Title: Positive geometries and the amplituhedron

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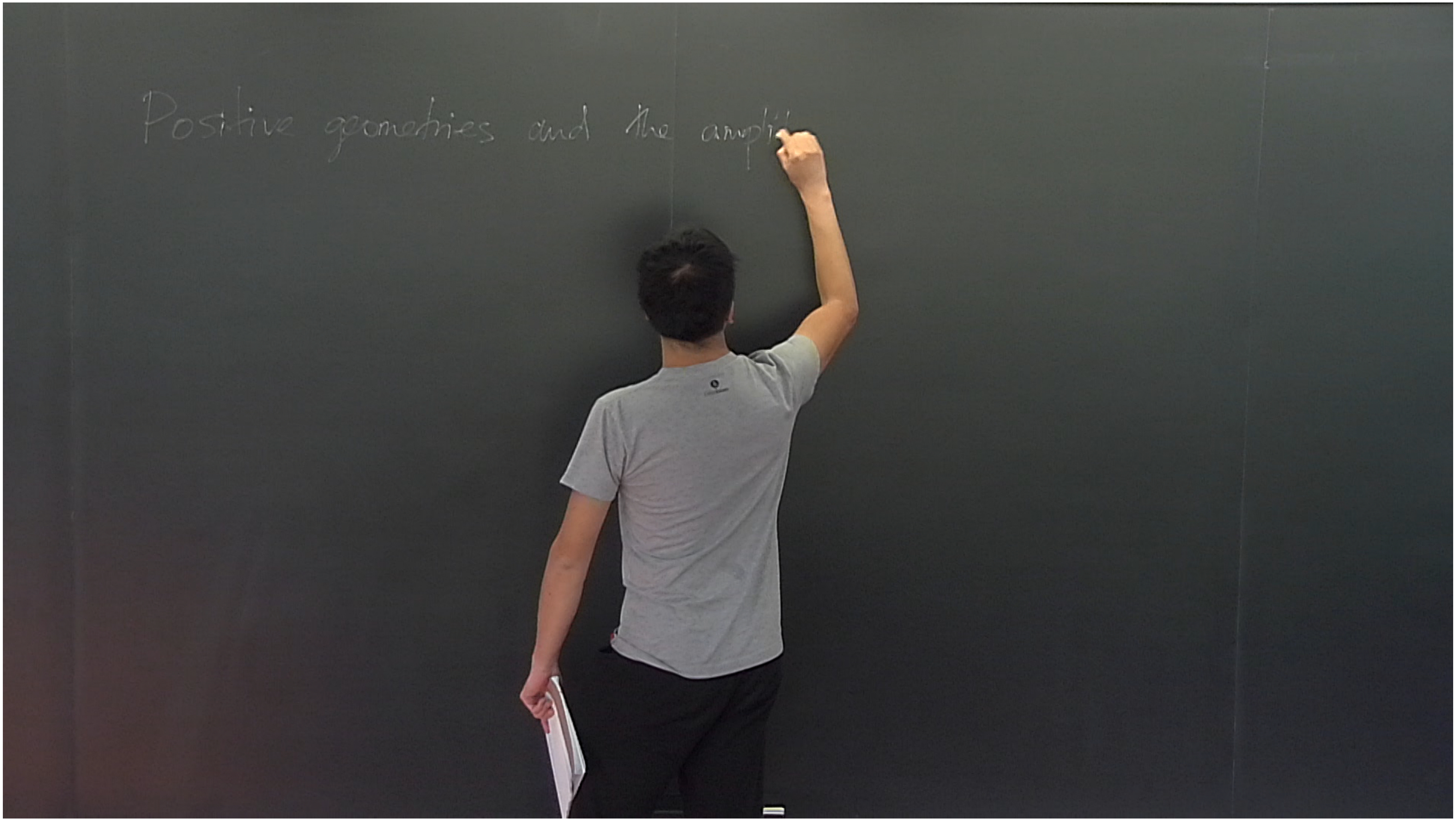
Abstract: <p>Positive geometries are real semialgebraic spaces that are
equipped with a meromorphic ``canonical form'' whose residues reflect
the boundary structure of the space. Familiar examples include
polytopes and the positive parts of toric varieties. A central, but
conjectural, example is the amplituhedron of Arkani-Hamed and Trnka.
In this case, the canonical form should essentially be the tree
amplitude of N=4 super Yang-Mills.

I will talk about the definition and examples of positive geometries,
and discuss what is known about the geometry and combinatorics of the
amplituhedron. The talk will be based on various joint works with
Arkani-Hamed, Bai, Galashin, and Karp.</p>


Positive geometries and the ampli

Positive geometries and the amplituhedron

Positive geometry (Arkani-Hamed, Bai, L.)

$(X, X_{\geq 0})$

X complex proj. variety dim d defined $/\mathbb{R}$

$X_{\geq 0} \subset X(\mathbb{R})$ closed semialgebraic set

$X_{>0} := \mathrm{Int}(X_{\geq 0})$ open oriented submanifold in $X(\mathbb{R})$ dim d.

$(X_{\geq 0} = \overline{X_{>0}})$

$\partial X_{\geq 0} = X_{\geq 0} - X_{>0}$

$\partial X = \overline{\partial X_{\geq 0}}$ Zariski closure

$= C_1 \cup C_2 \cup \ldots \cup C_r$ irred comp. of codim 1.

$C_{i, \geq 0} := C_i \cap \partial X_{\geq 0}$.

Positive geometries and the amplituhedron

Positive geometry (Arkani-Hamed, Bai, L.)

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normal irreducible X complex proj. variety dim d defined $/\mathbb{R}$

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$C_{i,\geq 0} := C_i \cap \partial X_{\geq 0}$.

$(\mathbb{R})$

If $d=0$.

$(X, X_{\geq 0}) = (pt, pt)$ and

define $\Omega(X, X_{\geq 0}) = \pm 1$

If $d>0$, then need to check

(P1) Each $(C_i, C_{i,\geq 0})$

is a positive geometry of dim $d-1$

(P2) There exists a unique rational (ie meromorphic)

d-form $\Omega(X, X_{\geq 0}) \neq 0$ satisfying.

$$\mathrm{Res}_{C_i}(\Omega(X, X_{\geq 0})) = \Omega(C, C_{\geq 0})$$

and $\Omega(X, X_{\geq 0})$ has no other poles.

$$\operatorname{Res}_{C_i}\left(\Omega(X, X_{\geq 0})\right) = \Omega(C, C_{\geq 0})$$

simple poles along C_i

and $\Omega(X, X_{\geq 0})$ has no other poles.

Example

- $d = 1$

no holomorphic top-forms $\Rightarrow X = \mathbb{P}^1$

$X_{\geq 0}$ = union of closed intervals

$X_{\geq 0} = [a,b]$ in chart $(1:x)$ of $\mathbb{P}^1(\mathbb{R})$

then $\Omega(\mathbb{P}^1, [a,b]) = \frac{dx}{x-a} - \frac{dx}{x-b} = \frac{(b-a)}{(b-x)(x-a)}\,dx$

$$\Omega\left(\mathbb{P}^1, \coprod_i [a_i, b_i]\right) = \sum_i \Omega(\mathbb{P}^1, [a_i, b_i])$$

simple poles along C_i

$$\mathrm{Res}_{C_i}\left(\Omega(X, X_{\geq 0})\right) = \Omega(C, C_{\geq 0})$$

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$$\Omega(\mathbb{P}^1, \coprod_i [a_i, b_i]) = \sum_i \Omega(\mathbb{P}^1, [a_i, b_i])$$

- $P \subset \mathbb{P}^n$ polytope $\dim P = n$

$(\mathbb{P}^n, P)$ are all positive geometries

Th'm $\exists$ unique family of rational top forms Ω_P s.t. • poles of Ω_P are along facet $F \subset P$, are simple and $\mathrm{Res}_F \Omega_P = \Omega_F$

• $P = \circ$ then $\Omega_P = \pm 1$

Pf. of existence of Ω_P

- Write down Ω_{Δ^n} for a simplex

If $\Delta^n \subset \mathbb{P}^n$ is the standard simplex

then $\Omega_{\Delta^n} = \prod_i d\log\alpha_i$ in chart $(1:\alpha_1:\alpha_2:\ldots:\alpha_n)$

- Triangulate P into simplices

$$P = \bigcup_i \Delta_i$$

then $\Omega_P = \sum_i \Omega_{\Delta_i}$

Pf. of existence of Ω_P

- Write down Ω_{Δ^n} for a simplex

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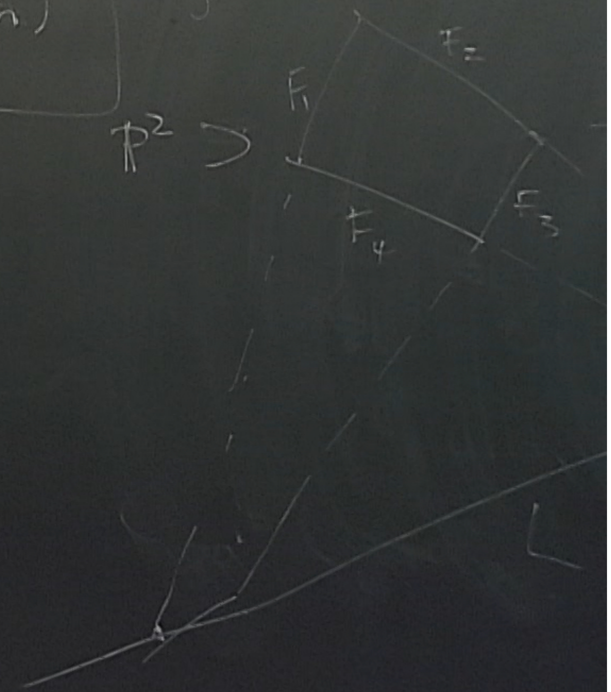
$$\Omega_P = \frac{?}{\prod \text{linear forms for facets}} d^n x \ldots$$

e.g.

$\mathbb{P}^2 \supset$



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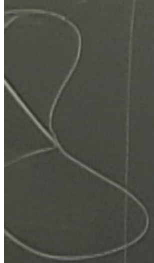

- Toric varieties

$$X_P \supset T = (\mathbb{C}^*)^d \qquad \Omega_X = \prod_i d\log x_i$$
$$\cup$$
$$X_{>0} := (\mathbb{R}_{>0})^d$$

"morphism of positive geometries"

$$(X_P, X_{P,\geq 0}) \xrightarrow[\text{map}]{\text{algebraic moment}} (\mathbb{P}^n, P)$$

degree = volume.

- Conjecturally, cluster varieties have compactifications that are positive geometries

- $\left(G/P, (G/P)_{\geq 0}\right)$

Main example

$$Gr(k,n)_{\geq 0} = \{ Y \in Gr(k,n)(\mathbb{R}) \mid$$

X_i

- Conjecturally, cluster varieties have compactifications that are positive geometries

- $\left(G/P, \left(G/P\right)_{\geq 0}\right)$

Main example

$$k\left[\overset{n}{}\right] \qquad \Delta_{134} = \text{minor using columns } 1,3,4$$

$$Gr(k,n)_{\geq 0} = \{ X \in Gr(k,n)(\mathbb{R}) \mid \Delta_I(X) \geq 0 \}$$

Δ_{134} = minor using columns 1,3,4

$) \mid \Delta_I(X) \geq 0 \}$

$$\Omega(Gr(k,n)_{\geq 0}) = \frac{dX^{kn}/GL(k)}{\Delta_{12\ldots k}(X)\,\Delta_{23\ldots k+1}(X)\cdots\Delta_{n12\ldots k-1}(X)}$$

has no zeroes on $Gr(k,n)$

has n poles

$Z : Gr(k,n) \dashrightarrow Gr(k,k+m)$

Main conjectural example

$k+m<n$.

$Z : \mathbb{R}^n \longrightarrow \mathbb{R}^{k+m}$ linear.

If $A = Z(Gr(k,n)_{\geq 0}) \subset Gr(k,k+m)$ is defined.

call A a Grassmann polytope.

If $k=1$,

$$Z: \begin{array}{ccc} Gr(1,n) \cong \mathbb{P}^{n-1} & \longrightarrow & \mathbb{P}^m \\ \cup \quad\quad \cup & & \\ Gr(1,n)_{\geq 0} \cong \Delta^{n-1} & & Z(\Delta^{n-1}) = \mathrm{conv}(\underset{\mathbb{P}^m}{\underset{\cap}{Z_1}}, \dots, Z_n) \end{array}$$

Q: Are Grassmann polytopes positive geometries?

Call Z <u>positive</u> if all $(k+m)\times(k+m)$ minors are positive.

In this case $\mathcal{A} = \mathcal{A}(Z)$ is called the <u>amplituhedron</u>

<u>Conjecture</u> (maybe

$(Gr(k,k+m), \mathcal{A}(Z$

a positi

Conjecture (maybe m even)

$(Gr(k,k+m), \mathcal{A}(Z))$ is a positive geometry

Conjecture (Arkani-Hamed, Trnka)
$m=4$

$\Omega(Gr(k,k+4), \mathcal{A}(Z)) =: \Omega_{n,k,4}$

is the amplitude form for $N=4$ SYM

Fix $Y_0 = [\underbrace{0}_{4} \,|\, Id_{k\times k}] \in Gr(k,k+4)$

$$Z = \begin{array}{c} \overbrace{\hphantom{linear combination}}^{n} \\ 4\left\{\begin{bmatrix} z \\ \hline \text{linear combination of fermionic variables} \end{bmatrix}\right. \end{array}$$

superamplitude $= \int d^4\phi_1 \ldots d^4\phi_k \dfrac{\Omega_{n,k,4}(Y,Z)}{\prod\langle Y_1 \, Y_k d^4 Y\rangle}$

• Conjecturally,
have compact
are positive

• $(G/P, ($

Main example

$Gr(k,n)$

If $k=1$,

$Gr(k,n) \cong Gr(n-k,n)$

$$Gr(1,n) \cong \mathbb{P}^{n-1} \xrightarrow{Z} \mathbb{P}^m$$

$$Gr(1,n)_{\geq 0} \cong \Delta^{n-1} \qquad Z(\Delta^{n-1}) = \mathrm{conv}(Z_1, \ldots, Z_n) \subset \mathbb{P}^m$$

Q: Are Grassmann polytopes positive geometries?

Z positive if all $(k+m)\times(k+m)$ minors are positive.

Case $\mathcal{A} = \mathcal{A}(Z)$ is called the amplituhedron

$$\mathcal{A}_{n,k,m} \longleftrightarrow \mathcal{A}_{n,n-k-m,m}$$

triangulation $\longleftrightarrow$ triangulation

Conjecture (maybe m even)

$(Gr(k,k+m), \mathcal{A}(Z))$ is a positive geometry

Conjecture (Arkani-Hamed, Trnka) $m=4$

$\Omega(Gr(k,k+4), \mathcal{A}(Z)) =: \Omega_{n,k,4}$ is the amplitude form for $N=4$ SYM

$$\Omega_{n,k,m} \longleftrightarrow \Omega_{n,n-k-m,m}$$

$Gr(k,n) \cong Gr(n-k,n)$

If $k=1$,

$$Gr(1,n) \cong \mathbb{P}^{n-1} \supset Gr(1,n)_{\geq 0} \cong \Delta^{n-1}$$

$$Z: \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^m$$

$$Z(\Delta^{n-1}) = \mathrm{conv}(Z_1, \ldots, Z_n) \subset \mathbb{P}^m$$

Q: Are Grassmann polytopes positive geometries?

Z positive if all $(k+m)\times(k+m)$ minors are positive.

Case $\mathcal{A} = \mathcal{A}(Z)$ is called the amplituhedron

$$\underset{\text{triangulation}}{\mathcal{A}_{n,k,m}} \longleftrightarrow \underset{\text{triangulation}}{\mathcal{A}_{n,n-k-m,m}}$$

$$\underset{km\text{-form}}{\Omega_{n,k,m}} \longleftrightarrow \underset{(n-k-m)\times m \text{ form}}{\Omega_{n,n-k-m,m}}$$

Conjecture (maybe m even)

$(Gr(k,k+m), \mathcal{A}(Z))$ is a positive geometry

Conjecture ($m=4$) (Arkani-Hamed, Trnka)

$\Omega(Gr(k,k+4), \mathcal{A}(Z)) =: \Omega_{n,k,4}$ is the amplitude form for $\mathcal{N}=4$ SYM