Title: Entanglement Branes, Modular Flow, and Extended TQFT

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Abstract: <p>In gauge theories, there is an inherent tension between locality and gauge invariance.   This is precisely expressed by the failure to factorize the physical Hilbert space into local tensor products.   The extended Hilbert space offers a resolution to this problem by embedding physical states into a larger Hilbert space containing gauge variant degrees of freedom. For two dimensional TQFT's, we show how this extension fits inside the framework of extended quantum field theory, as described by the Moore-Segal axioms. These are sewing axioms that impose a sort of spacetime covariance for the extended Hilbert space.   For the case of two-dimensional Yang Mills and it's string theory dual, we show how this framework leads to exact calculations of multi-interval modular flow, ,negativity, and entanglement entropy. Moreover, we given an axiomatic formulation of the "entanglement brane", which nicely compliments the worldsheet description given in previous work.</p>

Entanglement branes, Modular flow, and Extended quantum field theory

Gabriel Wong

Fudan University

Perimeter Institute

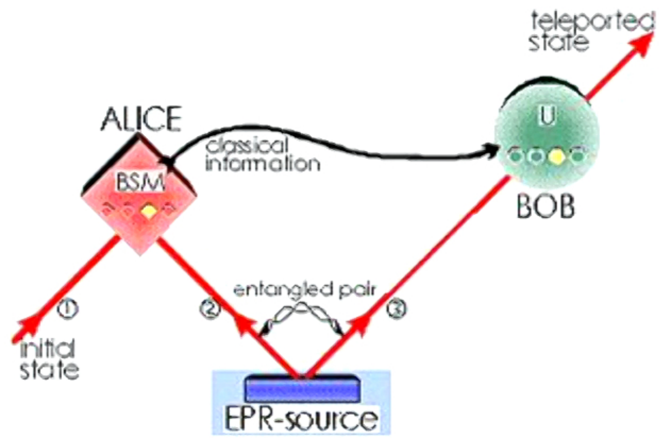
Based on forthcoming work with William Donnelly

and also arxiv 1610.01719

Quantum Entanglement is related to questions about **locality in QM**

Unentangled $|0\rangle|0\rangle$

Entangled EPR $\frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle)$

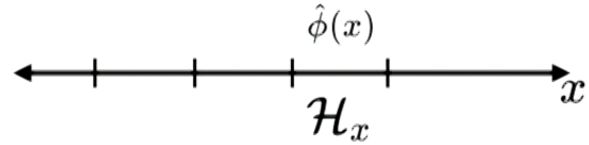


- Quantum Teleportation via EPR Pairs
- Non-local order parameter in Topological phases
- Many body localization
- Emergent smooth spacetimes from quantum mechanics

What do we mean by **locality in** quantum mechanics?

Hilbert Space factorization

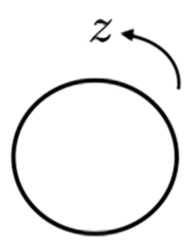
$$\mathcal{H} = \otimes_x \mathcal{H}_x$$



Different notions of locality can be assigned to the same Hilbert space

$$\mathcal{H} = \{\text{Anti-symmetric wavefunctions } \psi(z_1 \ldots, z_N),\ z_j = e^{i\theta_j}\}$$

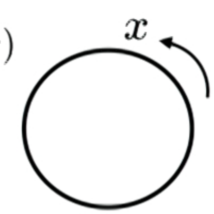
N non-relativisitic fermions in on a spatial circle



U(N) Yang Mills on a spatial circle .

$$U = \mathrm{P}\exp(i \int A_x \, dx)$$

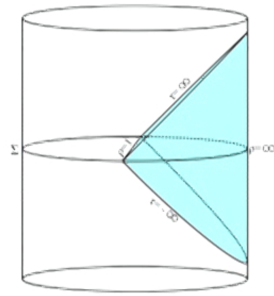
$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix}$$



AdS/CFT and Bulk locality

- AdS/CFT provides a QG Hilbert space at asymptotic infinity
- How does an approximately local bulk spacetime emerge?
- Subregion duality from QEC (Almeri, Dong,Harlow)
- Bulk relative entropy = boundary relative entropy

(Jafferis, Maldecena,Lewkowycz, Suh)



A free fermion Hilbert space in N=4 SYM

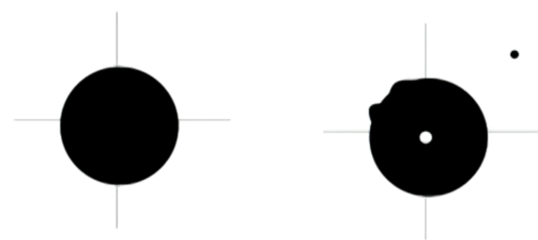
Ten- Dimensional Geometry for the IIB string

(Lin, Lunin, Maldacena)

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + y e^{G} d\Omega_3^2 + y e^{-G} d\tilde{\Omega}_3^2 \\
h^{-2} &= 2y \cosh G, \\
y\partial_y V_i &= \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z \\
z &= \frac{1}{2}\tanh G \\
F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B} \ , \\
\tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} \\
B_t &= -\frac{1}{4} y^2 e^{2G}, \qquad \tilde{B}_t = -\frac{1}{4} y^2 e^{-2G} \\
d\hat{B} &= -\frac{1}{4} y^3 *_3 d\left(\frac{z + \frac{1}{2}}{y^2}\right) , \qquad d\hat{\tilde{B}} = -\frac{1}{4} y^3 *_3 d\left(\frac{z - \frac{1}{2}}{y^2}\right)
\end{aligned}
$$

1/2 BPS sector of N=4 SYM

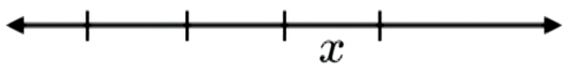
Free Fermions in a Magnetic field in the LLL



The extended Hilbert space construction

Hilbert Space factorization

$$\mathcal{H} = \otimes_x \mathcal{H}_x$$



Two obstructions

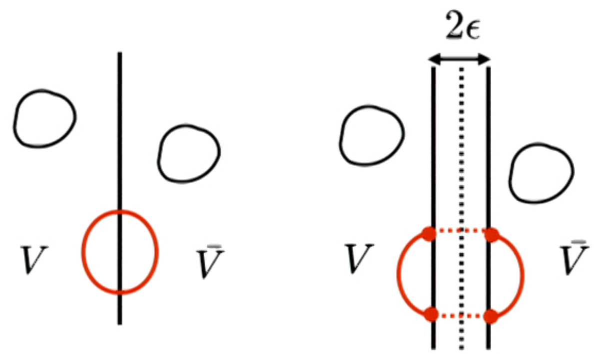
1 A subregion has a a boundary and therefore edge modes, even for a scalar field! (Agon, HeadrickJefferis,Kasko) (Campaglia, Freidal,et al)

2 Degrees of freedom in subregions are not independent

- e.g. continuity in a quantum field theory
- Gauss Law constraint in gauge theory. Even on a lattice !

The extended Hilbert space construction provides a solution by combining 1 and 2 (Donnelly, Freidel, Buividovich ,...)

Extended Hilbert space for gauge theories



$$\mathcal{H}_{\text{physical}} \subset \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$$

Contains edge modes transforming under boundary symmetry group $G_L \times G_R$

Gauss law $Q|\psi\rangle = 0 \quad \text{for} \quad |\psi\rangle \in \mathcal{H}_{\text{Physical}}$

Invariance under $G = \text{Diag}\ (G_L \times G_R)$

Entangling product $\mathcal{H}_{\text{physical}} = \mathcal{H}_V \otimes_G \mathcal{H}_{\bar{V}}$

Reduced density matrix $\rho_V = \text{tr}_{\bar{V}} |\psi\rangle\langle\psi|$

Entanglement Entropy $S_V = -\text{tr}\ \rho_V \log \rho_V$

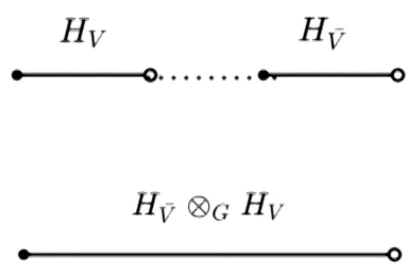
Extended Hilbert space and extended TQFT

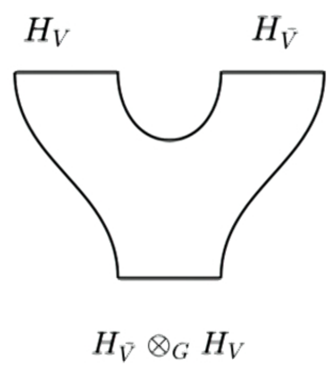
Edge modes are not unique (Cano, Cheng, Mulligan, …et. al) (Fliss,Wen,Parrikar,..et. al.)

What are the rules for determining the "correct" edge modes and their gluings?

In 2D, we provide constraints on the Hilbert space extension using the frame work of extended topological quantum field theory

Key insight: The path integral implements the entangling product via a spacetime process

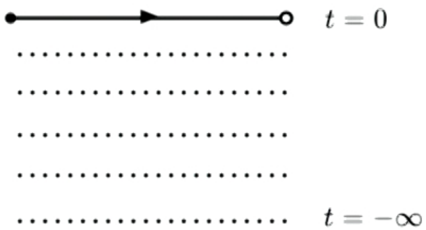




Modular flow is a Cobordism

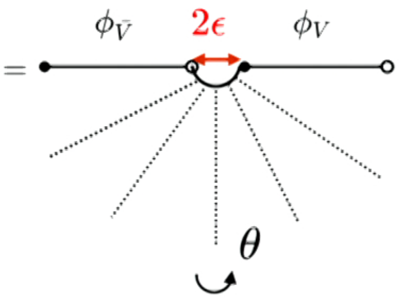
Euclidean path integral preparation of the (unnormalized) vacuum

$\langle \phi(x) | 0 \rangle$ =



Angular quantization

$\langle \phi_V \, \phi_{\bar{V}} | 0 \rangle =$

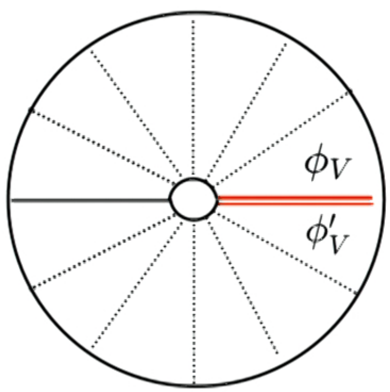


Modular Hamiltonian

CPT

Schmidt decomposition

$$\langle \phi_V \, \phi_{\bar{V}} | 0 \rangle = \langle \phi_V | \exp\left(\frac{-H_V}{2} \right) \mathcal{J} | \phi_{\bar{V}} \rangle = \sum_n \exp\left(-\frac{E_n}{2} \right) \langle \phi_V | n \rangle \langle \phi_{\bar{V}} | \bar{n} \rangle$$



$$\langle \phi_{V'} | \tilde{\rho}_V | \phi_V \rangle = \langle \phi_V | \exp(-H_V) | \phi_V \rangle$$

$$Z_V = \mathrm{Tr}_V \, \tilde{\rho}_V$$

Thermal Partition function on V

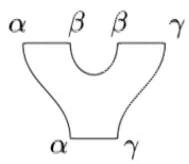
EE = Thermal entropy

Locality in Extended TQFT

Cut Path integral on a general spacetime along surfaces of increasing codimension

Extending previous work, Moore-Segal axiomatized cutting and glueing of path integrals.

They gave rules to classify D branes, given as a category of boundary conditions



Modular flow is also about cutting and glueing the circle in spacetime…

What we did:

- Interpret Moore-Segal as classifying extended Hilbert space and edge modes
- Formulate 2D Yang Mills as an extended TQFT a la Moore-Segal
- Introduce the Entanglement brane axiom
- Compute multi-interval modular flows and negativity

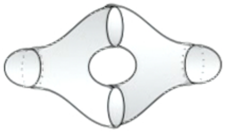
Outline

- Open-closed extended TQFT (Moore-Segal)
- Entanglement brane
- 2DYM as an open-closed TQFT
- Multi-interval Modular flows, Negativity
- Future works: CFT, higher dimensions, holography

Atiyah's formulation of Axiomatic TQFT

Axiomatize operations using the path integral

A TQFT in 2 dimension is a rule assigning:



1-dim closed manifolds =Hilbert space over $\mathbb{C}$

$\mathcal{H}_{S_1}$ | $\mathcal{H}_{-S^1} = \mathcal{H}^*_{S^1}$ | $\mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1}$ | $\emptyset$ $\mathbb{C}$

Cobordism between circles = Linear maps (quantum evolution)

$\mathbb{C} \rightarrow \mathcal{H}_{S^1}$

Wavefunction $\Psi[\phi(x)]$

$\mathcal{H}_{S^1} \xrightarrow{e^{-tH}} \mathcal{H}_{S^1}$

Propagator

$\mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \rightarrow \mathcal{H}_{S^1}$

Fusion/Multiplication

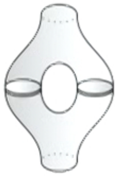
$\mathbb{C} \rightarrow \mathbb{C}$

Partition function

Gluing Cobordisms =Composing linear maps

$\langle \Psi | \Psi \rangle$

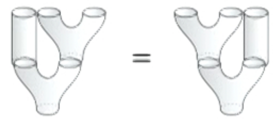
$$\int D\phi(x)\, \Psi[\phi(x)] \Psi^*[\phi(x)]$$

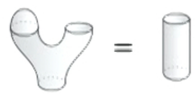


$\mathcal{H}_{S^1} \rightarrow \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \rightarrow \mathcal{H}_{S^1}$

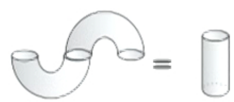
A 2D Closed TQFT is a commutative Frobenius algebra

Commutative


Associative


Unit


Symmetric bilinear form

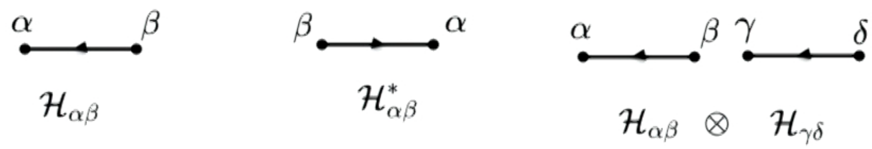

Invertible


Invariance

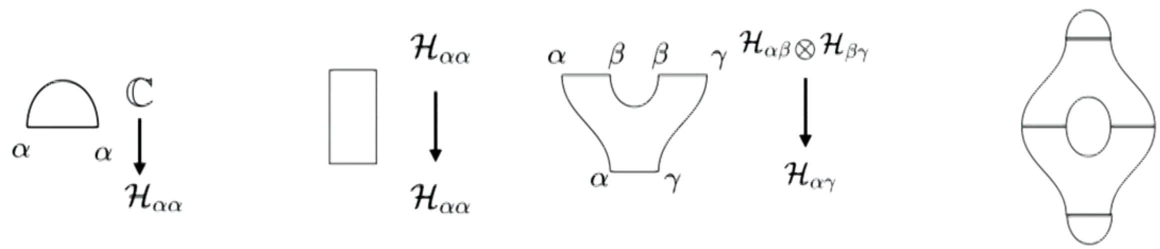

Open TQFT is a noncommutative Frobenius Algebra

An open TQFT assigns

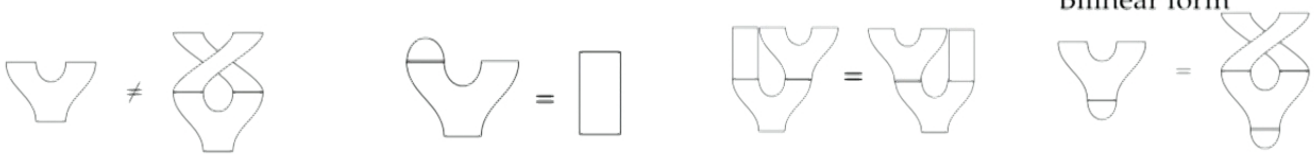
Hilbert space to oriented intervals with labelings :



Open cobordisms to linear maps

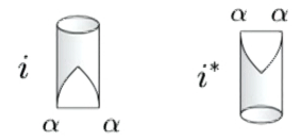


Non-commutative mult. | Unit | Associative | Invariant symmetric Bilinear form

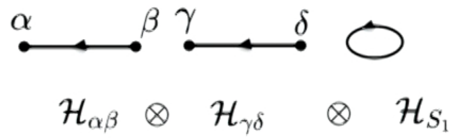


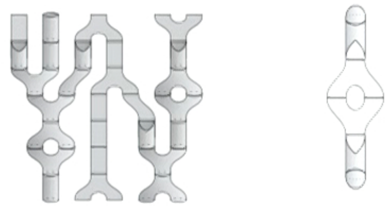
Open closed TQFT

The zipper relates the closed and open algebra...



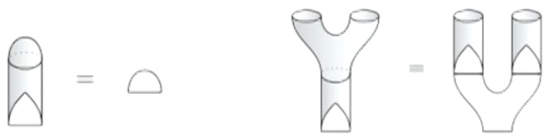
Open-closed hilbert spaces and cobordisms:



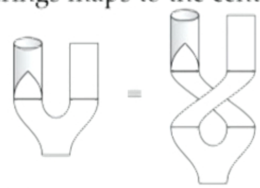


Moore-Segel Sewing rules : Ensures compatibility of gluing

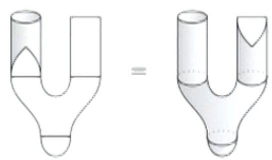
i is an algebra homomorphism



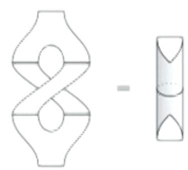
Closed strings maps to the center of open strings



i is the adjoint of i^*

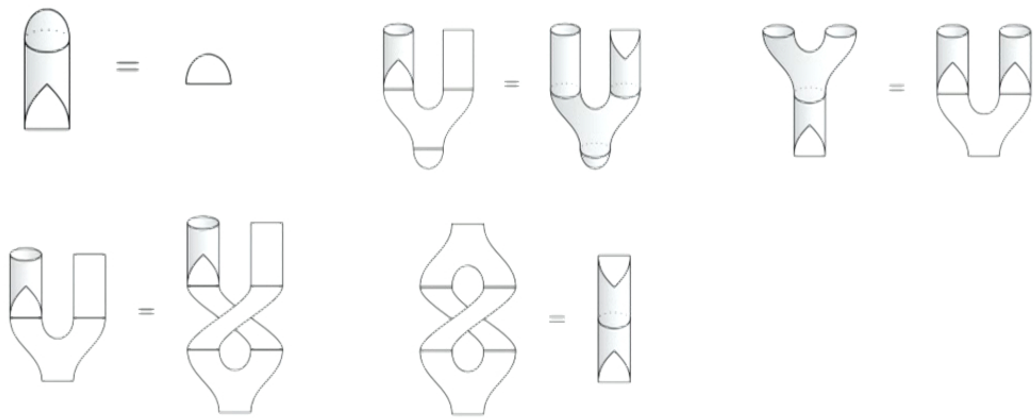


Cardy (Spacetime covariance)



Moore-Segal Sewing relations

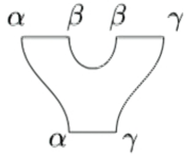
To ensure the compatibility of different gluings, five gluing axioms have to be satisfied



Moore-Segal:

Q: Given a closed string theory, what are the possible boundaries, i.e. D Branes?

A: D branes correspond to extensions to an open string algebra satisfying these constraints.

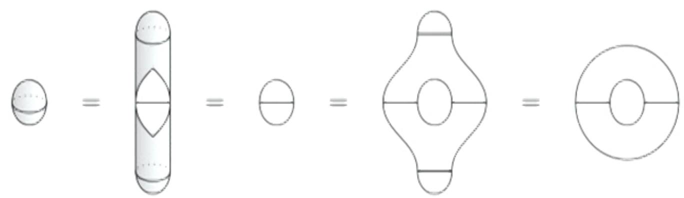


For us: **Open string algebra ~ choice Hilbert space extension i.e. edge modes**

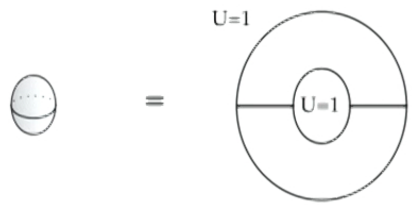
The Entanglement Brane axiom

Holes originating from splitting the Hilbert space can be sewed up

This axiom implies a choice of boundary conditions allowing us to relate the sphere to the annulus



In 2DYM this corresponds to a trivial holonomy along boundary circles:



$$\mathrm{tr}(\rho_V O_V) = \langle O_V \rangle$$

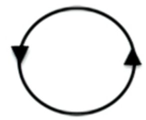
Outline

- Open-closed extended TQFT (Moore-Segal)
- Entanglement brane
- 2DYM as an open-closed TQFT
- Multi-interval Modular flows, Negativity
- Remarks about the Gross Taylor string theory
- Future works: CFT, higher dimensions, holography

2DYM Hilbert space on a Circle

Configuration space

$$U = \mathcal{P}\exp\left(i\int_0^L dx\, A_x(x)\right) \in G$$



$x \sim x + L$

Gauge transformation and invariant wave functions

$$U \to g(0)Ug(0)^{-1} \qquad\qquad \Psi[U] = \Psi[gUg^{-1}]$$

Hilbert space and representation basis

Class functions on G $\qquad \langle U|R\rangle = \mathrm{Tr}_R(U)$

Hamiltonian ~ Casimir

$$H|R\rangle = \frac{g_{\mathrm{YM}}^2 L}{2} C_2(R)|R\rangle \qquad\qquad C_2(R) = \text{Quadratic Casimir}$$

Propagator

$$\sum_R e^{-AC_2(R)}|R\rangle\langle R|$$

T

L

$A = LT$

2DYM as a Closed TQFT

Class functions on G

$$\sum_R \dim R \, e^{-AC_2(R)}$$

$$\sum_R \frac{1}{\dim R} e^{-AC_2(R)} |R\rangle\langle R|\langle R|$$

Path integral on Riemann surface M Rusakov 1993 Witten 1991

$$Z(M) = \sum_R (\dim R)^{\chi(M)} e^{-\frac{g_{\mathrm{YM}}^2 A}{2} C_2(R)}$$

$\chi(M) = $ Euler Characteristic of M

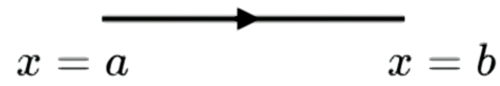
$A = $ Area of M

The path integral only depends on topology and area

Hilbert space on an interval

Configuration space

$$U = \mathrm{P}\exp\int_a^b A_x\, dx$$



Hilbert space

General functions on gauge group G

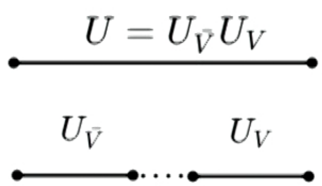
Basis Edge modes

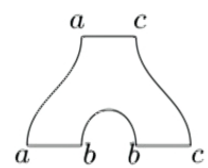
$$\langle U | R a b \rangle = \sqrt{\dim R}\; R_{ab}(U)$$

Boundary symmetry :

$$U \to g(a) U g^{-1}(b) \qquad R \to G R G^{-1}$$

Entangling product~ Matrix multiplication





$$\langle U | R\, a\, c \rangle \to \frac{1}{\sqrt{\dim R}} \sum_b \langle U_V | R\, a\, b \rangle \otimes \langle U_{\bar{V}} | R\, b\, c \rangle$$

2DYM as an Open-Closed TQFT

 = Class functions on G

 =Functions on G

$$|\Omega\rangle = \sum_R \dim R\,|R\rangle$$

$$\sum_R \frac{1}{\dim(R)}|R\rangle\langle R|\langle R|$$

$$\sum_{R,a,b,c} \frac{1}{\sqrt{\dim(R)}}\,|Rac\rangle\,\langle Rab|\,\langle Rbc|$$

$$\sum_{R,a} \sqrt{\dim(R)}\,|Raa\rangle$$

$$\sum_{R,a} \frac{1}{\sqrt{\dim(R)}}\,|Raa\rangle\,\langle R|$$

Satisfy Moore-Segal

Can insert Boltzmann factor $e^{-AC_2(R)}$ to recover area dependence

Single interval Modular flow and EE

Tensor product factorization

$$|\psi\rangle =$$ 

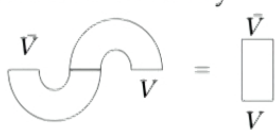
$$= \sum_{R,a,b} |Rab\rangle\, |Rba\rangle$$

Unnormalized reduced density matrix

$$\rho_V = \mathrm{tr}_{\bar{V}} |\psi\rangle\langle\psi| =$$ 

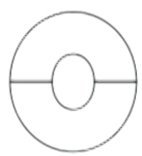
$$= \sum_{R,a,b} |R\,a\,b\rangle\langle R\,a\,b| = 1$$

State-Channel duality

$$\psi =$$ 

$$\rho_V = \psi\psi^\dagger = 1$$

Effective partition function

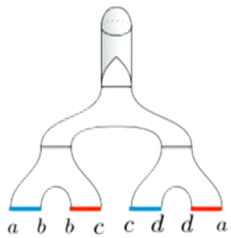
$$Z = \mathrm{tr}_V\, \rho_V =$$ 
$$= \sum_{R,a,b} 1 = \sum_R (\dim R)^2 =$$ 

Entanglement entropy

$$S = \mathrm{tr}(\rho_V H_V) + \log Z$$

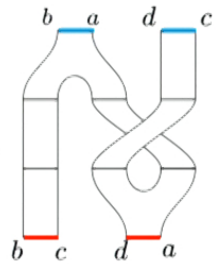
Multi-interval Modular flow

$|\psi\rangle =$



State-Channel duality

$\psi =$



$$= \sum_{R,a,b,c,d} \frac{1}{\dim(R)} |Rbc\rangle |Rda\rangle \langle Rba| \langle Rdc|$$

$$= \sum_{R,a,b,c} \frac{1}{\sqrt{\dim(R)}} |Rac\rangle \langle Rab| \langle Rbc|$$

The modular hamiltonian is non-zero

$$\rho_V = \psi\psi^\dagger = (\dim R)^{-2}\mathbf{1}$$

$$H_V = -2\log(\dim R)\mathbf{1}$$

Partition function

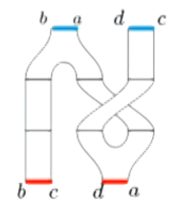
$$Z = \mathrm{tr}(\rho_V) = \sum_{R,a,b,c,d} (\dim R)^{-2} = \sum_R (\dim R)^2$$

Modular time is a morse function
This is saddle point

Entropy

$$S = -\sum_{R,a,b,c,d} \frac{1}{(\dim R)^2 Z} \log \frac{1}{(\dim R)^2 Z}$$

$$= \sum_R \frac{(\dim R)^2}{Z} \log(\dim R)^2 + \log Z$$



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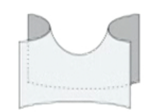
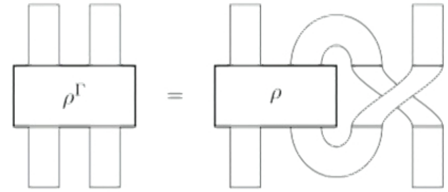


Negativity

A useful measure of entanglement at finite temperature, defined using the partial transpose

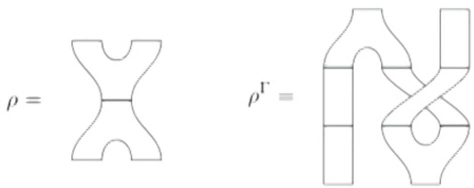


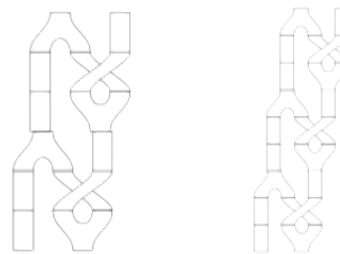
$$\mathcal{E} = \log \mathrm{tr}|\rho^\Gamma|$$

Replica trick for negativity

$$Z_{n_e} = tr(\rho^\Gamma)^{n_e} \qquad n_e \quad \text{even}$$
$$Z_{n_o} = tr(\rho^\Gamma)^{n_o} \qquad n_o \quad \text{odd}$$

$$\mathcal{E} = \lim_{n_o \to 1, n_e \to 1} \log \frac{Z_{n_e}}{Z_{n_o}}$$





$$P(R) = \frac{1}{Z_{n_o=1}} (\dim R)^2 e^{-AC_2(R)}$$

$$\mathcal{E} = \log \langle \dim(R) \rangle$$

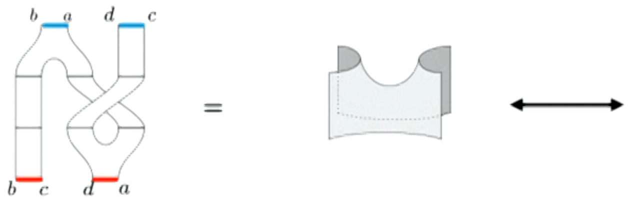
$$Z_{n_o} = \sum_R \dim(R)^{3-n_o} e^{-n_o A C_2(R)}$$
$$Z_{n_e} = \sum_R \dim(R)^{4-n_e} e^{-n_e A C_2(R)}$$

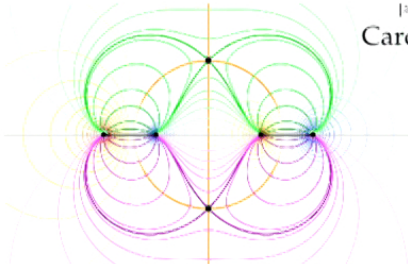
Conclusion and future directions

We can understand entanglement in 2D Yang Mills via the frame work of open-closed extended QFT

Can apply the same frame work to the string theory dual, where the entanglement brane has a worldsheet interpretation (Donnelly, Wong 2016)

2D CFT's? For free fermions, interpretation of modular flow as a cobordism provided a derivation of the multi-interval modular hamiltonian (Wong 2018)





Cardy-Tonni 2016

Extended TQFT are defined in higher dimensions. Apply same ideas to Chern Simons theory?

In AdS/CFT, bulk dual to the extended boundary CFT?
Relation to Raamsdonk's BC bits?