

Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 9

Date: Nov 26, 2018 09:00 AM

URL: <http://pirsa.org/18110014>

Abstract:

Perturbative Renormalization theory QFT in space-time

- UV singularities in ϕ^4 $d=4 \Rightarrow \beta(g)$ beta function

scale or dilatation invariance of theory is anomalous

- renormalizable or not renormalizable?

- OPE local products of operators

$$\left(d^d x \frac{1}{2}\right)$$

x-time

n

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]^{d-4}$

x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimension of g

Δ_g

$d-4$

- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

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- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$$d=4 \quad \Delta_g=0, \quad d<4 \quad \Delta_g<0, \quad d>4 \quad \Delta_g>0$$

$$\text{loop} \rightarrow \log \Lambda \quad \text{loop} \rightarrow \text{finite}$$

time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimension of g

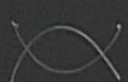
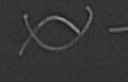
Δ_g

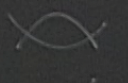
$\downarrow$

$d-4$

- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

 $\rightarrow \log \Lambda$,  $\rightarrow \text{finite}$

 $\sim \Lambda^{d-4}$


x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

- dimension

$$\phi = [\text{length}]^{\frac{2-d}{2}}$$

$$g = [\text{length}]$$

$$d = 4, \quad d < 4 \quad \Delta_g < 0, \quad d > 4 \quad \Delta_g > 0$$

$\rightarrow$ finite

$$\text{loop} \sim \Lambda^{d-4}$$

$$\text{triangle} \sim \Lambda^{d-6}$$

dimension of g

$$\Delta_g$$

$$d-4$$

x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimension of g

$$\Delta_g$$

$$d-4$$

- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$$d=4 \quad \Delta_g=0, \quad d<4 \quad \Delta_g<0, \quad d>4 \quad \Delta_g>0$$

$$\text{loop diagram} \rightarrow \log \Lambda, \quad \text{tadpole diagram} \rightarrow \text{finite}$$

$$\text{triangle diagram} \sim \Lambda^{d-4}$$

$$\text{bubble diagram} \sim \Lambda^{d-6} \Rightarrow \text{new counterterm} \Lambda^{d-6} \phi^6$$

time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

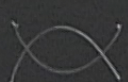
dimension of g

Δ_g

$d-4$

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$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

 $\rightarrow \log \Lambda$,  $\rightarrow \text{finite}$

 $\sim \Lambda^{d-4}$

 $\sim \Lambda^{d-6} \Rightarrow \text{new counterterm}$
 $\Lambda^{d-6} \phi^6$

only $\partial_\mu \phi$, ϕ^4 ,
 $(\partial \phi)^2$ and ϕ^2

x-time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimension of g

Δ_g

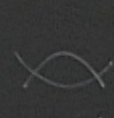
$d-4$

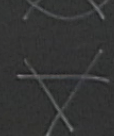
- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

 $\rightarrow \log \Lambda$

 $\rightarrow \text{finite}$

 $\sim \Lambda^{d-4}$

 $\sim \Lambda^{d-6} \Rightarrow \text{new counterterm}$
 $\Lambda^{d-6} \phi^6$

only reT if ϕ^4 ,

Superrenormalizable

$(\partial\phi)^2$ and ϕ^2
 renormalizable

time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

dimension of g

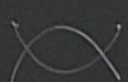
Δ_g

↓

$d-4$

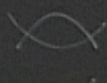
- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

 $\rightarrow \log \Lambda$,  $\rightarrow \text{finite}$

only CT if ϕ^4 ,
 $(\partial\phi)^2$ and ϕ^2
renormalizable

Superrenormalizable
no CT but a few
for ϕ^2

 $\sim \Lambda^{d-4}$
 $\sim \Lambda^{d-6}$

$\Rightarrow$ new counterterm
 $\Lambda^{d-6} \phi^6$

time

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

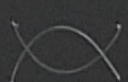
dimension of g

Δ_g

$d-4$

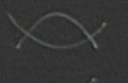
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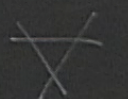
$d=4$ $\Delta_g=0$, $d<4$ $\Delta_g<0$, $d>4$ $\Delta_g>0$

 $\rightarrow \log \Lambda$,  $\rightarrow \text{finite}$

only CT for ϕ^4 ,
 $(\partial\phi)^2$ and ϕ^2
renormalizable

Superrenormalizable
no CT but a few
for ϕ^2

 $\sim \Lambda^{d-4}$

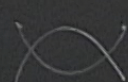
 $\sim \Lambda^{d-6} \Rightarrow$ new counterterm
 $\Lambda^{d-6} \phi^6$

non-renormalizable
 ∞ # of new CT associated to new operators

$$\int_{\mathbb{R}^d} d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right) = S[\phi]$$

- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]^{\frac{4-d}{2}}$

$d=4$ $\Delta_g = 0$, $d < 4$ $\Delta_g < 0$, $d > 4$

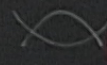
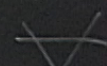
 $\rightarrow \log \Lambda$,  $\rightarrow \text{finite}$

only CT for ϕ^4 ,

$(\partial\phi)^2$ and ϕ^2

renormalizable

Superrenormalizable
no CT but a few
for ϕ^2

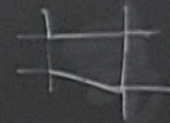


non
 ∞

new counter term no control

lead to new operators

dimension of g

Δ_g



$$\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 = S[\phi]$$

dimension of g

$$x = [\text{length}] \quad \phi = [\text{length}]^{\frac{2-d}{2}}$$

$$\Delta_g = 0, \quad d < 4 \quad \Delta_g < 0, \quad d > 4$$

$$\Lambda \rightarrow \log \Lambda, \quad \text{loop} \rightarrow \text{finite}$$

Super renormalizable
no CT but a few
for ϕ^2

$$g =$$

$\times$

$\times$

$\times$

$\times$

$\times$

$\times$

$\times$

$\times$

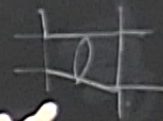
$\times$

$\times$

new counterterm no control

$$\Lambda^{d-6} \phi^6$$

restricted to new operators



div in 8 dim

space-time

$$\int_{\mathbb{R}^d} d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 \right] = S[\phi]$$

dimension of g

$$\Delta_g \downarrow \quad \textcircled{d-4}$$

- dimensions $x = [\text{length}]$ $\phi = [\text{length}]^{\frac{2-d}{2}}$ $g = [\text{length}]$

$$d=4 \quad \Delta_g=0, \quad d<4 \quad \Delta_g<0, \quad d>4 \quad \Delta_g>0$$

$$\text{loop} \rightarrow \log \Lambda$$

$$\text{loop} \rightarrow \text{finite}$$

$$\text{loop} \sim \Lambda^{d-4}$$

$$\text{loop} \sim \Lambda^{d-6} \Rightarrow \text{new counterterm } \Lambda^{d-6} \phi^6 \quad \text{no control}$$

only CT for ϕ^4 ,

$(\partial\phi)^2$ and ϕ^2
renormalizable

Superrenormalizable
no CT but a few
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non-renormalizable

∞ # of new CT associated to new operators

$$\frac{1}{2}(\partial_\mu \phi)^2 + \frac{g}{4} \phi^4 = S[\phi]$$

$$x = [\text{length}] \quad \phi = [\text{length}]^{\frac{2-d}{2}}$$

$$\Delta_g = 0, \quad d < 4 \quad \Delta_g < 0, \quad d > 4$$

$$\Lambda \rightarrow \log \Lambda, \quad \text{loop} \rightarrow \text{finite}$$

ϕ^4
 ϕ^2
 and ϕ^2
 renorm

Superrenormalizable
 no CT but $\leftarrow$ for
 for ϕ^2

dimension of g

Δ_g

$d-4$

$d-4$

$d-6 \Rightarrow$ new counterterm no control

$\Lambda^{d-6} \phi^6$

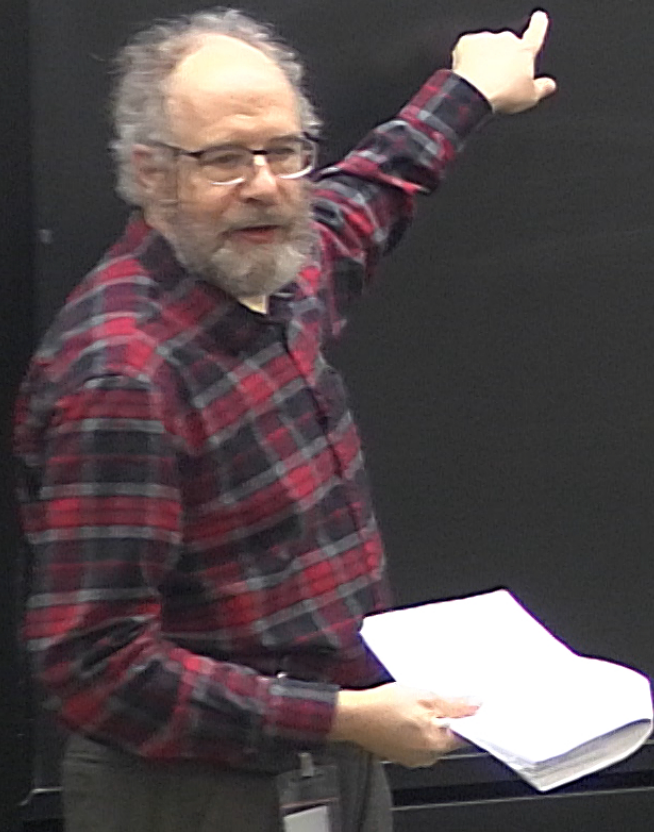
renormalizable

new CT associated to new operators

Fermi theory

General Relativity

2] Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y}$



2

Free Field

$$\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$$

↑ requires renormalization

2

Free Field

$$\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{-2-d} \mathbb{I} + \phi^2(x)$$

Operator Product Expansion

↑
new term

↑ requires renormalization

2] * Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$

Operator Product Expansion

$\uparrow$ new term

$\uparrow$ requires renormalization

* ϕ^4 theory, Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$

OPE for $\phi^4(x) \phi^4(y)$

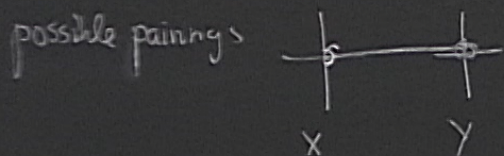
2] * Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$

Operator Product Expansion

$\uparrow$ new term $\uparrow$ requires renormalization

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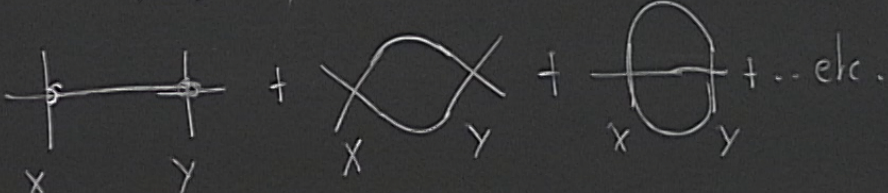
Operator Product Expansion

$\uparrow$ new term $\uparrow$ requires renormalization

* ϕ^4 theory, Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$

OPE for $\phi^4(x) \phi^4(y)$

possible pairings



$+ \dots$ etc.

2] * Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \cdot \mathbb{I} + \phi^2(x)$

Operator Product Expansion

↑ requires renormalization

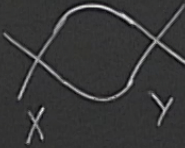
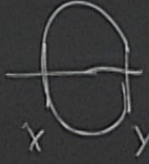
new term

* Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$

$\phi^4(x) \phi^4(y)$

pairings

$|x-y|^{2-d}$

+  +  + ... etc.

2] * Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$

Operator Product Expansion

$\uparrow$ new term $\uparrow$ requires renormalization

* ϕ^4 theory, Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$

OPE for $\phi^4(x) \phi^4(y)$

possible pairings

$\text{---} \times \text{---} + \text{---} \text{---} + \text{---} \text{---} + \dots \text{etc.}$

$|x-y|^{2-d} \times \text{---} \times \text{---} = \phi^6$

2] * Free Field $\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$

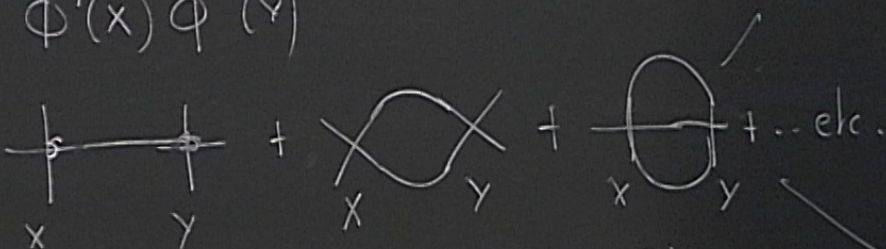
Operator Product Expansion

new term $\uparrow$ requires renormalization

* ϕ^4 theory, Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$

OPE for $\phi^4(x) \phi^4(y)$

possible pairings



$|x-y|^{2-d} \times$ (diagram of two vertices connected by a single line) $= \phi^6 + (x-y)^{4-2d} \times$ (diagram of two vertices connected by two lines) $= \phi^4$

ϕ^2

$$\phi^i(x) \phi^j(y) = |x-y|^{(2-d) \times i} \prod + |x-y|^{(2-d) \times 3} \phi^2(x) + |x-y|^{(2-d) \times 2} \phi^4(x) + |x-y|^{(2-d)} \phi^6(x)$$

$$\begin{aligned}
 \phi^i(x) \phi^j(y) \underset{x \rightarrow y}{=} & |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + |x-y|^{(2-d) \times 2} \phi^4(x) + |x-y|^{(2-d)} \phi^6(x) \\
 & + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}
 \end{aligned}$$

$$\begin{aligned}
 \phi^i(x) \phi^j(y) \underset{x \rightarrow y}{=} & |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \phi^4(x)} + |x-y|^{(2-d)} \phi^6(x) \\
 & + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}
 \end{aligned}$$

2] * Free Field

$$\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \cdot \mathbb{I} + \phi^2(x)$$

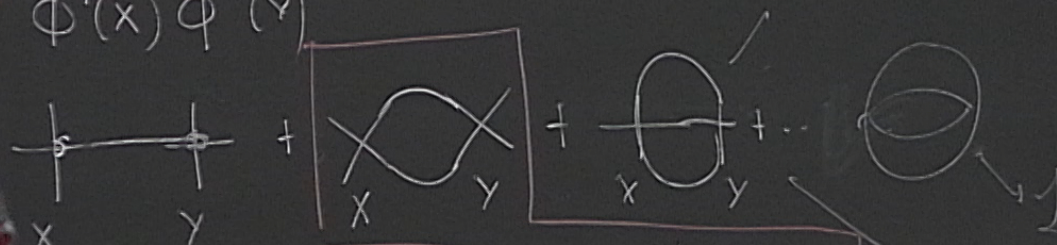
$\uparrow$ new term $\uparrow$ requires renormalization

$$\phi^4(x) \phi^4(y) \xrightarrow{x \rightarrow y} \dots$$

Operator Product Expansion

$$\phi^4 \text{ Theory } \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

$$\phi^4(x) \phi^4(y)$$



$$|x-y|^{2-d} \times \text{diagram} = \phi^6 + \dots$$

2] * Free Field

$$\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$$

Operator Product Expansion

new term

requires renormalization

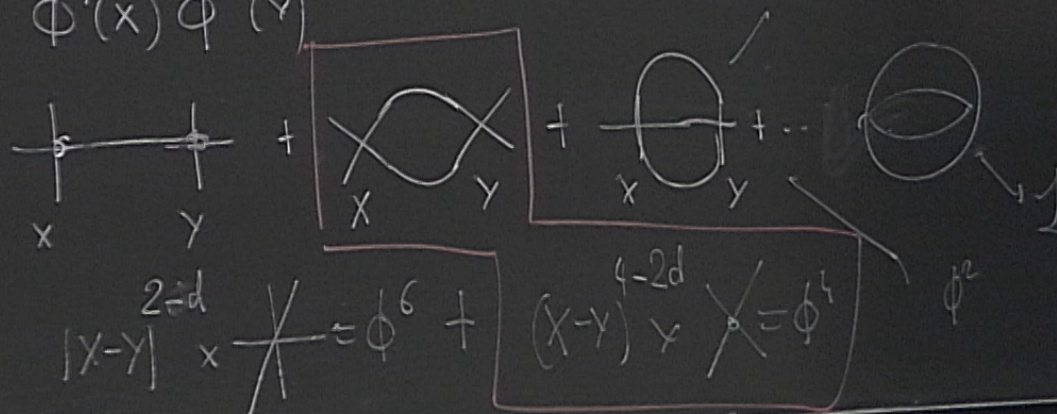
$$\phi^4(x) \phi^4(y) =$$

$$\int d$$

$$* \phi^4 \text{ theory, Per. Theory } \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle dx_1 \dots dx_2 \dots$$

OPE for $\phi^4(x) \phi^4(y)$

possible pairings



$$\phi^4(x) \phi^4(y) \underset{x \rightarrow y}{=} |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \dot{\phi}^4(x)} + |x-y|^{(2-d)} \phi^6(x) + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}$$

$$\int dx_2 \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

↑

$$\phi^4(x) \phi^4(y) \underset{x \rightarrow y}{=} |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \phi^4(x)} + |x-y|^{(2-d)} \phi^6(x) + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}$$

divergent

$$\int dx_2 \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

$\uparrow |x_2 - x_1| \rightarrow 0 \Rightarrow (x_1 - x_2)^{4-2d} \cdot d^d x_2 \Rightarrow \log \Lambda \text{ if } d=4$
 divergent if $d \geq 4$

$$\phi^4(x) \phi^4(y) \underset{x \rightarrow y}{=} |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \phi^4(x)} + |x-y|^{(2-d)} \phi^6(x) + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}$$

divergent

$$\int dx_2 \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

$$\uparrow |x_2 - x_1| \rightarrow 0 \Rightarrow (x_1 - x_2)^{4-2d} \cdot d^d x_2 \Rightarrow \log \Lambda \text{ if } d=4$$

$$\Downarrow$$

$$\text{divergent if } d \geq 4$$

$$\log \Lambda \langle \dots \phi^4(x_1) \dots \rangle$$

$$\phi^4(x) \phi^4(y) \underset{x \rightarrow y}{=} |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \phi^4(x)} + |x-y|^{(2-d)} \phi^6(x) + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}$$

divergence

$$\int dx_2 \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

$$\uparrow |x_2 - x_1| \rightarrow 0 \Rightarrow (x_1 - x_2)^{4-2d} \cdot d^d x_2 \Rightarrow \log \Lambda \text{ if } d=4$$

divergent if $d \geq 4$

$$\Downarrow$$

$$\log \Lambda \langle \dots \phi^4(x_1) \dots \rangle$$

short divergences $\log \Lambda$
are proportional to
local operators !

2] * Free Field

$$\phi(x) \phi(y) \xrightarrow{x \rightarrow y} |x-y|^{2-d} \mathbb{I} + \phi^2(x)$$

$\uparrow$ new term $\uparrow$ requires renormalization

Operator Product Expansion

$$\phi^4(x) \phi^4(y) \xrightarrow{x \rightarrow y}$$

* ϕ^4 theory, Per. Theory $\langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle dx_1 \dots dx_2 \dots$

OPE for $\phi^4(x) \phi^4(y)$

possible pairings

$$\begin{array}{c}
 \text{diagram 1: } \text{---} \text{---} \text{---} \\
 \text{diagram 2: } \text{---} \text{---} \text{---} \\
 \text{diagram 3: } \text{---} \text{---} \text{---} \\
 \text{diagram 4: } \text{---} \text{---} \text{---}
 \end{array}$$

$|x-y|^{2-d} \times \text{diagram 1} = \phi^6 + \dots$

$$\begin{array}{c}
 \int dx_2 \langle \dots \phi^4(x) \dots \rangle \\
 \uparrow |x_2 - x_1| \rightarrow 0 \\
 \Downarrow \\
 \log \Lambda \langle \dots \phi
 \end{array}$$

$$\phi^4(x) \phi^4(y) \underset{x \rightarrow y}{=} |x-y|^{(2-d) \times 4} \mathbb{I} + |x-y|^{(2-d) \times 3} \phi^2(x) + \boxed{|x-y|^{(2-d) \times 2} \phi^4(x)} + |x-y|^{(2-d)} \phi^6(x) + |x-y|^{(2-d) \times 2} (\partial \phi)^2(x) \leftarrow \text{not discussed here}$$

divergent

$$\int dx_2 \langle \dots \phi^4(x_1) \phi^4(x_2) \dots \rangle$$

$$\uparrow |x_2 - x_1| \rightarrow 0 \Rightarrow (x_1 - x_2)^{4-2d} \cdot d^d x_2 \Rightarrow \log \Lambda \text{ if } d=4$$

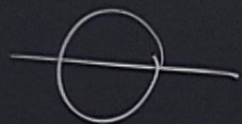
divergent if $d \geq 4$

$$\Downarrow$$

$$\log \Lambda \langle \dots \phi^4(x_1) \dots \rangle$$

shl divergences $\log \Lambda$ are proportional to local operators $\Rightarrow$ local counterterms





$$\phi^4 \times \phi^4 \rightarrow \phi^2, \phi^6$$



$$\phi^4 \times \phi^4 \times \phi^4 \rightarrow \phi^4$$

fluctuation is fixed Ising model $\Lambda \sim 1/a$

Now Iterate those renormalization transformations

N times $\Leftrightarrow$ rescaling by a factor $S = B^N$

$$\phi(x) \rightarrow \phi'(x') = B^{\Delta} \tilde{\phi}(x) \rightarrow \phi''(x'') = B^{\Delta + \Delta'} \tilde{\tilde{\phi}}(x) \rightarrow \dots$$

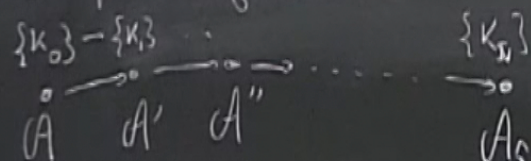
Δ might depend on the step, and on the $\{K_i\}$ (usually not)

$\phi_S(x_S)$ renormalized field

$$\phi^{(N)}(x^{(N)}) = B^{\Delta + \Delta^{(1)} + \dots + \Delta^{(N-1)}} \tilde{\tilde{\tilde{\phi}}}(x)$$

$X_S = X/S$ new distance

In the "space" of actions



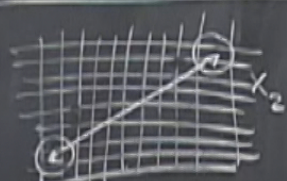
A_S renormalized action
 $\{K_S\}$ renormalized couplings

in coupling space microscopic couplings $\{K_0\} \rightarrow \{K_S\}$ renormalized couplings

lized Z_s field renormalization factor if $\xi \gg a$ (or $\sim aS$ at least)

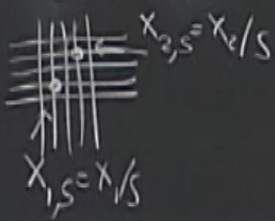
$\Delta(x) \propto \phi(x)$

distance



x_1
in the lattice L

$\downarrow S$



$$\langle \phi(x_1) \phi(x_2) \rangle_A = Z(s)^{-2} \langle \phi_s(x_{1,s}) \phi_s(x_{2,s}) \rangle_{A_s}$$

$$\langle \phi(sx_1) \phi(sx_2) \rangle_A = Z(s)^{-2} \langle \phi(x_1) \phi(x_2) \rangle_{A_s}$$

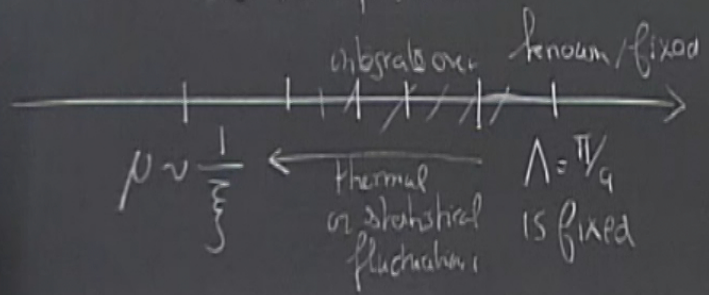
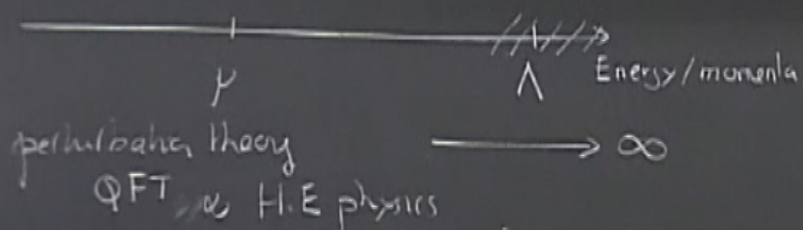
large distance in
microscopic theory

$\Rightarrow$ distances of $O(1)$
in the renormalized theory

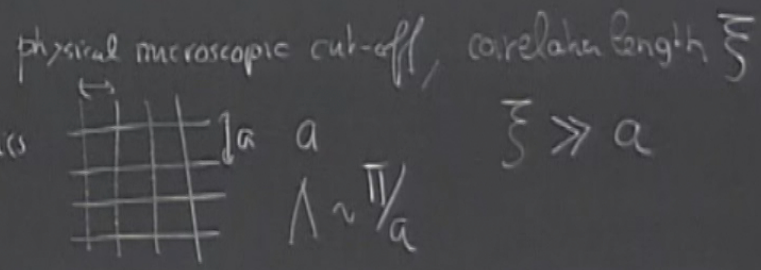
$\times \rightarrow \log \Lambda$
 only terms ϕ^4 ,
 $(\partial\phi)^2$ and ϕ^2
 renormalizable

II) Wilsonian Renormalization $\Rightarrow$ statistical mechanics quantum fluctuations

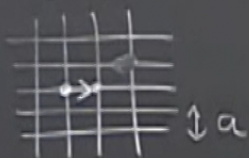
Action $S[\phi]$: ϕ^4
 cut-off $|k| < \Lambda$
 define the theory at scale μ
 let $\Lambda \rightarrow \infty$



statistical mechanics
 Ising model



$\text{non-renormalizable} \Rightarrow \Lambda^{d-6} \Rightarrow \text{new counter term} \quad \Lambda^{d-6} \cdot \phi^6 \quad \text{no control}$
 $\infty \# \text{ of new CT associated to new operators}$



A scalar field $\phi(\vec{x})$
real variable

$$\mathcal{L} = \mathbb{Z}^d$$

$$\vec{x} = a \vec{n} \quad \text{discrete}$$

$$\vec{n} = (n_z, n_d) \quad \text{vertex positions}$$

vertex $\bar{t} = (t_1, t_2) \Rightarrow \phi_{\bar{t}} = \phi(\bar{x} - a\bar{t})$

$$\phi = \{ \phi_{\tau}; \text{all } \tau \in \mathbb{Z}^d \}$$

$$\uparrow_{\text{microscopic}} = \sum_{i \in \mathcal{I}} \sum_{\alpha=1}^K k_{\alpha}^i \partial_{\alpha} [\phi_i, \nabla_{\mu} \phi_i]$$

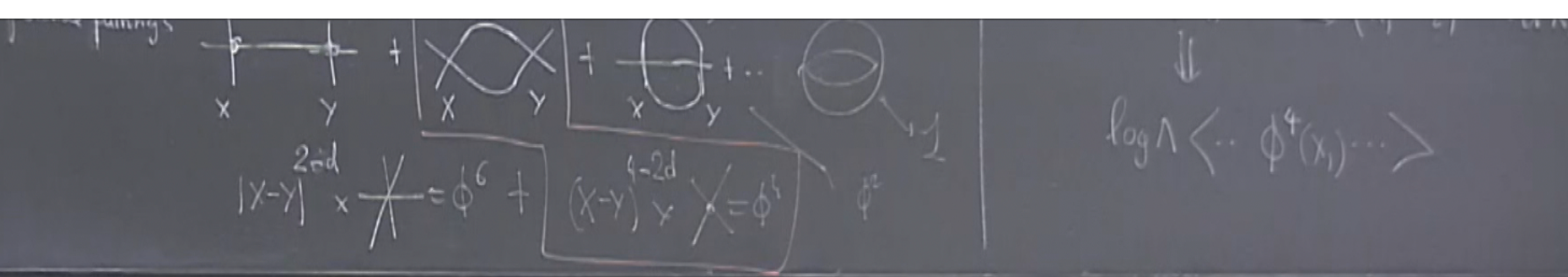
microscopic
"action" or
"Hamiltonian"

numbers
"coupling constants"
or the parameter
of the model

Memorandum

Value of ϕ at i

1 finite
differenze
"derivati"
in discrete



ϕ^4 theory on a lattice

$$O_0[\phi_i] = 1$$

$$O_1[\phi_i] = \phi_i$$

$$O_2[\phi_i] = \phi_i^2$$

$$O_3[\phi_i] = \phi_i^3$$

$$O_4[\phi_i] = \phi_i^4$$

$$O_5[\phi_i] = \sum_{\mu} (\nabla_{\mu} \phi_i)^2 = \sum_{\mu=1}^d (\phi_{i+e_{\mu}} - \phi_i)^2$$

$$A = \sum_i K_2 O_2 + K_4 \phi_i^4 + K_5 \phi_i^5$$

discretized ϕ^4 theory

$$K_2 \rightarrow m^2 \quad K_4 \rightarrow g$$

$$K_5 \rightarrow 1 \quad \text{normalization of } \phi$$

μ (component)

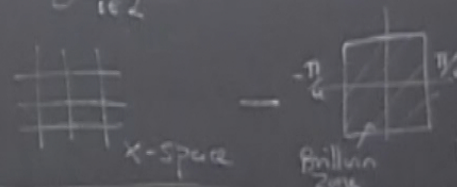
$$e_{\mu} = (0, \dots, 1, \dots, 0)$$

$$i = (i_1, \dots, i_d)$$

Partition function

Functional integral

regularized in real space

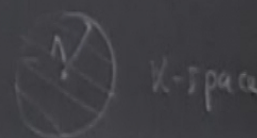


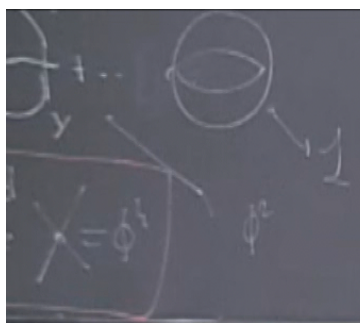
Regularize in momentum space

$$Z_{\Lambda} = \int \mathcal{D}[\phi_{\Lambda}] \exp(-A[\phi_{\Lambda}])$$

chop off the modes $|k| > \Lambda$

$$\phi(x) = \int_{\Lambda} \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \hat{\phi}(k)$$





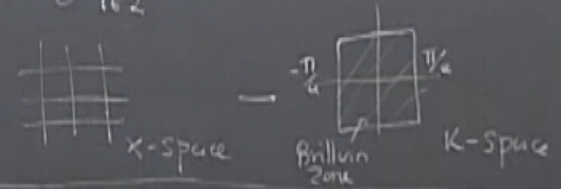
$\Rightarrow (\Lambda_1, \Lambda_2) \cdot dx_2 \Rightarrow \log \Lambda$ if $d=4$
 divergent if $d \geq 4$
 $\log \Lambda < \dots \phi^4(x_1) \dots$

are proportional to local operators $\Rightarrow$ local counterterms

Partition function
Functional integral

$$Z = \int \prod_{i \in \mathcal{I}} d\psi_i \exp(-\mathcal{A}[\psi])$$

regularized in real space

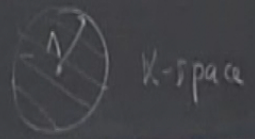


Regularize in momentum space

$$Z_\Lambda = \int \mathcal{D}[\phi_\Lambda] \exp(-\mathcal{A}[\phi_\Lambda])$$

chop off the modes $|k| > \Lambda$

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d} e^{i k \cdot x} \hat{\phi}(k)$$



Renormalization

Step 1. "integrate" over high energy/short distance fluctuations by a "decimation" or "block-spin" or "averaging" procedure

Step 2 averaged field $\tilde{\phi}$
averaged action or effective action $\mathcal{A}_{\text{eff}}[\tilde{\phi}]$

Step 3 rescale the distances, and the field to get renormalized field ϕ_R , action $\mathcal{A}_R$

Renormalized action

compare $\mathcal{A}$ and $\mathcal{A}'$

$$i = (i_1 \dots i_d) \quad \varphi_i = (0 \dots 1 \dots 0) \quad k_5 \rightarrow 1 \quad \text{normalization of } \phi \quad \Lambda \quad \sqrt{|k| \Lambda} (2\pi)^d$$

Explicit example (but no calculations)



blocks b of size B

average field $\tilde{\phi}_b$ in block b

$$\tilde{\phi}_b = \frac{1}{B^d} \sum_{i \in b} \phi_i$$

sites inside block b

$$e^{-A_{\text{eff}}[\tilde{\phi}]} = \int \prod_{i \in \mathbb{Z}} d\phi_i \prod_{b \in \tilde{\mathbb{Z}}} \delta[\tilde{\phi}_b - \frac{1}{B^d} \sum_{i \in b} \phi_i]$$

1 constraint per b

defines the effective action for block variables

Step 1 and 2

Step 3

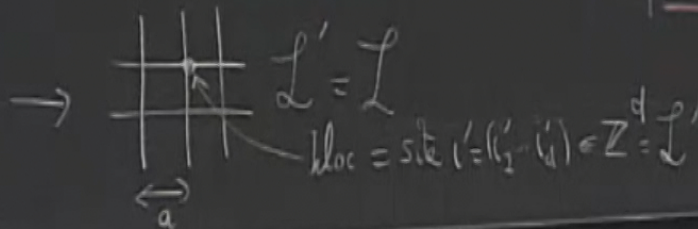
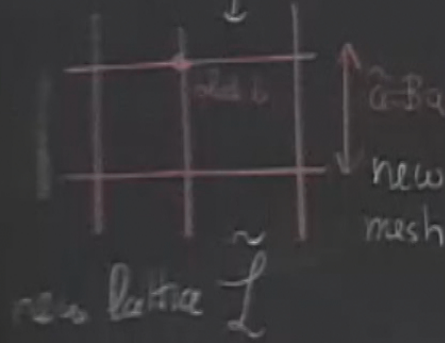
Space rescaling $X' = X/B$

block $\rightarrow$ new sites i'

Field rescaling, $\tilde{\phi}$ lives on $\tilde{\mathbb{Z}}$

$$\phi' = B^\Delta \tilde{\phi}$$

with Δ some scaling dim



$\int \frac{d^d k}{(2\pi)^d} \phi(k)$

k -space

to get renormalized field $\tilde{\phi}$, action $\mathcal{A}$

$$Z = \int \prod_{i \in \tilde{I}} d\phi_i \prod_{b \in \tilde{I}} \delta[\tilde{\phi}_b - \frac{1}{S^d_{ab}} \sum_{i \in b} \phi_i] e^{-\mathcal{A}[\phi]}$$

1 constraint per block

effective action for block variables

Renormalized action

$$\mathcal{A}'[\phi'] = \mathcal{A}_{\text{eff}}[\tilde{\phi}]$$

$$\mathcal{A}[\phi'] = \sum_{i \in \tilde{I}} \sum_{\alpha} K'_{\alpha} \mathcal{O}_{\alpha}[\phi'_i, \nabla_{\mu} \phi'_i, \dots]$$

new couplings renormalized

Field rescaling $\tilde{\phi} \rightarrow \phi'$

lives on $\tilde{I}'$

$$\phi'_i = B^{\Delta} \tilde{\phi}_i$$

with Δ some scaling dimension

why? because the fields have dimension

$x' = x/B$