Title: Towards building a Lorentzian space-time tensor network

Date: Oct 16, 2018 10:30 AM

URL: http://pirsa.org/18100090

Abstract: <p>We discuss some algebraic quantum field theory (AQFT) ingredients that should be useful in defining a tensor network describing a Lorentzian space-time.</p>

<p>We look into toy models that approximate Minkowski space and show how Lorentz boosts are approximately recovered, and obtain Rindler modes that can be compared with the entanglement spectrum. </p>

<p>We make connections of these approximations of Lorentz boosts with the corner transfer matrices in integrable models, and comment on the discrete realization of the Reeh-Schlieder theorem that governs entanglement in a Hilbert space with lower bounded energy. </p>

Tensor network and Lorentzian Spacetime

Perimeter Institute, 16th October, 2018

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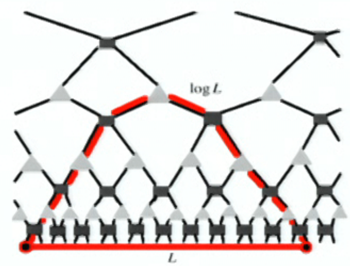
Work in collaboration with Arpan Bhattacharya, Long Cheng, Sirui Ning, Zhi Yang

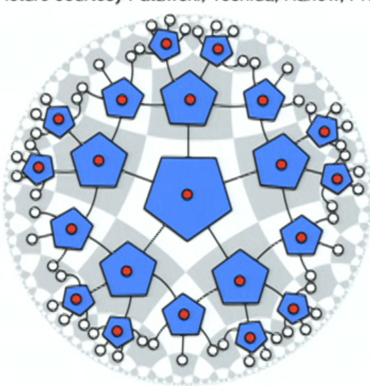
Motivation

- Tensor network vs AdS/CFT
- Mostly a snapshot at a given time
- or Euclidean spacetime

Picture courtesy Orus



Picture courtesy Patawski, Yoshida, Harlow, Preskill



- Black holes, Hawking radiation etc
  physics of the horizon is best understood in Lorentzian signature.

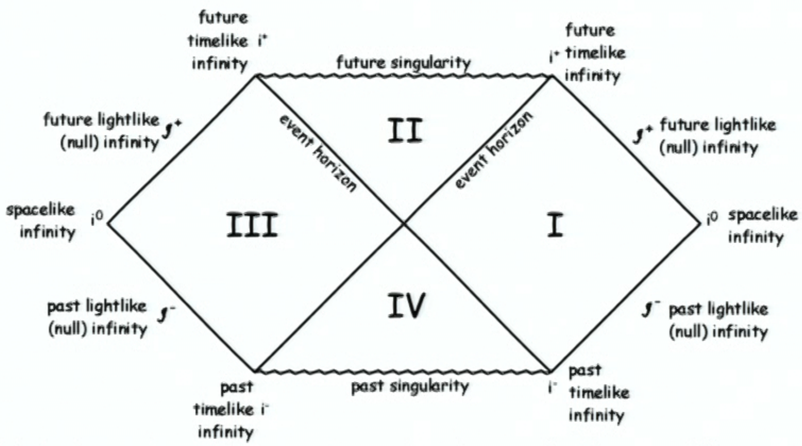


diagram courtesy Prof. J. Norton

- Obtain a tensor network description of Lorentzian spacetime?
  modest version : a toy version of Minkowski space at least?

- a discrete version would also allow for experimental simulations, opening up many interesting possibilities

AQFT — a rough guide

- AQFT is a set of rules to attach an algebra to a spacetime M
- There are some basic assumptions about M:
    - M as a topological space is Hausdorff, connected and paracompact
    - M has a metric defining a casual structure with causal curves (time-like/null-like)
      2 points are space-like separated if they cannot be connected by a causal curve
    - Cauchy surface foliation, locally $\Sigma \times R$
    - collection $B \subset M$ forms a directed set. i.e.

$$O_1, O_2 \in B, \ \exists O : O_1 \subseteq O; O_2 \subseteq O$$

AQFT

- 1) AQFT maps a region to an algebra $O \subset M,\ O \to \mathcal{U}(O)$
  - (This algebra is usually taken to be a C* algebra)
  - think of this as having a Hilbert space H with an inner product < , >
    norms of the operators in U(O) : $\sup ||a\xi|| \mid \xi \in H$
  - adjoint : $A \in \mathcal{U}(O),\ A^\dagger : \langle A^\dagger \alpha, \beta \rangle = \langle \alpha, A\beta \rangle$

AQFT

- 2) Isotony

$$O \subset O', \, \mathcal{U}(O) \subset \mathcal{U}(O')$$

- 3) Locality (Einstein causality)
  for causally disconnected regions O and O'

$$A \in \mathcal{U}(O), \; A' \in \mathcal{U}(O') \quad \Longrightarrow \; [A, A'] = 0$$

- 4) time slice axiom

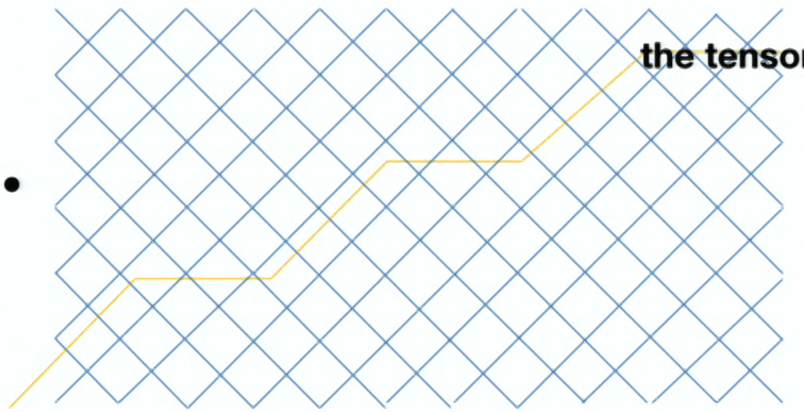
$$\mathcal{U}(N) \text{ is isomorphic to } \mathcal{U}(M)$$

- N a causally convex neighbourhood of a Cauchy surface

AQFT vs Tensor Network

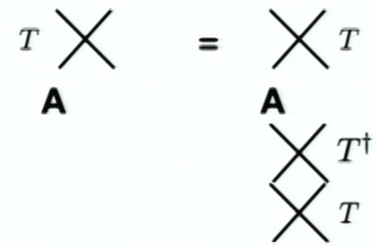
- There is a vector space on each link. There is an operator acting on this space. i.e. We can define $\mathcal{U}(O)$ to be the operator algebra acting on the collection of links O

the tensor network is a map between links

- 

AQFT vs Tensor Network

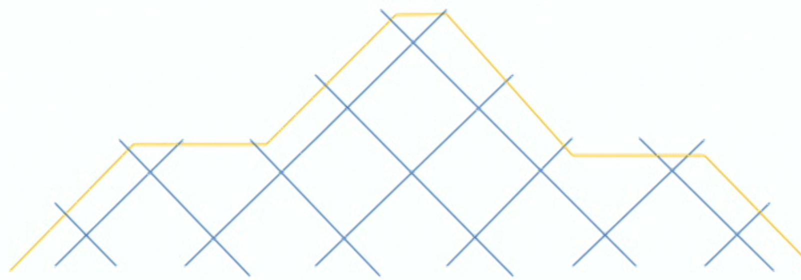
- operator pushing:



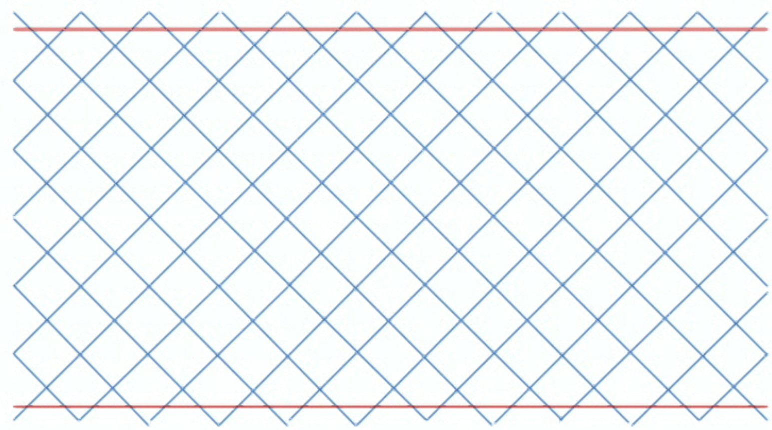
- causal structure : requires that T is unitary in 1 direction.
- two legs related by a (series) of unitary transformations are causally connected
- Cauchy surface ~ maximal set of legs such that not two legs are causally connected

AQFT vs Tensor Network

- Time-slice axiom:

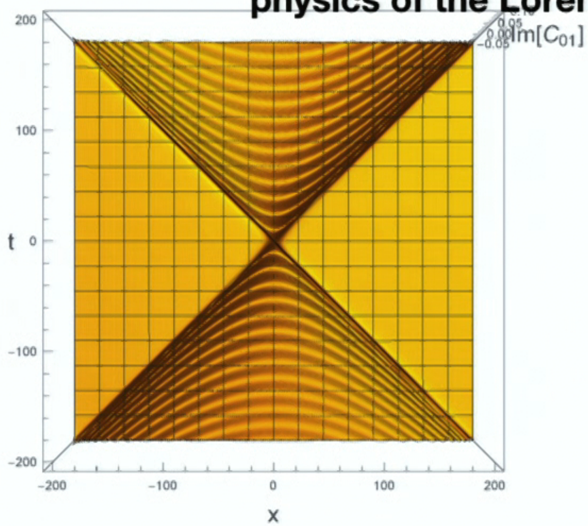


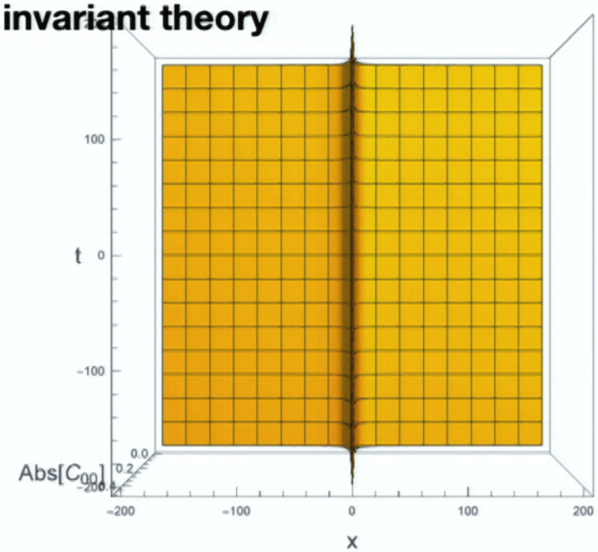
A given observer:
a collection of Cauchy surfaces



Correlation functions

note that small c is NOT the usual continuous time limit. Yet it still recovers the expected physics of the Lorentz invariant theory





c=0.1 **c=1**

When c approaches 0 the dispersion relation becomes linear. Correlation function looks like a Lorentz invariant theory!

(It is pointed out to us: our fermion model has a non-standard kinetic term. When the space/time spacing matches, it gives a linear spectrum without any doubling problem.)

Toy example: Free Fermions

- Strategy
  - 1. Spectra of an "inertial observer"
  - 2. Defining a Boost operator
  - 3. Spectra of the Boost operator (Comparison with the Rindler observers)
  - 4. Compare with entanglement Hamiltonian

1. Spectra

- Solved by recursion relation by using
- $a_p = \sum_{i=-L+\frac{1}{2}}^{L-\frac{1}{2}} \left(f_{2i} a_{2i} + g_{2i+1} a_{2i+1} \right)$

$$U\, a_p U^{-1} = \lambda\, a_p. \qquad c = \cos(\tilde{\alpha}\Delta t), \qquad s = \sin(\tilde{\alpha}\Delta t).$$

$$f_{2i}\, c^2 + g_{2i+1}\, c\, s + f_{2i-2} s^2 - g_{2i-1}\, c\, s = \lambda\, f_{2i},$$
$$g_{2i+1} c^2 + g_{2i+3} s^2 + f_{2i+2}\, s\, c - f_{2i}\, s\, c = \lambda\, g_{2i+1}.$$

Correlation functions

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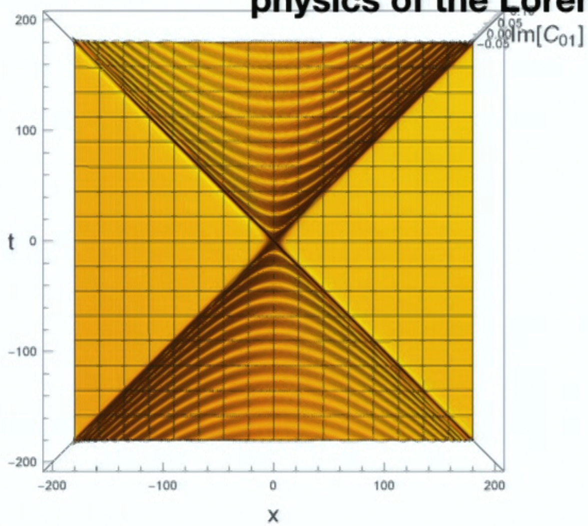

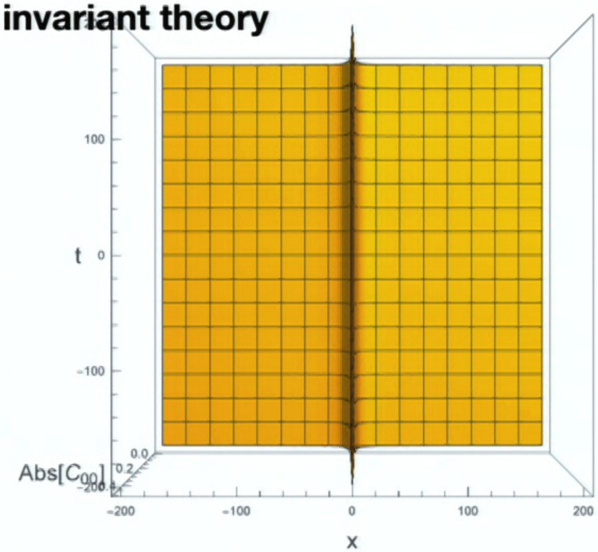

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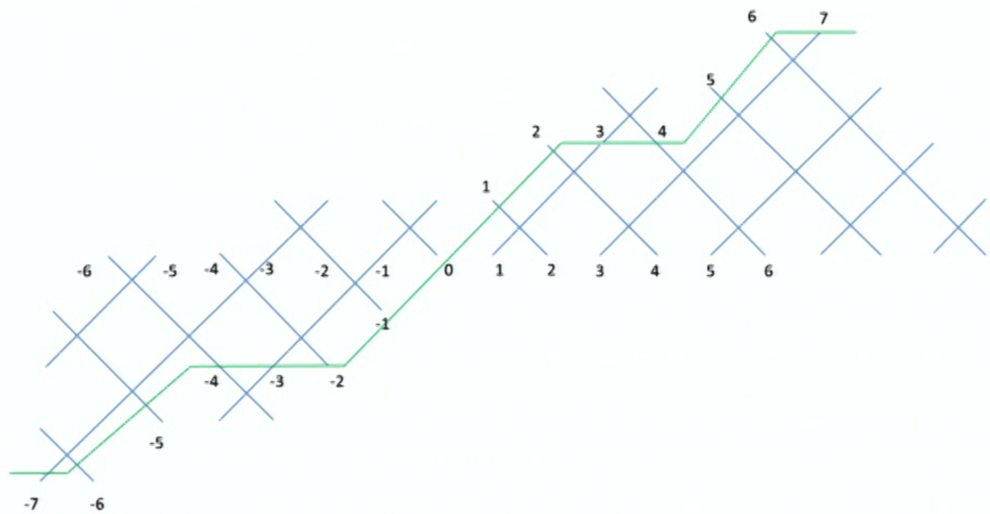
2.0-Guess of a boost



odd sites $x = 2s - 1, \qquad \Delta x = ws, \text{ where } w = 2$

2.1-Action of boost

Explicit solution of the spectrum of the boost operator at c=0

$$\hat{B} a_{q>0} \hat{B}^{-1} = \mathcal{N}^2 \sum_{p>0,s} e^{i((1+w/2)s)p-iqs} a_p + e^{-i((1+w/2)s)p-iqs} b_p^\dagger$$

$$= \sum_{p>0} e^{i(q-(1+w/2)p)/2} \frac{\sin(L(q-(1+w/2)p))}{2L\sin((q-(1+w/2)p)/2)} a_p + \qquad (1)$$

$$e^{i(q+(1+w/2)p)/2} \frac{\sin(L(q+(1+w/2)p))}{2L\sin((q+(1+w/2)p)/2)} b_p^\dagger.$$

Almost preserve ground state

- for small q it is very close to

$$\hat{B}(\Lambda) a_p \hat{B}^{-1}(\Lambda) = a_{\Lambda^{-1}p}, \qquad \Lambda = e^\eta$$

$$\Lambda = (1 + w/2) = 2$$ **in the previous diagram**

3.Spectra of boost

$$A^R_\kappa = \sum_{x>0} \psi_\kappa(x) a_{2x-1}. \qquad \hat{B} A^R_\kappa \hat{B}^{-1} = \eta_R(\kappa) A^R_\kappa.$$

$$\tilde{\psi}_\kappa(p) = p^\kappa, \qquad \tilde{\psi}_\kappa(p) \equiv \sum_{x>0} \psi_\kappa(x) e^{ipx}$$

$$\eta^R(\kappa) = (1 + w/2)^\kappa = e^{\eta\kappa} = e^{-i\eta\epsilon}$$

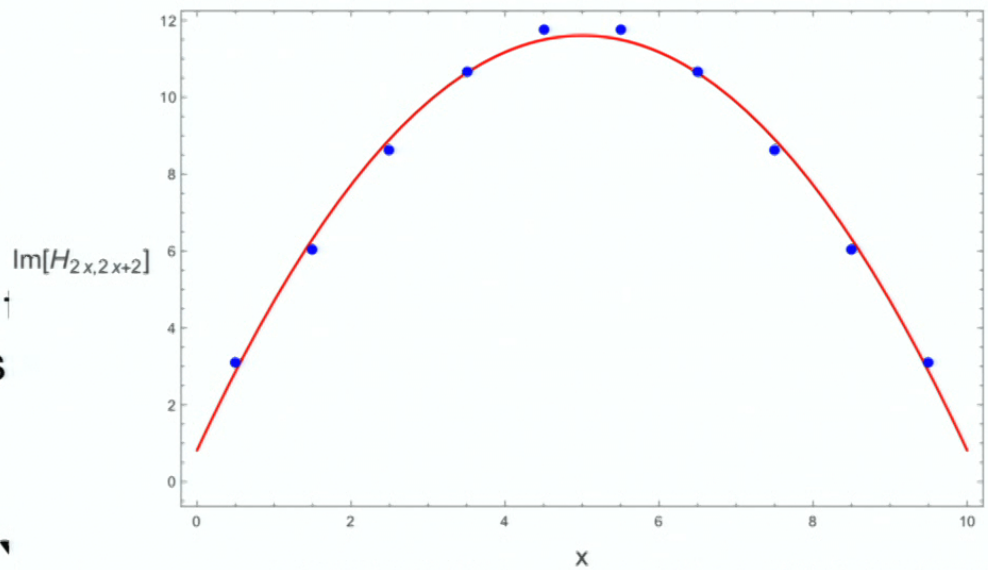
can define $\kappa = -i\epsilon$ **to be the positive energy modes where** $\epsilon > 0$

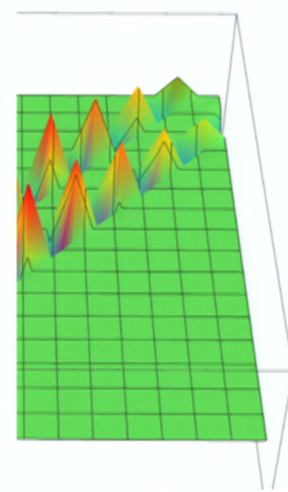
need to do the same for the x<0 modes. We recognise they are related to the "inertial" modes by a Bogoliubov transformation just like in the Rindler observers

Comparison with the entanglement Hamiltonian

- For any finite size system having constructed the “ground state” we can obtain the actual entanglement Hamiltonian
  Casini, Huerta
- Obtain the reduced density matrix and take log.

Actual Entanglement Hamiltonian





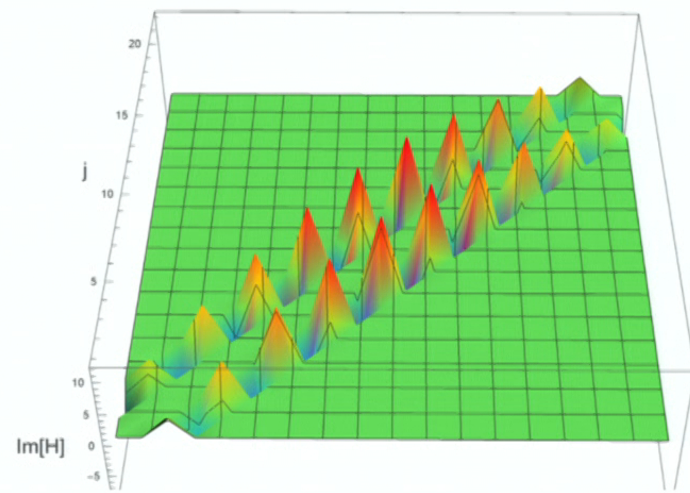
- Near t
  it bas

$$\mathcal{H} = \frac{\mathcal{N}}{i}\sum_x \quad 2L \quad \cdots e) \cdots , \ \nu_0 \sim \frac{1}{10}$$

compare with $K = \int dx\, x\, T_{00}$

Actual Entanglement Hamiltonian

- Near to the boundary, it basically looks like

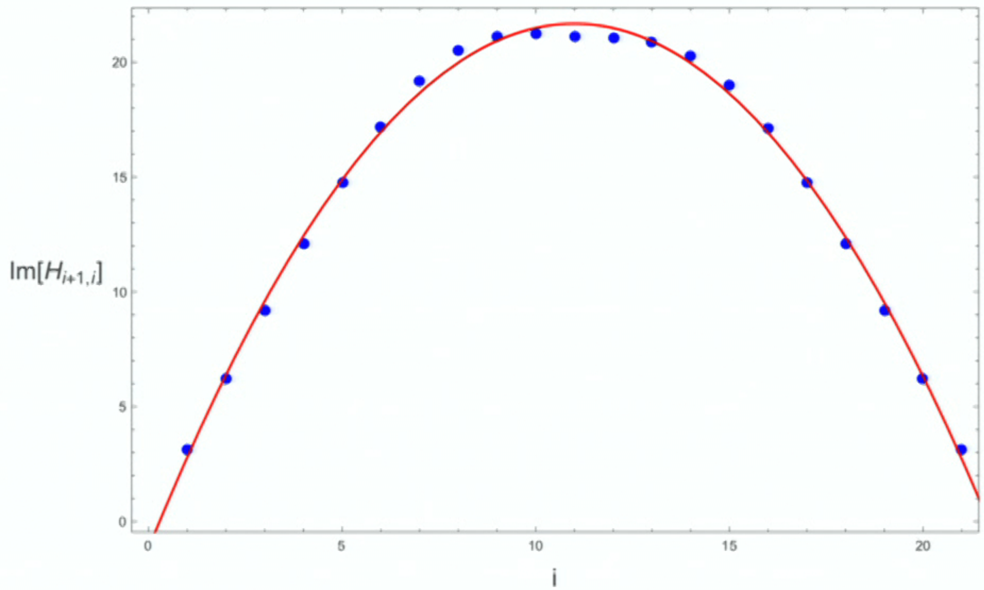


$$\mathcal{H} = \frac{\mathcal{N}}{i}\sum_x \left(\frac{x(L-x)}{2L} + \nu_0\right)(a^\dagger_{2x}a_{2x+2} - a^\dagger_{2x+2}a_{2x}) + (\text{odd sublattice})\cdots, \; \nu_0 \sim \frac{1}{10}$$

compare with $K = \int dx\, x\, T_{00}$

Actual Entanglement Hamiltonian

For c approaching 1



$$\mathcal{H} = \frac{\mathcal{N}}{i} \sum_x \left(\frac{x(2L-x)}{4L} + \nu_1\right)(a_x^\dagger a_{x+1} - a_{x+1}^\dagger a_x), \qquad \nu_1 \sim \nu_0,$$

Actual Entanglement Hamiltonian

Full modular Hamiltonian

$$\Gamma \equiv (\mathcal{H} - \bar{\mathcal{H}})_{c\to 0, L\to\infty} = \frac{\mathcal{N}}{2i} \sum_{x=-\infty}^{\infty} (x-\nu)(a^\dagger_{2x} a_{2(x+1)} - a^\dagger_{2(x+1)} a_{2x}),$$

Commutation relations with the momenta eigenmodes

$$[\Gamma, a_p] = \mathcal{N} \sum_{x=-\infty}^{\infty} a_{2x} e^{ixp} (\sin px + \frac{1}{2i}((1-\nu)e^{-ip} + \nu e^{ip})).$$

The x sin p term behaves like $\partial_p a_p$

The entanglement Hamiltonian behaves pretty much like our guess.
It does generate something like a boost that ~shifts p, which is what our guess does at least for c=0. There is subtlety with bc at infinity to account for the extra terms.

4. Generalization to other integrable models

- discrete space time has been considered in the literature based on statistical integrable lattice models
  Fadeev et al

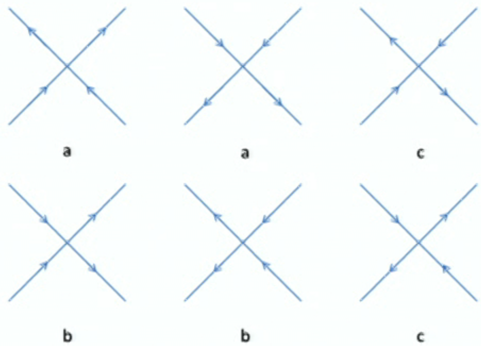
- can think of our tensor network as one building from the "diagonal-to-diagonal" basis of the transfer matrices (or inhomogeneous transfer matrix) in integrable models

To describe a quantum Lorentzian model: there is the issue of analytic continuation!

The Lorentzian model is obtained by analytically continuing the "spectral parameter"

6 vertex model

- example: the 6 vertex model



$$a = \sin(\lambda - u), \qquad b = \sin u, \qquad c = \sin \lambda, \qquad \Delta = -\cos \lambda.$$

Analytic continuation

$$R_{f_1,f_2}(a-b)L_{n,f_1}(a)L_{n,f_2}(b) = L_{n,f_2}(b)L_{n,f_1}(a)R_{f_1,f_2}(a-b),$$

transfer matrix **taking the solution** $R = L$

$$T_f(v) = (L_{N,f}(u)L_{N-1,f}(u)\cdots L_{1,f}(u)),\ \ T(v) = \mathrm{tr}_f(L_{N,f}(u)L_{N-1,f}(u)\cdots L_{1,f}(u))$$

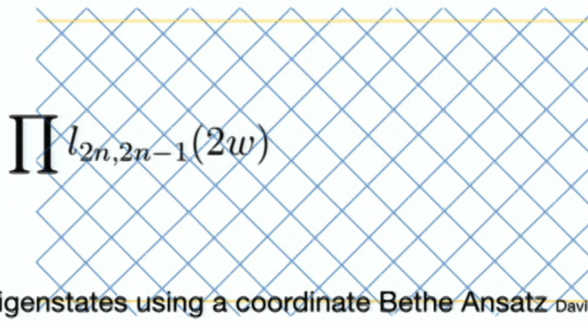
Lorentzian transfer matrix $T(u) = T^E(iu).$

Looking ahead, to build a network, Fadeev introduced "inhomogenous transfer matrix"

$$T_f(u,v) = L_{2N,f}(u+v)L_{2N-1,f}(u-v)\cdots L_{2,f}(u+v)L_{1,f}(u-v),$$

6 vertex model

$$U_+ = \mathrm{tr}_f T_f(w,w), \qquad U_- = \mathrm{tr}_f T_f(-w,w).$$

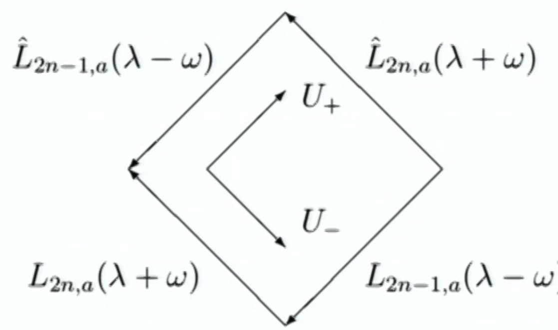


$$\exp(-iH) = U_+ U_-^{-1} = V \prod l_{2n,2n-1}(2w) V^{-1} \prod l_{2n,2n-1}(2w)$$
$$= \prod l_{2n+1,2n}(2w) \prod l_{2n,2n-1}(2w),$$

- solve all eigenstates using a coordinate Bethe Ansatz Davies

$$L_{n,f}(u) = P_{n,f} l_{n,f}(u),$$

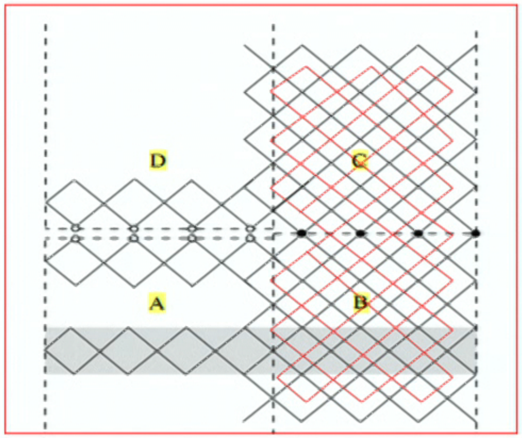
$$U_+ = \exp(-i(H-P)/2), \qquad U_-$$



Flatness condition: $L_{2n,a}(\lambda+\omega)L_{2n-}$ $-\omega)U_+.$

Boost using the classical model: Corner Transfer Matrix

- It is observed that there is exact symmetry rotating the graph by 90 degrees. The rotation transformation is effected by the "corner transfer matrix" Baxter



picture courtesy Thacker

- The corner transfer matrix is the Euclidean version of our Lorentz transformation generating an exact symmetry of the Euclidean lattice. (A generates 90 deg rotation)

$$\hat{A}^E(u) = \exp(-uK).$$

Corner Transfer Matrix

- It has been shown that

$$K = \sum_{n=-\infty}^{\infty} n H_{XXZ}(n, n+1)$$

- $H_{XXZ}(n, n+1) = -\frac{1}{2}(J\sigma_n^x\sigma_{n+1}^x + J\sigma_n^y\sigma_{n+1}^y + K\sigma_n^z\sigma_{n+1}^z)$

It has been noted that the reduced density matrix Cardy etc... $\rho_h = A.B.C.D$

Some symmetry algebra related to the CTM

$$K_c \equiv K - \bar{K} = \sum_{n=-\infty}^{\infty} n H_{XYZ}(n, n+1).$$

$$[K_c, T^E_{N,f}(v)] = \partial_v T^E_{N,f}(v),$$ **This expression follows from the YB equation**

$$e^{ip} = \frac{e^{\lambda} e^{iv} - 1}{e^{\lambda} - e^{iv}}$$

it's a shift of the momentum. how does that compare with the free fermion case? it has to do with some boundary conditions.....

(The boost operator can be used to build a Virasoro algebra.)

Corner Transfer Matrix

- The corner transfer matrix gives rise to a “boost operator”

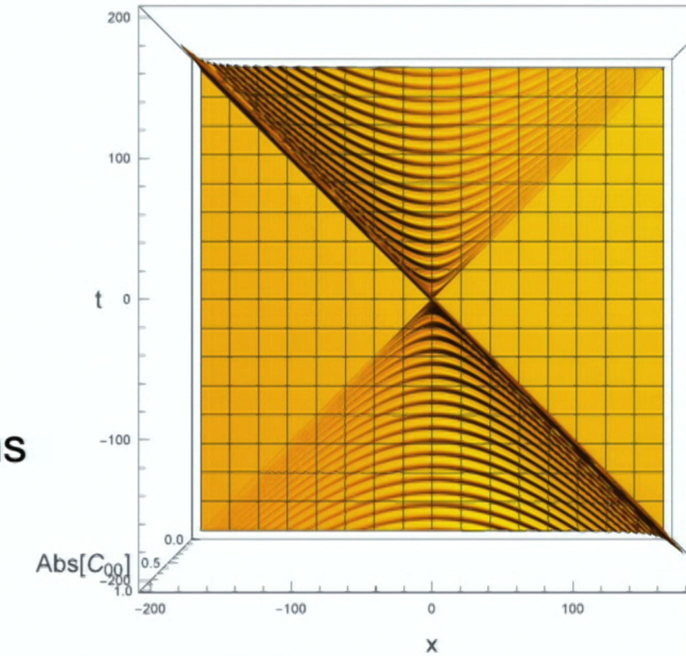
K : it generates shifts in the rapidity

in the limit $c \to 0$ **the dispersion relation also becomes linear.**

eigen-wavefunctions of K can be compared with the wavefunctions we have for free fermions. We find that it approaches our result.

Final comment on the Reeh-Schlieder theorem

- In the ferromagnetic state, the ground state only has short range entanglement. in fact the reference ground state is simply all spin up = direct product state. In some limit dispersion of spinons becomes linear like the fermions



- the reference state remains exactly invariant under the action of this boost. GS no entanglement?!?!

Final comment on the Reeh-Schlieder theorem

For a spectrum that is lower bounded in energy:

$$Q(t,x) = Q^+ + Q^- + Q^0, \qquad Q^-|\Omega\rangle = Q^{+\,*}|\Omega\rangle = 0,$$

$$Q_i = \int d^d p\, e^{-ip_0 t + ip_i x^i}\, \tilde{Q}_i(p_0, p_i), \qquad \tilde{Q}_+(p_0,p_i)|\Omega\rangle = \tilde{Q}^*_-(p_0,p_i)|\Omega\rangle = 0.$$

Boundedness of energy implies that the ground state is annihilated by a set of non-local operators

In the ferromagnetic model: there is no notion of positive vs negative energy. Violates the boundedness of energy assumed in the Reeh-Schlieder Theorem. No entanglement even with translation and (approximate) Lorentz invariance

Conclusion and Outlook

- We have attempted to put the tensor network in the framework of the AQFT.
- We show very simple examples that basic features, including the Unruh effect, can be recovered in a discrete tensor network without actually taking the continuous limit.
- This is a broader feature that can be readily investigated in other simple models such as integrable models.
- We are looking toward generalization to curved backgrounds, and also to higher dimensions.

(There is a mystery about Bosons.)