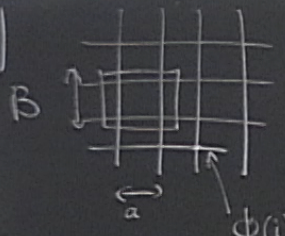
Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 10

Date: Nov 27, 2018 09:00 AM

URL: http://pirsa.org/18110015

Abstract:

1]



UV. regulator $|k| < \Lambda$

or $\Lambda = \pi/a$

$\mathcal{A}[\phi]$ $\quad Z = \int \Pi d\phi_i \, e^{-\mathcal{A}[\phi]}$

Block Transformations, Ren. Transf

- Integrate over fluctuations of ϕ inside a "block" Block size B

$\mathcal{A}[\phi] \longrightarrow \mathcal{A}_{eff}[\hat{\phi}]$, $\hat{\phi}$ = averaged field

mesh, cut off $\Lambda' = \Lambda/B$, $a' = a \times B$

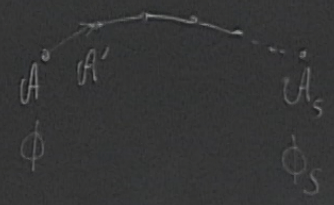
- Rescale distances $x' \to x = x'/B$

$k' \to k = k' \cdot B$

and the field $\tilde{\phi} \to \phi'$

so that $\mathcal{A}'[\phi]$ renormalized action

- Iterate this N times $\quad S = B^N$ scaling factor, $x_s = x/s$



$\phi_s(x_s) = Z(s)\phi(x)$

↑ field ren. factor

$x' \to x = x'/B$

$k' \to k = k' \cdot B$

field $\tilde{\phi} \to \phi'$

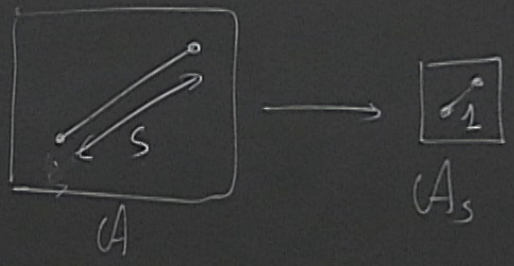
renormalized action

$= B^N$ scaling factor, $x_s = x/s$

$\phi_s(x_s) = Z(s)\,\phi(x)$

↑ field ren. factor

$$\langle \phi(x_1 s)\, \phi(x_2(s)) \cdots \rangle_{\mathcal{A}} = Z(s)^{-2} \langle \phi(x_1)\, \phi(x_2) \rangle_{\mathcal{A}_s}$$



coupling or parameter

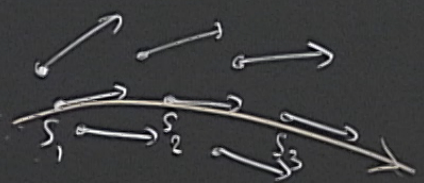
local operators

mapping in space of "actions" $\mathcal{A}[\phi] = \sum_i \sum_\alpha K_\alpha O_\alpha[\phi(i), \nabla_\mu \phi(i), \cdots]$

or space of parameters $\{K_\alpha\} \to \{K_\alpha(s)\}$

Treat s as a continuous paramter: flow local

2] RG Flow is generated by a Vector Field in space of couplings



$$W = \{ W_\alpha(\{K\}) \}_\alpha$$ Vector

$K(s) = \{ K_\alpha(s) \}$ as $s \nearrow$

$$s\frac{d}{ds} K_\alpha(s) = W_\alpha(\{K(s)\})$$ differential flow equation

Knowing the W_α's

$\Updownarrow$

Know $\mathcal{A} \to \mathcal{A}_s$

space of couplings

ector

differential flow
equation

A simple but important example:

Euclidean space d = dim

$$* \quad \mathcal{A}[\phi] = \int \left[\frac{1}{2} (\partial_\nu \phi)^2 + V(\phi) \right] d^d x$$

keep this term

↑ general function

Landau-Ginzburg-Wilson Actions or Hamiltonians

ϕ is a scalar field

* as a regulator: Momentum Shell regulator

$$\phi(x) \xrightarrow[FT]{} \hat{\phi}(k) \leftarrow \text{defined only for } |k| < \Lambda$$



couplings

A simple but important example.

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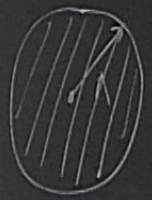
↑ general function (local potential)

Landau-Ginzburg-Wilson Actions or Hamiltonians

ϕ is a scalar field

* as a regulator: Momentum Shell regulator

$$\phi(x) \xrightarrow[FT]{} \hat{\phi}(k) \quad \text{defined only for } |k| < \Lambda$$

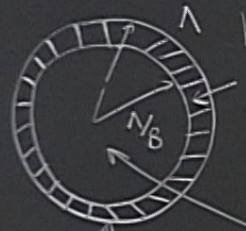


RG calculation using the Local Potential Approximation LPA

- Action $\longleftrightarrow$ $V(\phi)$
- one loop calculation

mesh, cutoff $\Lambda' = \Lambda/B$, $a' = a \times B$ | ϕ ϕ_s

Renormalization transformation. integrate over $\frac{\Lambda}{B} < |k| < \Lambda$



high momentum shell

$$\hat{\phi}(k) = \hat{\phi}_{slow}(k) + \hat{\phi}_{fast}(k)$$

$|k| < \frac{\Lambda}{B}$, $\frac{\Lambda}{B} < |k| < \Lambda$

very standard

$$e^{-\mathcal{A}_{eff}[\phi_{slow}]} = \int \mathcal{D}[\phi_{fast}]\, e^{-\mathcal{A}[\phi_{slow} + \phi_{fast}]}$$

ϕ_{slow} is defined for momenta $|k| < \frac{\Lambda}{B}$

Treat S as a continuous parameter: flow local

Think about the effective action at 1 loop

$$\Gamma[\varphi] = \mathcal{A}[\varphi] + \frac{1}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(-\Delta + V''(\varphi)\right)\right]$$

diff operator acting on function

$$\int d^d x \,\langle x|\, \log(-\Delta + V''(\varphi))\,|x\rangle$$

this kernel is complicated because $\varphi(x)$ is not a constant function

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↓ replace this by ← here, treat φ as if it was a constant: value of φ at x

$$\int d^d x \,\langle x|\, \log\left(-\Delta + V''(\varphi(x))\right)|x\rangle$$

now a number

linear trace-class operator $\mathcal{O}$ acting on functions $\psi(x)$

$$[\mathcal{O}\psi](x) = \int d^d y\, O(x,y)\psi(y)$$

$O(x,y)$: kernel of $\mathcal{O}$

$|x\rangle$ is a basis

$$1 = \int d^d x\, |x\rangle\langle x|$$

$$O(x,y) = \langle x|\mathcal{O}|y\rangle$$ bra-ket notation of QM

$$\mathrm{Tr}[\mathcal{O}] = \int d^d x \langle x|\mathcal{O}|x\rangle$$ definition of the trace

Treat S as a continuous parameter: flow local

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↑ now a number

$$V(\varphi) = \frac{K_2}{2}\varphi^2 + \frac{K_4}{4!}\varphi^4 + \frac{K_6}{6!}\varphi^6 \dots$$

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eigenfunctions $\left(-\Delta + V''(\varphi(x_0))\right) e^{ikx} = \left[k^2 + V''(\varphi(x_0))\right] e^{ikx}$

x_0 is fixed

e^{ikx} ← eigen-state

$k^2 + V''(\varphi(x_0))$ ← eigen-value

Treat S as a continuous parameter: flow local

Think about the effective action at 1 loop

$$\Gamma[\varphi] = A[\varphi] + \frac{1}{2}\,\mathrm{Tr}\left[\mathrm{Log}\left(-\Delta + V''(\varphi)\right)\right]$$

diff operator acting on function

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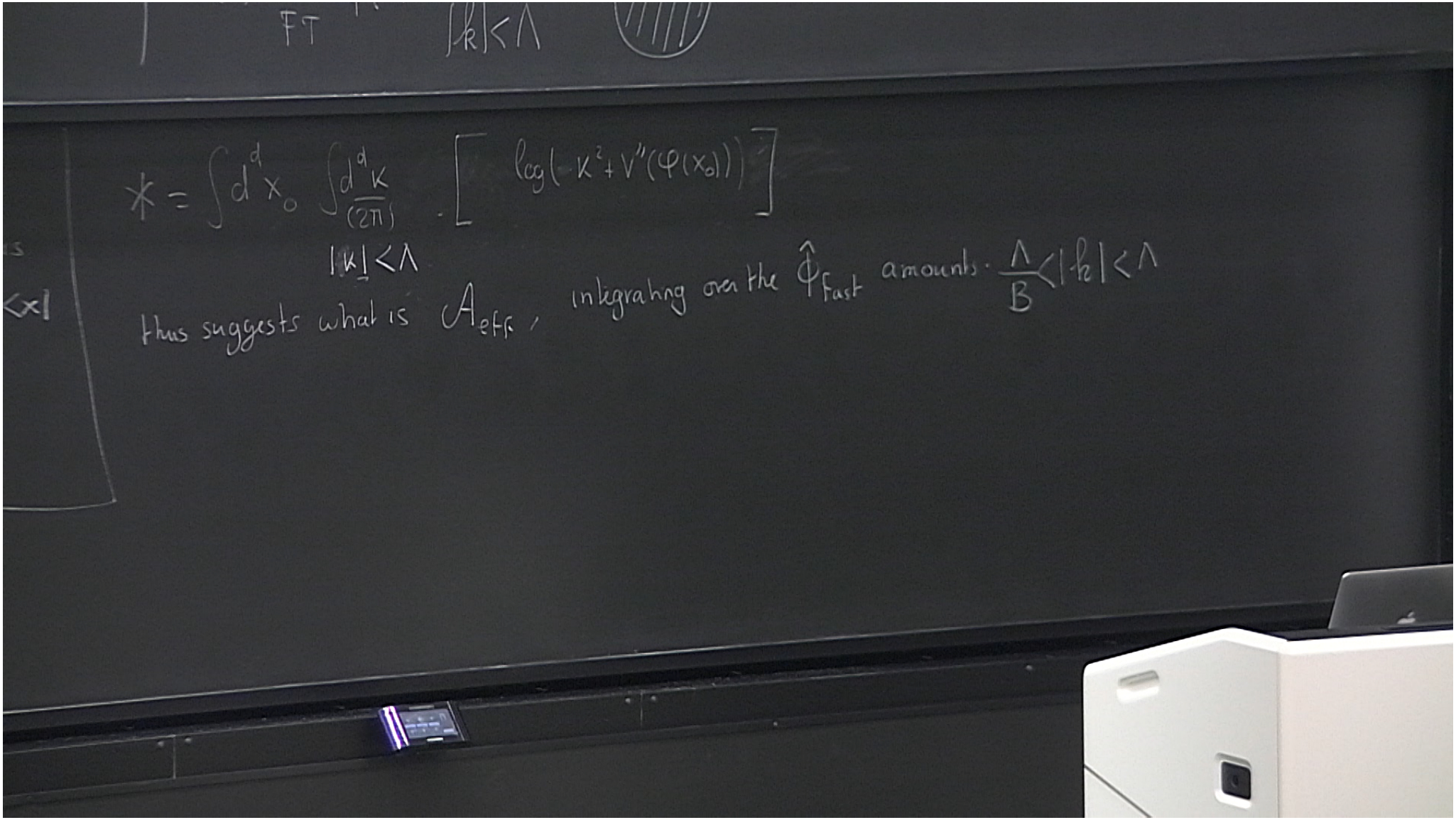
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$$* = \int d^d x \langle x | \log(-\Delta + V''(\varphi(x))) | x \rangle$$

now a number

$$V(\varphi) = \frac{K_2}{2}\varphi^2 + \frac{K_4}{4!}\varphi^4 + \frac{K_6}{6!}\varphi^6 \ldots$$

$$V''(\varphi) = K_2 + \frac{K_4}{2}\varphi^2 + \frac{K_6}{4!}\varphi^4 \ldots$$


FT
$|k|<\Lambda$
$* = \int d^d x_0 \int \frac{d^d k}{(2\pi)} \left[\log(-k^2 + V''(\varphi(x_0))) \right]$
$|k|<\Lambda$
thus suggests what is $\mathcal{A}_{eff}$, integrating over the $\hat{\phi}_{fast}$ amounts. $\frac{\Lambda}{B}<|k|<\Lambda$

* as a regulator: Momentum Shell regulator

$\phi(x) \xrightarrow[FT]{} \hat{\phi}(k)$ ← defined only for $|k| < \Lambda$



$$* = \int d^d x_0 \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d} \cdot \left[\log\left(k^2 + V''(\varphi(x_0)) \right) \right]$$

thus suggests what is A_{eff}, integrating over the $\hat{\phi}_{fast}$ amounts. $\frac{\Lambda}{B} < |k| < \Lambda$

A function of $\phi_{slow}(x_0)$

Result

$$A_{eff}[\phi_{slow}] = A[\phi_{slow}] + \frac{1}{2} \int d^d x_0 \int_{\frac{\Lambda}{B}<|k|<\Lambda} \frac{d^d k}{(2\pi)^d} \log\left[k^2 + V''(\phi_{slow}(x_0)) \right]$$

Γ_S

$\dot{\Gamma}_S$

Σ field ren. factor

... space of parameters $\{K_q\} \longrightarrow \{K_q(S)\}$

Treat S as a continuous parameter: flow local

Λ, $V(\varphi) \to \Gamma$

$\Lambda' = \frac{\Lambda}{B}$, $V_{eff}(\varphi) \to \Gamma^{II}_{eff}$

$\frac{\Lambda}{B}$

Think about the effective action at 1 loop

$$\Gamma[\varphi] = \mathcal{A}[\varphi] + \frac{1}{2} \operatorname{Tr}_{\Lambda}\left[\operatorname{Log}\left(-\Delta + V''(\varphi)\right)\right]$$

diff operator acting on function

$$\int d^d x \, \langle x | \log(-\Delta + V''(\varphi)) | x \rangle$$

this kernel is complicated because $\varphi(x)$ is not a constant function

here, treat φ as if it was a constant: value of φ at x

$\downarrow$ replace this by

$$* = \int d^d x \, \langle x | \log(-\Delta + V''(\varphi(x))) | x \rangle$$

$\varphi(x)$ now a number

$$V(\varphi) = \frac{K_2}{2}\varphi^2 + \frac{K_4}{4!}\varphi^4 + \frac{K_6}{6!}\varphi^6 \ldots$$

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eigenfunctions $\left(-\Delta + V''(\varphi(x_0))\right) e = \left[k^2 + V''(\varphi(x_0))\right] e$

x_0 is fixed

eigen-state — eigen-value

$$A_{eff}[\phi_{slow}] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2_{slow} + \underbrace{V[\phi_{slow}(x)] + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \log\left[k^2 + V''(\phi_{slow}(x_0))\right]}_{\frac{\Lambda}{B} < |k| < \Lambda} \right] = \text{new } V_{eff}(\phi_{slow})$$

Result of integrating over the ϕ_{fast} modes

$V[\phi] \to V_{eff}[\phi_{slow}]$

$|k| < \Lambda \qquad |k| < \Lambda/B$

Rescale the distances and the field

$x \to x' = x/B$

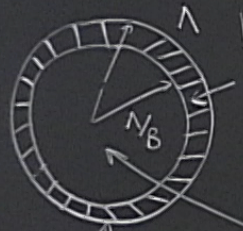
$k \to k' = k \cdot B$

$\phi_R(x') = B^{\Delta} \phi_{slow}(x)$ with $\Delta = \frac{d-2}{2}$ dimension of the field

renormalized field

mesh, cutoff $\Lambda' = \Lambda/B$, $a' = a \times B$ | ϕ ϕ_s

Renormalization transformation. integrate over $\frac{\Lambda}{B} < |k| < \Lambda$



high momentum shell

$$\hat{\phi}(k) = \hat{\phi}_{slow}(k) + \hat{\phi}_{fast}(k)$$

$|k| < \frac{\Lambda}{B}$, $\frac{\Lambda}{B} < |k| < \Lambda$

very standard

$$e^{-\mathcal{A}_{eff}[\phi_{slow}]} = \int \mathcal{D}[\phi_{fast}]\, e^{-\mathcal{A}[\phi_{slow} + \phi_{fast}]}$$

ϕ_{slow} is defined for momenta $|k| < \frac{\Lambda}{B}$

neglected $(\nabla\phi)^2_{slow}$

$(\nabla^2\phi)^2_{slow}$, $(\nabla\phi_{slow})^4$, ... $\Leftarrow$ *

Think ab

$\Gamma[$

$\int d^d x <$

$\int d^d x <$

eigenfunctions $(-\Delta + V(\varphi(x_0)))\,e = k^2 + V(\varphi(x_0))\,e$

x_0 is fixed

eigen-state — eigen-value

$$\mathcal{A}_{eff}[\phi_{slow}] = \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)^2_{slow} + \left[V[\phi_{slow}(x)] + \frac{1}{2}\int_{\frac{\Lambda}{B}<|k|<\Lambda} \frac{d^d k}{(2\pi)^2} \log\left[k^2 + V''(\phi_{slow}(x))\right] \right] \right] = \text{new } V_{eff}(\phi_{slow})$$

result of integrating over the ϕ_{fast} modes

$$V[\phi] \to V_{eff}[\phi_{slow}]$$

$|k|<\Lambda \qquad |k|<\Lambda/B$

Rescale the distances and the field

$$x \to x' = x/B$$

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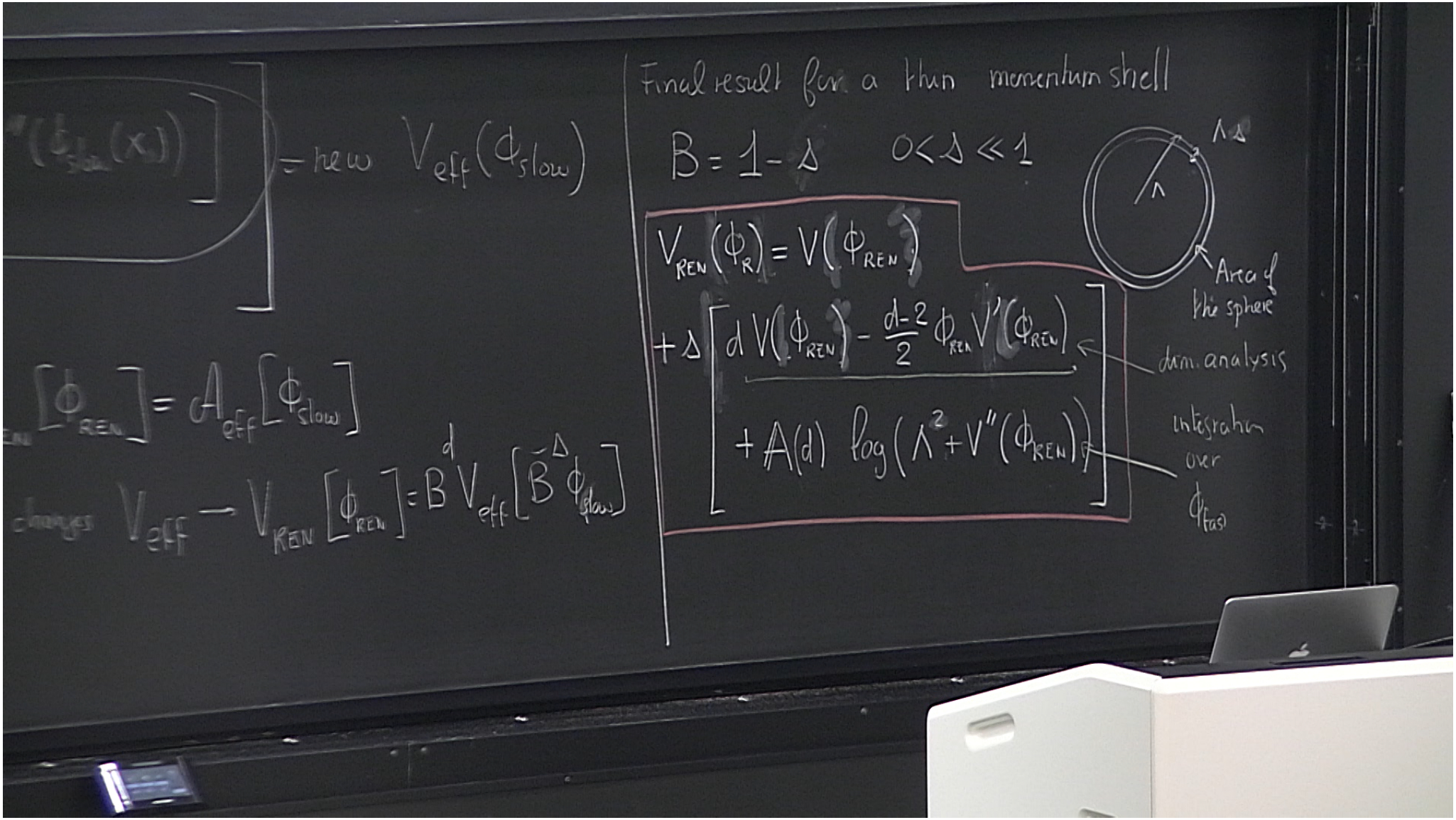
$$\phi_R(x') = B^{\Delta}\,\phi_{slow}(x)$$

renormalized field

with $\Delta = \frac{d-2}{2}$ dimension of the field

$$\mathcal{A}_{REN}[\phi_{REN}] = \mathcal{A}_{eff}[\phi_{slow}]$$

this changes $V_{eff} \to V_{REN}[\phi_{REN}] = B^d V_{eff}[B^{\Delta}\phi_{slow}]$


$= \text{new } V_{eff}(\phi_{slow})$
$\left[\phi_{REN}\right] = A_{eff}\left[\phi_{slow}\right]$
changes $V_{eff} \to V_{REN}\left[\phi_{REN}\right] = B^{d} V_{eff}\left[\tilde{B}^{\Delta}\phi_{slow}\right]$
Final result for a thin momentum shell
$B = 1 - \Delta \qquad 0 < \Delta \ll 1$
$V_{REN}(\phi_R) = V(\phi_{REN})$
$+\Delta\left[d\,V(\phi_{REN}) - \frac{d-2}{2}\phi_{REN}V'(\phi_{REN}) + A(d)\,\log\left(\Lambda^2 + V''(\phi_{REN})\right)\right]$
$\Lambda\Delta$
Λ
Area of the sphere
dim. analysis
integration over ϕ_{fast}

$(\nabla^2\phi_{slow})^2, (\nabla\phi_{slow})^2, \ldots$

Iterate to rescaling S

$\boxed{\Lambda = 1}$ ← normalization

$A = \left[(4\pi)^{d/2}\,\Gamma(d/2)\right]^{-1}$ Euler Γ function

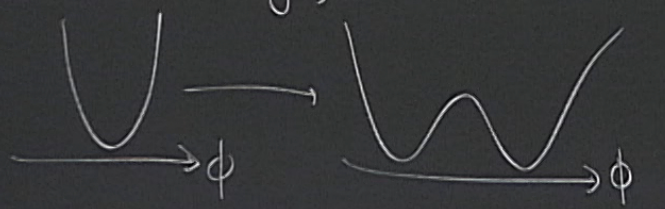
$V(\phi) \longrightarrow V_s(\phi)$ renormalized local potential

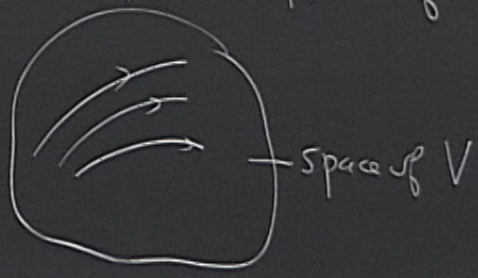
!

$$\boxed{S\frac{d}{dS}V_s(\phi) = d\,V_s(\phi) - \frac{d-2}{2}\phi\frac{\partial}{\partial\phi}V_s(\phi) + A\left[\log\left(1+\frac{\partial^2}{\partial\phi^2}V_s(\phi)\right)\right]}$$

Flow equation on the function V_s . At each step, Λ is fixed

linear in $\log S$, but non linear in ϕ





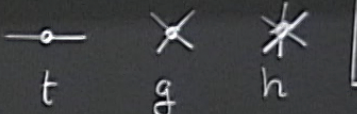
now a number

$$V''(\varphi) = K_2 + \frac{K_4}{2}\varphi^2 + \frac{K_6}{4!}\varphi^4 \ldots$$

Γ functional

$$V(\phi) = \frac{t}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{h}{6!}\phi^6$$

3 parameters

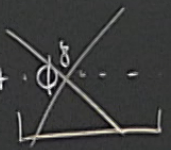


$A = 1$

$$\phi\frac{\partial}{\partial\phi}V = t\phi^2 + \frac{g}{3!}\phi^4 + \frac{h}{5!}\phi^6$$

$$\frac{\partial^2}{\partial\phi^2}V = t + \frac{g}{2}\phi^2 + \frac{h}{4!}\phi^4$$

$$\log\left(1 + \frac{\partial^2 V}{\partial\phi^2}\right) = \cdot + \cdot\phi^2 + \cdot\phi^4 + \cdot\phi^6 + \cdot\phi^8 \ldots$$



expand it in powers of ϕ and truncate

at order ϕ^6

$d = 4$

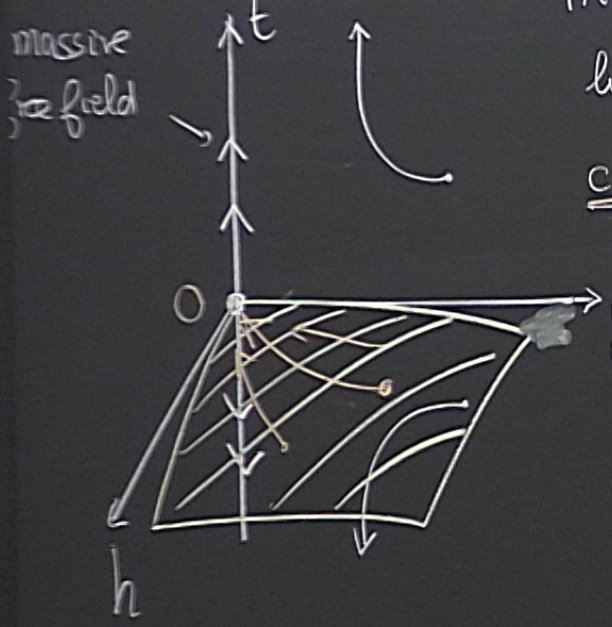
RG $\rightarrow t_s, g_s, h_s$

$$s\frac{d}{ds}t_s = W_t = 2t_s + g_s - t_s g_s$$

$$s\frac{d}{ds}g_s = W_g = h_s - 3g_s^2 - t_s h_s$$

$$s\frac{d}{ds}h_s = W_h = -2h_s - 15 g_s h_s$$

flow in a 3dim space of couplings



trivial fixed point $0 = (0,0,0)$ massless free field

line flowing out of 0

critical surface : separates UP $t_5 \nearrow +\infty$

DOWN $t_5 \searrow -\infty$

on the surface flow toward 0

attractive domain of the fixed point

$$t_{crit} = t(g,h)$$

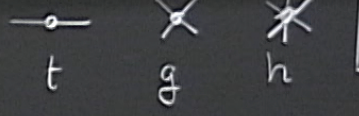
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3 parameters

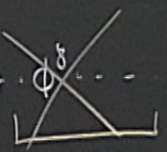


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expand it in powers of ϕ and truncate

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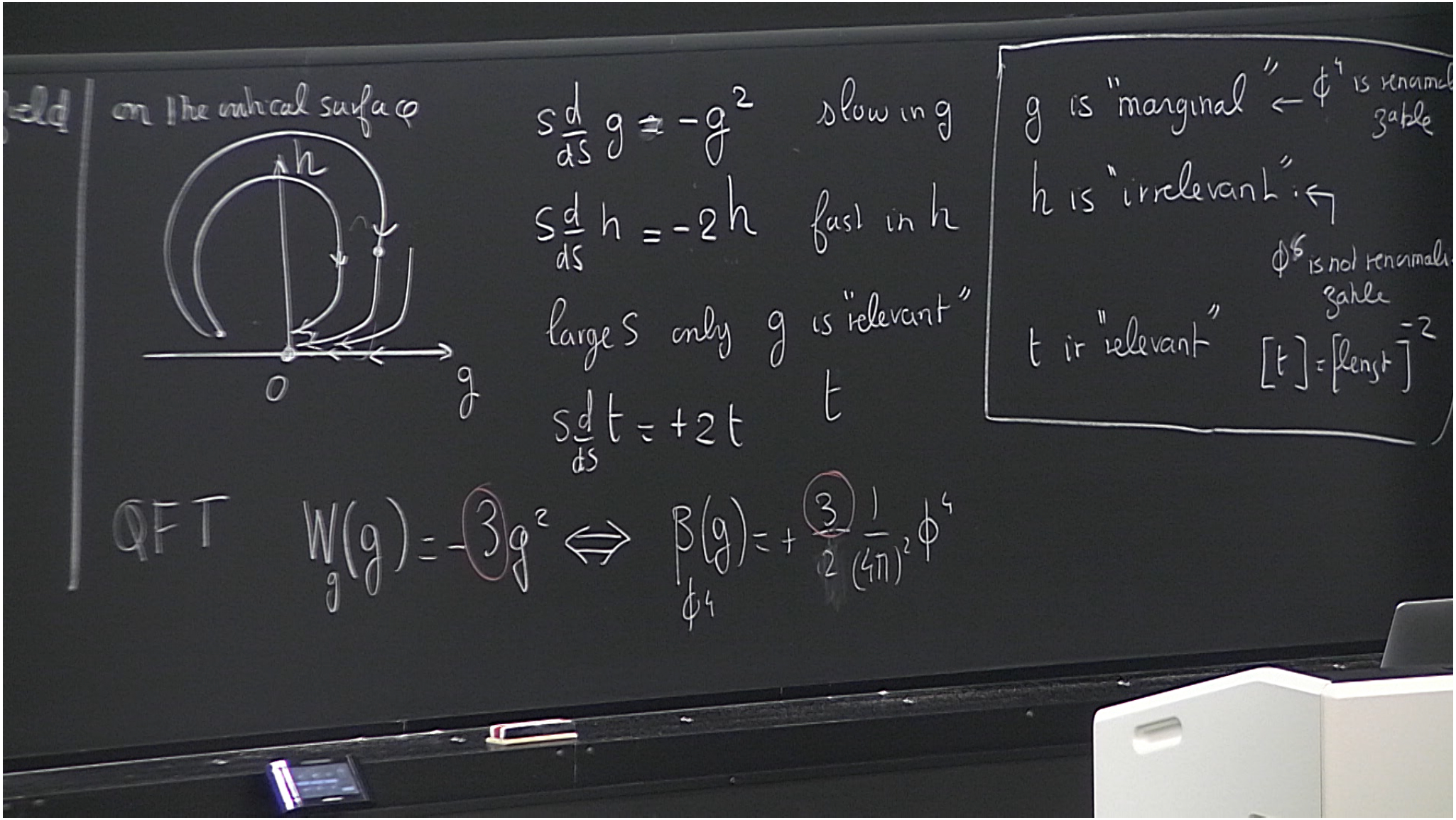
RG → t_s, g_s, h_s

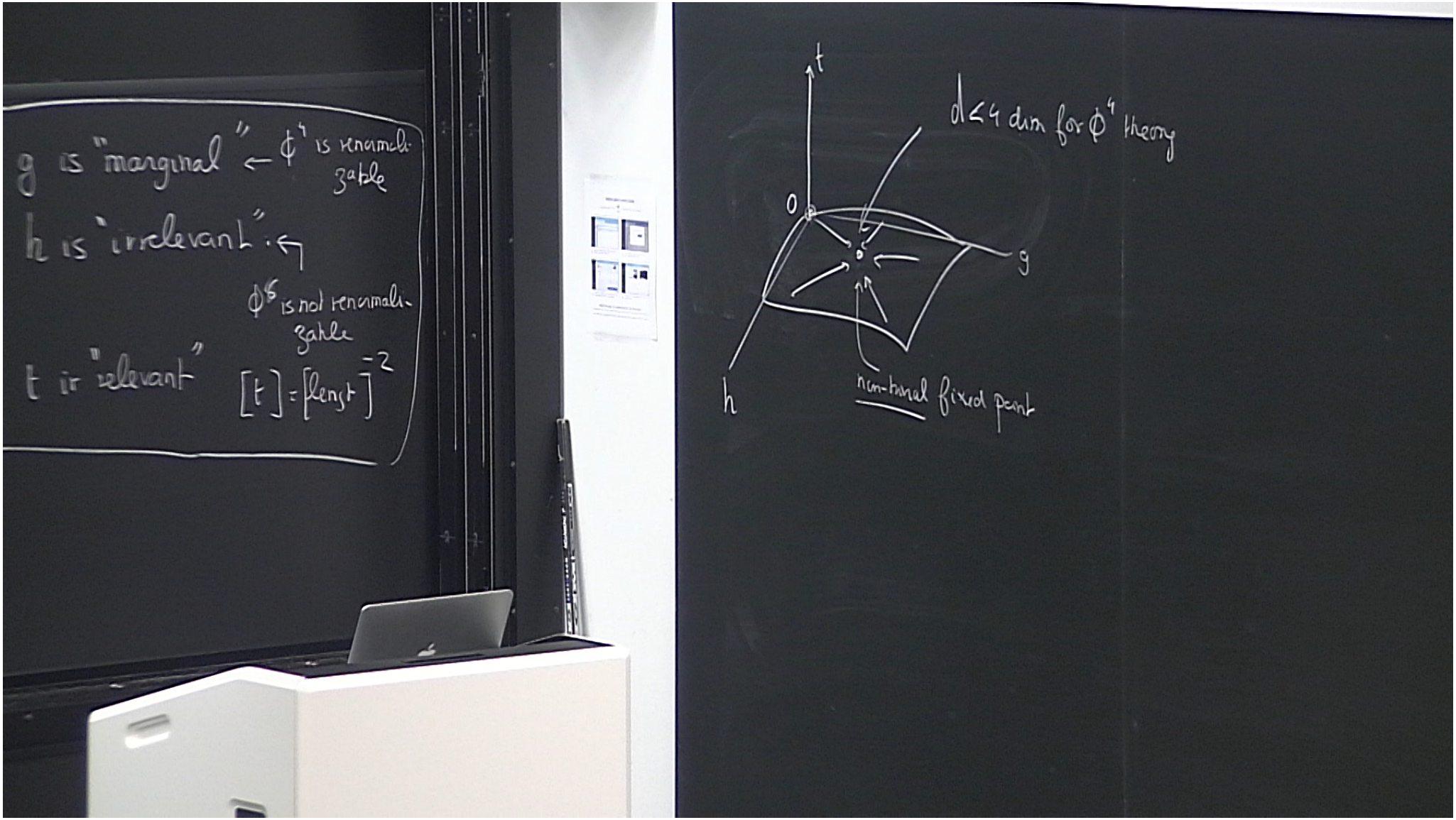
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$$s\frac{d}{ds}g_s = W_g = h_s - 3g_s^2 - t_s h_s$$

$$s\frac{d}{ds}h_s = W_h = -2h_s - 15 g_s h_s$$

flow in a 3dim space of couplings


eld
on the critical surface
h
0
g
$s\frac{d}{ds}g = -g^2$ slowing g
$s\frac{d}{ds}h = -2h$ fast in h
large s only g is "relevant"
$s\frac{d}{ds}t = +2t$ t
g is "marginal" ← ϕ^4 is renormalizable
h is "irrelevant" ←
ϕ^6 is not renormalizable
t is "relevant"
$[t] = [\text{length}]^{-2}$
QFT
$W_g(g) = -3g^2 \Leftrightarrow \beta_{\phi^4}(g) = +\frac{3}{2}\frac{1}{(4\pi)^2}\phi^4$


g is "marginal" ← ϕ^4 is renormalizable
h is "irrelevant" ↩
ϕ^5 is not renormalizable
t is "relevant"
$[t] = [\text{length}]^{-2}$
t
$d \leq 4$ dim for ϕ^4 theory
0
g
h
non-trivial fixed point