Title: Entanglement Complexity and Scrambling via Braiding of Nonabelions

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Abstract: <p>Entanglement spectrum (ES) contains more information than the entanglement entropy, a single number. For highly excited states, this can be quantified by the ES statistics, i.e. the distribution of the ratio of adjacent gaps in the ES. I will first present examples in both random unitary circuits and Hamiltonian systems, where the ES signals whether a time-evolved state (even if maximally entangled) can be efficiently disentangled without precise knowledge of the time evolution operator. This allows us to define a notion of entanglement complexity that is not revealed by the entanglement entropy.</p>

<p>In the second part, I will discuss how quantum states are scrambled via braiding in systems of non-Abelian anyons through the lens of ES statistics. We define a distance between the entanglement level spacing distribution of a state evolved under random braids and that of a Haar-random state, using the Kullback-Leibler divergence D_{KL}. We study D_{KL} numerically for random braids of Majorana fermions (supplemented with random local four-body interactions) and Fibonacci anyons. Our results reveal a hierarchy of scrambling among various models --- even for the same amount of entanglement entropy --- at intermediate times, whereas all models exhibit the same late-time behavior. In particular, we find that braiding of Fibonacci anyons scrambles more efficiently than the universal H+T+CNOT set. Our results promote D_{KL} as a quantifiable metric for scrambling and quantum chaos, which applies to generic quantum systems.</p>

Entanglement Complexity and Scrambling via Braiding of Nonabelions

arXiv: 1804.01097

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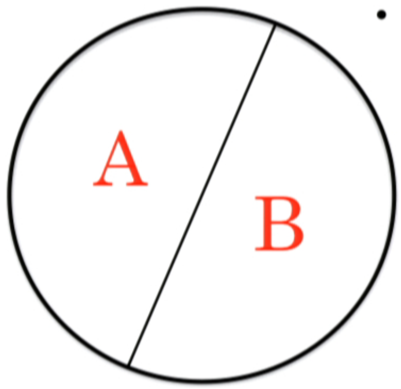
Entanglement spectrum (ES)



- A pure state $|\psi\rangle$, under bipartitioning:

$$|\psi\rangle = \sum_{\sigma} \psi(\sigma)|\sigma\rangle$$
$$= \sum_{\sigma_A,\sigma_B} \Psi(\sigma_A,\sigma_B)|\sigma_A\rangle|\sigma_B\rangle$$
$$= \sum_{k} \lambda_k |\phi_A\rangle|\chi_B\rangle$$

Schmidt decomposition (SVD)

- Entanglement spectrum: $\{\lambda_k\}$
- Relation to the reduced density matrix: $\rho_A = \mathrm{tr}_B|\psi\rangle\langle\psi|$

$$p_k = \lambda_k^2$$

Entanglement spectrum (ES)

- von-Neumann entropy: $S_{vN} = -\sum_k \lambda_k^2 \ln \lambda_k^2$
- Alternative definition: $\rho_A = e^{-H_E}$

  H_E "entanglement Hamiltonian"

  "energy spectrum" of H_E : $\{\xi_k\}$ "entanglement energy"

- Ground state ES: fingerprints to identify topological order

  e.g. Moore-Read wave function and $\nu = 5/2$ FQH

  H. L and F. D. M. Haldane, PRL **101**, 010504 (2008)

- ***Question: does ES of highly excited states encode more information than entanglement entropy?***

Extend to highly excited states

- ***How to extract the information contained in the ES of highly excited states?***

- Ground state of gapped systems, entanglement is low, ES decays fast (often exponentially);

→ low energy sector of ES encodes universal physics

- Highly excited states typically feature volume law entangle-ment;
- ES are usually continuous and decay slowly;

→ Information encoded in (a fraction of) the full ES

level spacing statistics

Entanglement spectrum (ES)

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Level spacing statistics of ES

- Ratio of adjacent gaps: $r_k = \dfrac{\lambda_{k-1} - \lambda_k}{\lambda_k - \lambda_{k+1}}$
- Probability distribution $P(r)$

$$P(r) = \frac{1}{(1+r)^2}$$ Poisson distribution

$$P(r) = \frac{1}{Z}\frac{(r+r^2)^\beta}{(1+r+r^2)^{1+3\beta/2}}$$ Wigner-Dyson distribution (level repulsion)

$Z = \frac{8}{27},\ \beta = 1$ GOE $\qquad Z = \frac{4\pi}{81\sqrt{3}},\ \beta = 2$ GUE

- Tails of the distributions are markedly different:

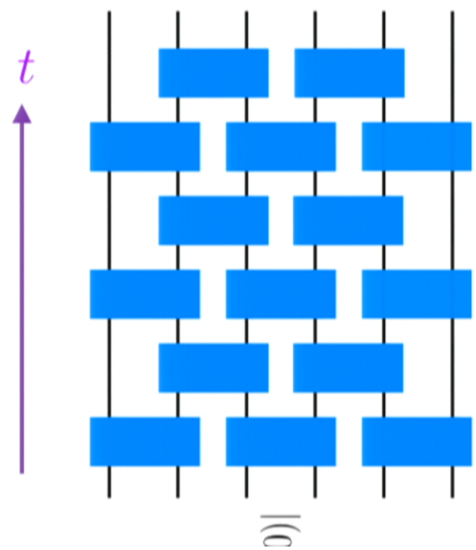
$P(r) \sim r^{-2}$ (Poisson) $\qquad P(r) \sim r^{-\beta-2}$ (W-D)

ES statistics and entanglement complexity

- Question: does ES of highly excited states encode more information than entanglement entropy?

  Yes!

- Example 1: random unitary circuits



C. Chamon *et al.* PRL **112**, 240501 (2014)
D. Shaffer *et al.* JStat P12007 (2014)

time evolution generated by random circuits consisting of local gates;

initial state: random product state

ES statistics and entanglement complexity

- Universal quantum circuit: H+T+CNOT;
- Example of non-universal quantum circuit: Clifford circuit, generated by Hadamard (H), $\pi/4$ (S) and CNOT gates;

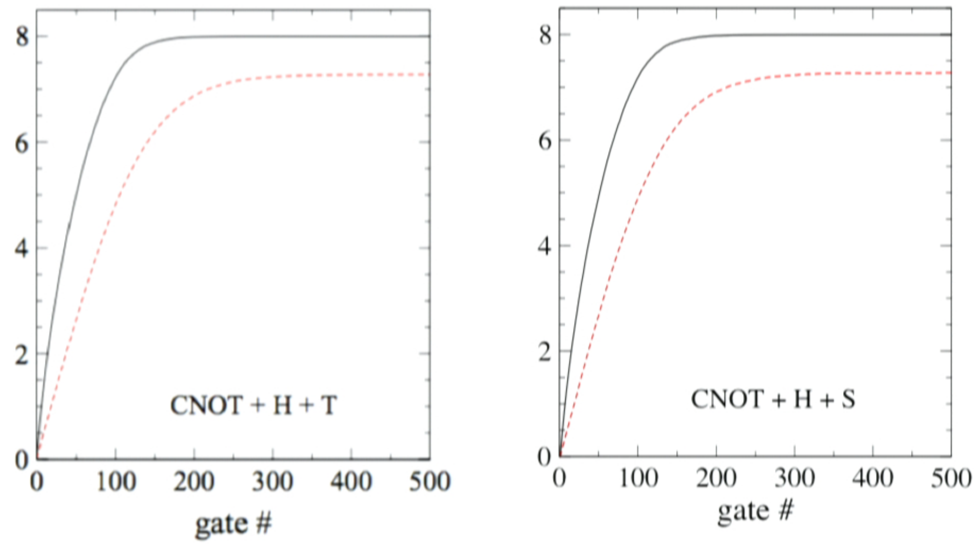
$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

- Clifford circuit can be simulated efficiently on a classical computer; (Gottesman-Knill theorem)
- ***What about entanglement properties?***
  random product states evolved under random circuits of:
  (1) H+T+CNOT (universal gates);
  (2) H+S+CNOT (Clifford gates);

ES statistics and entanglement complexity

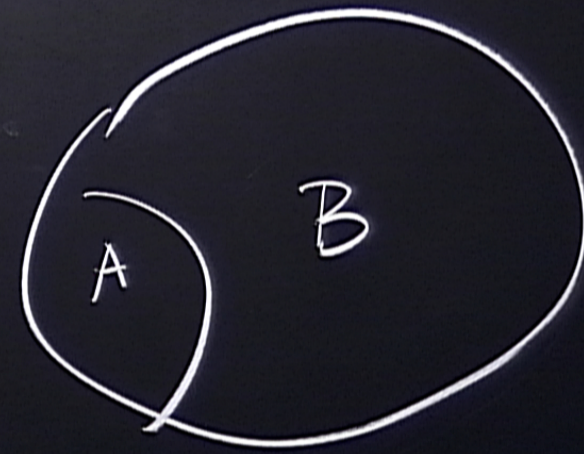
- Growth of entanglement entropy in both cases:

D. Shaffer *et al.* JStat P12007 (2014)



The entanglement growth behaviors are identical in both cases; Clifford circuit can maximally entangle random product states.

$$S^{(n)} = \frac{1}{1-n} \ln \sum_k p_k^n$$



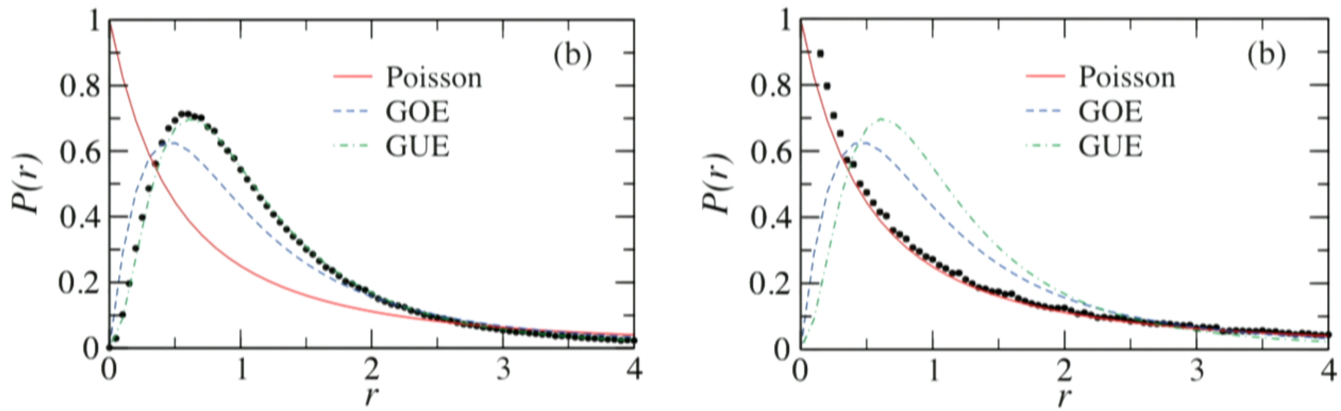
$$\log 2^{n_A} = n_A \ln 2.$$

$$S_{vN} \sim n_A \ln 2 - \frac{1}{2}$$

ES statistics and entanglement complexity

- The ***amount*** of entanglement is not a good indicator;
- What about the ES?

D. Shaffer *et al.* JStat P12007 (2014)



H+T+CNOT

Clifford circuit

The ES statistics are drastically different for the two cases!

same result holds for other choices of non-universal gates;
e.g. H+CNOT+NOT

ES statistics and entanglement complexity

- In this example of random unitary circuit, ES does reveal additional information not captured by entropy;
- How do we interpret the difference in the ES statistics?

Entanglement complexity:

Wether or not one can efficiently **disentangle** the state without precise knowledge of the time evolution operator.

- For generic maximally entangled states (generated by chaotic systems), this is a highly non-trivial task!

ES statistics and entanglement complexity

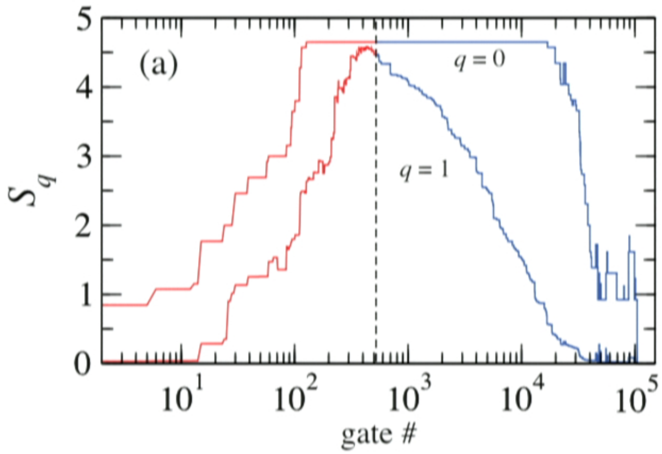
Disentangling algorithm: C. Chamon *et al.* PRL **112**, 240501 (2014)

1. Randomly pick a gate/site, apply to the final state;
2. Compute the entanglement entropy of the new state;
3. Accept such an attempt with probability $\min\{1, \exp(-\beta\Delta\bar{S})\}$.

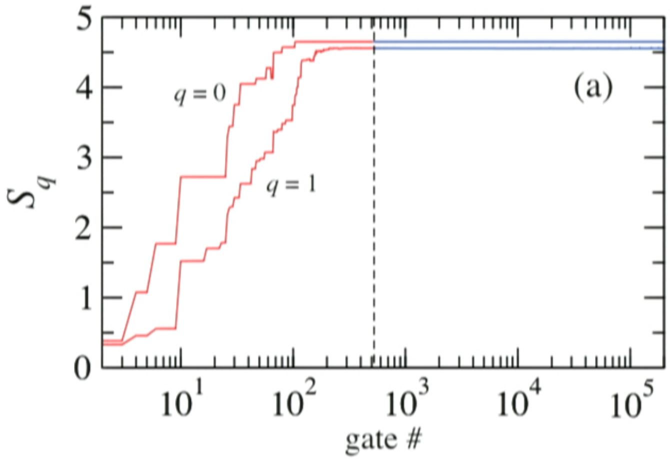
$\Delta\bar{S}$ averaged over all possible bipartitions.

This procedure is analogous to the Metropolis algorithm in thermal annealing: entanglement "cooling" algorithm!

ES statistics and entanglement complexity



Clifford circuit



H+T+CNOT

D. Shaffer *et al.* JStat P12007 (2014)

Using the disentangling algorithm, one can efficiently disentangle states generated by Clifford circuits (maximally entangled!), but not for the universal computation circuits!

Entropic scrambling/randomness complexity

Z.-W. Liu *et al.* PRL **120**, 130502 (2018)
Z.-W. Liu *et al.* JHEP 2018, 41

- Renyi entropies and randomness of pure states;
- Pseudo-random ensembles: quantum designs;
- Renyi- α entropy is almost maximal for α-designs;

→ A gap of complexity between entropic diagnostics and complete randomness.

- Clifford group: unitary 3-design, von-Neumann entropy is nearly maximal;
- ES statistics probes the degree of randomness similar to the Renyi entropies;

Entropic scrambling/randomness complexity

Z.-W. Liu *et al.* PRL **120**, 130502 (2018)
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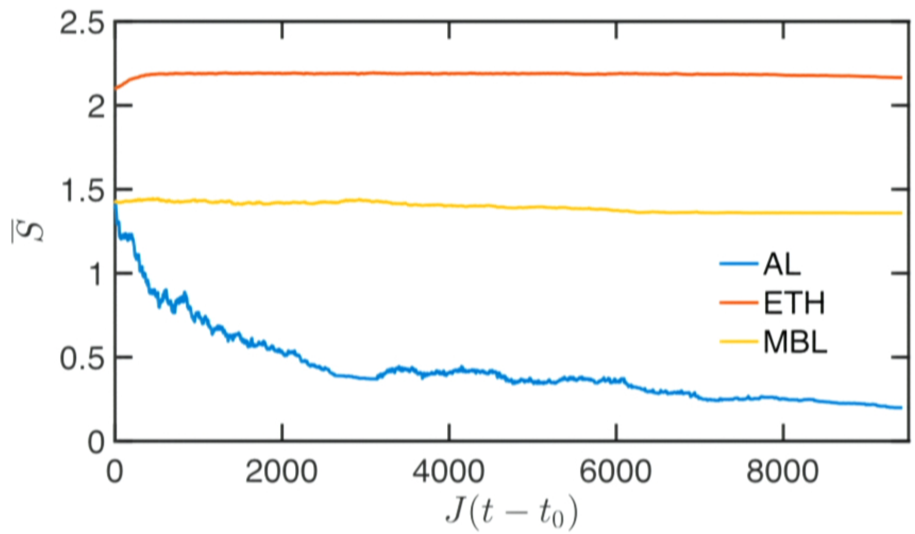
ES statistics and entanglement complexity

- Example 2: Hamiltonian evolution

Z.-C. Yang *et al.* PRB **96**, 020408(R) (2017)

$$\mathcal{H} = \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z + \Gamma \sigma_i^x)$$



Similar result holds for Hamiltonian dynamics as well.

ES statistics and entanglement complexity

Summary:

- ES statistics encodes additional information not captured by the entanglement entropy;
- ES reveals the *pattern* of entanglement, not the *amount*;
- This pattern can be interpreted as the complexity of entanglement;

ES described by random matrix theory (Wigner-Dyson distributed level spacings): → complex entanglement, cannot be disentangled efficiently

Poisson distributed ES → simple entanglement, can be disentangled efficiently

Quantum Chaos and Scrambling

- A precise definition of quantum chaos is still lacking;

attempts: random matrix theory description of energy levels;
out-of-time-ordered correlation functions;
quantum Lyapunov exponents;
etc.

- A related notion: scrambling (quantum information viewpoint)

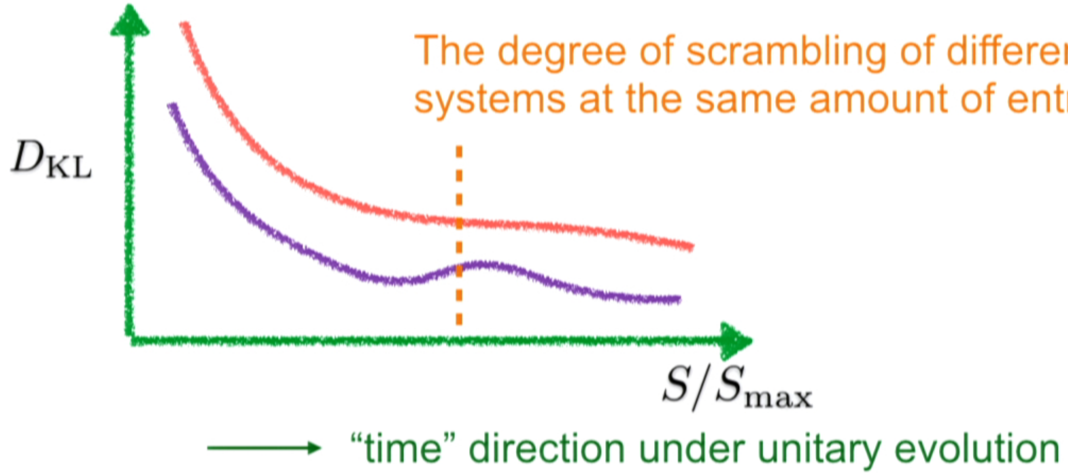
initially localized information becomes undetectable without measuring a significant fraction of the system;

- Unitary channels: quantum butterfly effect necessarily implies scrambling.

P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, JHEP 4 (2016)

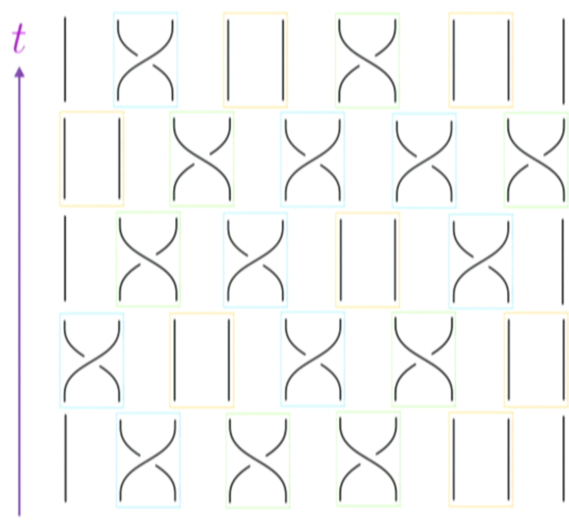
Quantum Chaos and Scrambling

- Use KL divergence as a measure of distance from the W-D distribution;
- How do we compare different chaotic systems? *Time unit between different systems is ambiguous!*
- Use Page limit-normalized entanglement entropy as a universal "clock": S/S_{max}



Non-Abelian anyon models

- We study scrambling in systems of non-abelian anyons under braiding (supplemented with interactions when necessary);
- We focus only on truly chaotic systems: ES of final states will all reach W-D distribution;
- Related to topological quantum computation;



Model 1: Majorana fermions

Model 2: Fibonacci anyons

↑

universal quantum computation!

Z.-C. Yang *et al.* arXiv: 1804.01097

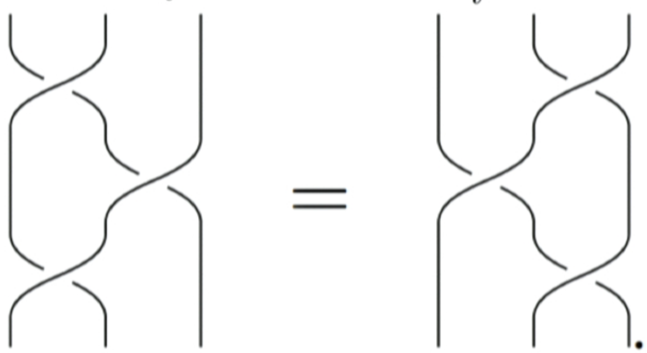
The braid group

$$\mathcal{B}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \; i = 1, 2, \ldots, n-1 \rangle$$

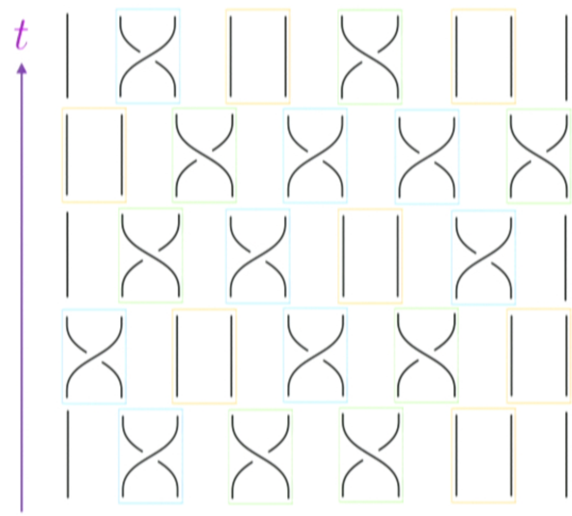
Represent braid operations pictorially on n strands:

σ_i

σ_i^{-1}



$\sigma_i \sigma_{i+1} \sigma_i \quad = \quad \sigma_{i+1} \sigma_i \sigma_{i+1}$.



Model 1: Majorana fermions

- Quasiparticle excitation in $SU(2)_2$ Chern-Simons theory;
- Physical systems: $p + ip$ topological superconductors;
  possibly $\nu = 5/2$ FQH fluid;
- Fusion rule:

Majorana	Ising
$i\gamma_{2k-1}\gamma_{2k} = 1 - 2n_k$	$\sigma \times \sigma = 1 + \psi$

- Hilbert space dimension of $2n$ Majoranas is 2^{n-1}

quantum dimension $d = \sqrt{2}$, nonlocal!

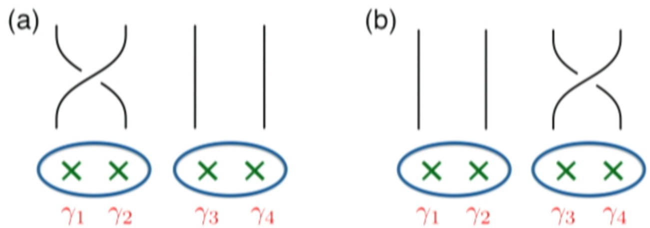
Model 1: Majorana fermions

- Representation of the braid group:

D. A. Ivanov, PRL **86**, 268 (2001)

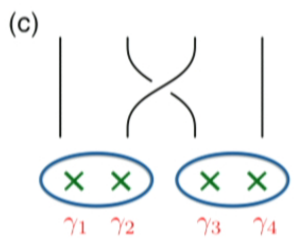
$$\sigma_i = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}} = \frac{1}{\sqrt{2}}(1+\gamma_i\gamma_{i+1})$$

- Action of braiding on Hilbert space:



$$\rho(T_1)|n_1,n_2\rangle = e^{i\frac{\pi}{4}(1-2n_1)}|n_1,n_2\rangle$$

$$\rho(T_3)|n_1,n_2\rangle = e^{i\frac{\pi}{4}(1-2n_2)}|n_1,n_2\rangle$$



$$\rho(T_2)|n_1,n_2\rangle = \frac{1}{\sqrt{2}}\left(|n_1,n_2\rangle + i|1-n_1,1-n_2\rangle\right)$$

Model 1: Majorana fermions

- Braidings of Majorana fermions are insufficient to create circuits capable of universal quantum computation (not truly chaotic);

- The unitary operator resembles a system of free fermions:

$$\sigma_i = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}} = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1})$$

- In order to reach maximal entropy and W-D distributed ES: supplement with random local four-body interactions:

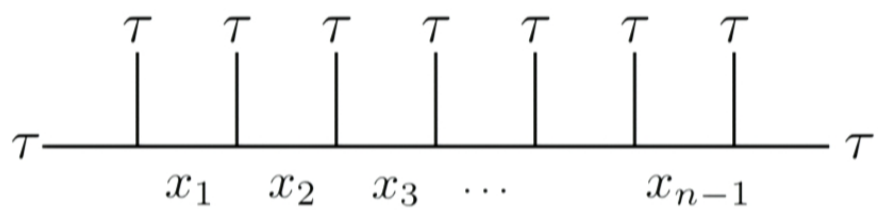
$$U_j = \exp\left(-i\alpha_j\gamma_j\gamma_{j+1}\gamma_{j+2}\gamma_{j+3}\right)$$

Model 2: Fibonacci anyons

- Quasiparticle excitation in $SU(2)_3$ Chern-Simons theory;
- Physical systems: ??? possibly $\nu = 12/5$ FQH fluid;
- Fusion rule:

$$\tau \times \tau = 1 + \tau$$

- Hilbert space: fusion chain



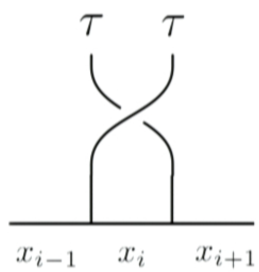
$$|x_1, x_2, \ldots, x_{n-1}\rangle \qquad \dim(\mathcal{H}) = \mathrm{Fib}(n+1)$$

quantum dimension $d = \phi$, nonlocal! $\phi = \frac{1+\sqrt{5}}{2}$

Model 2: Fibonacci anyons

- Braidings of Fibonacci anyons alone are sufficient for universal quantum computation;

- Action of braiding on Hilbert space:



can be derived using the F and R matrices of the corresponding TQFT

$$\rho(T_i)\ |101\rangle = -e^{-i\pi/5}/\phi\ |101\rangle - ie^{-i\pi/10}/\sqrt{\phi}\ |111\rangle$$

$$\rho(T_i)\ |111\rangle = -ie^{-i\pi/10}/\sqrt{\phi}\ |101\rangle - 1/\phi\ |111\rangle$$

$$\rho(T_i)\ |110\rangle = -e^{-2\pi i/5}\ |110\rangle$$

$$\rho(T_i)\ |011\rangle = -e^{-2\pi i/5}\ |011\rangle$$

$$\rho(T_i)\ |010\rangle = e^{-4\pi i/5}\ |010\rangle$$

Comparison to other chaotic models

- As a comparison, we would like to compare our results of non-Abelian anyon models with more familiar chaotic models:

  Model 3: H + T + CNOT

  Model 4: Two-qubit Haar-random unitary circuits

  Model 5: Sachdev-Ye-Kitaev (SYK) model

- SYK model: Majorana fermions with all-to-all interactions

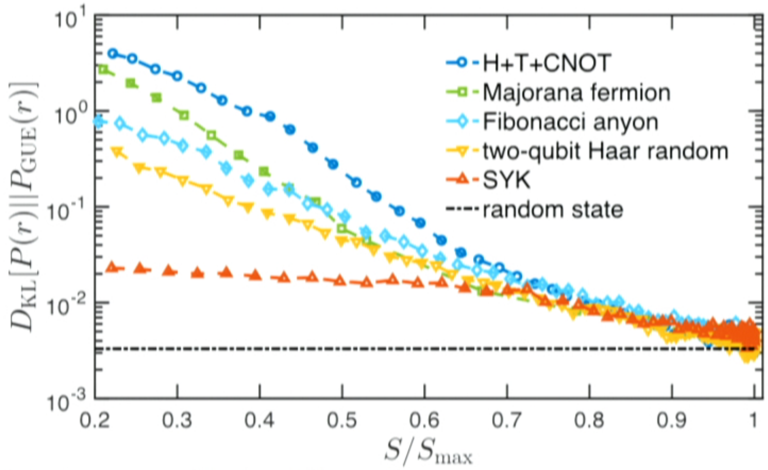
$$\mathcal{H} = \sum_{ijkl} J_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$$

$$\overline{J_{ijkl}} = 0 \qquad \overline{J_{ijkl}^2} = 3!J^2/N^3$$

maximally chaotic at large-N

J. Maldacena and D. Stanford, PRD **94**, 106002 (2016)

The degree of scrambling



- There is a clear hierarchy of D_{KL} even at the same amount of entropy!
- Real time (circuit depth or evolution time) has been eliminated from the plot.

Z.-C. Yang *et al.* arXiv: 1804.01097

At intermediate "time":

- H+T+CNOT turns out to be the least efficient scrambler;
- Braiding Fibonacci anyons scrambles almost as fast as Haar-random unitaries;
- SYK remains the fastest scrambler from this perspective;

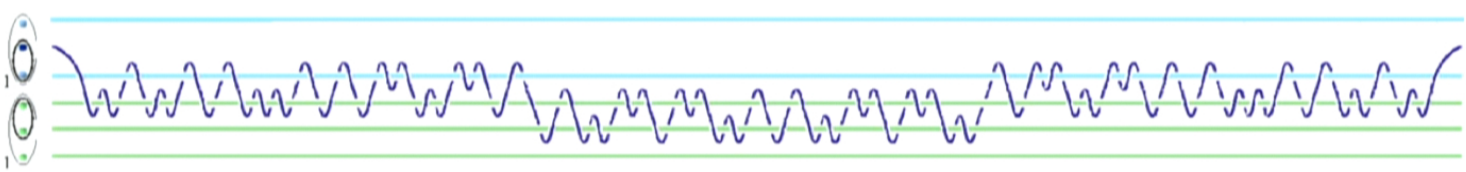
At late "time":

- All models show similar behavior.

The degree of scrambling

- The hierarchy should not be thought of as merely an effect of the length of a sequence or circuit depth;
- Quantum chaos ⟷ computational complexity;
- Braiding non-Abelian anyons may be a faster quantum computer?
- Conventional methods for “compiling” standard gates out of braids:

CNOT



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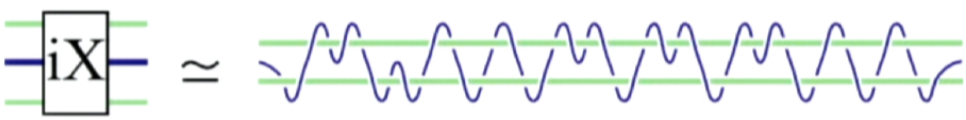

Is there a more clever way of utilizing the computational power of nonabelions?

L. Hormozi *et al.* PRB **75**, 165310 (2007)

Summary and conclusion

- ES statistics of states under chaotic quantum dynamics are described by random matrix theory (W-D level spacing distribution);
- Random matrix behavior of the ES reveals the complexity of entanglement, defined by the efficiency of disentangling the state;
- The distance to W-D distribution of the ES during quantum evolution can be used as a measure of the degree of scrambling;
- There is a hierarchy of degree of scrambling for various chaotic systems even at the same amount of entropy;
- Braiding of non-Abelian anyons may be a fast scrambler and fast quantum computer, comparing to standard universal gates;