Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 2

Date: Nov 13, 2018 09:00 AM

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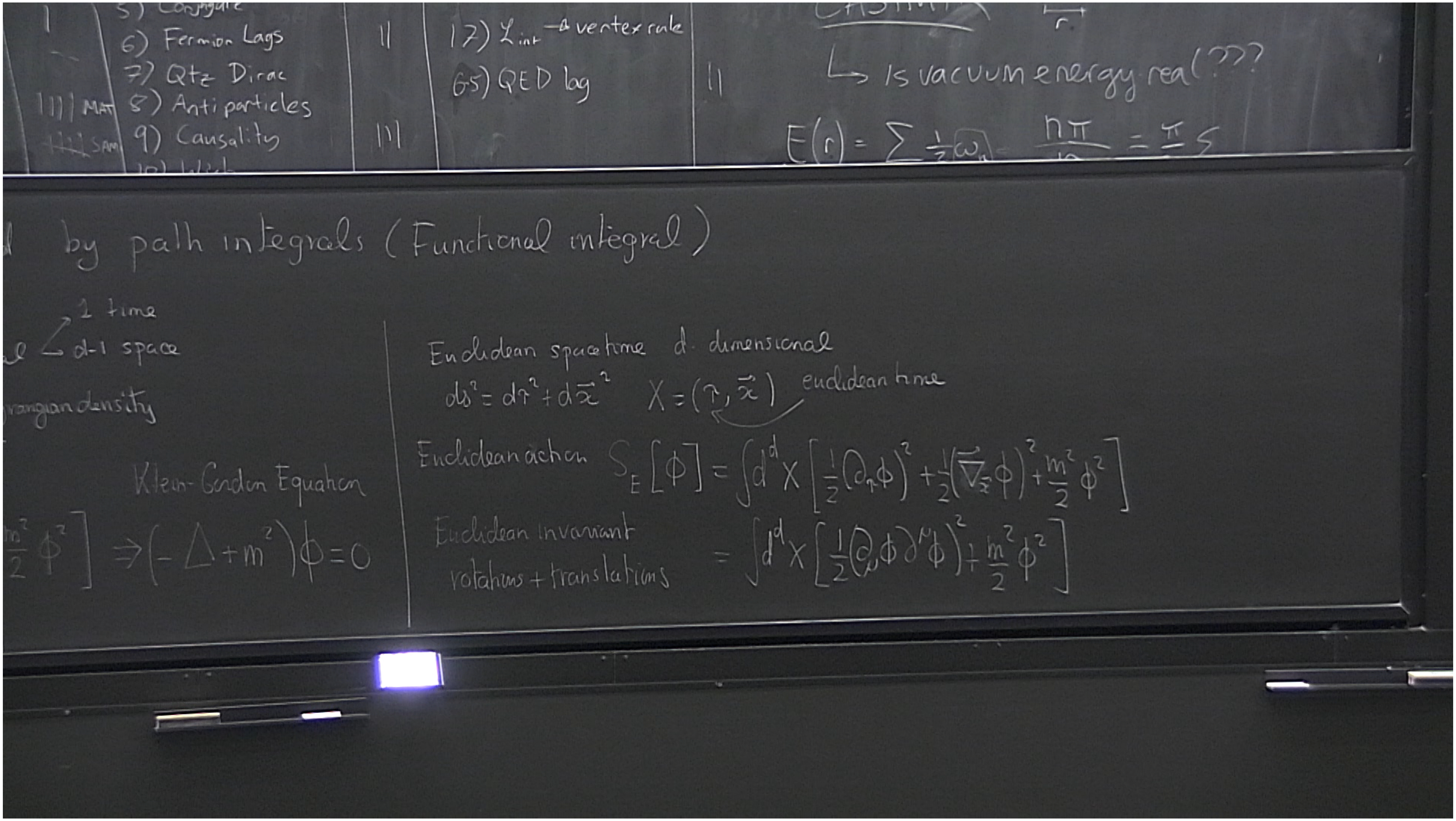
Abstract:



5) Conjugate
6) Fermion Lags
7) Qtz Dirac
8) Antiparticles
9) Causality
10) Wick

MAT

SAM

17) $\mathcal{L}_{int} \rightarrow$ vertex rule
6.5) QED lag

CASIMIR

r

↳ Is vacuum energy real???

$E(r) = \sum \frac{1}{2}\omega_n \quad \frac{n\pi}{r} \quad = \frac{\pi}{} \sum$

by path integrals (Functional integral)

$\nearrow$ 1 time
$\searrow$ d-1 space

grangian density

Klein-Gordon Equation

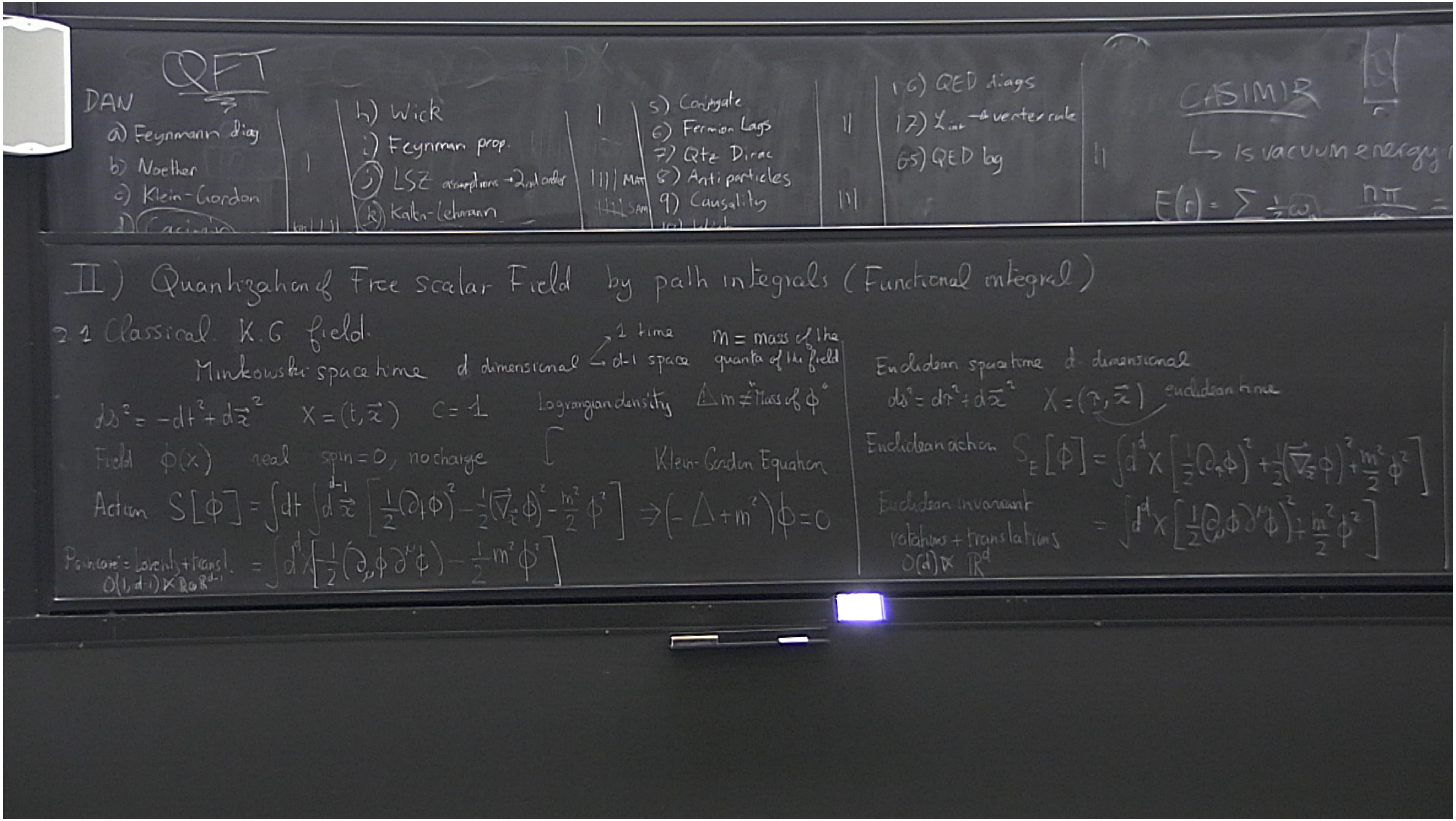
$\frac{m^2}{2}\phi^2 \Big] \Rightarrow (-\Delta + m^2)\phi = 0$

Euclidean spacetime d. dimensional

$ds^2 = d\tau^2 + d\vec{x}^2 \qquad X = (\tau, \vec{x})$ euclidean time

Euclidean action $S_E[\phi] = \int d^dX \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2 \right]$

Euclidean invariant
rotations + translations $= \int d^dX \left[\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi)^2 + \frac{m^2}{2}\phi^2 \right]$



QFT

DAN
- a) Feynmann diag
- b) Noether
- c) Klein-Gordon
- d) Casimir

| ||||

- h) Wick
- i) Feynman prop.
- j) LSZ assumptions → 2nd order
- k) Kallen-Lehmann

| |||| MAT |||| SAM

- 5) Conjugate
- 6) Fermion Lags
- 7) Qtz Dirac
- 8) Antiparticles
- 9) Causality

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- 16) QED diags
- 17) $\mathcal{L}_{int} \to$ vertex rule
- 65) QED lag

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CASIMIR

↳ is vacuum energy

$E(r) = \sum \frac{1}{2}\omega_n \quad \frac{n\pi}{r} =$

II) Quantization of Free scalar Field by path integrals (Functional integral)

2.1 Classical K.G field.

Minkowski spacetime d dimensional $\begin{cases} 1 \text{ time} \\ d\text{-}1 \text{ space} \end{cases}$

m = mass of the quanta of the field

$ds^2 = -dt^2 + d\vec{x}^2 \qquad X = (t, \vec{x}) \qquad c = 1$

Lagrangian density $\int$

⚠ m ≠ "Mass of ϕ"

Field $\phi(x)$ real spin = 0, no charge

Klein-Gordon Equation

Action $S[\phi] = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\vec{\nabla}_x\phi)^2 - \frac{m^2}{2}\phi^2\right] \Rightarrow (-\Delta + m^2)\phi = 0$

Poincaré = Lorentz + transl.
$O(1,d-1) \ltimes \mathbb{R} \otimes \mathbb{R}^{d-1}$

$= \int d^d x \left[-\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right]$

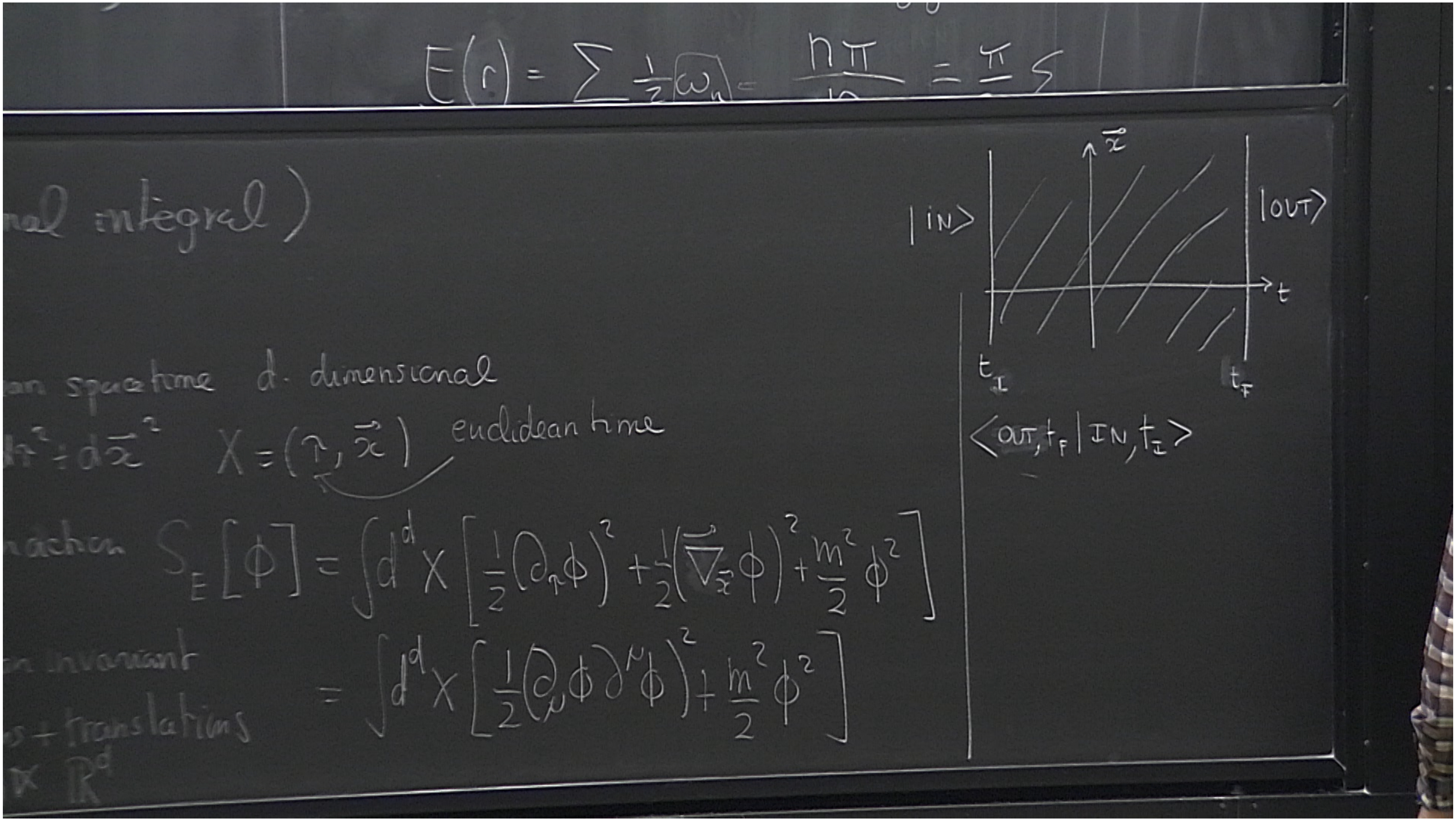
Euclidean spacetime d dimensional

$ds^2 = d\tau^2 + d\vec{x}^2 \qquad X = (\tau, \vec{x})$ euclidean time

Euclidean action $S_E[\phi] = \int d^d x \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\vec{\nabla}_x\phi)^2 + \frac{m^2}{2}\phi^2\right]$

Euclidean invariant
rotations + translations
$O(d) \ltimes \mathbb{R}^d$

$= \int d^d x \left[\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi)^2 + \frac{m^2}{2}\phi^2\right]$



$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{\ldots} = \frac{\pi}{\ldots}\sum$

nal integral)

an spacetime d. dimensional

$d\tau^2 + d\vec{x}^2 \quad X = (\tau, \vec{x})$ euclidean time

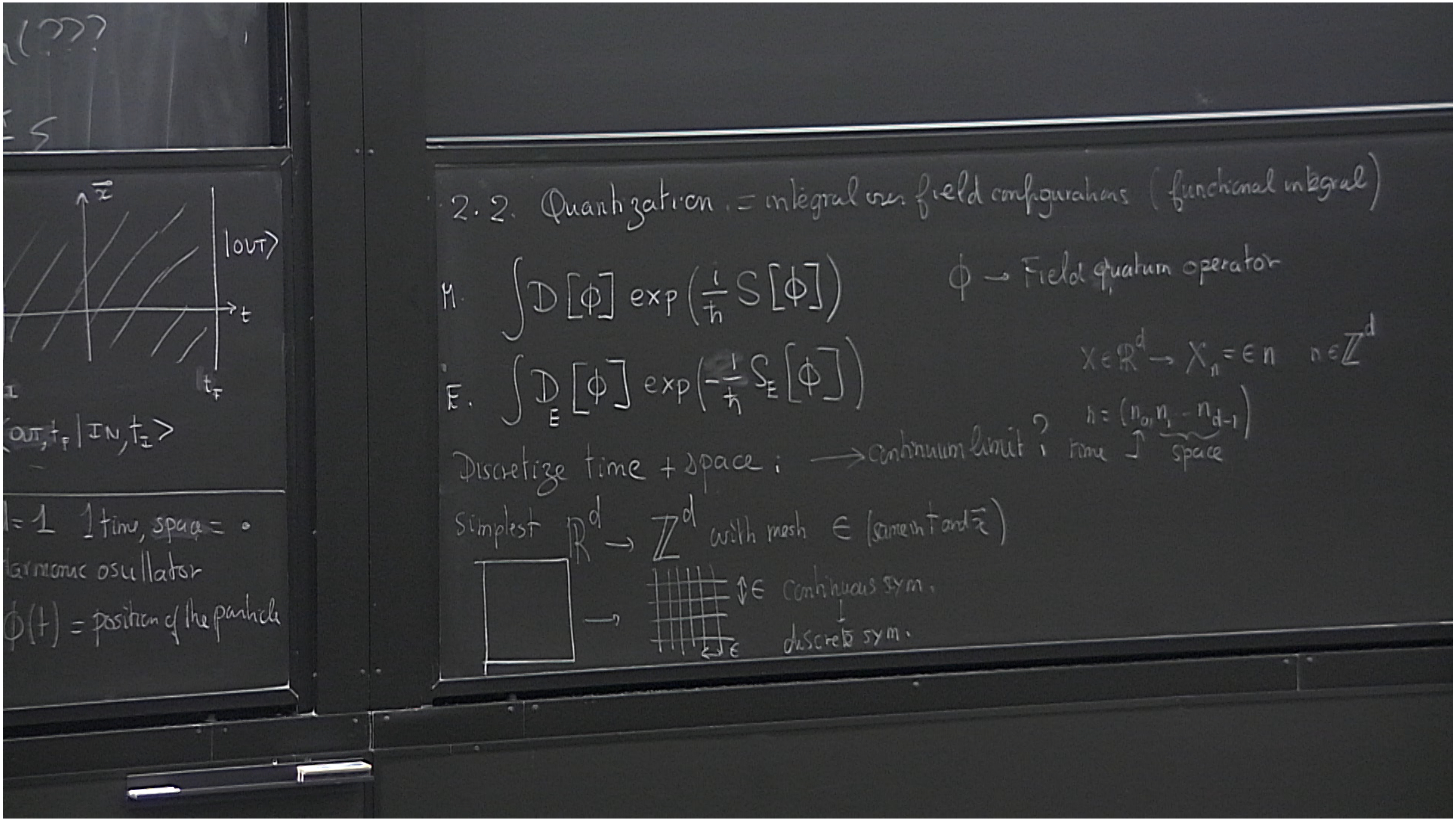
action $S_E[\phi] = \int d^dX \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2\right]$

n invariant $= \int d^dX \left[\frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi)^2 + \frac{m^2}{2}\phi^2\right]$

s + translations

$\ltimes\ \mathbb{R}^d$

$\langle OUT, t_F | IN, t_I \rangle$



(???

$\langle OUT, t_F | IN, t_I \rangle$

$=1$ 1 time, space = ·

armonic oscillator

$\phi(t)$ = position of the particle

2.2. Quantization = integral over field configurations (functional integral)

M. $\int D[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$ $\qquad \phi \to$ Field quantum operator

E. $\int D_E[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$

$x \in \mathbb{R}^d \to x_n = \epsilon n \quad n \in \mathbb{Z}^d$

$n = (\underset{\text{time}}{n_0}, \underbrace{n_1 \ldots n_{d-1}}_{\text{space}})$

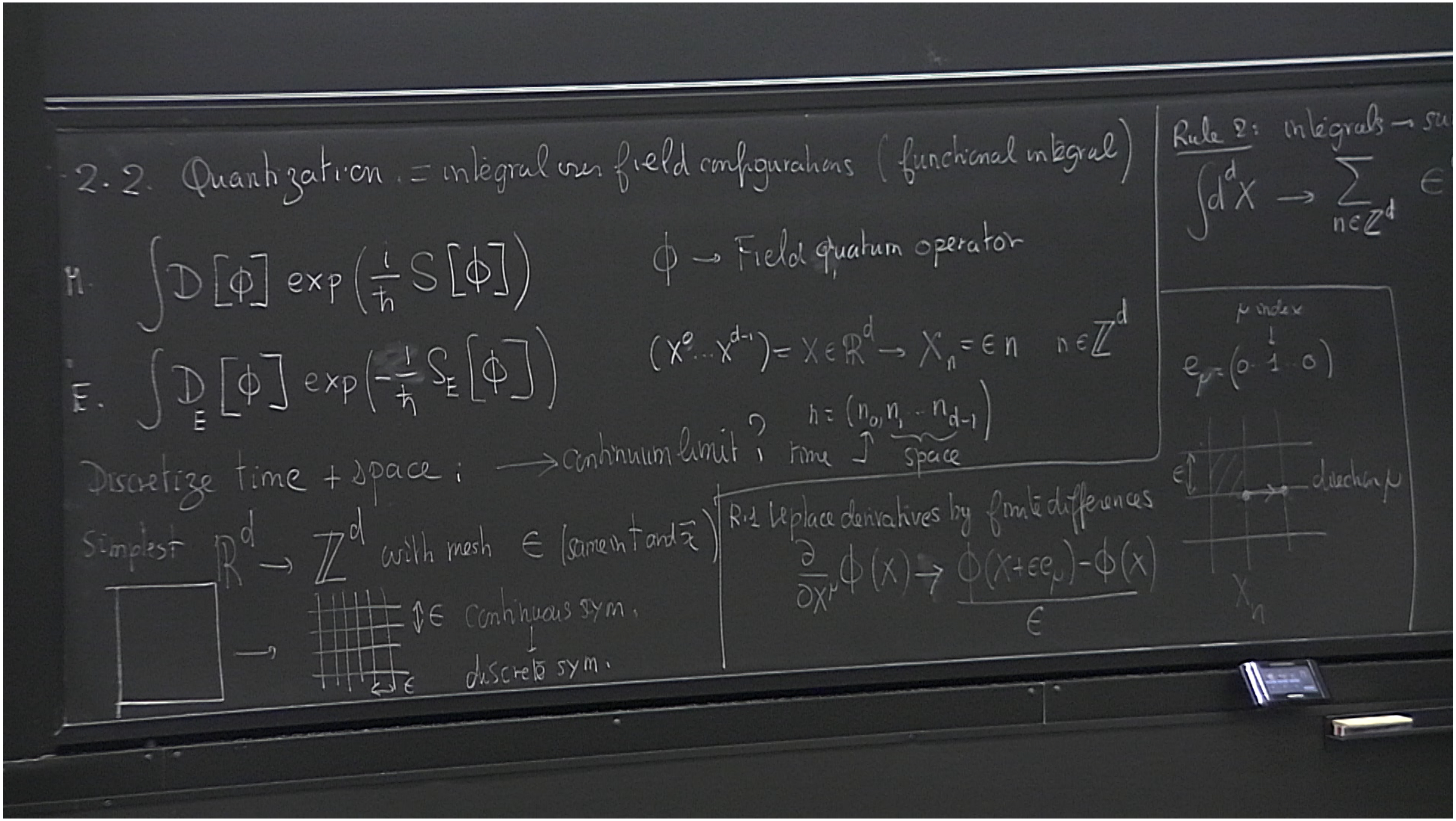
Discretize time + space: $\longrightarrow$ continuum limit?

Simplest $\mathbb{R}^d \to \mathbb{Z}^d$ with mesh ϵ (same in t and $\vec{x}$)

$\updownarrow \epsilon$ continuous sym.

$\downarrow$

discrete sym.



2.2. Quantization = integral over field configurations (functional integral)

M. $\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$

E. $\int \mathcal{D}_E[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$

$\phi \to$ Field quatum operator

$(x^0 \dots x^{d-1}) = x \in \mathbb{R}^d \to x_n = \epsilon n \quad n \in \mathbb{Z}^d$

$n = (\underset{\text{time}}{n_0}, \underbrace{n_1 \dots n_{d-1}}_{\text{space}})$

Discretize time + space: $\longrightarrow$ continuum limit?

Simplest $\mathbb{R}^d \to \mathbb{Z}^d$ with mesh ϵ (same in t and $\vec{x}$)

$\updownarrow \epsilon$ continuous sym.

$\downarrow$

discrete sym.

$\leftrightarrow \epsilon$

R.1 Replace derivatives by finite differences

$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\phi(x+\epsilon e_\mu)-\phi(x)}{\epsilon}$$

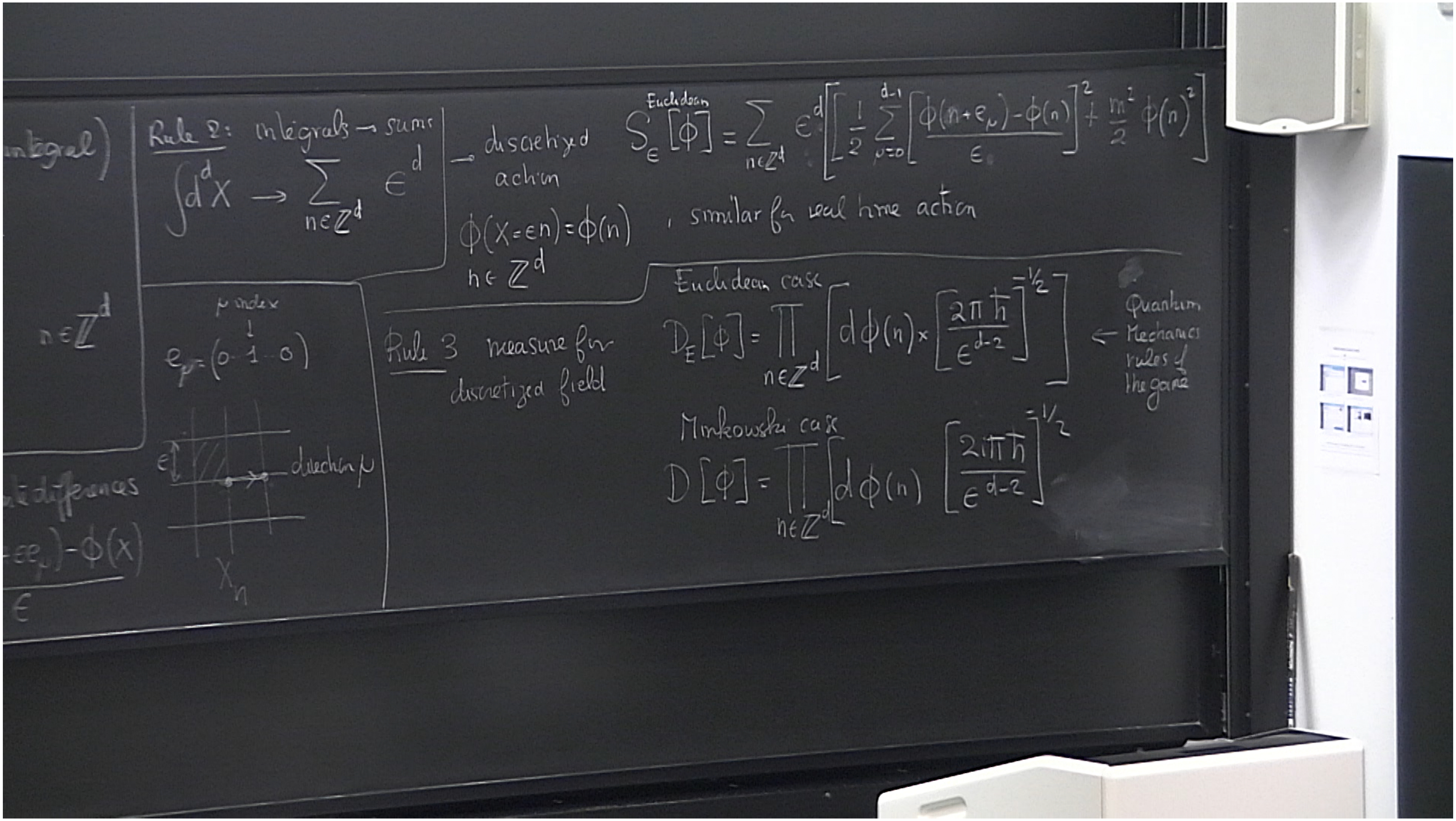
Rule 2: integrals → su

$$\int d^d x \to \sum_{n \in \mathbb{Z}^d} \epsilon$$

μ index

$\downarrow$

$e_\mu = (0 \cdots 1 \cdots 0)$



integral)

$n \in \mathbb{Z}^d$

difference

$\epsilon e_\mu) - \phi(x)$

ϵ

Rule 2: integrals → sums

$$\int d^d x \longrightarrow \sum_{n \in \mathbb{Z}^d} \epsilon^d$$

→ discretized action

$$\phi(x = \epsilon n) = \phi(n)$$
$$n \in \mathbb{Z}^d$$

$$S^{\text{Euclidean}}_{\epsilon}[\phi] = \sum_{n \in \mathbb{Z}^d} \epsilon^d \left[\frac{1}{2} \sum_{\mu=0}^{d-1} \left[\frac{\phi(n+e_\mu) - \phi(n)}{\epsilon} \right]^2 + \frac{m^2}{2} \phi(n)^2 \right]$$

, similar for real time action

μ index

$$e_\mu = (0 \cdots \overset{\downarrow}{1} \cdots 0)$$

Rule 3 measure for discretized field

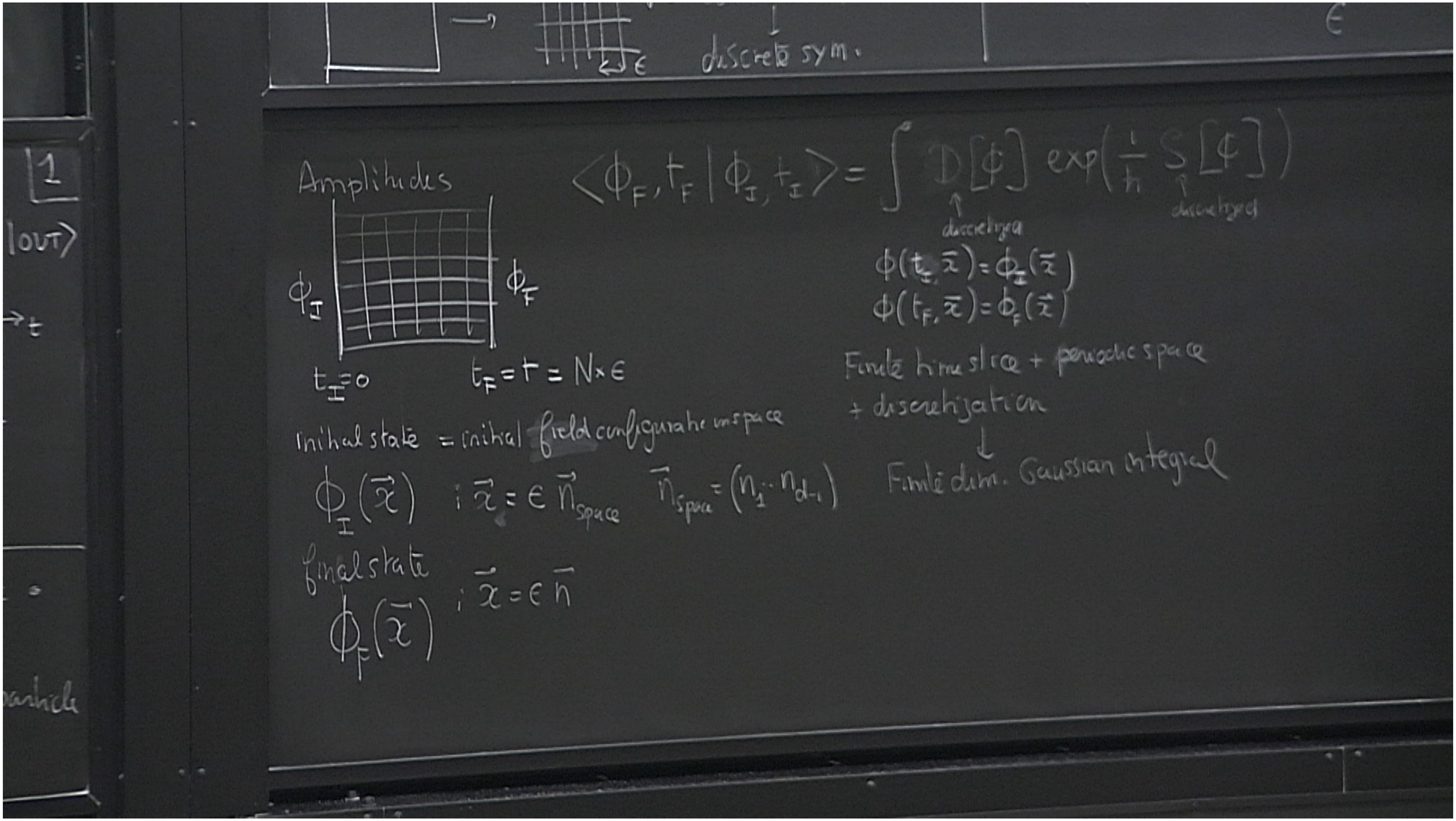
Euclidean case

$$D_E[\phi] = \prod_{n \in \mathbb{Z}^d} \left[d\phi(n) \times \left[\frac{2\pi\hbar}{\epsilon^{d-2}} \right]^{-1/2} \right]$$

← Quantum Mechanics rules of the game

Minkowski case

$$D[\phi] = \prod_{n \in \mathbb{Z}^d} \left[d\phi(n) \left[\frac{2i\pi\hbar}{\epsilon^{d-2}} \right]^{-1/2}\right.$$



Amplitudes

$$\langle \phi_F, t_F \mid \phi_I, t_I \rangle = \int \underset{\substack{\uparrow \\ \text{discretized}}}{D}[\phi] \exp\left(\frac{i}{\hbar} \underset{\substack{\uparrow \\ \text{discretized}}}{S}[\phi]\right)$$

$$\phi(t_I, \vec{x}) = \phi_I(\vec{x})$$
$$\phi(t_F, \vec{x}) = \phi_F(\vec{x})$$

$t_I = 0$ $\quad$ $t_F = t = N \times \epsilon$

Finite time slice + periodic space
+ discretization

$\downarrow$

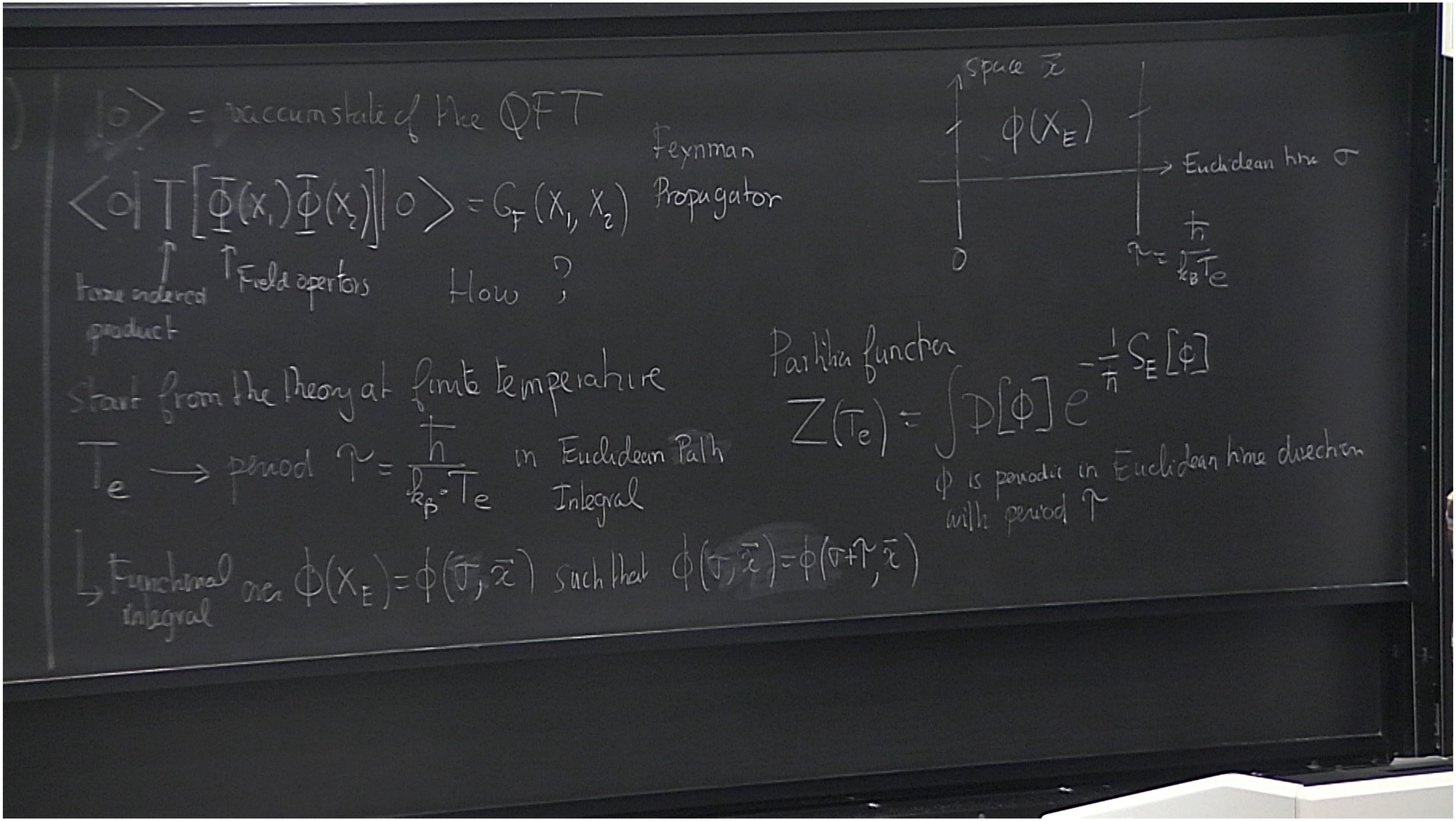
Finite dim. Gaussian integral

initial state = initial field configuration in space

$\phi_I(\vec{x})$; $\vec{x} = \epsilon \, \vec{n}_{space}$ $\quad$ $\vec{n}_{space} = (n_1 \cdots n_{d-1})$

final state

$\phi_F(\vec{x})$; $\vec{x} = \epsilon \, \vec{n}$



$|0\rangle$ = vaccum state of the QFT

$$\langle 0| T[\Phi(x_1)\Phi(x_2)]|0\rangle = G_F(x_1, x_2)$$ Feynman Propagator

↑ time ordered product ↑ Field opertors

How ?

Start from the theory at finite temperature

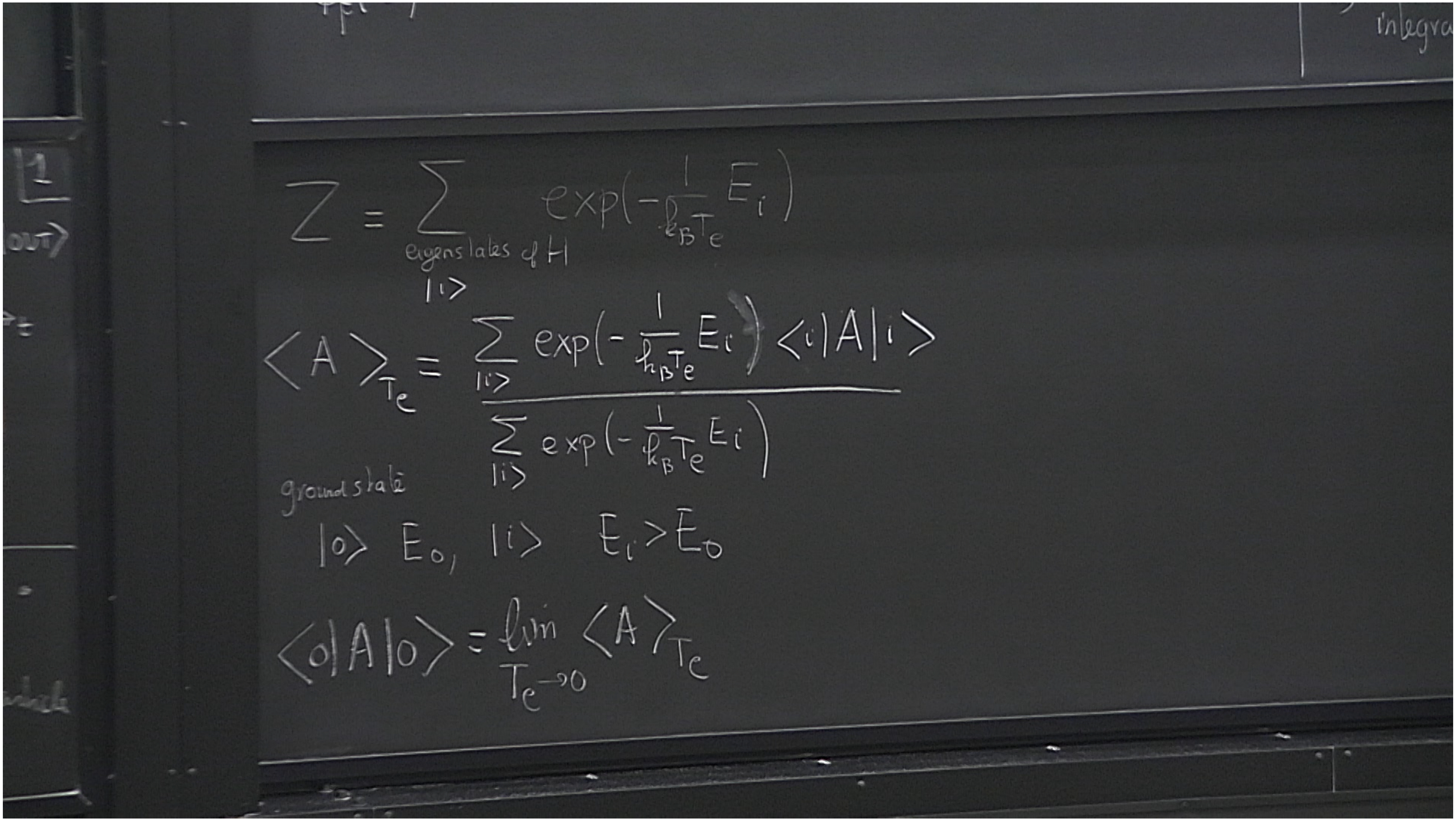
$$T_e \longrightarrow \text{period } \tau = \frac{\hbar}{k_\beta \cdot T_e}$$ in Euclidean Path Integral

↳ Functional integral over $\phi(X_E) = \phi(\sigma, \vec{x})$ such that $\phi(\sigma, \vec{x}) = \phi(\sigma + \tau, \vec{x})$

Partition function

$$Z(T_e) = \int D[\phi]\, e^{-\frac{1}{\hbar} S_E[\phi]}$$

ϕ is periodic in Euclidean time direction with period τ



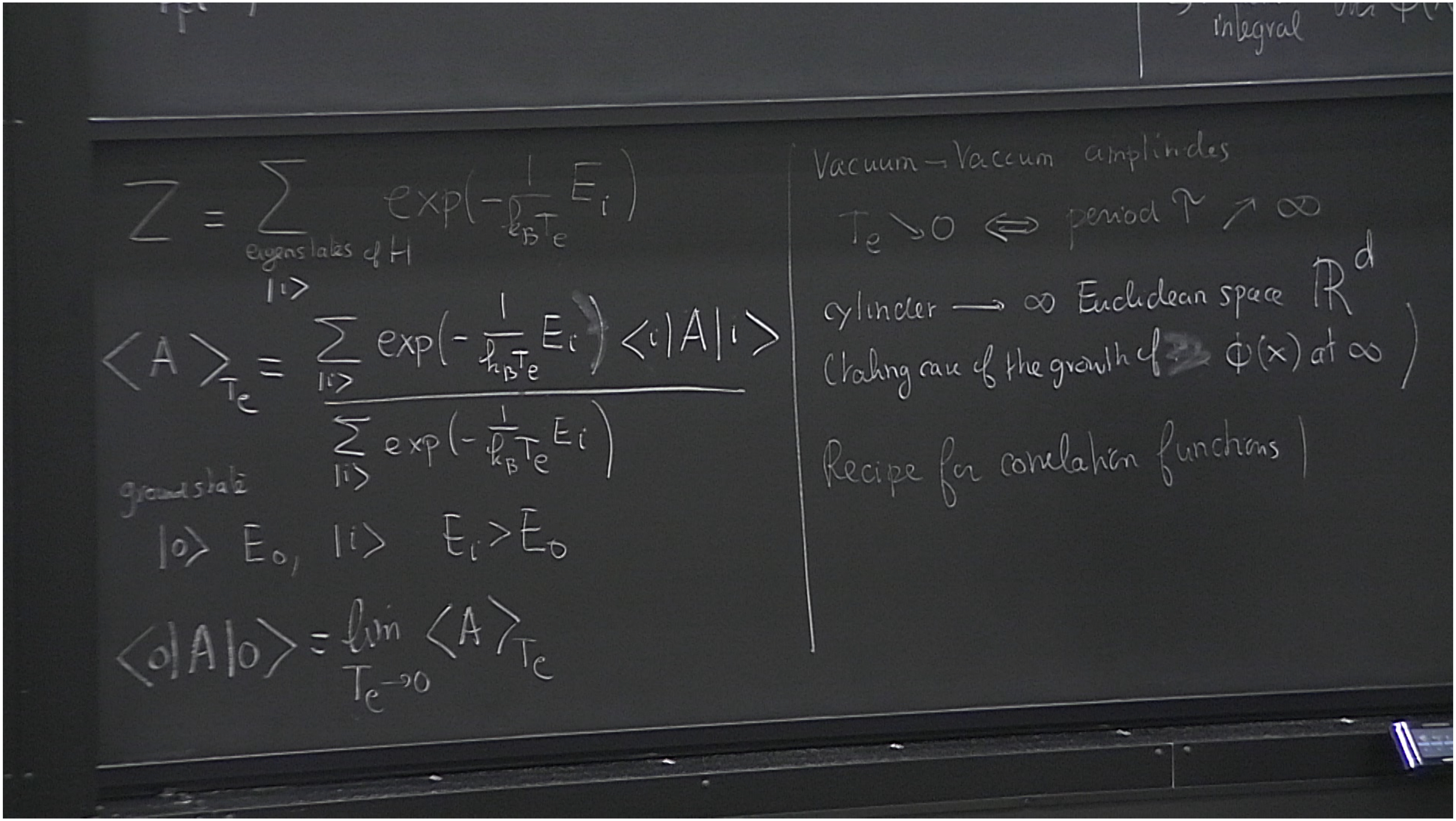
$$Z = \sum_{\substack{\text{eigenstates of } H \\ |i\rangle}} \exp\left(-\frac{1}{k_B T_e} E_i\right)$$

$$\langle A \rangle_{T_e} = \frac{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right) \langle i|A|i\rangle}{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right)}$$

ground state

$|0\rangle$ E_0, $|i\rangle$ $E_i > E_0$

$$\langle 0|A|0\rangle = \lim_{T_e \to 0} \langle A \rangle_{T_e}$$



integral

$$Z = \sum_{\substack{\text{eigenstates of } H \\ |i\rangle}} \exp\left(-\frac{1}{k_B T_e} E_i\right)$$

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ground state

$|0\rangle$ E_0, $|i\rangle$ $E_i > E_0$

$$\langle 0 | A | 0 \rangle = \lim_{T_e \to 0} \langle A \rangle_{T_e}$$

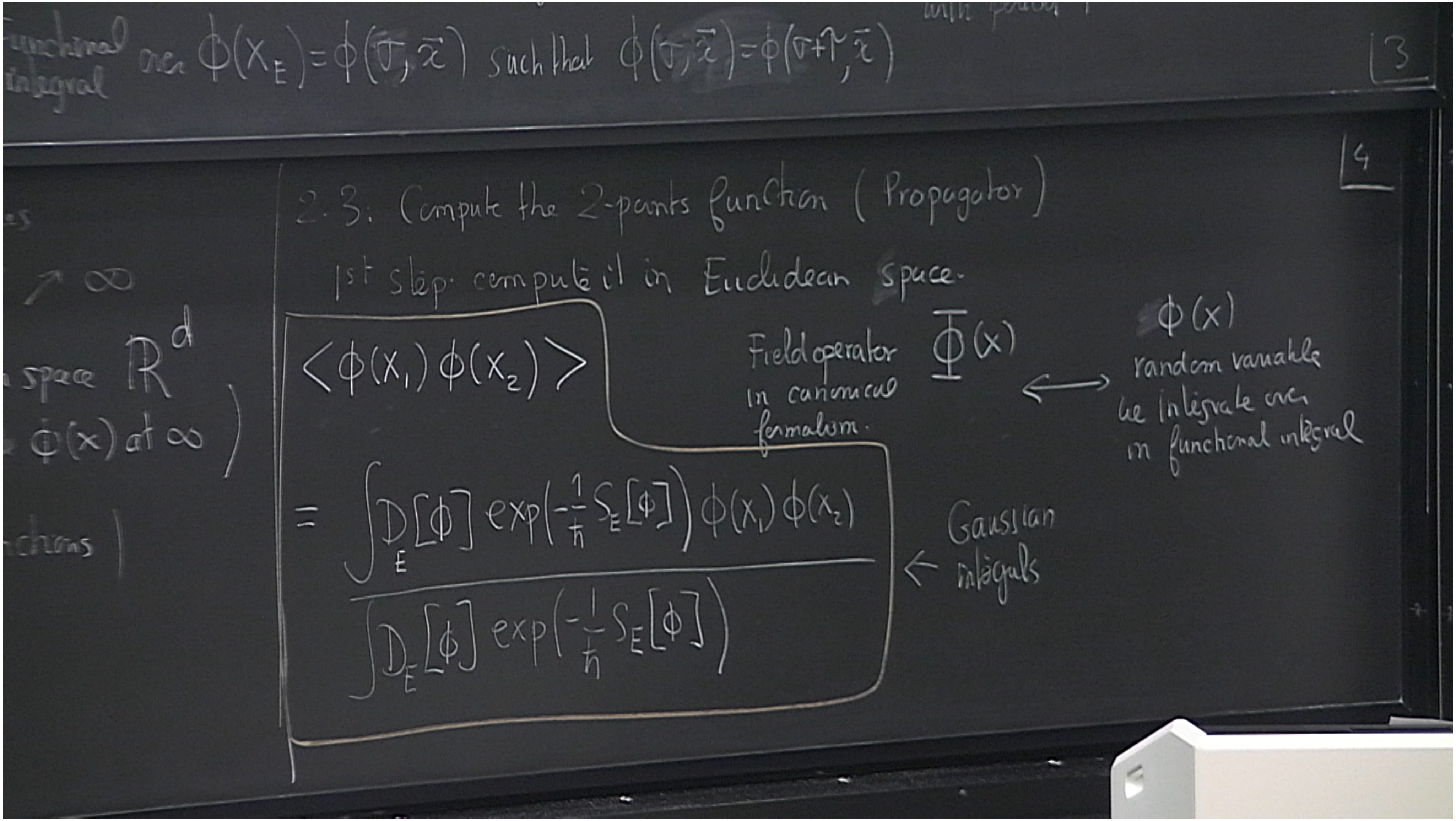
Vacuum → Vaccum amplitudes

$T_e \searrow 0 \iff$ period $\tau \nearrow \infty$

cylinder ⟶ ∞ Euclidean space $\mathbb{R}^d$

(Taking care of the growth of $\phi(x)$ at ∞)

Recipe for correlation functions)



with perio

unctional integral over $\phi(x_E)=\phi(\tau,\vec{x})$ such that $\phi(\tau,\vec{x})=\phi(\tau+T,\vec{x})$

3

es

$\nearrow \infty$

space $\mathbb{R}^d$

$\Phi(x)$ at ∞)

ctions)

4

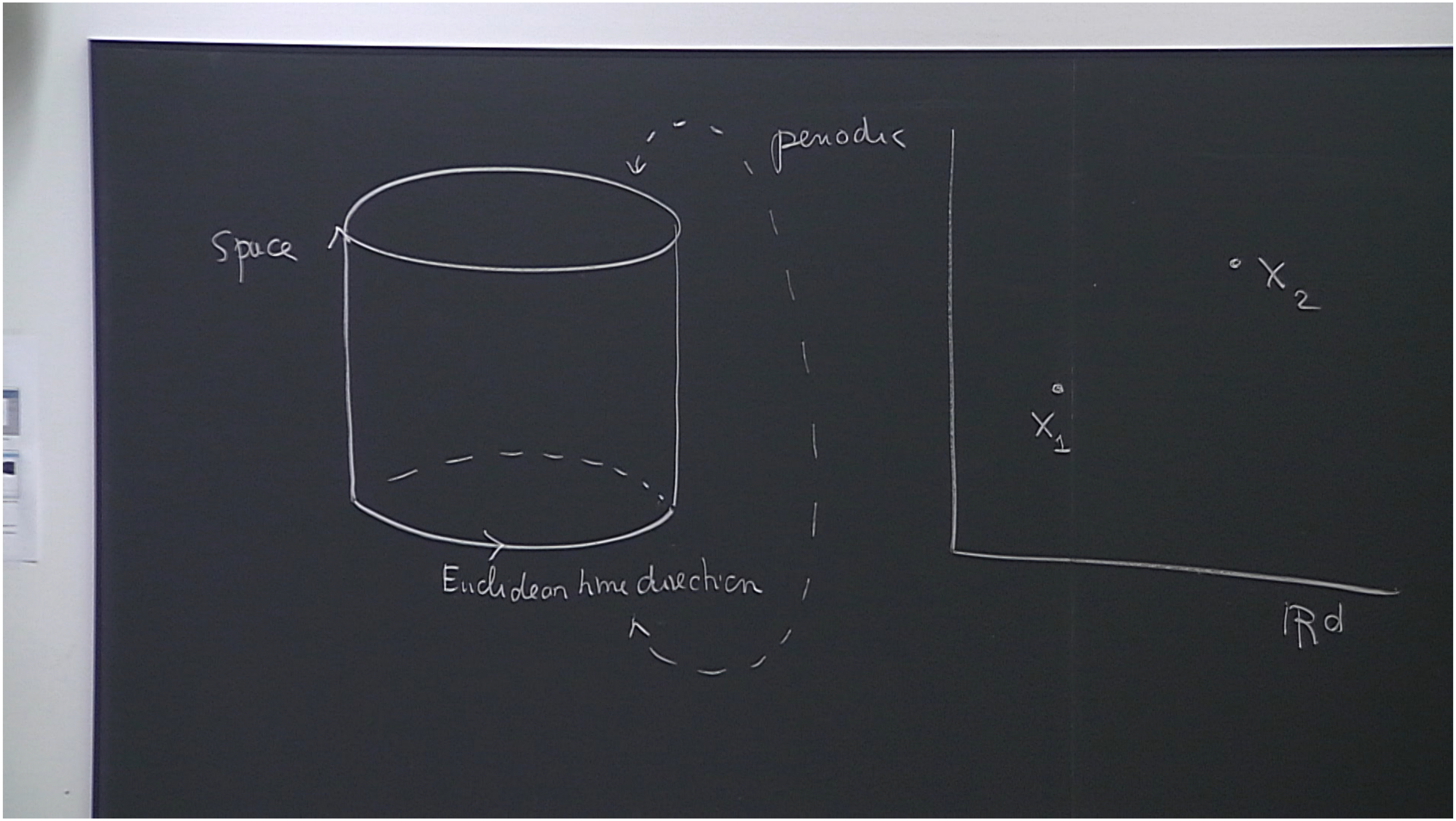
2.3: Compute the 2-points function (Propagator)

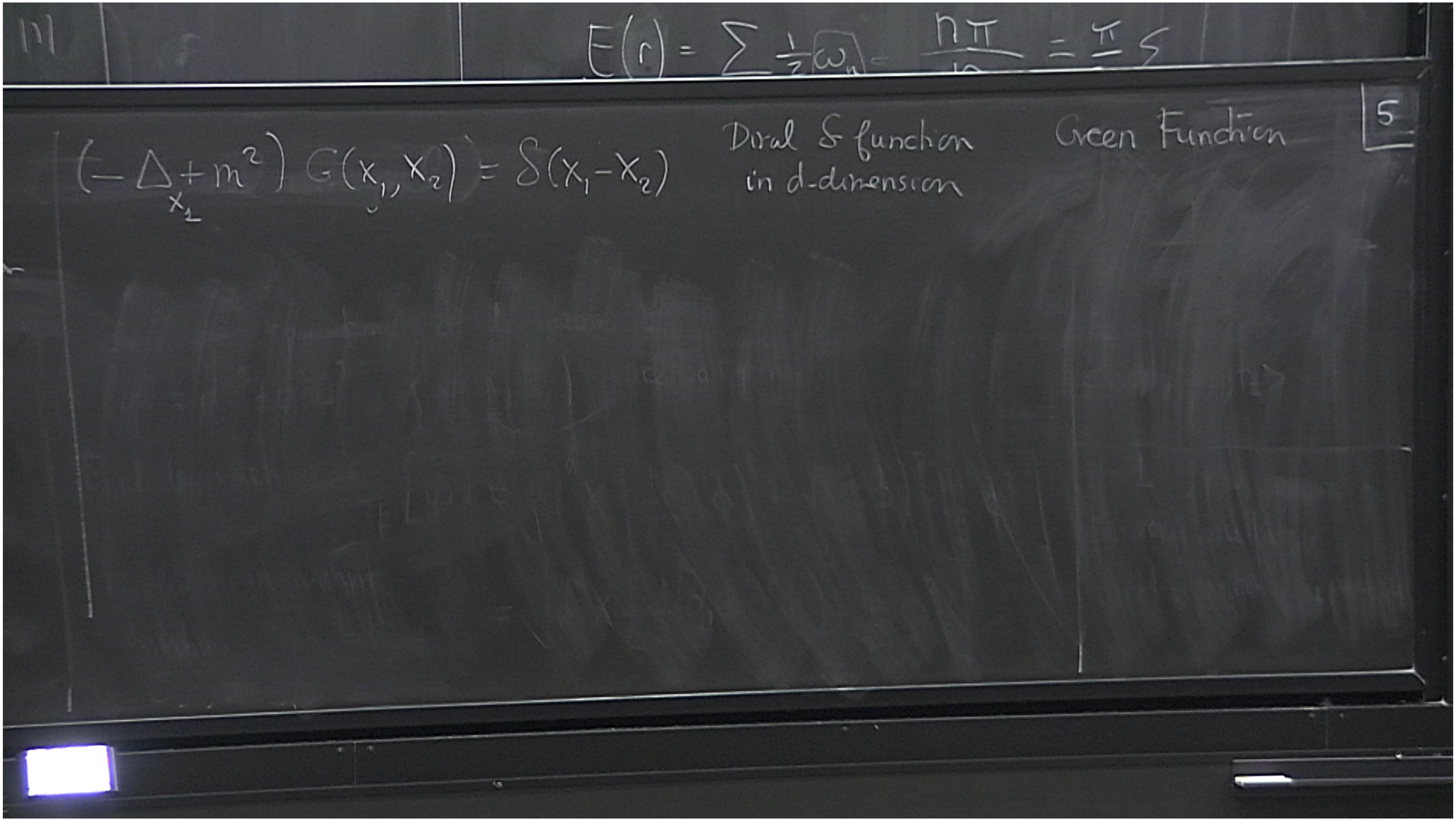
1st step: compute it in Euclidean space.

$$\boxed{\begin{aligned}&\langle \phi(x_1)\,\phi(x_2)\rangle \\ &= \frac{\int \mathcal{D}_E[\phi]\,\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)\phi(x_1)\,\phi(x_2)}{\int \mathcal{D}_E[\phi]\,\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)}\end{aligned}}$$

Field operator $\Phi(x)$ in canonical formalism. $\longleftrightarrow$ $\phi(x)$ random variable we integrate over in functional integral

$\leftarrow$ Gaussian integrals


periodic
Space
Euclidean time direction
x_2
x_1
$\mathbb{R}^d$



111

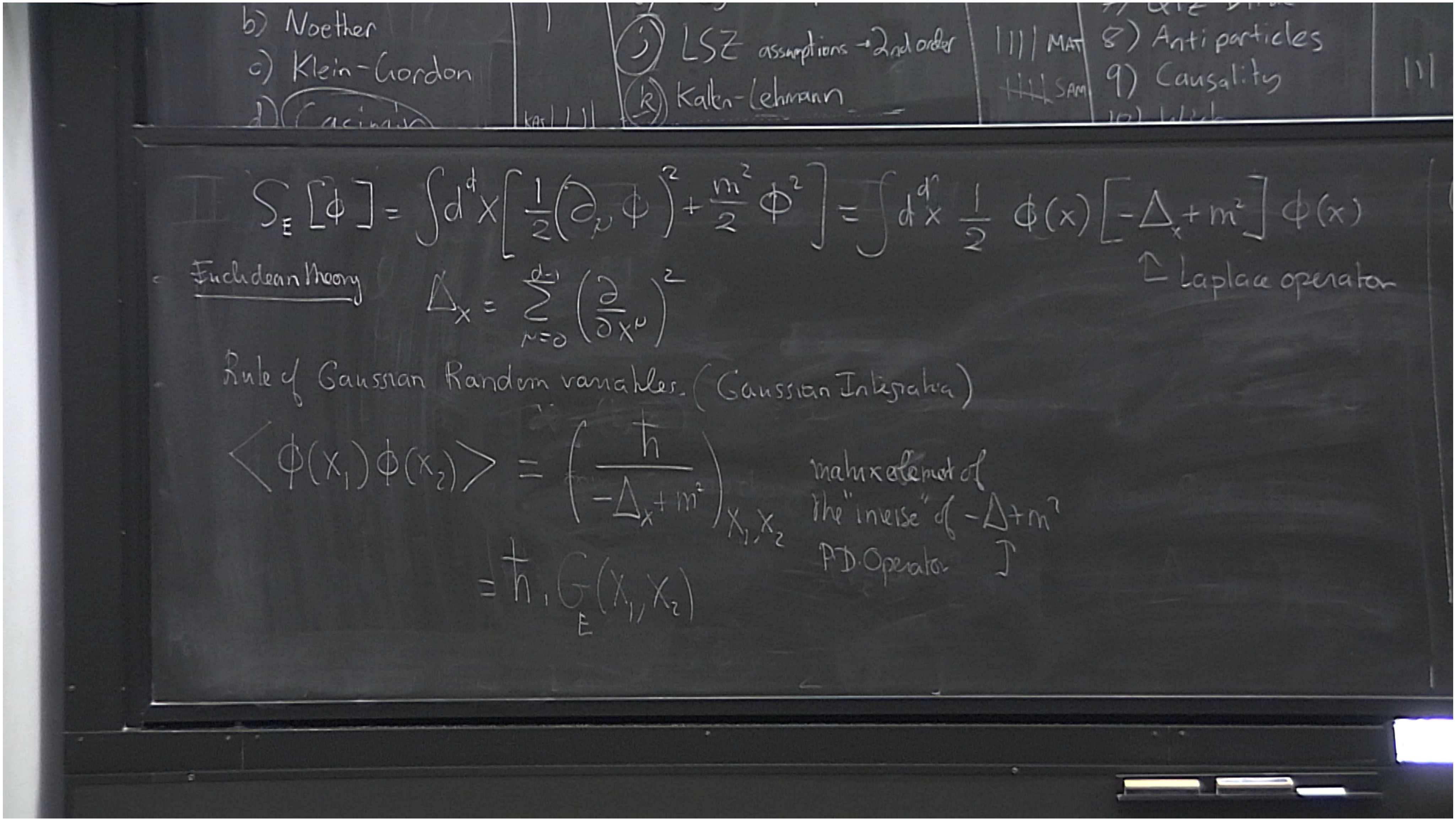
$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{} = \frac{\pi}{} \Sigma$

5

$(-\Delta_{x_1} + m^2)\, G(x_1, x_2) = \delta(x_1 - x_2)$

Dirac δ function
in d-dimension

Green Function



b) Noether

c) Klein-Gordon

d) Casimir

KAS ||||

j) LSZ assumptions → 2nd order

k) Kallen-Lehmann

|||| MAT

SAM

8) Antiparticles

9) Causality

10)

||

II $$S_E[\phi] = \int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2\right] = \int d^dx\, \frac{1}{2}\, \phi(x)\left[-\Delta_x + m^2\right]\phi(x)$$

↑ Laplace operator

Euclidean theory

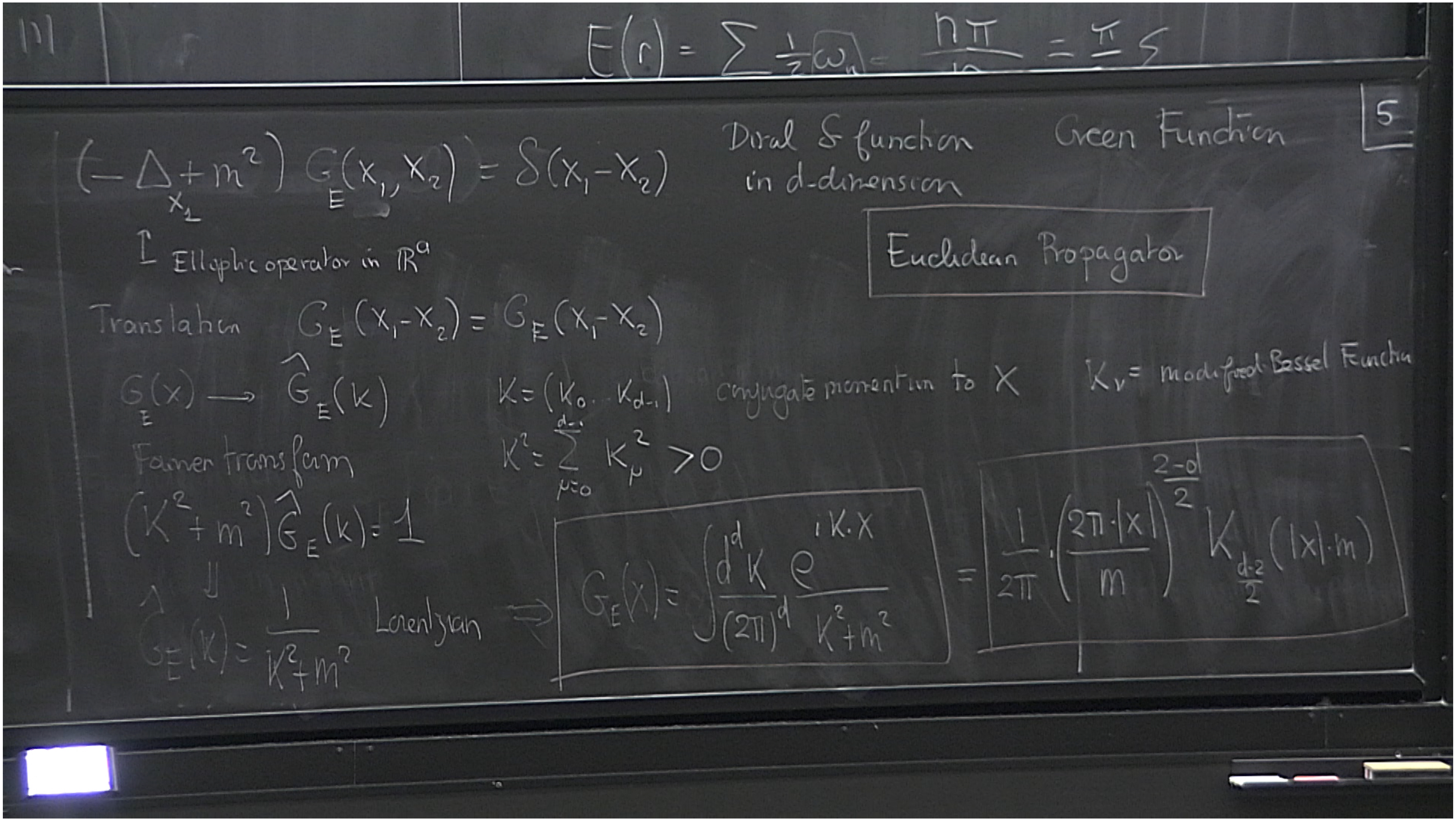
$$\Delta_x = \sum_{\mu=0}^{d-1}\left(\frac{\partial}{\partial x^\mu}\right)^2$$

Rule of Gaussian Random variables. (Gaussian Integration)

$$\langle \phi(x_1)\phi(x_2)\rangle = \left(\frac{\hbar}{-\Delta_x + m^2}\right)_{x_1,x_2}$$

matrix element of the "inverse" of $-\Delta+m^2$ [P.D. Operator]

$$= \hbar\, G_E(x_1,x_2)$$



111

$E(r) = \sum \frac{1}{2}\omega_n = n\pi = \pi \sum$

5

$(-\Delta_{x_1} + m^2)\, G_E(x_1, x_2) = \delta(x_1 - x_2)$ Dirac δ function in d-dimension

Green Function

↑ Elliptic operator in $\mathbb{R}^d$

$\boxed{\text{Euclidean Propagator}}$

Translation $G_E(x_1 - x_2) = G_E(x_1 - x_2)$

$G_E(x) \longrightarrow \hat{G}_E(k)$

Fourier transform

$K = (K_0 \ldots K_{d-1})$ conjugate momentum to X

K_ν = modified Bessel Function

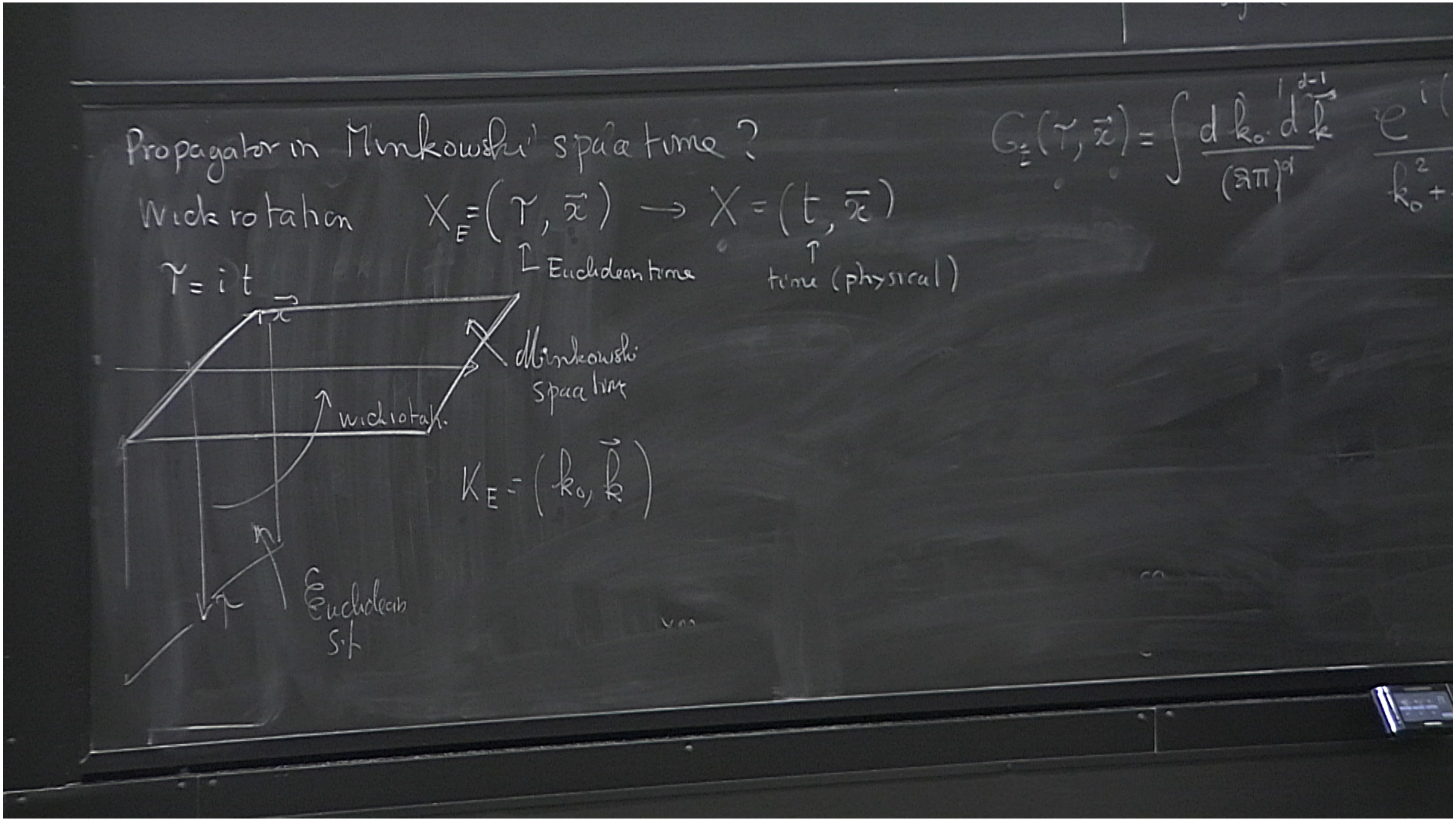
$K^2 = \sum_{\mu=0}^{d-1} K_\mu^2 > 0$

$(K^2 + m^2)\,\hat{G}_E(k) = 1$

$\Uparrow \quad \Downarrow$

$\hat{G}_E(K) = \frac{1}{K^2 + m^2}$ Lorentzian $\Longrightarrow$

$$\boxed{G_E(x) = \int \frac{d^d K}{(2\pi)^d} \frac{e^{iK\cdot X}}{K^2 + m^2}} = \boxed{\frac{1}{2\pi} \cdot \left(\frac{2\pi \cdot |x|}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(|x| \cdot m)}$$



Propagator in Minkowski space time ?

Wick rotation $X_E = (\tau, \vec{x}) \rightarrow X = (t, \vec{x})$

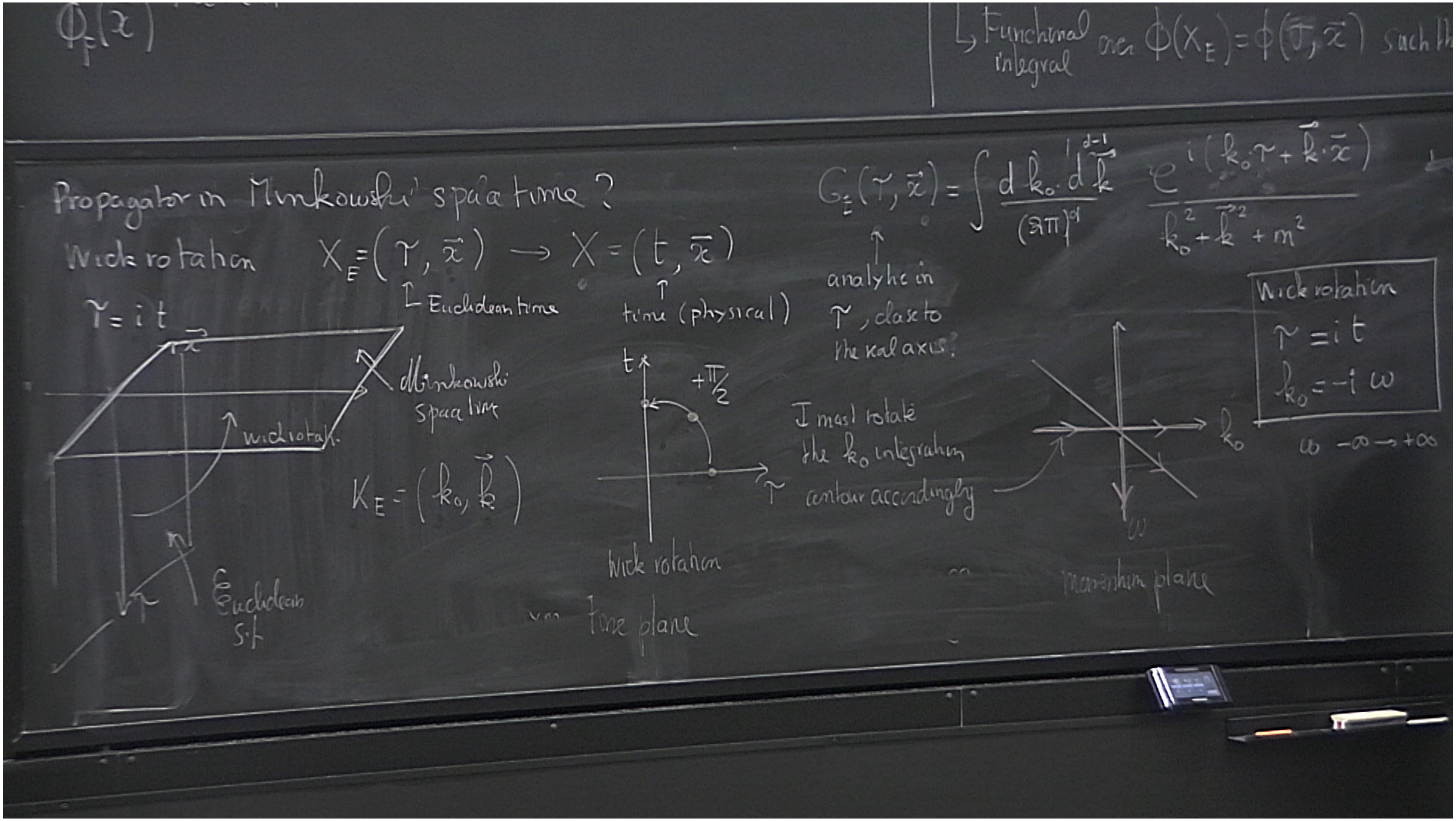
↑ Euclidean time

↑ time (physical)

$\tau = i t$

$K_E = (k_0, \vec{k})$

$$G_E(\tau, \vec{x}) = \int \frac{d k_0 \, d^{d-1}\vec{k}}{(2\pi)^d} \quad \frac{e^{i(}}{k_0^2 +}$$



$\phi_F(x)$

↳ Functional integral over $\phi(X_E)=\phi(\tau,\vec{x})$ such th

Propagator in Minkowski spacetime?

Wick rotation $X_E=(\tau,\vec{x}) \longrightarrow X=(t,\vec{x})$

└ Euclidean time

↑ time (physical)

$\tau = i\,t$

$K_E=(k_0,\vec{k})$

$$G_E(\tau,\vec{x}) = \int \frac{dk_0\, d^{d-1}\vec{k}}{(2\pi)^d}\, \frac{e^{i(k_0\tau+\vec{k}\cdot\vec{x})}}{k_0^2+\vec{k}^2+m^2}$$

↑ analytic in τ, close to the real axis?

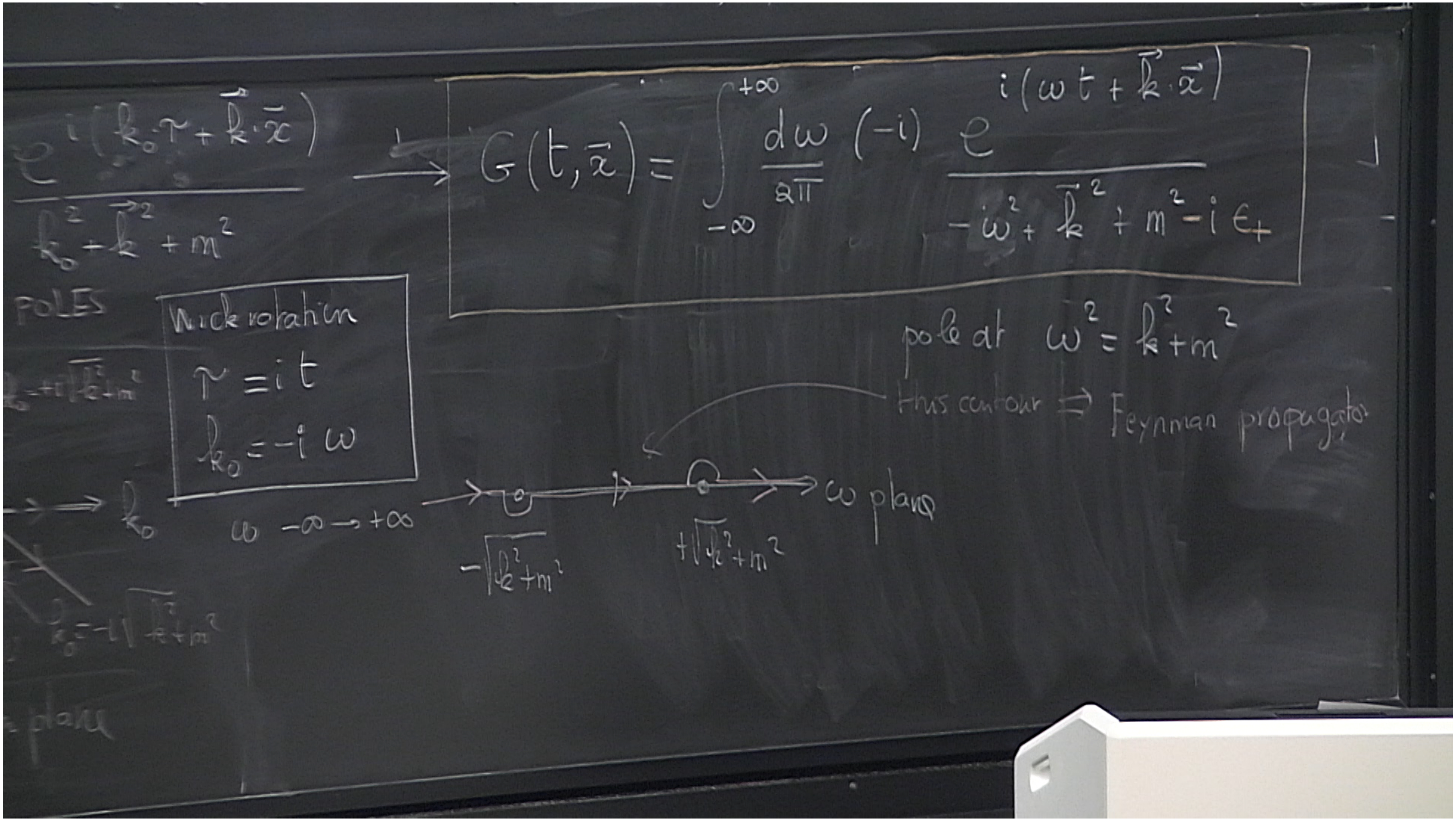
I must rotate the k_0 integration contour accordingly

Wick rotation

$\tau = i\,t$

$k_0=-i\,\omega$

ω −∞ → +∞



$$\frac{e^{i(k_0\tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2} \longrightarrow \boxed{G(t,\vec{x}) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,(-i)\,\frac{e^{i(\omega t + \vec{k}\cdot\vec{x})}}{-\omega^2 + \vec{k}^2 + m^2 - i\epsilon_+}}$$

POLES

$k_0 = +i\sqrt{\vec{k}^2+m^2}$

Wick rotation

$\tau = i t$

$k_0 = -i\,\omega$

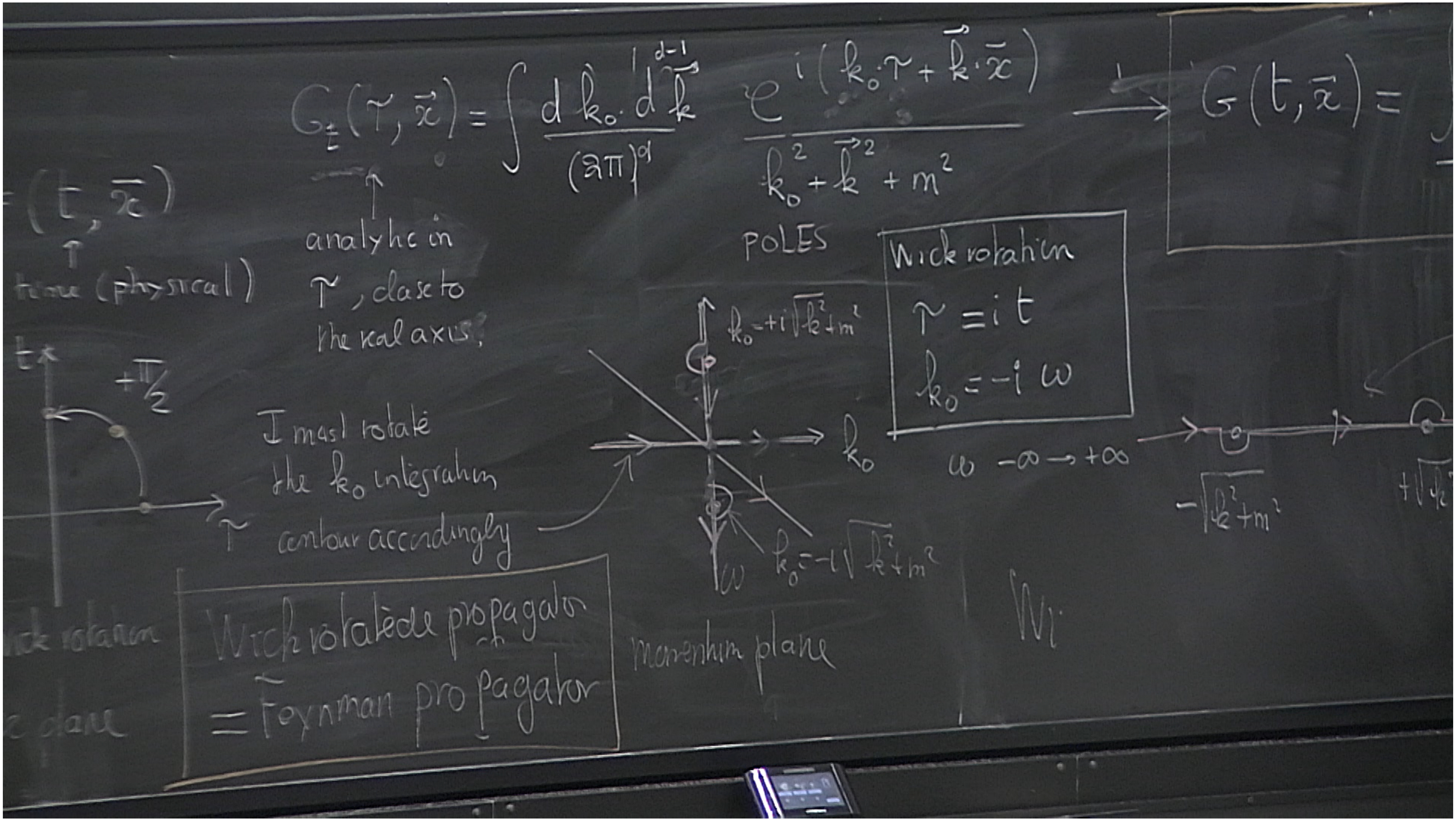
$\omega\ -\infty \to +\infty$

$k_0 = -i\sqrt{\vec{k}^2+m^2}$

plane

pole at $\omega^2 = \vec{k}^2 + m^2$

This contour ⇒ Feynman propagator



$$G_E(\tau, \vec{x}) = \int \frac{dk_0 \cdot d^{d-1}\vec{k}}{(2\pi)^d} \frac{e^{i(k_0\tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2} \longrightarrow G(t, \bar{x}) =$$

$(t, \bar{x})$ ↑ time (physical)

↑ analytic in τ, close to the real axis?

POLES

Wick rotation

$\tau = i t$

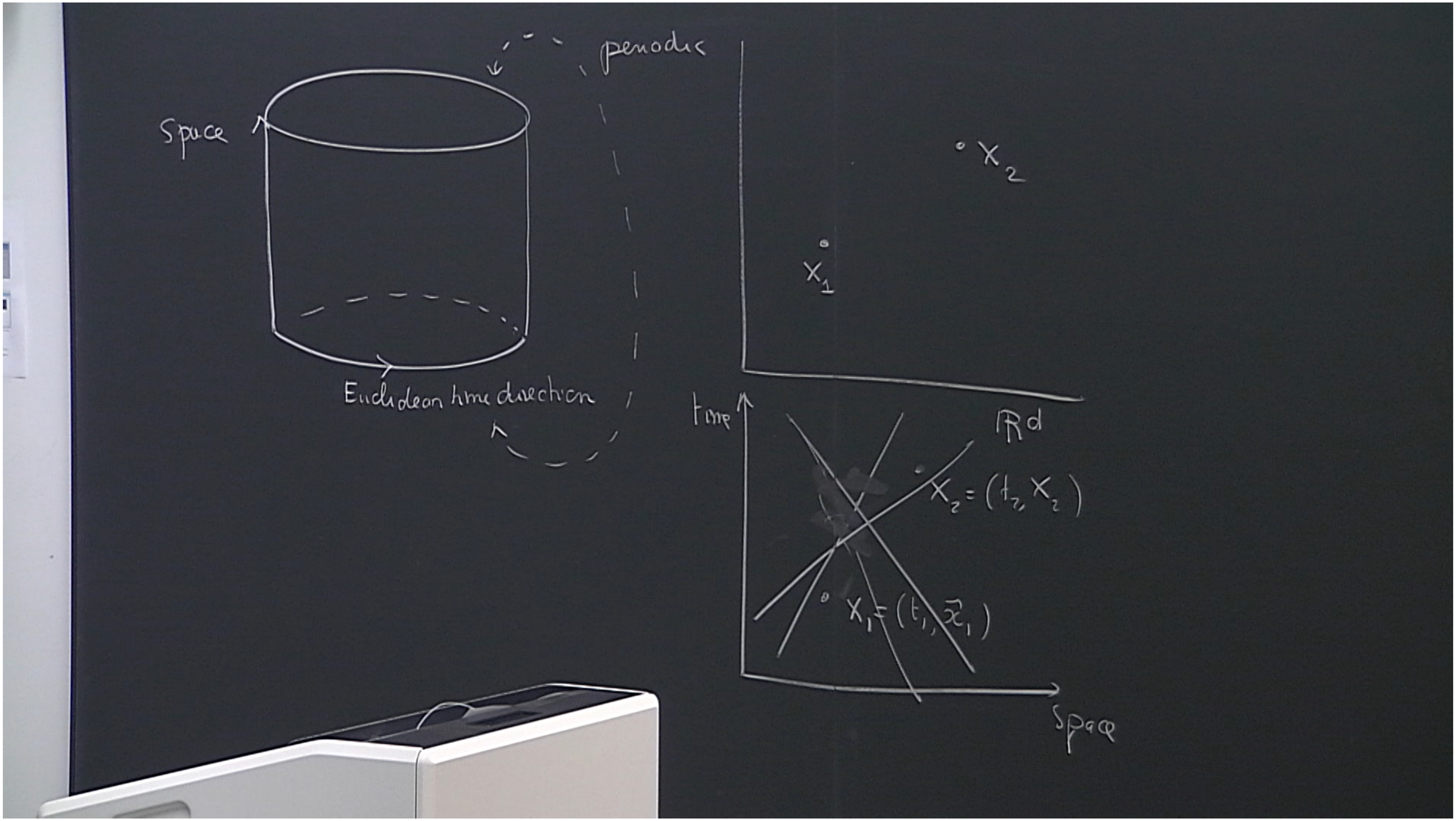
$k_0 = -i\,\omega$

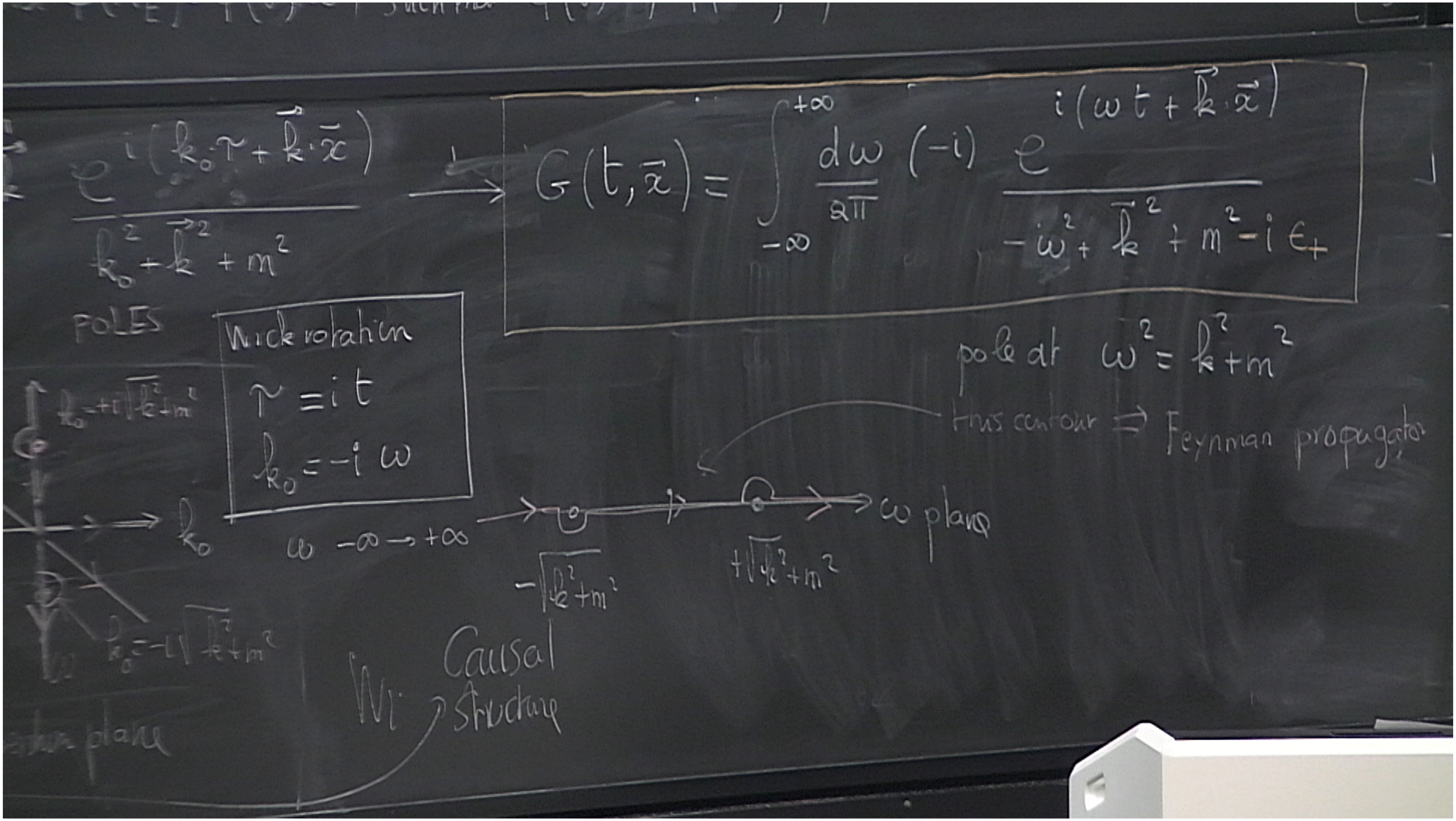
ω $-\infty \to +\infty$

I must rotate the k_0 integration contour accordingly

Wick rotatede propagator = Feynman propagator

momentum plane


periodic
Space
Euclidean time direction
X_2
X_1
$\mathbb{R}^d$
time
$X_2 = (t_2, X_2)$
$X_1 = (t_1, \vec{x}_1)$
Space



$$\frac{e^{i(k_0\tau+\vec{k}\cdot\vec{x})}}{k_0^2+\vec{k}^2+m^2} \longrightarrow \boxed{G(t,\vec{x})=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}(-i)\,\frac{e^{i(\omega t+\vec{k}\cdot\vec{x})}}{-\omega^2+\vec{k}^2+m^2-i\epsilon_+}}$$

POLES

$$\boxed{\begin{array}{l}\text{Wick rotation}\\ \tau=it\\ k_0=-i\omega\end{array}}$$

pole at $\omega^2=\vec{k}^2+m^2$

this contour ⇒ Feynman propagator

Wi

Causal structure