

Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 7

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URL: <http://pirsa.org/18110012>

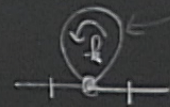
Abstract:

Renormalization at 1 loop ϕ^4 (massless), $d=4$

1) 2 and 4 pt functions: UV divergences

Irreducible Functions

2pt



tadpole

$$I_2 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}$$

momentum
space

$$\hat{\Gamma}_{(2)}(p_1, p_2) = (2\pi)^d \delta(p_1 + p_2) \cdot \tilde{\Gamma}_{(2)}(p_1), \quad \tilde{\Gamma}_{(2)}(p) = p^2 + m^2 - \frac{g}{2} \mathcal{O}$$

Renormalization at 1 loop ϕ^4 (massless), $d=4$ $S[\phi]=$

1) 2 and 4 pt functions: UV divergences internal momentum amplitude
 Irreducible Functions 2pt  tadpole $\Gamma_2 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} = T(m)$

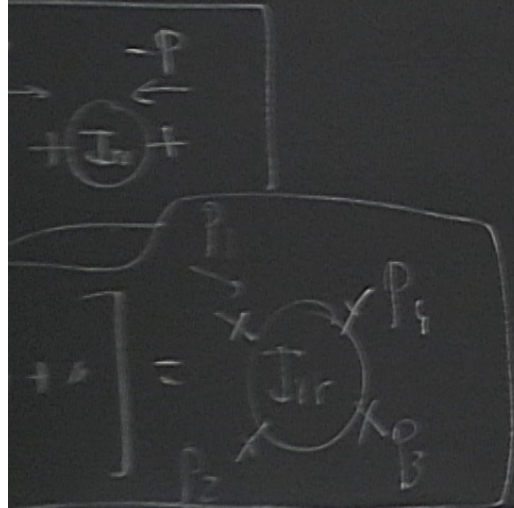
momentum space 2pts $\hat{\Gamma}_{(2)}(p_1, p_2) = (2\pi)^d \delta(p_1 + p_2) \cdot \tilde{\Gamma}_{(2)}(p_1)$, $\tilde{\Gamma}_{(2)}(p) = \frac{p^2 + m^2 - \frac{g}{2} \text{loop}}{p^2 + m^2} = +(\text{Ir}) +$

4pts $\hat{\Gamma}_{(4)}(p_1, \dots, p_4) = (2\pi)^d \delta(p_1 + \dots + p_4) \hat{\Gamma}_{(4)}(\{p_i\})$ 3 indept momenta
 $= \left[g \text{XXXX} - \frac{g}{2} \left[\text{XXXX} + \dots \right] \right] = \text{Ir}$

$$S[\phi] = \int d^4x \left[\frac{1}{2} (\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 \right]$$

amplitude

$$= \frac{1}{T} (m)$$



$$\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

General property: in momentum space

Connected Correlation Function

$$G_{(2)}(p) = \frac{1}{\Gamma_{(2)}(p)}$$

→ Legendre Transform property

$$G_{(2)}(p) = \text{diagram: two external lines with momenta } p \text{ and } -p \text{ connected by a circle labeled 'Conn'}$$

contributions of Irr. Graphs
with $B \geq 1$ loops
↓

$$\Gamma_{(2)}(p) = \text{diagram: two external lines connected by a circle labeled 'Irr.'} + \text{diagram: two external lines connected by a circle labeled } \Sigma$$

$$G_{(2)}(p) = \text{diagram: two external lines connected by a circle labeled 'Conn'} + \text{diagram: two external lines connected by a circle labeled } \Sigma + \text{diagram: two external lines connected by a circle labeled } \Sigma$$

$$\propto \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4$$

General property: in momentum space

Connected Correlation Function

$$G_{(2)}(p) = \text{diagram: a circle labeled 'Conn' with incoming momentum } p \text{ and outgoing momentum } -p$$

contributions of Irr. Graphs
with $B \geq 1$ loops

$$G_{(2)}(p) = \frac{1}{\Gamma_{(2)}(p)}$$

$$\Gamma_{(2)}(p) = \text{diagram: a circle labeled 'Irr.'} + \text{diagram: a circle with a cross} + \text{diagram: a circle with a cross and a loop} + \dots$$

→ Legendre Transform property
→ diagram.

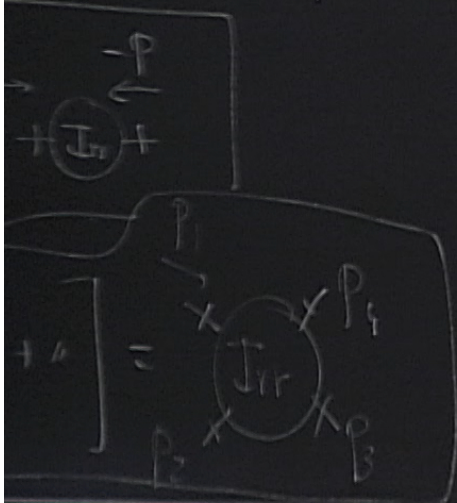
$$G_{(2)}(p) = \text{diagram: a circle labeled 'Conn'} = \text{diagram: a circle labeled 'Irr.'} + \text{diagram: a circle with a cross} + \text{diagram: a circle with a cross and a loop} + \dots$$

$$\Gamma_{(2)} = p^2 + m^2 + \sum (p, m)$$

↑ some function

$$G_2(p) = \frac{1}{p^2 + m^2} - \frac{\Sigma(p)}{(p^2 + m^2)^2} + \frac{\Sigma(p)^2}{(p^2 + m^2)^3} + \dots$$

amplitude

$$\frac{1}{T}(\text{m})$$


Connected Correlation Function

2pt
function

$$G_{(2)}(P) = \begin{array}{c} \xrightarrow{P} \\ \circ - \text{Conn} - \circ \\ \xleftarrow{-P} \end{array}$$

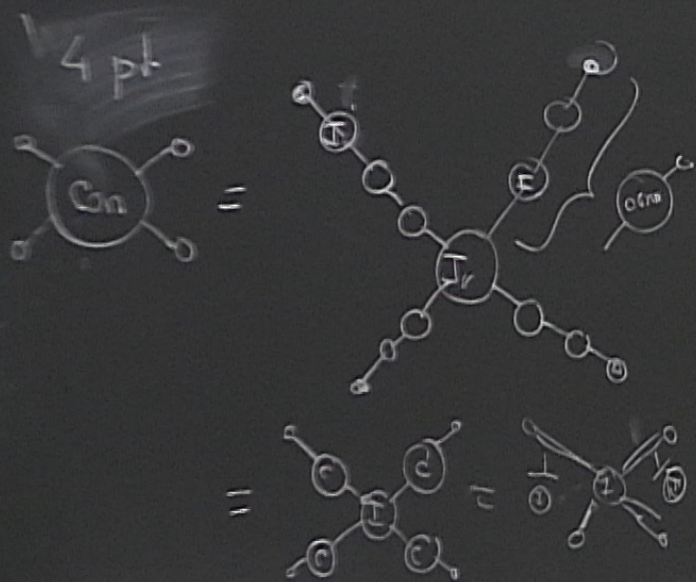
$$\Gamma_{(2)}(p) = \text{diagram with a circle labeled Irr.} + 0 = \text{diagram with two vertical lines and a circle in the middle}$$

$$\Gamma[\varphi] = j \varphi \equiv W[j]$$

$\downarrow \quad \downarrow$
 $\sum$ connected

$$\varphi = \frac{\delta W}{\delta j} = j \circ \text{Conn}$$

$$j = \frac{\delta W}{\delta \varphi} = \varphi \times \text{Irr}$$



$$\text{C} = G_{(4)}(p_1, \dots, p_4) = \text{Irr}$$

connected

$$\text{Irr} = \Gamma_{(4)}(p_1, \dots, p_4)$$

$$1 \text{ } \overline{\overline{p}} \text{ } 2 = G_{(2)}(p) = \frac{1}{\Gamma_{(2)}(p)}$$

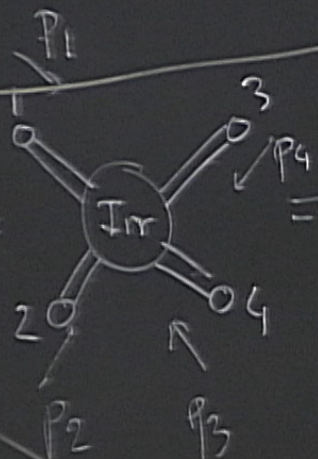
$$\varphi = \frac{\delta W}{\delta j} = j \circ \text{Conn}$$

$$j = \frac{\delta W}{\delta \varphi} = \varphi \times \text{Irr}$$

$$\begin{aligned} \text{C} &= G_{(4)}(p_1, p_4) = \text{Irr} = \frac{1}{\Gamma_{(2)}(p_1)} \cdot \frac{1}{\Gamma_{(2)}(p_2)} \cdot \frac{1}{\Gamma_{(2)}(p_3)} \cdot \frac{1}{\Gamma_{(2)}(p_4)} \Gamma_{(4)}(p_1, p_2, p_3, p_4) \\ &\quad \text{with } p_1 + p_2 + p_3 + p_4 = 0 \end{aligned}$$

$$\text{Irr} = \Gamma_{(4)}(p_1, p_2, p_3, p_4)$$

$$1 \xrightarrow{p} 2 = G_{(2)}(p) = \frac{1}{\Gamma_{(2)}(p)}$$

$\Gamma_{(4)}(p_1, p_2, p_3, p_4) =$


$= \frac{1}{\Gamma_{(2)}(p_1)} \cdot \frac{1}{\Gamma_{(2)}(p_2)} \cdot \frac{1}{\Gamma_{(2)}(p_3)} \cdot \frac{1}{\Gamma_{(2)}(p_4)} \Gamma_{(4)}(p_1, p_2, p_3, p_4)$

with $p_1 + p_2 + p_3 + p_4 = 0$

$\Gamma_{(2)}(p)$

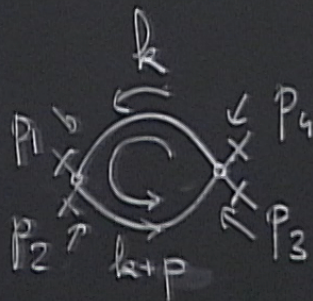
$G_{(2)}(p) = \frac{1}{\Gamma_{(2)}(p)}$

↳ LSZ red. formula

Back to renormalization

$$\text{Loop } T(m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2} = \langle \phi(0)^2 \rangle \text{ is } \infty \quad d \geq 2$$

↑ divergent at $|k| \rightarrow \infty$ UV divergence



$$\Rightarrow B(p, m) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)} \frac{1}{(p+k)^2 + m^2}$$

$$p_1 + p_2 + p_3 + p_4 = 0$$

divergent when $d \geq 4$

$p = p_1 + p_2$, k are d-components vectors momenta

- Radiative corrections (quantum) are seemingly ∞

- Regularization: • Lattice  a = "minimal distance" non-perturbative
mesh a $\Lambda = \frac{\pi}{a}$ "maximal energy/momentum"

* change the propagator
sharp regulator

$ k > \Lambda$	$\rightarrow$ zero
$ k < \Lambda$	$\rightarrow \frac{1}{p^2 + m^2}$

• Pauli-Villars

• change the dimension of space time $d \rightarrow$ complex ??
dimensional regularization (very useful) '1

energy/momentum

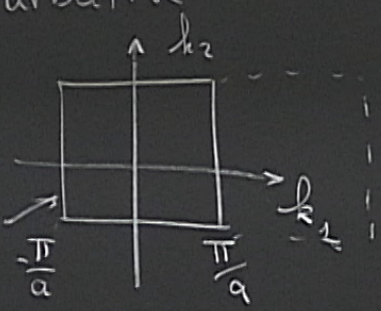
→ zero
 $\frac{1}{p^2 + m^2}$

→ complex
 (useful)

non-perturbative

Lattice

Brillouin zone



$$\frac{1}{p^2 + m^2} \rightarrow \frac{1}{\frac{4}{a^2} \sum_{\mu=1}^d \sin^2(k_{\mu} a) + m^2}$$

propagator on a lattice

- Radiative corrections (quantum) are seemingly ∞
- Regularization: • Lattice  $a =$ "minimal distance"
 $\Lambda = \frac{\pi}{a}$ "maximal momentum"

* change the propagator
 sharp regulator

$|k| > \Lambda$
 $|k| < \Lambda$

• Pauli-Villars

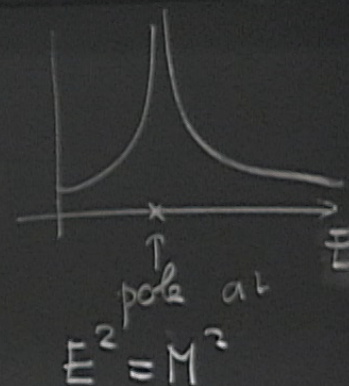
• change the dimension of spacetime d -
 dimensional regularization (very useful)

$$\frac{1}{k^2 + m^2}$$

$$k = (k_1, \dots, k_d)$$

- What are we calculating? QFT I $G_{(2)}(p)$

pole at $E^2 = M^2$ where M is called the effective mass
(self energy), physical mass



Euclidean regime : M_{phys}^2 : $\Gamma_{(2)}(p^2) = 0$ at $\boxed{p^2 + M_{\text{phys}}^2 = 0}$

$$M_{\text{phys}}^2 = m^2 + \frac{g}{2} \text{ (loop diagram) } + \text{higher loop corrections} \quad m \text{ "bare mass"}$$

$$M_{\text{phys}}^2 = m^2 + \frac{g}{2} T(m, \Lambda) \quad \begin{array}{l} \text{continuum limit} \\ \Lambda \rightarrow \infty \end{array} \quad \begin{array}{l} \text{adjust } m \text{ so that} \\ M_{\text{phys}} \text{ is fixed} \end{array}$$

4 | Massless ϕ^4 theory (High energy) $d=4$

$\rightarrow M_{\text{phys}} = 0 \Rightarrow m=0$ only for the classical theory

$$T(m=0, \Lambda) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{1}{(4\pi)^2} \Lambda^2$$

sharp
momentum

$$|k| < \Lambda$$

we need that

$$m^2 = -\frac{g}{2} \frac{1}{(4\pi)^2} \Lambda^2$$

1 loop counterterm for the mass

energy) $d=4$

by for the classical theory

$$T(m, \Lambda) = \frac{1}{(4\pi)^2} \Lambda^2 + m^2 \left(-\frac{1}{(4\pi)^2} \right) \log(\Lambda^2) + \dots$$

$m \neq 0$

expansion in m

$$\Gamma_{(2)}(p) = p^2 - \frac{g}{2} \frac{1}{(4\pi)^2} \Lambda^2 + \frac{g}{2} \left(\frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} \left(\frac{-g}{2} \frac{1}{(4\pi)^2} \Lambda^2 \right) \log(\Lambda^2) + \dots \right)$$

$O(g^2)$ at next order (2 loops)

$\rightarrow M_{\text{phys}} = 0 \Rightarrow m = 0$ only for the classical theory

$$T(m=0, \Lambda) = \int_{|k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{1}{(4\pi)^2} \Lambda^2$$

sharp
momentum

We need that $m^2 = -\frac{g}{2} \frac{1}{(4\pi)^2} \Lambda^2$

1 loop counterterm for the mass
enough at 1 loop order

$$T(m, \Lambda) = \frac{1}{(4\pi)^2} \Lambda^2 + m^2 \left(-\frac{1}{(4\pi)^2} \right) \log(\Lambda^2) +$$

$m \neq 0$

expansion in m

$$\Gamma_{(2)}(p) = p^2 - \frac{g}{2} \frac{1}{(4\pi)^2} \Lambda^2 + \frac{g}{2} \left(\frac{1}{(4\pi)^2} \Lambda^2 - \frac{1}{(4\pi)^2} \left(-\frac{g}{2} \frac{1}{(4\pi)^2} \right) \right)$$

$O(g^2)$ at next order

with this counterterm

$$\Gamma_{(2)}(p) = \left(p^2 + 0 \right) + g \times \underbrace{0}_{\text{zero}} + o(g^2)$$

↑ finite and massless

1 loop counterterm for the mass
enough at 1 loop order

$O(g^2)$ at next order

5] 4pt function and coupling constant renormalization

$$\Gamma_{(4)} = \text{diagram with 4 external lines and a loop} = g - \frac{g^2}{2} \left[B(p_1+p_2, m; \Lambda) + B(p_2+p_3, m; \Lambda) + B(p_1+p_4, m; \Lambda) \right]$$

$$\text{with } B(p, m, \Lambda) = \int \frac{d^d k}{(2\pi)^2} \frac{1}{k^2 + m^2} \frac{1}{(p+k)^2 + m^2} \rightarrow \text{at } d=4 \quad \log \Lambda$$

we can set $m=0$
inside B , up to
2 loop corrections
(i.e. g^3)

$|k| < \Lambda$
 $|k+p| < \Lambda$

divergent $d=4$
for large $|k|$

coefficient pure
number

ization.

$$B(p_1 + p_3, m; \Lambda) + B(p_1 + p_4, m; \Lambda)$$

at $d=4$

$$\left(\log(\Lambda^2) \right) \frac{1}{(4\pi)^2} + \text{finite term}(p) = B(p, m=0, \Lambda)$$

efficient plus
number

in fact

$$B(p, m=0, \Lambda) = \frac{1}{(4\pi)^2} \log\left(\frac{\Lambda^2}{|p|^2}\right) + \text{Some number} + \text{terms} \rightarrow 0 \text{ when } \Lambda \rightarrow \infty$$

this is universal
independent of
the regularization
(dimensional analysis)

universal
of
ization
analysis)

So

$$\Gamma_{(4)} = g - \frac{g^2}{2} \frac{1}{(4\pi)^2} \left[\log \frac{\Lambda^2}{(p_1 + p_2)^2} + p_2'' p_3'' + p_2'' p_4'' + \text{cst} \right] + O(g^3)$$

so the 4p1 function is not well defined

physical coupling constant

How do we define a "physical coupling constant"



elastic scattering amplitude

$$S(\theta, E) \quad \text{energy}$$

↑
4 pt function (on shell) $\leftarrow$ LSZ

↑
Irreducible 4 pt function

A (possible) definition of g_{phys}

reference point in momentum space where (symmetric in the S, T, U channels)

$$|p_1 + p_2| = |p_1 + p_3| = |p_1 + p_4| = \mu$$

reference mass/energy point ← renormalization condition

$$g_{\text{phys}} = g_{\text{renormalized}} = \Gamma_{(4)}(p_1, p_2, p_3, p_4) \Big|_{\text{reference point}}$$

$$g_{\text{phys}} = g - \frac{g^2}{2} \frac{3}{(4\pi)^2} \log\left(\frac{\Lambda^2}{\mu^2}\right) + O(g^3)$$

in the S, T, U channels)
 mass/energy renormalization
 ← condition

$$\Gamma_4(p_1, p_2, p_3, p_4)$$

is finite

arbitrary p_1, p_2, p_3, p_4
 reexpressed in term of g_R

$$g_R = g_R^2 \frac{1}{(4\pi)^2} \left[\log \left[\frac{\mu^2}{(p_1 + p_2)^2} \right] + "p_2 - p_3" + "p_2 - p_4" \right]$$

$$g_R = g_{\text{phys}}$$

- mass renormalization

- coupling constant renormalization

Renormalization Group

$$m_{\text{phys}} = 0$$

g_{phys} is defined at a renormalization scale μ

$\Rightarrow$ 2pt finite
4pt "
Npts are also finite

at 1 loop
order