Title: Renormalization and Effective Field Theory - Lecture 7

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Abstract:

BV formalism in finite dimensions

Consider

$$\int_{\mathbb{R}^n} e^{-W/\hbar} d^n x \, F(x)$$

W a polynomial

F a polynomial

$$\mathrm{Div}_{e^{-W/\hbar}d^n x} \quad \mathrm{Vect}(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$$

If $V = \sum f_i(x)\partial_{x_i}$ is a vector field

Then

$$\mathrm{Div}_{e^{-W/\hbar}d^n x} V = \sum \frac{\partial f_i}{\partial x_i} - \frac{1}{\hbar}\sum \frac{\partial W}{\partial x_i} f_i(x)$$

We say

observables $=$ polynomials in x_i

modulo $\hbar \cdot \mathrm{Div}_{e^{-W/\hbar} d^n x} V$, for arbitrary V.

$\underline{\hbar = 0}$

Image of $\hbar\,\mathrm{Div}$ at $\hbar = 0$ is spanned by

$$f_i(x) \frac{\partial W}{\partial x_i}$$

∂x_i

Image of $\hbar \mathrm{Div}$ = those fns which vanish on

$$\mathrm{Crit}(W) = \left\{ \vec{x} \in \mathbb{R}^n, \ \frac{\partial W}{\partial x_i}(x) = 0 \ \text{for } i = 1 \ldots n \right\}$$

At $\hbar = 0$, observables = functions on $\mathrm{Crit}\, W$

Homological way of removing divergence

With vol. form $d^n x$

$$C^\infty(\mathbb{R}^n) \overset{\simeq}{\longleftrightarrow} \Omega^n(\mathbb{R}^n)$$

$$f \longmapsto f\, dx_1 \dots dx_n$$

$$= f\, d\mathrm{Vol}$$

$$\mathrm{Vect}(\mathbb{R}^n) \longleftrightarrow \Omega^{n-1}(\mathbb{R}^n)$$

$$f_i \partial_{x_i} \longmapsto \sum f_i(x)(-1)^i dx_1 \ldots \widehat{dx_i} \ldots dx_n$$

$$= \left(\sum f_i \partial_{x_i}\right) \lrcorner\, d\mathrm{Vol}$$

The diagram

$$\begin{tikzcd}
\mathrm{Vect}(\mathbb{R}^n) \arrow[r] \arrow[d, "\mathrm{Div}"'] & \Omega^{n-1}(\mathbb{R}^n) \arrow[d, "d"] \\
C^\infty(\mathbb{R}^n) \arrow[r] & \Omega^n(\mathbb{R}^n)
\end{tikzcd}$$

Extend to involve the whole de Rham complex:

$$C^\infty(\mathbb{R}^n, \Lambda^k T\mathbb{R}^n) \simeq \Omega^{n-k}(\mathbb{R}^n)$$

$$f_{i_1\dots i_k}\partial_{x_{i_1}}\cdots\partial_{x_{i_k}} \longmapsto \pm f_{i_1\dots i_k}\, dx_1\cdots\widehat{dx_{i_1}}\cdots\widehat{dx_{i_k}}\cdots dx_n$$

Define the divergence complex to be the complex:

where we have

$$\begin{tikzcd}
\mathrm{Div}: C^{\infty}(\mathbb{R}^n, \Lambda^k T\mathbb{R}^n) \arrow[r] \arrow[d, "\wr"] & C^{\infty}(\mathbb{R}^n, \Lambda^{k-1} T\mathbb{R}^n) \arrow[d, "\wr"] \\
\Omega^{n-k}(\mathbb{R}^n) \arrow[r, "d"] & \Omega^{n-k+1}(\mathbb{R}^n)
\end{tikzcd}$$

this commutes

$$\mathrm{Div}\left(f_{i_1 \dots i_k} \partial_{x_{i_1}} \cdots \partial_{x_{i_k}} \right) = \sum (-1)^r \left(\partial_{x_{i_r}} f_{i_1 \dots i_k} \right) \partial_{x_{i_1}} \cdots \widehat{\partial_{x_{i_r}}} \cdots \partial_{x_{i_k}}$$

$$\mathrm{Div} \circ \mathrm{Div} = 0$$

If we use the vol. form $e^{-W/\hbar} d^n x$ then

Usual de Rham $\Longrightarrow$ twisted de Rham

$$
\begin{tikzcd}
\mathrm{Vect}(\mathbb{R}^n) \arrow[r] \arrow[d, "\mathrm{Div}_{d^n x} - \frac{1}{\hbar} dW \vee"'] & \Omega^{n-1}(\mathbb{R}^n) \arrow[d, "d - \frac{1}{\hbar} dW \wedge"] \\
C^\infty(\mathbb{R}^n) \arrow[r] & \Omega^n(\mathbb{R}^n)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\Omega^\bullet(\mathbb{R}^n),\ d - \frac{dW}{\hbar}\right) \\
\left(\Omega^\bullet(\mathbb{R}^n),\ d\right) \arrow[u, "e^{-W/\hbar}"']
\end{tikzcd}
$$

$$\mathrm{Div}\left(f_{i_1\dots i_k}\,\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}\right) = \sum (-1)^r \left(\partial_{x_{i_r}} f_{i_1\dots i_k}\right)\partial_{x_{i_1}}\cdots\widehat{\partial_{x_{i_r}}}\cdots\partial_{x_{i_k}}$$

$$\mathrm{Div}\circ\mathrm{Div} = 0$$

$-\frac{dW}{\hbar}$ term sends

$$f_{i_1\dots i_k}\,\partial_{x_{i_1}}\cdots\partial_{x_{i_k}} \longrightarrow -\sum (-1)^r \frac{1}{\hbar} f_{i_1\dots i_k}\frac{\partial W}{\partial x_{i_r}}\,\partial_{x_{i_1}}\cdots\widehat{\partial_{x_{i_r}}}\cdots\partial_{x_{i_k}}$$

Anti-Field formalism

Consider

$\mathbb{R}[x_i, \theta^i]$ θ^i are odd variables of coh. degree -1 (complex)

$$\theta^i\theta^j = -\theta^j\theta^i$$

"functions of fields (x) + anti-fields (θ)"

There's an iso

$$\mathbb{R}[x_i, \theta^i] \simeq \bigoplus_k \Gamma(\mathbb{R}^n, \Lambda^k T\mathbb{R}^n)$$

$$f(x)\,\theta^{i_1}\cdots\theta^{i_k} \longrightarrow f(x)\,\partial_{x_{i_1}}\cdots\partial_{x_{i_k}}$$

The divergence operator for $d^n x$ is $\Delta = \sum \frac{\partial}{\partial x_i}\frac{\partial}{\partial \theta^i}$ = "BV Laplacian"

$$\Delta\left(f\,\theta^{i_1}\cdots\theta^{i_k}\right) = \sum \frac{\partial f}{\partial x_{i_r}}(-1)^r\, \theta^{i_1}\cdots\widehat{\theta^{i_r}}\cdots\theta^{i_k}$$

Other term is

$$-\frac{1}{\hbar}\sum \frac{\partial W}{\partial x_i}\,\partial_{\theta^i}$$

Coho. of $\Delta + \partial$ gives observables; $H^0 = \mathbb{R}[x_i]/\text{divergences}$

Define a Poisson bracket of degree +1

on $\mathbb{R}[x_i, \theta^i]$ by

$$\{F(x,\theta), G(x,\theta)\} = \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \theta^i} \pm \frac{\partial F}{\partial \theta^i}\frac{\partial G}{\partial x_i}$$

Characterized by

1) $\{F, G\} = (-1)^{|F||G|}\{G, F\}$

2) $\{FG, H\} = F\{G, H\} \pm G\{F, H\}$

3) $\{x_i, \theta^j\} = \delta_i^j$

Then, the differential in divergence complex has nice form

$$\Delta + \frac{1}{\hbar}\{W, -\}$$

Multiply by $\hbar$ $\quad \hbar\Delta - \{W, -\}$

BV formalism in ∞ dimensions

Consider a free scalar field $\varphi \in C^\infty(M)$

with Lagrangian $\int \varphi \Delta \varphi$

To put this in BV formalism, introduce an anti-field φ^*, which is a scalar, but anti-commuting.

$M = \mathbb{R}$ Observables at $0 \in \mathbb{R}$

…bservables at $0 \in \mathbb{R}$ …

are $\varphi(0)$, $\partial\varphi(0)$, $\partial^2\varphi(0)$ … or polynomials

$\mathbb{R}[\varphi(0), \partial\varphi, \ldots]$

With anti-fields, we also have polys. in $\varphi^*(0), \partial\varphi^*(0) \ldots$

What is $\{S, -\}$?

1) $\{F, G\} = (-1)^{\cdots}\{G, F\}$

$$\{S, \varphi^*(0)\} = (\partial^2\varphi)(0)$$

observables at $0 \in \mathbb{R}$

are $\varphi(0)$, $\partial\varphi(0)$, $\partial^2\varphi(0)$ … or polynomials

$\mathbb{R}[\varphi(0), \partial\varphi, \dots]$

$$\underset{\varphi}{C^\infty(\mathbb{R})} \oplus \underset{\varphi^*}{\Pi C^\infty(\mathbb{R})}$$

With anti-fields, we also have polys. in $\varphi^*(0), \partial\varphi^*(0) \dots$

What is $\{S, -\}$? BV symplectic pairing is $\int_{\mathbb{R}} \varphi\varphi^*$

$$\{S, \varphi^*(0)\} = (\partial^2 \varphi)(0)$$

$$\mathcal{O}^{(n)} = \text{obs. } \partial^n \varphi(0)$$

$$\mu^{(n)} \quad \partial^n \varphi^*(0)$$

$$\{S, -\} = \sum \mathcal{O}^{(n+2)} \frac{\partial}{\partial \mu^{(n)}}$$

H^0 of this complex

$$= \mathbb{R}[\underbrace{\mathcal{O}^{(0)}}_{\text{position}}, \underbrace{\mathcal{O}^{(1)}}_{\text{momentum}}]$$

No other cohomology