = \frac{1}{\Omega} \int d^3r\, V(\bar{r})\, e^{i\bar{r}\cdot\bar{q}}$$



$$\Psi(\bar{r}) = \sum_{k} \varphi_{\bar{k}}(\bar{r})\, c_{\bar{k}}$$

$$\Psi^\dagger(\bar{r}) = \sum_{k} \varphi^*_{k}(\bar{r})\, c^\dagger_{k}$$

$$\Psi^\dagger(\bar{r})\,|0\rangle = \sum_{k} \varphi^*_{k}(\bar{r})\, c^\dagger_{\bar{k}}\,|0\rangle$$

$$\{\Psi(\bar r), \Psi^\dagger(\bar r')\} = \sum_{kk'} \varphi_k(\bar r)\,\varphi^*_{k'}(\bar r')\,\underbrace{\{c_{\bar k'}, c^\dagger_{k}\}}_{\delta_{kk'}} = \frac{1}{\Omega}\sum_k e^{ik\cdot(r-r')}$$

$$= \delta(\bar r - \bar r')$$

$$\frac{1}{\Omega}\sum_k \longrightarrow \int \frac{d^3k}{(2\pi)^3}$$

$$\Psi(\bar r) = \sum_{\bar k} \varphi_{\bar k}(\bar r)\, c_{\bar k} = \sum_k \frac{1}{\sqrt{\Omega}} e^{i\bar k\cdot\bar r}\, c_k \qquad \left(\times\, e^{-iq\cdot r} \text{ and } \int d^3r\right)$$

$$\int d^3r\, e^{-i\bar q\cdot\bar r}\,\Psi(\bar r) = \sum_k \sqrt{\Omega}\, c_{\bar q} \Rightarrow \boxed{c_{\bar q} = \frac{1}{\sqrt{\Omega}}\int d^3r\, e^{-i\bar q\cdot\bar r}\,\Psi(\bar r)} \quad \boxed{c^\dagger_{\bar q} = \frac{1}{\sqrt{\Omega}}\int d^3r\, e^{i\bar q\cdot\bar r}\,\Psi^\dagger(\bar r)}$$

$$\hat T_{tot} = \int d^3r\, \Psi^\dagger(\bar r)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\Psi(\bar r)$$

$$\hat V_{tot} = \int d^3r\, d^3r'\, \Psi^\dagger(\bar r)\,\Psi^\dagger(\bar r')\, V(|\bar r - \bar r'|)\,\Psi(\bar r')\,\Psi(\bar r)$$