

Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 12

Date: Nov 30, 2018 09:00 AM

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Abstract:

Path & Functional Integral for Fermions

Grassmann Algebra

anticommuting "numbers"

$2N$ generators
on $\mathbb{C}$

$\theta_i, \bar{\theta}_i$
conjugate $\dagger$

$$\theta_i \theta_j + \theta_j \theta_i = 0$$

$$\int d\theta_i = \frac{\partial}{\partial \theta_i} \quad \text{anticommutate}$$

$$\int d\bar{\theta} d\theta \exp(-\bar{\theta} \cdot A \cdot \theta) = \det(A)$$

Maître QT (not in my notes ← French lectures)

Fermion with a single mode

$$|0\rangle, |1\rangle, \quad H|0\rangle = 0, \quad H|1\rangle = \overset{\omega}{\downarrow} \bar{E}_0 |1\rangle$$

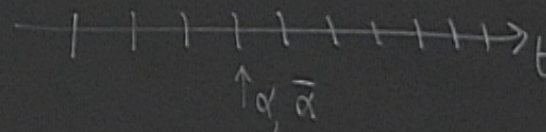
energy of the mode

$\mathcal{H} \otimes G$
 $\mathbb{C}^2$ algebra

$$a^\dagger |0\rangle = |1\rangle, \quad a^\dagger |1\rangle = 0$$

$$a |1\rangle = |0\rangle, \quad a |0\rangle = 0$$

path integral $U(t) = e^{\frac{i}{\hbar} t \cdot H}$



coherent states $|\alpha\rangle = \left(1 - \frac{\bar{\alpha}\alpha}{2}\right) |0\rangle + \alpha |1\rangle \in \mathcal{H} \otimes G$

$\int_t \alpha, \bar{\alpha} \} \text{anticommute} \rightarrow \text{anticommuting functions } \alpha(t)$

Path integral

$$\langle \alpha(t) | \beta(0) \rangle = \int \prod_s d\bar{\alpha}_s d\alpha_s \exp\left(\frac{i}{\hbar} S[\bar{\alpha}(s), \alpha(s)]\right)$$

↑ final state ↑ initial state ↑ action ↑ collection
 "history" or "path" in "Grassmann space"

Action

$$S[\bar{\alpha}(s), \alpha(s)] = \int_0^t ds \frac{\hbar}{i} \left(\frac{d}{ds} \bar{\alpha}(s) \alpha(s) - \bar{\alpha}(s) \frac{d}{ds} \alpha(s) \right) - E_0 \bar{\alpha}(s) \alpha(s)$$

$$\frac{d}{ds} \alpha(s) \approx \frac{\alpha(s+\epsilon) - \alpha(s)}{\epsilon}$$

kinetic term potential

$$|1\rangle \rightarrow e^{\frac{i}{\hbar} t E_0} |1\rangle$$

$$t=0 \quad |\alpha\rangle \rightarrow t, \quad |\alpha(t)\rangle = e^{\frac{i}{\hbar} t E_0} |\alpha\rangle$$

$$\psi(t) = e^{\frac{i}{\hbar} t E_0} \alpha$$

$$\bar{\alpha}(t) = e^{-\frac{i}{\hbar} t E_0} \bar{\alpha}$$

dynamics of coherent state

$\psi(\pm) = |e^{\pm i E_0} - \alpha\rangle$

Dirac Field

Dirac Equation
 $\uparrow$ action

$$S[\bar{\Psi}, \Psi] = \int d^4x \bar{\Psi}(x) (i \not{\partial} - m) \Psi(x)$$

$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$ γ^μ Dirac matrices

$\{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu}$ $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$

west coast γ -matrices east coast

$\Psi(x) = \{\psi_a(x)\}$
 Dirac indices: 2 $[d/2]$
 2 in $d=2$
 4 in $d=4$

$\bar{\Psi}(x) = \{\bar{\psi}_a(x)\}$
 Dirac particles $\{\psi, \bar{\psi}\}$
 $\Psi, \bar{\Psi}$ are classical fields

Dirac Field

$$\Psi(x) = \left\{ \Psi_a(x) \right\}$$

↑
Dirac indices: $2^{\lfloor d/2 \rfloor}$

2 in $d=2$
 4 in $d=4$

Dirac Equation
↑ action

$$S[\bar{\Psi}, \Psi] = \int d^4x \bar{\Psi}(x) (i \not{\partial} - m) \Psi(x)$$

$$\not{\partial} = \gamma^\mu \frac{\partial}{\partial x^\mu}$$

γ^μ Dirac matrices

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu} \quad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

west coast γ matrices

east coast

$$\bar{\Psi}(x) = \left\{ \bar{\Psi}_a(x) \right\}$$

Dirac particle (1st quantization)

$\Psi, \bar{\Psi}$ are wave functions

classical action: Euler Lagrange

$$\rightarrow \text{Dirac Equation } (i \not{\partial} + m) \Psi(x) = 0$$

$$\frac{\delta S}{\delta \bar{\Psi}(x)} = 0$$

2nd quantization of particle theory $\Leftrightarrow$ 1st quantization of a field

spin = 0 boson

$\longleftrightarrow$ KG field $\phi(x)$

spin $1/2$ fermion

$\longleftrightarrow$ spinor field

How to treat Ψ and $\bar{\Psi}$ as quantum Fermi Dirac Fields?

apply Grassmann / Berezin Functional integration to the Dirac Equation

$$\Psi = \{ \Psi^a(x) \}_{x \in \text{Spacetime}}$$

$a = 1, 4$
Dirac Index

each $\Psi^a(x)$ and $\bar{\Psi}^a(x)$
is an independent
Grassmann number

$$Z = \int \mathcal{D}[\bar{\Psi}, \Psi] e$$

$$\bar{\Psi} = \{ \bar{\Psi}^a(x) \}$$

$$\hbar = 1$$

$$\theta_i \longrightarrow \Psi_{i=\{x,a\}}$$

Action

$$S[\bar{\Psi}, \Psi] = \int_{\text{Spacetime}} d^4x \bar{\Psi}(x) (i \not{\partial}_x - m) \Psi(x)$$

$$= \int d^4x \sum_{a,b} \bar{\Psi}_a(x) \left(i \gamma_\mu^{ab} \frac{\partial}{\partial x^\mu} - m \delta^{ab} \right) \Psi_b(x)$$

Grassmann number

quadratic form

$\delta\psi(x)$
 path integral
 path integral

$$Z = \int \mathcal{D}[\bar{\psi}, \psi] \exp(iS[\bar{\psi}, \psi]) = \det[i\not{D} - m] \leftarrow \text{agrees with canonical quantization}$$

partition function
 functional determinant

v.e.v or correlation
 functions

$$\langle \psi_a(x) \bar{\psi}_b(y) \rangle = \langle x | \left(\frac{1}{i\not{D} - m} \right) | y \rangle = \int \frac{d^4 p}{(2\pi)^4} \left(\frac{i}{-\not{p} - m - i\epsilon} \right) e^{ip(x-y)} = G_F(x, y) \text{ Feynman Propagator}$$

$$= \frac{b}{y} \rightarrow \frac{a}{x}$$

quadratic form

Need Grassmann numbers

$$\langle \psi_a(x) \bar{\psi}_b(y) \psi_c(x') \bar{\psi}_d(y') \rangle = \text{diagram with four vertices and two fermion lines}$$

$$-m\delta^{ab} \psi_b(x)$$

Unitarity + Poincare invariance



Spin-statistics

integer spin B E

$\frac{1}{2}$ integer F D

II: Introduction to non-abelian Gauge theories

Maxwell theory

QED

spin 1 vector field

$$\vec{E}, \vec{B} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

vector potential $A_\mu \quad \mu = 1, \dots, 4$

e.m. tensor

Action principle + $U(1)$ gauge invariance

$$A_\mu \rightarrow A_\mu + \partial_\mu \alpha \quad \alpha = \alpha(x) \text{ gauge transformation}$$

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \text{ so } \vec{E} \text{ and } \vec{B}$$

Langkah-langkah formulasi dan uji formulasi

$SU(2)$ Yang Mills

$$g \in SU(2)$$

$$g = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \text{ unitary}$$
$$g \cdot g^+ = 1$$

$$\overline{\Phi} = (\overline{\Phi}^1, \overline{\Phi}^2)$$

$- = \text{complex conjugate}$

$$\phi \rightarrow g \cdot \phi$$

Indices ϕ^1 and ϕ^2

$$\overline{\phi} \rightarrow \overline{\phi}, g^+$$

Scalar product

$$\overline{\phi} \cdot \phi = \overline{\phi}^1 \phi^1 + \overline{\phi}^2 \phi^2$$

state $|N\rangle = (1 - \frac{\alpha}{2})|0\rangle + \alpha|1\rangle \in \mathcal{H}_G$
 anticommuting $\rightarrow$ anticommuting fields $\psi(t)$ $\{a, a^\dagger\} = 1$

$\frac{d}{ds} \alpha(s) = \frac{q(s) - \alpha(s)}{\epsilon}$ kinetic term
 'Lagrangian formulation' for fermions

54 Yang-Mills $\rightarrow$ ZC + Higgs mechanism. Non-Abelian GT renormalizable

$SU(2)$ Yang Mills

Gauge Fields coupled to matter

$$\phi = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = (\phi^i)$$

$i = 1, 2$ color indices

$SU(2)$ group of color symmetry

Sym. transf $\boxed{\phi \rightarrow g \cdot \phi}$

mixes ϕ^1 and ϕ^2

$$\boxed{\bar{\phi} \rightarrow \bar{\phi} g^\dagger}$$

$$g \in SU(2)$$

$$g = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \text{ unitary}$$

$$g \cdot g^\dagger = 1$$

conjugate field

$$\bar{\phi} = (\bar{\phi}^1, \bar{\phi}^2)$$

$- = \text{complex conjugate}$

Scalar product

$$\bar{\phi} \cdot \phi = \bar{\phi}^1 \phi^1 + \bar{\phi}^2 \phi^2$$

$$\bar{\phi} \phi \rightarrow \bar{\phi} \phi$$

$SU(2)$ renormalizable action

$$S[\phi] = \int d^4x \left[\frac{1}{2} \partial_\mu \bar{\phi} \cdot \partial^\mu \phi + \frac{m^2}{2} \bar{\phi} \phi + \frac{g}{8} (\bar{\phi} \phi)^2 \right]$$

currents conserved

$SU(2)$ has 3 generators

$$t_a = \frac{1}{2} \sigma_a \quad \sigma_a \text{ Pauli matrices}$$

$a = 1, 3$

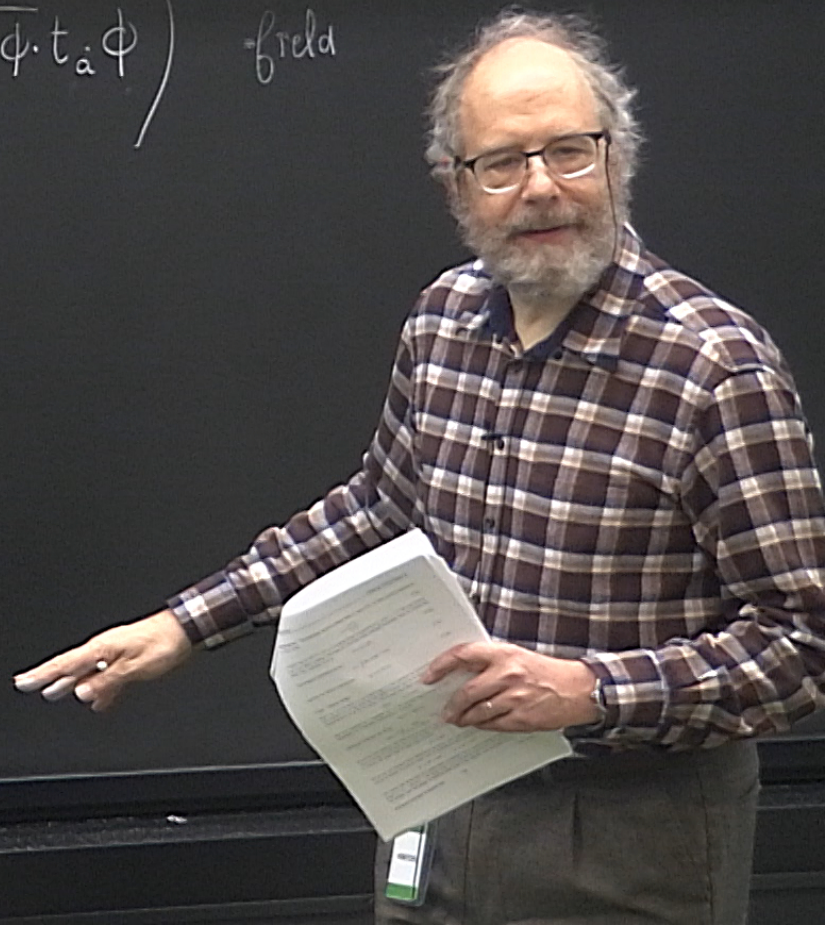
Currents are

$$J^a = (J_\mu^a) \quad , \quad J_\mu^a = \frac{i}{2} \left(\bar{\psi} t_a \partial_\mu \psi - \partial_\mu \bar{\psi} \cdot t_a \psi \right) \quad \text{vector field}$$

they are conserved
e.o.m $\Rightarrow \partial^\mu J_\mu^a = 0$

Group them into a SU(2) multiplet

$$J_\mu = J_\mu^a t_a \quad \begin{array}{l} 2 \times 2 \text{ traceless antisym matrix} \\ \text{element of } \text{Lie}(SU(2)) \end{array}$$



$$\bar{\psi} \cdot t_a \psi$$

vector
field

Spinor Dirac Field

$$\Psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \rightarrow \text{a Dirac}$$

4 component
spinor

$$\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2)$$

$$S = \int d^4x \bar{\Psi} \cdot (i \not{\partial} - m) \Psi$$

currents

$$J_\mu^a = \bar{\Psi} \gamma_\mu t_a \Psi$$

$$J_\mu^a = \bar{\psi}_\alpha (\gamma_\mu)^{\alpha\beta} (t_a)_{ij} \psi_j$$

adjoint repr
of SU(2)

Dirac, spin $\frac{1}{2}$ representation
of Lorentz Group

$$\psi_\alpha^i$$

$i = 1, 2$ group (Fund. Rep. of SU(2))
 $\alpha = 1, 2$ Dirac (spin $\frac{1}{2}$ repr. of Lorentz)
 $a = 1, 2, 3$ group (Adj. Rep. of SU(2))

$SU(2)$
- Lorentz
 $SU(2)$

Spin 1 Vector Fields $\leftarrow$ coupling by currents

$A_\mu^a(x)$ $a=1, 2, 3$ 3 vector fields

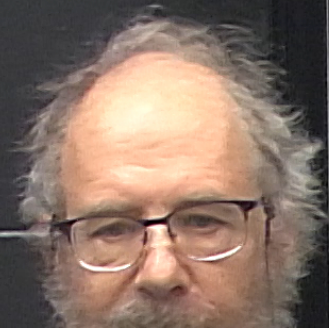
belong to Adj Representation of $SU(2)$

multiplet of $SU(2)$

$$A_\mu(x) = \sum_a A_\mu^a(x) t_a$$

2×2

2×2 antisym.
traceless matrix



(Fund. Rep. of $SU(2)$)
 (Spin $\frac{1}{2}$ rep. of Lorentz)
 (Adj. Rep. of $SU(2)$)

Spin 1 Vector Fields $\leftarrow$ coupling by currents

$A_\mu^a(x)$ $a=1, 2, 3$ 3 vector fields

belong to Adj. Representation of $SU(2)$

multiplet of $SU(2)$

$$A_\mu(x) = \sum_a A_\mu^a(x) t_a$$

2×2

2×2 antisym.
 traceless matrix

Lorentz 1-form connection $SU(2)$

$$A(x) = \sum_\mu A_\mu(x) \cdot dx^\mu$$

$\uparrow$ 1-form

1-form spacetime

Adj($SU(2)$)

$J_\mu = J_\mu^a t_a$ 2×2 traceless antisymmetric matrix
element of $\text{Lie}(SU(2))$

$$J_\mu^a = \bar{\Psi} \gamma_\mu t_a \Psi$$

$$J_\mu^a = \bar{\Psi}_\alpha (\gamma_\mu)^{\alpha\beta} (t_a)_{ij} \Psi_j$$

$\leftarrow$ Dirac spin $1/2$ representation of Lorentz Group

Lorentz $\mathbb{R}^{1,3}$

$$A(x) = \sum_\mu A_\mu(x)$$

Unitarity $\leftrightarrow$ Gauge Invariance

rigid symmetry $\phi \rightarrow g \phi, \psi \rightarrow g \psi$ Fundamental repr

$A \rightarrow g A g^{-1}$ Adjoint representation

$J \rightarrow g J g^{-1}$

↓
local symmetry

$A \cdot J \quad \partial_\mu J^\mu = 0$

current - vector coupling $\rightarrow$ covariant derivative

element of $\text{Lie}(SU(2))$

$$J_\mu^\alpha = \bar{\psi}_\alpha \left(\gamma_\mu \right)_{\beta\gamma} (t_a)_{ij} \psi_j$$

Dirac spin $1/2$ representation of Lorentz group

Unitarity $\leftrightarrow$ Gauge Invariance

rigid symmetry $\phi \rightarrow g \phi, \psi \rightarrow g \psi$ Fundamental repr

$A \rightarrow g A g^{-1}$ Adjoint representation

$J \rightarrow g J g^{-1}$



local symmetry

g is a local function of x

$\phi(x) \rightarrow g(x) \phi(x)$

$A(x) \rightarrow$

$g \rightarrow g(x)$ gauge transformation

$A \cdot J \quad \partial_\mu J^\mu = 0$

current-vector coupling $\rightarrow$ covariant derivative

$g(x) = 1 + i \alpha(x)$

2×2 antisym. traceless matrix
Adjoint representation of $SU(2)$

infinitesimal gauge transformation

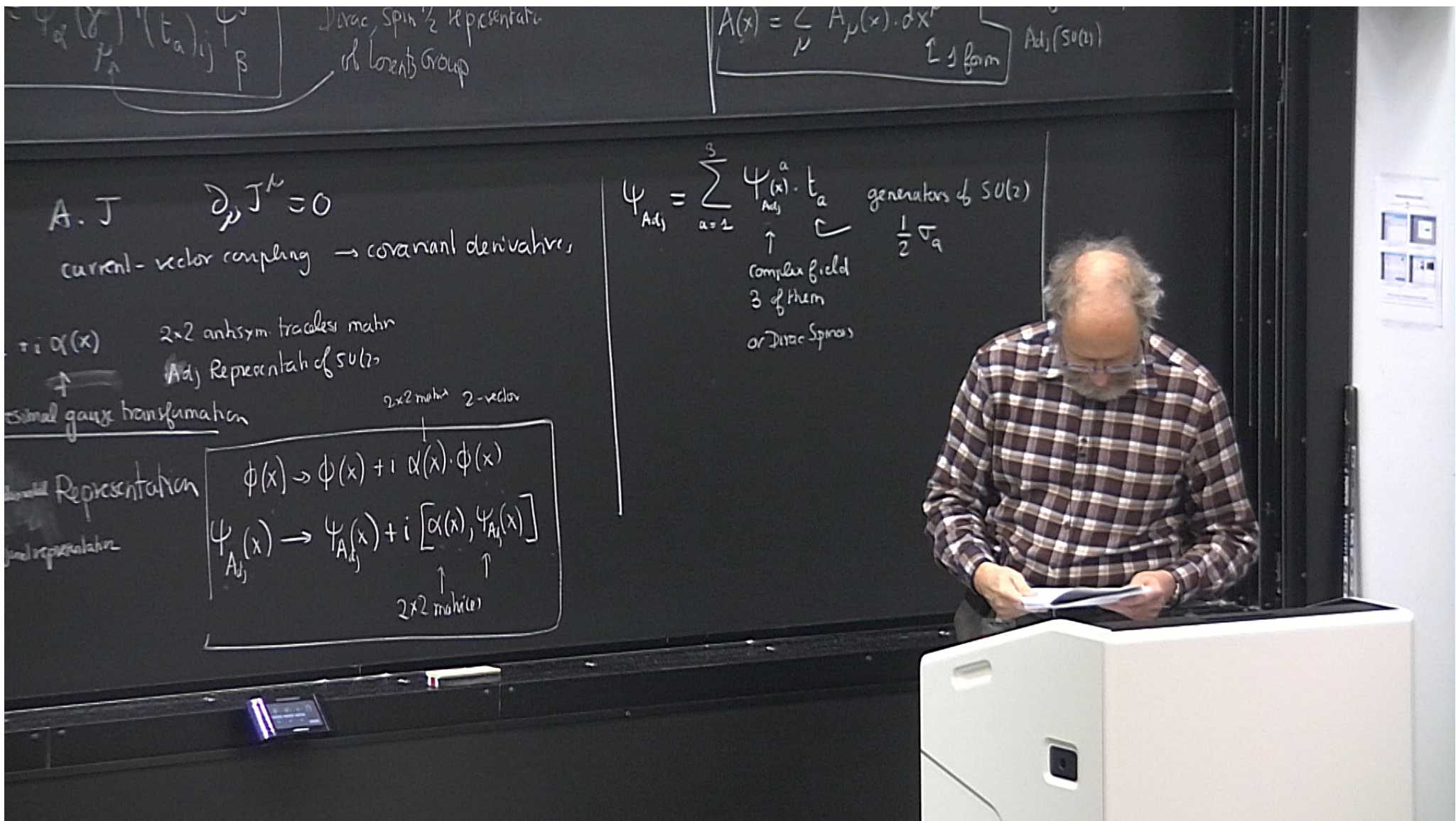
2×2 matrix 2-vector

$\phi(x)$ Fundamental Representation

$\psi_A(x)$ Adjoint representation

$$\begin{aligned} \phi(x) &\rightarrow \phi(x) + i \alpha(x) \cdot \phi(x) \\ \psi_A(x) &\rightarrow \psi_A(x) + i [\alpha(x), \psi_A(x)] \end{aligned}$$

2×2 matrix



$\psi_\alpha(x) \quad (t_a)_{ij} \quad \beta$
 Dirac, Spin 1/2 representation
 of Lorentz Group

$$A(x) = \int_{\mu} A_{\mu}(x) \cdot dx^{\mu} \quad \text{1 form} \quad \text{Adj}(SU(2))$$

$A \cdot J \quad \partial_{\mu} J^{\mu} = 0$
 current-vector coupling $\rightarrow$ covariant derivative,

$\rightarrow i \alpha(x)$ 2x2 antisym. traceless matrix
 Adj Representation of $SU(2)$
 infinitesimal gauge transformation

Representation
 adjoint representation

$$\begin{aligned}
 \phi(x) &\rightarrow \phi(x) + i \alpha(x) \cdot \phi(x) \\
 \psi_{A_j}(x) &\rightarrow \psi_{A_j}(x) + i [\alpha(x), \psi_{A_j}(x)]
 \end{aligned}$$

$\uparrow$ 2x2 matrix $\uparrow$ 2-vector

$$\psi_{A_j} = \sum_{a=1}^3 \psi_{A_j}^a \cdot t_a \quad \text{generators of } SU(2)$$

$\uparrow$ complex field
 3 of them
 or Dirac Spinors

$\frac{1}{2} \sigma_a$