Title: PSI 2018/2019 - Quantum Field Theory II - Lecture 2

Date: Nov 13, 2018 09:00 AM

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Abstract:

6) Fermion Lags
7) Qtz Dirac
8) Anti particles
9) Causality

17) $\mathcal{L}_{int}$ → vertex rule
65) QED lag

↳ Is vacuum energy real(??

$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{r} = \frac{\pi}{r}$

by path integrals (Functional integral)

2 time
d-1 space

Euclidean spacetime d dimensional
$ds^2 = d\tau^2 + d\vec{x}^2 \quad X = (\tau, \vec{x})$ euclidean time

Euclidean action $S_E[\phi] = \int d^d X \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2\right]$

Euclidean invariant
rotations + translations $= \int d^d X \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) + \frac{m^2}{2}\phi^2\right]$

Klein-Gordon Equation

$\frac{m^2}{2}\phi^2\Big] \Rightarrow (-\Delta + m^2)\phi = 0$

QFT

DAN

a) Feynmann diag
b) Noether
c) Klein-Gordon
d) Casimir

h) Wick
i) Feynman prop.
j) LSZ assumptions → 2nd order
k) Kallen-Lehmann

5) Conjugate
6) Fermion Lags
7) Qte Dirac
8) Anti particles
9) Causality

10) QED diags
12) $\mathcal{L}_{int}$ → vertex rule
65) QED lag

CASIMIR

↳ is vacuum energy

$E(0) = \sum \frac{1}{2}\omega$ $\frac{n\pi}{}$ =

II) Quantization of Free scalar Field by path integrals (Functional integral)

2.1 Classical K.G field.

Minkowski space time d dimensional ← 1 time, $d-1$ space

m = mass of the quanta of the field

$ds^2 = -dt^2 + d\vec{x}^2$ $\quad X = (t, \vec{x})$ $\quad c = 1$ $\quad$ Lagrangian density $\quad \Delta$ $m \neq$ Mass of ϕ^6

Field $\phi(x)$ real spin = 0, no charge

Klein-Gordon Equation

Action $S[\phi] = \int dt \int d^{d-1}\vec{x} \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 - \frac{m^2}{2}\phi^2 \right] \Rightarrow (-\Delta + m^2)\phi = 0$

Poincaré = Lorentz + transl. $= \int d^d x \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right]$

$O(1, d-1) \times_R \mathbb{R}^{d-1}$

Euclidean spacetime d dimensional

$ds^2 = d\tau^2 + d\vec{x}^2$ $\quad X = (\tau, \vec{x})$ euclidean time

Euclidean action $S_E[\phi] = \int d^d X \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2 \right]$

Euclidean invariant rotations + translations $= \int d^d x \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) + \frac{m^2}{2}\phi^2 \right]$

$O(d) \ltimes \mathbb{R}^d$

$$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{r} = \frac{\pi}{r}\zeta$$

nal integral)

an spacetime d. dimensional

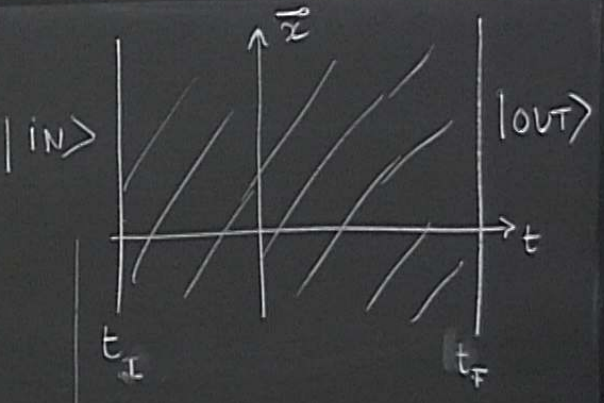
$dt^2 + d\vec{x}^2$ $\quad X = (\tau, \vec{x})$ euclidean time

action $S_E[\phi] = \int d^d X \left[\frac{1}{2}(\partial_\tau \phi)^2 + \frac{1}{2}(\vec{\nabla}_{\vec{x}}\phi)^2 + \frac{m^2}{2}\phi^2\right]$

n invariant $= \int d^d X \left[\frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) + \frac{m^2}{2}\phi^2\right]$

s + translations

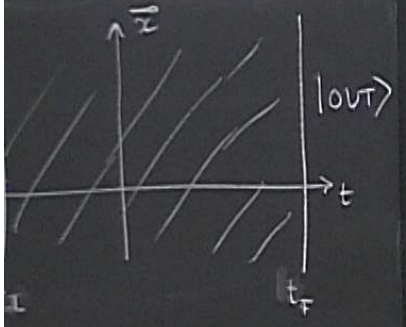
$\mathbb{R}^d$



$\langle OUT, t_F | IN, t_I \rangle$

(??)

$= S$



$\langle OUT, t_F | IN, t_I \rangle$

$\hbar = 1$ 1 time, space = ∘

harmonic oscillator

$\phi(t)$ = position of the particle

2.2. Quantization = integral over field configurations (functional integral)

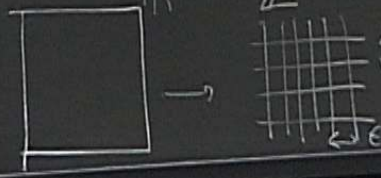
M. $\int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$ $\quad \phi \to$ Field quatum operator

E. $\int \mathcal{D}_E[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$ $\quad x \in \mathbb{R}^d \to x_n = \epsilon n \quad n \in \mathbb{Z}^d$

$n = (n_0, n_1 \dots n_{d-1})$ time ↑ space

Discretize time + space; → continuum limit?

Simplest $\mathbb{R}^d \to \mathbb{Z}^d$ with mesh $\in$ (segment and $\vec{x}$)



Continuous sym. → discrete sym.

2.2 Quantization = integral over field configurations (functional integral)

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E. $\int \mathcal{D}_E[\phi] \exp\left(-\frac{1}{\hbar} S_E[\phi]\right)$

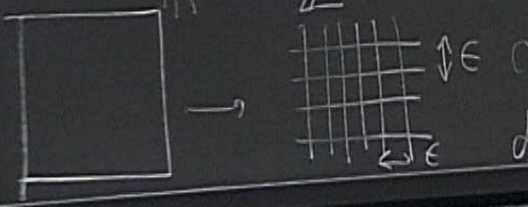
$\phi \to$ Field quantum operator

$(x^0 \dots x^{d-1}) = x \in \mathbb{R}^d \to x_n = \epsilon n \quad n \in \mathbb{Z}^d$

$n = (n_0, n_1 \dots n_{d-1})$; n_0: time, $n_1 \dots n_{d-1}$: space

Discretize time + space: $\to$ Continuum limit ?

Simplest $\mathbb{R}^d \to \mathbb{Z}^d$ with mesh ϵ (same in time and space)



$\updownarrow \epsilon$ Continuous sym.

$\downarrow$

discrete sym.

Rule 1: Replace derivatives by finite differences

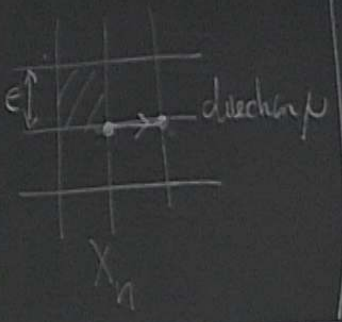
$$\frac{\partial}{\partial x^\mu}\phi(x) \to \frac{\phi(x+\epsilon e_\mu) - \phi(x)}{\epsilon}$$

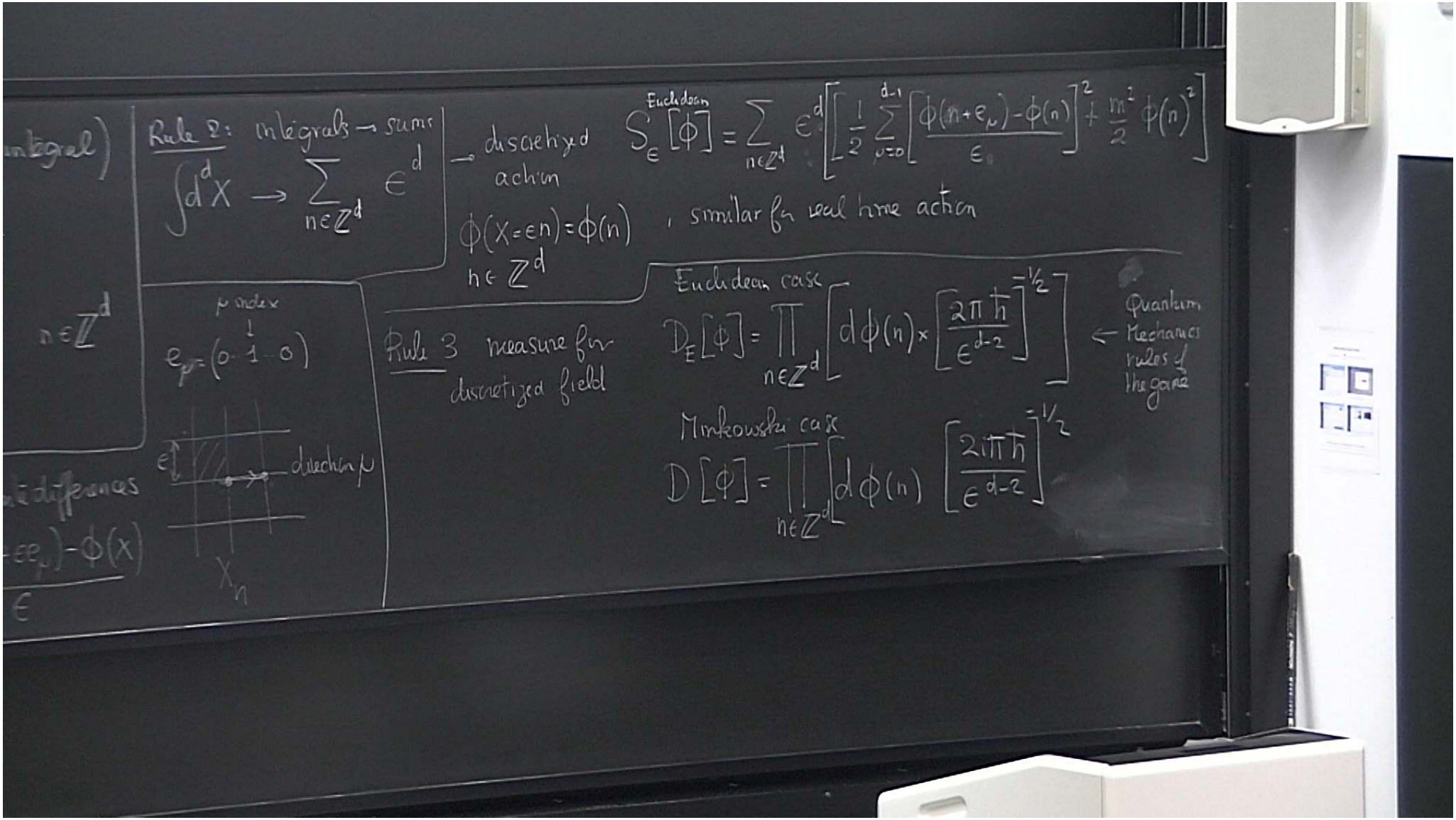
Rule 2: integrals $\to$ sums

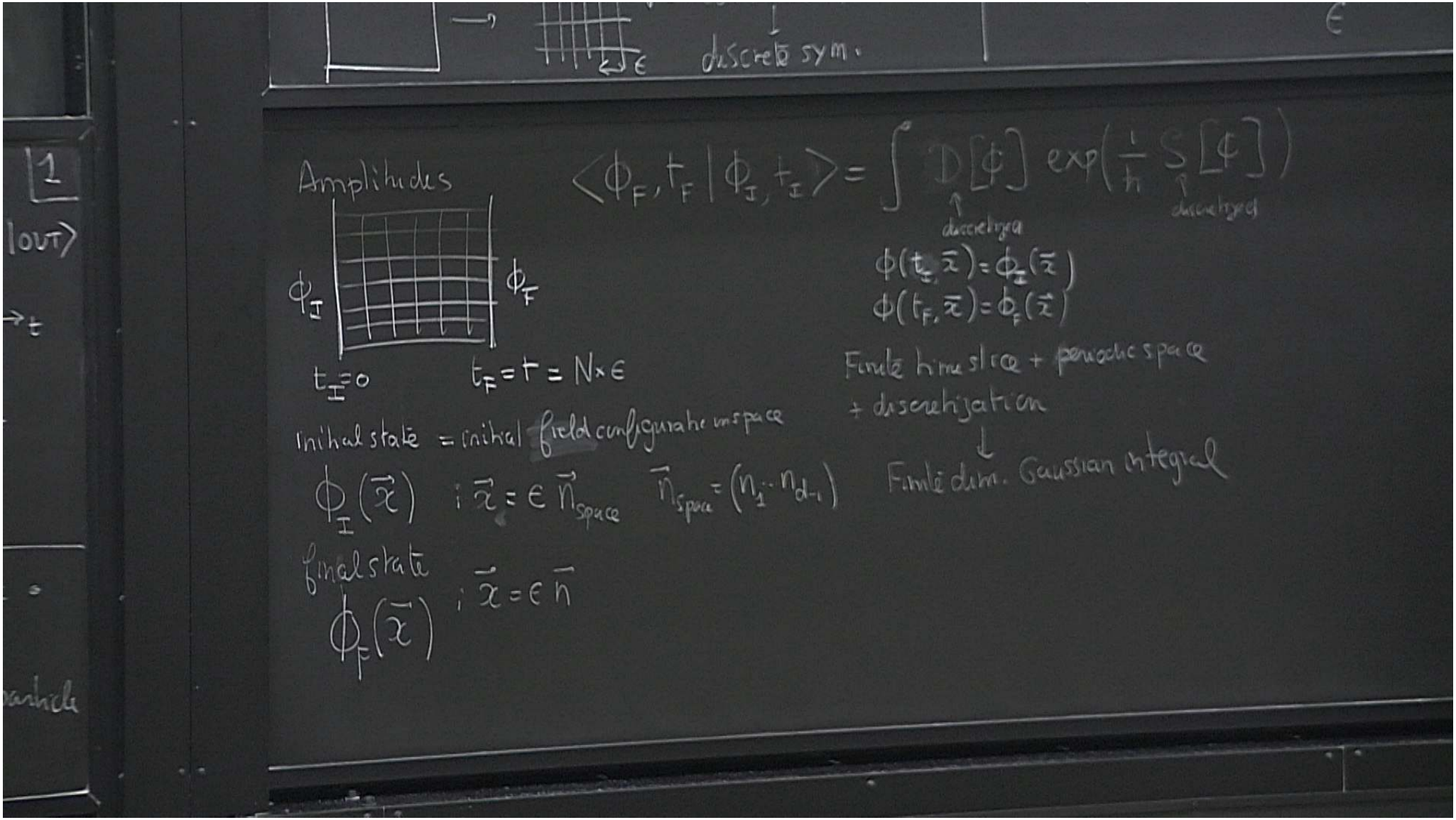
$$\int d^d x \to \sum_{n \in \mathbb{Z}^d} \epsilon^d$$

μ index

$e_\mu = (0 \dots \underset{\mu}{1} \dots 0)$




Rule 2: integrals → sums
$\int d^d x \to \sum_{n\in\mathbb{Z}^d} \epsilon^d$
→ discretized action
$\phi(x=\epsilon n)=\phi(n)$
$n\in\mathbb{Z}^d$
Euclidean
$S_\epsilon^{E}[\phi] = \sum_{n\in\mathbb{Z}^d} \epsilon^d \left[\left[\frac{1}{2}\sum_{\mu=0}^{d-1}\left[\frac{\phi(n+e_\mu)-\phi(n)}{\epsilon}\right]^2 + \frac{m^2}{2}\phi(n)^2\right]\right.$
, similar for real time action
Euclidean case
$D_E[\phi] = \prod_{n\in\mathbb{Z}^d}\left[d\phi(n)\times\left[\frac{2\pi\hbar}{\epsilon^{d-2}}\right]^{-1/2}\right]$
← Quantum Mechanics rules of the game
Minkowski case
$D[\phi] = \prod_{n\in\mathbb{Z}^d}\left[d\phi(n)\left[\frac{2i\pi\hbar}{\epsilon^{d-2}}\right]^{-1/2}\right.$
Rule 3 measure for discretized field
μ index
$e_\mu = (0\ 1\ 0)$
direction μ
x_n
integral)
$n\in\mathbb{Z}^d$
differences
$\phi(x+\epsilon e_\mu)-\phi(x)$
ϵ


discrete sym.
Amplitudes
$\langle \phi_F, t_F \mid \phi_I, t_I \rangle = \int \mathcal{D}[\phi] \exp\left(\frac{i}{\hbar} S[\phi]\right)$
discretized
discretized
$\phi(t_I, \vec{x}) = \phi_I(\vec{x})$
$\phi(t_F, \vec{x}) = \phi_F(\vec{x})$
ϕ_I
ϕ_F
$t_I = 0$
$t_F = T = N \times \epsilon$
Finite time slice + periodic space
+ discretization
Finite dim. Gaussian integral
Initial state = initial field configuration in space
$\phi_I(\vec{x})$; $\vec{x} = \epsilon \vec{n}_{space}$ $\vec{n}_{space} = (n_1 \cdots n_{d-1})$
final state
$\phi_F(\vec{x})$; $\vec{x} = \epsilon \vec{n}$
1
|OUT⟩
→ t

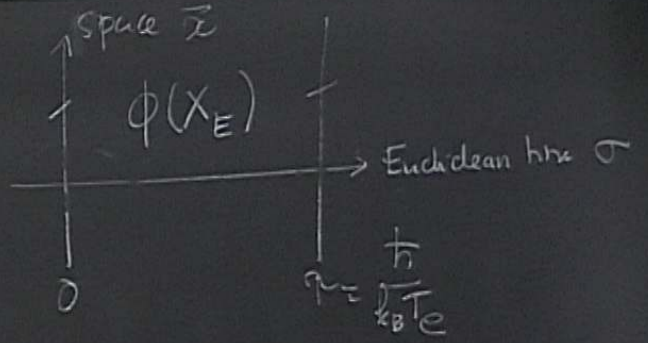
$|0\rangle$ = vacuum state of the QFT

$\langle 0|\, T\left[\underline{\Phi}(x_1)\underline{\Phi}(x_2)\right]|0\rangle = G_F(x_1, x_2)$ Feynman Propagator

T: time ordered product

$\underline{\Phi}$: Field operators

How ?



Start from the theory at finite temperature

$T_e \longrightarrow$ period $\tau = \frac{\hbar}{k_B \cdot T_e}$ in Euclidean Path Integral

↳ Functional integral over $\phi(x_E) = \phi(\sigma, \vec{x})$ such that $\phi(\sigma, \vec{x}) = \phi(\sigma + \tau, \vec{x})$

Partition function

$$Z(T_e) = \int \mathcal{D}[\phi]\, e^{-\frac{1}{\hbar} S_E[\phi]}$$

ϕ is periodic in Euclidean time direction with period τ

$$Z = \sum_{\substack{\text{eigenstates of } H \\ |i\rangle}} \exp\left(-\frac{1}{k_B T_e} E_i\right)$$

$$\langle A \rangle_{T_e} = \frac{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right) \langle i|A|i\rangle}{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right)}$$

ground state

$|0\rangle$ E_0, $|i\rangle$ $E_i > E_0$

$$\langle 0|A|0\rangle = \lim_{T_e \to 0} \langle A \rangle_{T_e}$$

integral

$$Z = \sum_{\substack{\text{eigenstates of } H \\ |i\rangle}} \exp\left(-\frac{1}{k_B T_e} E_i\right)$$

$$\langle A \rangle_{T_e} = \frac{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right) \langle i|A|i\rangle}{\sum_{|i\rangle} \exp\left(-\frac{1}{k_B T_e} E_i\right)}$$

ground state

$|0\rangle$ E_0, $|i\rangle$ $E_i > E_0$

$$\langle 0|A|0\rangle = \lim_{T_e \to 0} \langle A \rangle_{T_e}$$

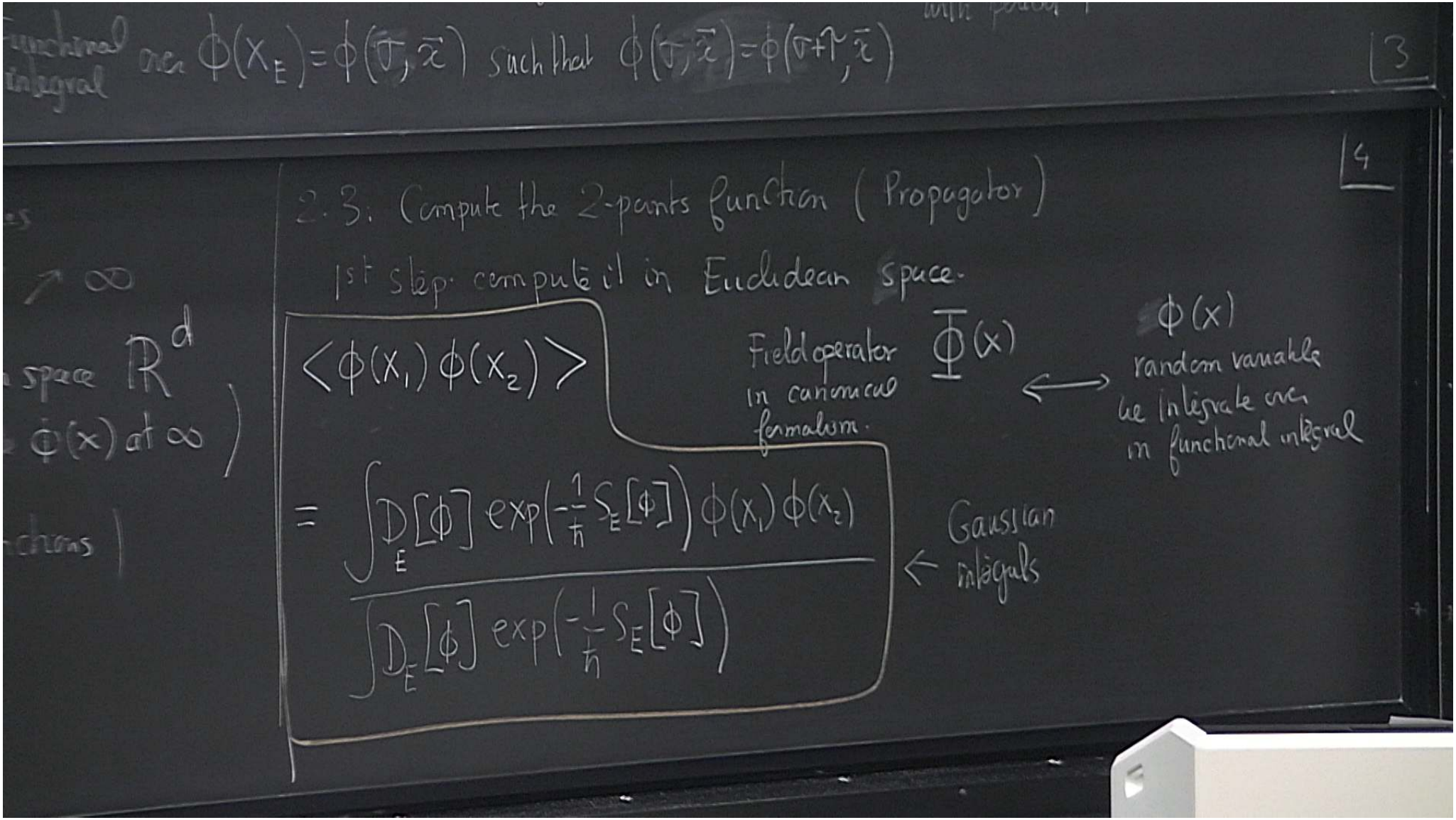
Vacuum → Vacuum amplitudes

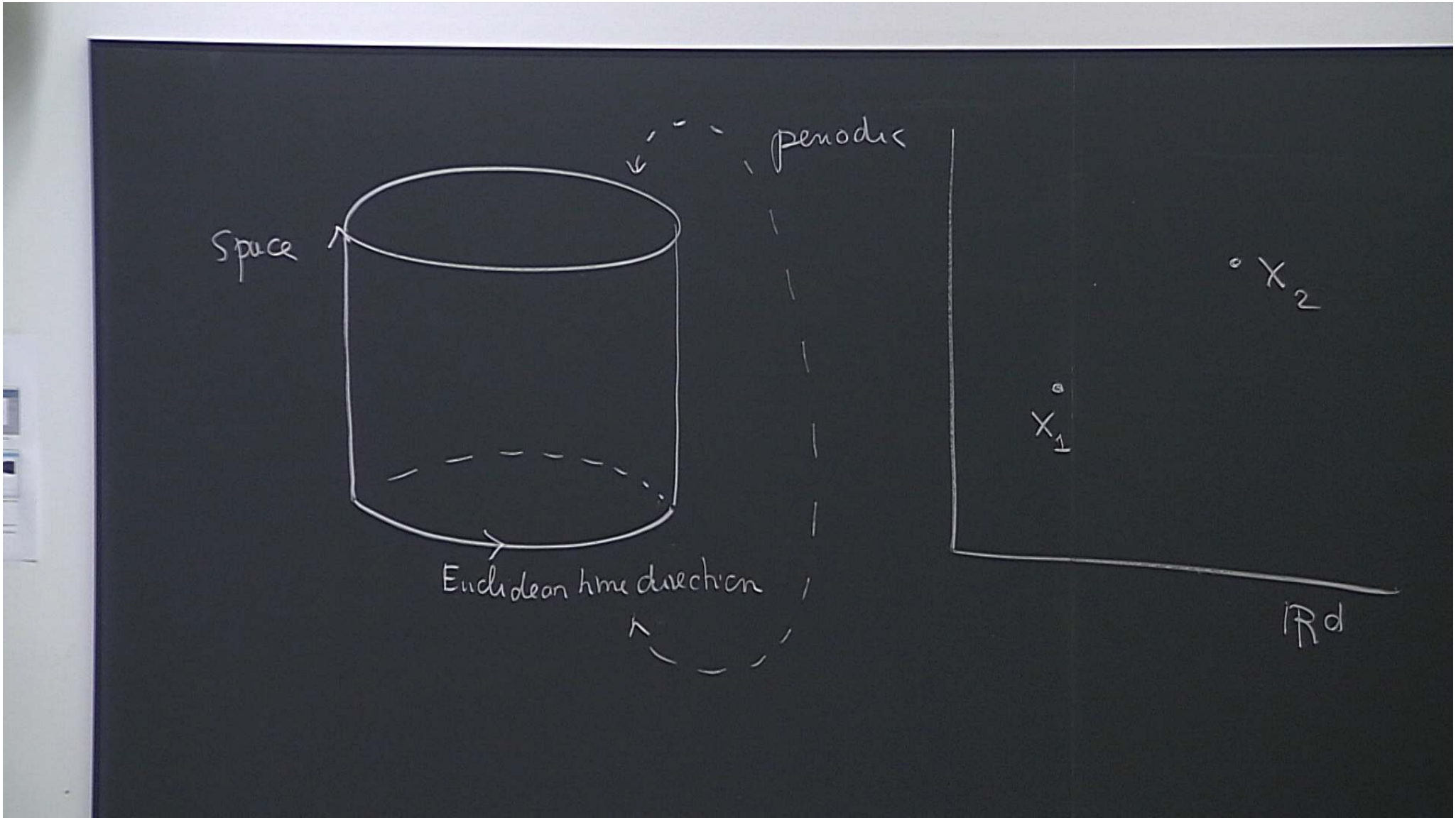
$T_e \searrow 0 \iff$ period $\tau \nearrow \infty$

cylinder $\longrightarrow \infty$ Euclidean space $\mathbb{R}^d$

Cooling case of the growth of $\Phi(x)$ at ∞

Recipe for correlation functions


functional integral over $\phi(X_E)=\phi(\vec{\tau},\vec{x})$ such that $\phi(\vec{\tau},\vec{x})=\phi(\vec{\tau}+\vec{\gamma},\vec{x})$
3
4
2.3. Compute the 2-points function (Propagator)
1st step: compute it in Euclidean space.
$\langle\phi(X_1)\phi(X_2)\rangle$
$= \dfrac{\int D_E[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)\phi(X_1)\phi(X_2)}{\int D_E[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)}$
Field operator in canonical formalism. $\hat{\Phi}(x)$
$\phi(x)$ random variable we integrate over in functional integral
Gaussian integrals
$\mathbb{R}^d$
$\phi(x)$ at ∞
∞


periodic
Space
Euclidean time direction
X_2
X_1
$\mathbb{R}^d$

11

$E(r) = \sum \frac{1}{2}\omega_n = \frac{n\pi}{n} = \frac{\pi}{5}$

5

$(-\Delta_{x_2} + m^2)\, G(x_1, x_2) = \delta(x_1 - x_2)$

Dirac δ function in d-dimension

Green Function

b) Noether
c) Klein-Gordon
d) Casimir

i) LSZ assumptions → 2nd order
k) Kallen-Lehmann

III MAT
IIII SAM

8) Anti particles
9) Causality

III

$$S_E[\phi] = \int d^d x \left[\frac{1}{2}(\partial_\nu \phi)^2 + \frac{m^2}{2}\phi^2 \right] = \int d^d x \, \frac{1}{2}\, \phi(x) \left[-\Delta_x + m^2 \right] \phi(x)$$

↑ Laplace operator

Euclidean theory $\quad \Delta_x = \sum_{\nu=0}^{d-1} \left(\frac{\partial}{\partial x^\nu} \right)^2$

Rule of Gaussian Random variables. (Gaussian Integrals)

$$\langle \phi(x_1)\phi(x_2) \rangle = \left(\frac{\hbar}{-\Delta_x + m^2} \right)_{x_1, x_2}$$

matrix element of the "inverse" of $-\Delta + m^2$ P.D. Operator]

$$= \hbar \cdot G_E(x_1, x_2)$$

$$E(r) = \sum \tfrac{1}{2}\omega_n = \frac{n\pi}{r} = \frac{\pi}{r} S$$

$$(-\Delta_{x_1} + m^2)\, G_E(x_1, x_2) = \delta(x_1 - x_2)$$

Dirac δ function in d-dimension Green Function

Δ Elliptic operator in $\mathbb{R}^d$

Euclidean Propagator

Translation $G_E(x_1 - x_2) = G_E(x_1 - x_2)$

$G_E(x) \longrightarrow \hat{G}_E(k)$ $K = (K_0 \dots K_{d-1})$ conjugate momentum to X K_ν = modified Bessel Function

Fourier transform $K^2 = \sum_{\mu=0}^{d-1} K_\mu^2 > 0$

$$(K^2 + m^2)\hat{G}_E(k) = 1$$

$$\hat{G}_E(K) = \frac{1}{K^2 + m^2}$$ Lorentzian $\Rightarrow$

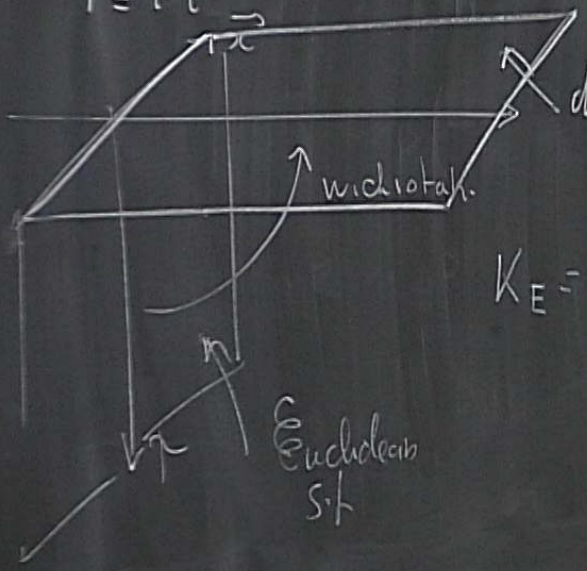
$$G_E(X) = \int \frac{d^d K}{(2\pi)^d} \frac{e^{iK\cdot X}}{K^2 + m^2} = \frac{1}{2\pi} \cdot \left(\frac{2\pi \cdot |x|}{m}\right)^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(|x| \cdot m)$$

Propagator in Minkowski spacetime?

Wick rotation $X_E = (\tau, \vec{x}) \rightarrow X = (t, \vec{x})$

τ Euclidean time — t time (physical)

$\tau = it$



$K_E = (k_0, \vec{k})$

$$G_E(\tau, \vec{x}) = \int \frac{dk_0 \, d^{d-1}\vec{k}}{(2\pi)^d} \frac{e^{i(}}{k_0^2 +}$$

$\phi_E(x)$

↳ Functional integral over $\phi(X_E) = \phi(\tau, \vec{x})$ such th

Propagator in Minkowski spacetime?

Wick rotation $X_E = (\tau, \vec{x}) \longrightarrow X = (t, \vec{x})$

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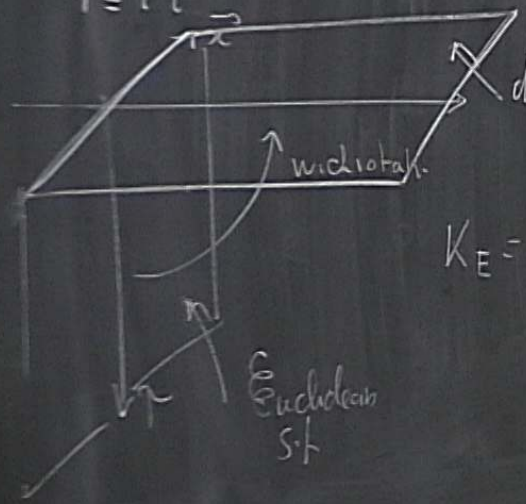

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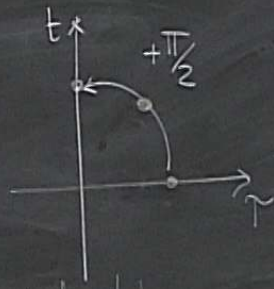
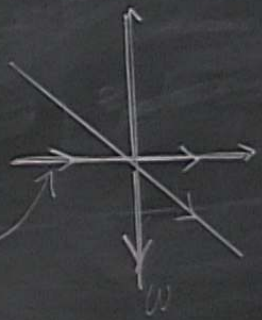


Wick rotation

$G_E(\tau, \vec{x}) = \int \frac{dk_0 \, d^{d-1}\vec{k}}{(2\pi)^d} \frac{e^{i(k_0 \tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2}$

↑ analytic in τ, close to the real axis!

I must rotate the k_0 integration contour accordingly



momentum plane

Wick rotation

$\tau = i t$

$k_0 = -i \omega$

$\omega \; -\infty \to +\infty$

$$\frac{e^{i(k_0\tau+\vec{k}\cdot\vec{x})}}{k_0^2+\vec{k}^2+m^2} \longrightarrow G(t,\vec{x}) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}(-i)\,\frac{e^{i(\omega t+\vec{k}\cdot\vec{x})}}{-\omega^2+\vec{k}^2+m^2-i\epsilon_+}$$

POLES

Wick rotation

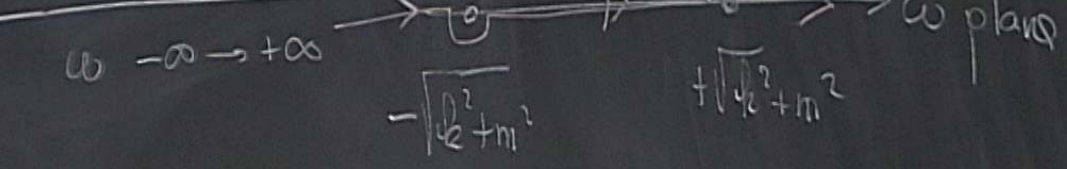
$\tau = it$

$k_0 = -i\omega$

$k_0 = \pm i\sqrt{\vec{k}^2+m^2}$

poles at $\omega^2 = \vec{k}^2+m^2$

this contour $\Rightarrow$ Feynman propagator



$k_0 = -i\sqrt{\vec{k}^2+m^2}$

k_0 plane

$$G_E(\tau, \vec{x}) = \int \frac{d k_0 \, d^{d-1}\vec{k}}{(2\pi)^d} \frac{e^{i(k_0 \tau + \vec{k}\cdot\vec{x})}}{k_0^2 + \vec{k}^2 + m^2}$$

τ ↑ analytic in τ, close to the real axis?

$$\longrightarrow \quad G(t, \vec{x}) = \int$$

$(t, \vec{x})$ ↑ time (physical)

POLES

Wick rotation
$\tau = i t$
$k_0 = -i\omega$

$k_0 = +i\sqrt{\vec{k}^2 + m^2}$

k_0

ω $-\infty \to +\infty$

$-\sqrt{\vec{k}^2 + m^2}$

$+\sqrt{\vec{k}^2 + m^2}$

$+\pi/2$

I must rotate the k_0 integration contour accordingly

$k_0 = -i\sqrt{\vec{k}^2 + m^2}$

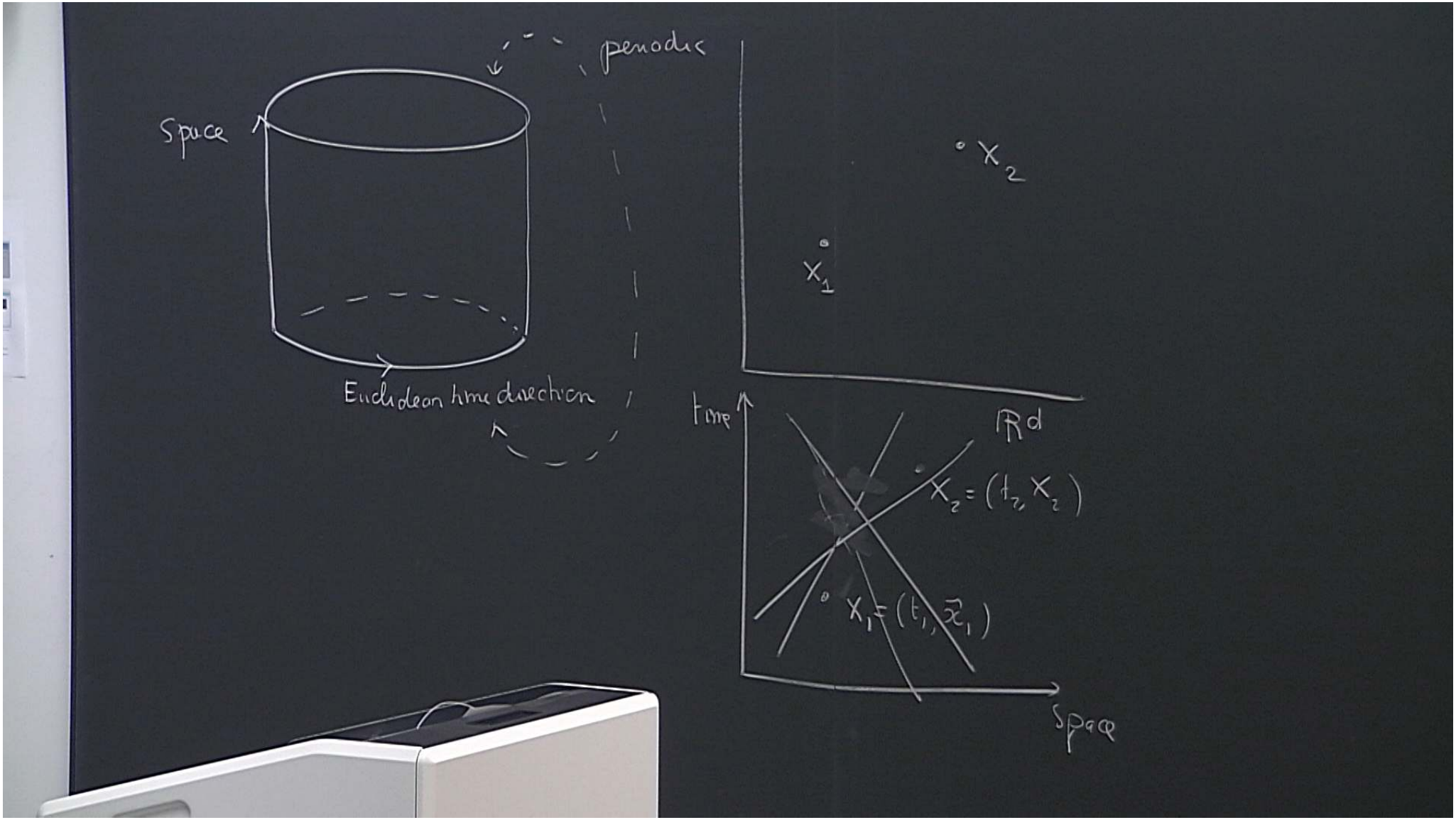
momentum plane

Wick rotation

e plane

Wick rotated propagator = Feynman propagator

Wi


periodic
Space
Euclidean time direction
X_2
X_1
time
$\mathbb{R}^d$
$X_2 = (t_2, \vec{x}_2)$
$X_1 = (t_1, \vec{x}_1)$
Space

$$\frac{e^{i(k_0\tau+\vec{k}\cdot\vec{x})}}{k_0^2+\vec{k}^2+m^2} \longrightarrow G(t,\vec{x}) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}(-i)\,\frac{e^{i(\omega t+\vec{k}\cdot\vec{x})}}{-\omega^2+\vec{k}^2+m^2-i\epsilon_+}$$

POLES

Wick rotation

$\tau = it$

$k_0 = -i\omega$

pole at $\omega^2 = \vec{k}^2+m^2$

this contour $\Rightarrow$ Feynman propagator

$k_0 = +i\sqrt{\vec{k}^2+m^2}$

k_0

ω plane

$\omega: -\infty \to +\infty$

$-\sqrt{\vec{k}^2+m^2}$

$+\sqrt{\vec{k}^2+m^2}$

$k_0 = -i\sqrt{\vec{k}^2+m^2}$

Causal structure