Title: TBA

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Abstract: <p>Abstract TBD.</p>

Continuous Tensor Network States of Quantum Fields

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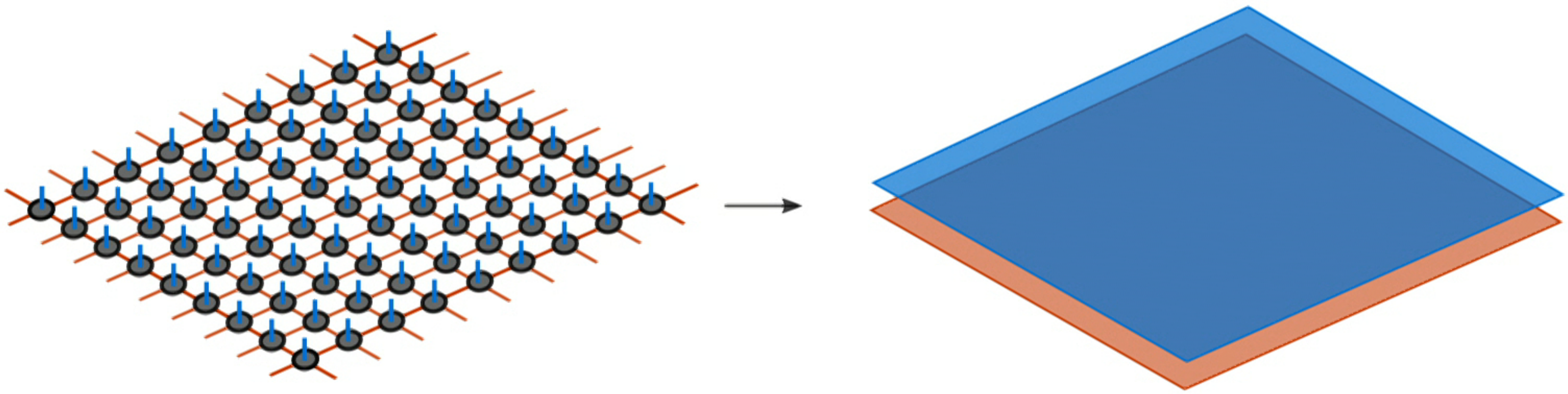

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MPQ

Problem

Many-body states are complicated.

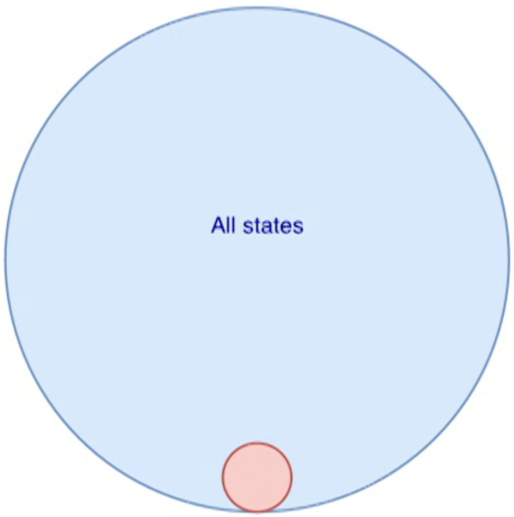
$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_n} c_{i_1,i_2,\cdots,i_n} |i_1,\cdots,i_n\rangle$$

2^n parameters $c_{i_1,i_2,\cdots,i_n}$.

Typical many-body Hamiltonians are simple.

$$H = \sum_{k=1}^{n} h_k$$

$\sim$ const $\times\, n$ parameters.



Variational optimization

To find the ground state:

$$|\text{ground}\rangle = \min_{|\psi\rangle\in\mathscr{S}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

Can we find a subspace $\mathscr{S}$ s. t.:

- $|\mathscr{S}| \propto n^k \ll e^n$
- $\mathscr{S}$ approximates well interesting states
- *bonus* $\langle\psi|\mathcal{O}(x)|\psi\rangle$ is computable

An idea popular in many fields

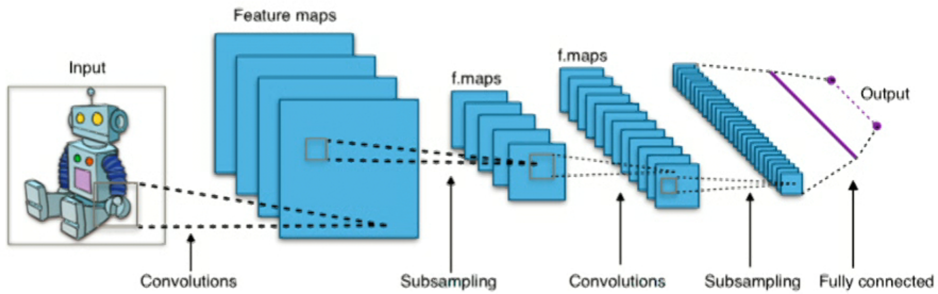
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- **Moore-Read wavefunctions** in the study of the quantum Hall effect

$$\psi(x_1, x_2, \cdots, x_n) = \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n)\right\rangle_{\mathrm{CFT}}$$

- Fully connected and convolutional **neural networks** used in machine learning



Matrix product states

$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

Matrix Product States (MPS)

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$$

- A_i are $D \times D$ complex matrices
- A is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are D-vectors.

Remark: actually equivalent with the density matrix renormalization group (DMRG)

◇ $n \times 2 \times D^2$ parameters instead of 2^n

◇ D is the **bond dimension** and encodes the size of the variational class

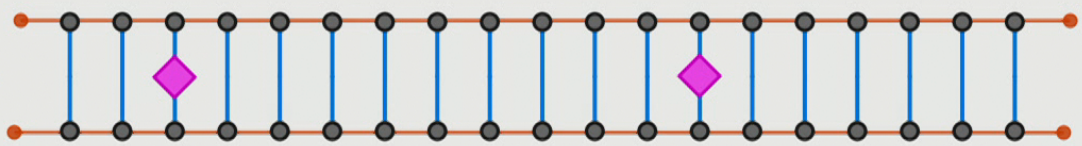
Graphical notation

$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L}| A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) |\mathrm{R}\rangle \; |i_1, \cdots, i_n\rangle$

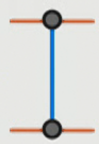
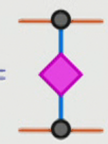
Notation: $[A_i]_{\mathbf{k},\mathbf{l}} =$  and k ——— $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$ 

Example: computation of correlations

$\langle A|\mathcal{O}(i_k)\mathcal{O}(i_\ell)|A\rangle =$ 

can be done by iteration 2 maps:

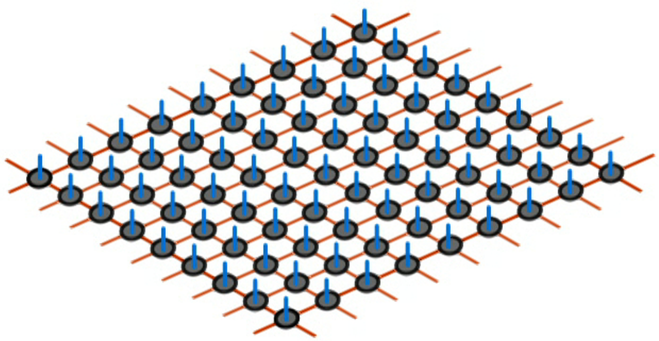
$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

The contraction for a $d = 1$ system, can be seen as an open-system dynamics in $d = 0$.

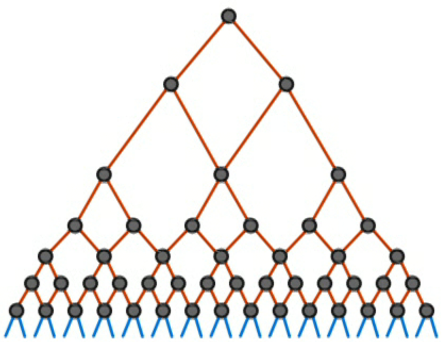
Generalizations: different tensor networks

Matrix Product States (MPS)

Projected Entangled Pair States (PEPS)



Multi-scale Entanglement Renormalization Ansatz (MERA)



Some facts

A list of theorems [very colloquially]:

- **Expressiveness** [trivial] Tensor Network States cover $\mathscr{H}$ when $D \propto 2^n$
- **Area law** The entanglement of a subregion of space scales as its area for a TNS
- **Efficiency** [gapped] Matrix Product States approximate well the ground states of gapped systems in 1 spatial dimension
- **Efficiency** [critical] Multi-scale Entanglement Renormalization Ansatz (MERA) approximate well the ground states of critical systems in 1 spatial dimension.
- **Symmetries** Physical symmetries can be implemented locally on the bond space
- **Inverse problem** TNS are the ground state of a local parent Hamiltonian

Successes and limits

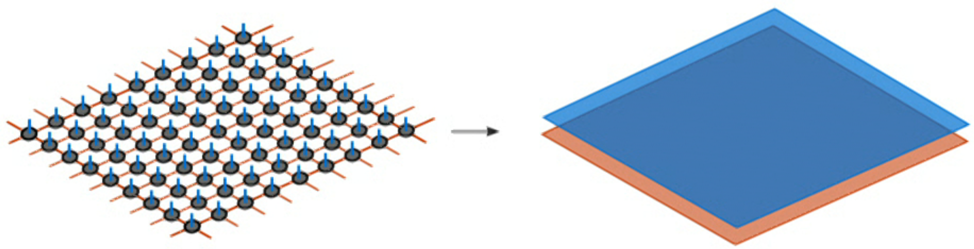
Successes

- ♡ Arbitrary precision for $1d$ quantum systems
- ♡ Classification of topological phases in $1d$ and $2d$
- ♡ Progress on non-Abelian lattice Gauge theories
- ♡ AdS/CFT toy models

Limits

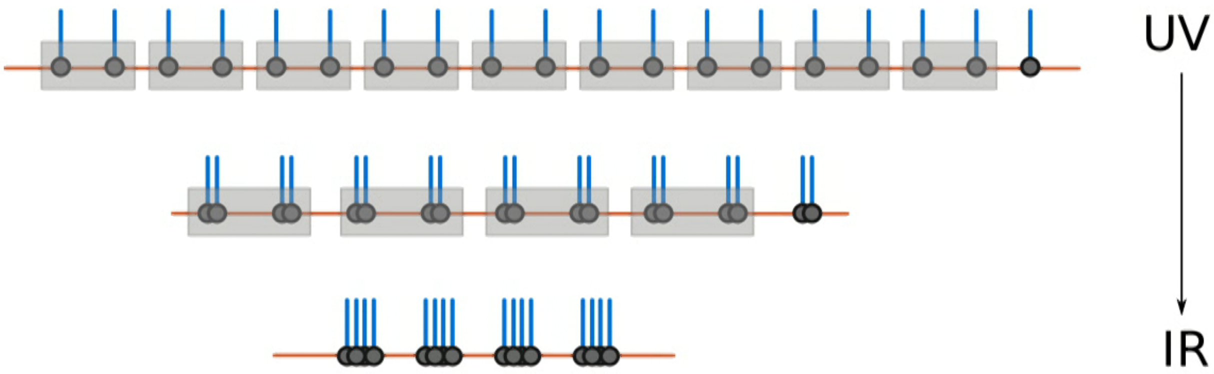
- ♠ Hard to contract in $d \geqslant 2$
- ♠ No continuum limit in $d \geqslant 2$
- ♠ Lack of analytic techniques

Can one apply tensor network techniques directly in the continuum, to QFT?



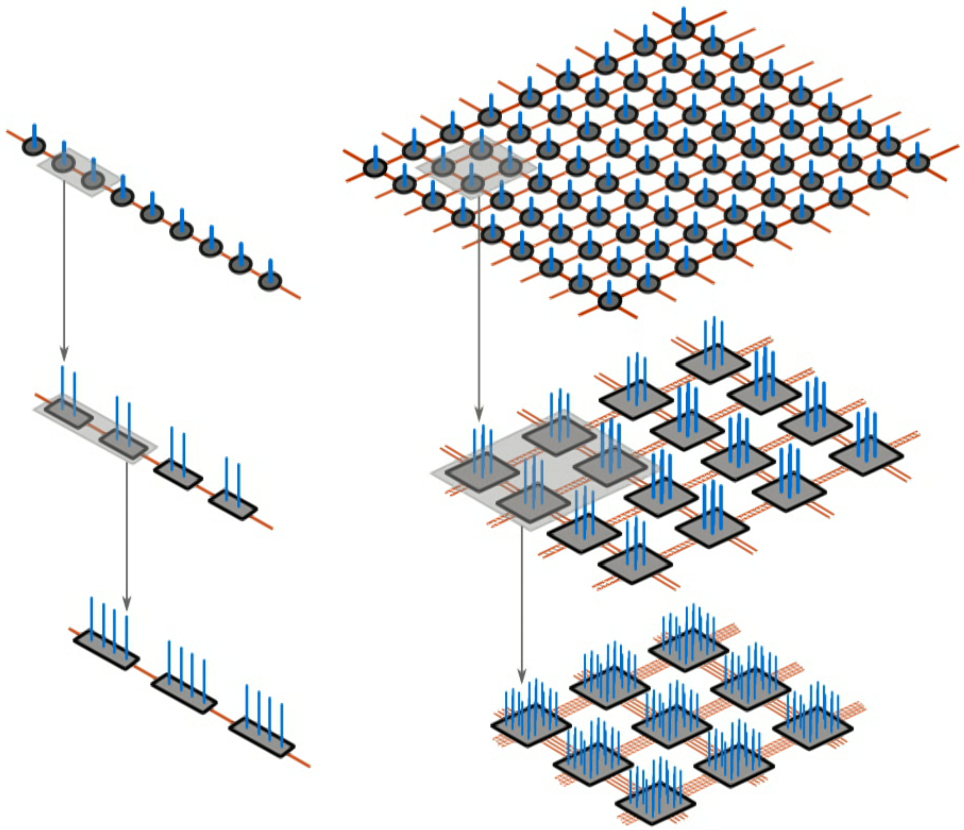
Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



- the bond dimension D stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.
  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$ quantum noises $\mathrm{d}\xi$, $\mathrm{d}\xi^\dagger$)
- A cMPS is a quantum field state parameterized by finite dimensional matrices:
  $|Q, R, \omega\rangle = \langle\omega_L|\mathcal{P}\exp\left\{\int_0^L dx\ Q(x)\otimes \mathbb{1} + R(x)\otimes\psi^\dagger(x)\right\}|\omega_R\rangle\ |0\rangle$

Continuous Tensor Networks: blocking



Upon blocking:

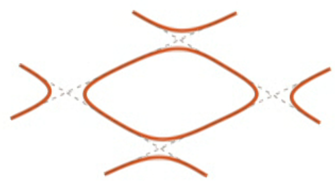
- The **physical** Hilbert space dimension d increases (idem cMPS $\Longrightarrow$ physical field)
- The **bond** dimension D increases too

Choice of trivial tensor

For **MPS**, not much choice:

For **TNS** in $d \geqslant 2$, many options:

1. Take a δ between all legs ~ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry
2. Take two identities $T^{(0)} =$ 
   $\Longrightarrow$ breakdown of Euclidean invariance
3. Take the sum of pairs of identities in both directions $T^{(0)} =$ 



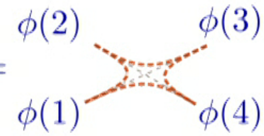
We will consider a softer modification of the first version:

$T^{(0)} \sim$ 

Ansatz

1 – Take a "Trivial" tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



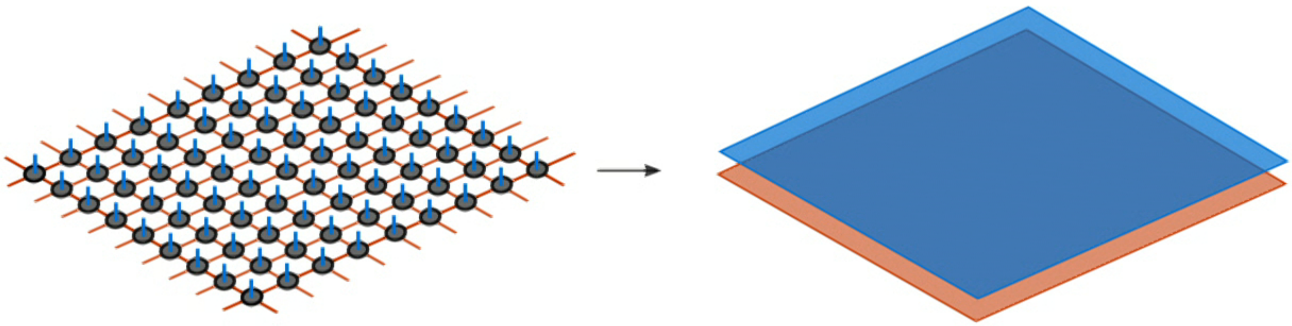
$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices ϕ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

2 – And add a "correction":

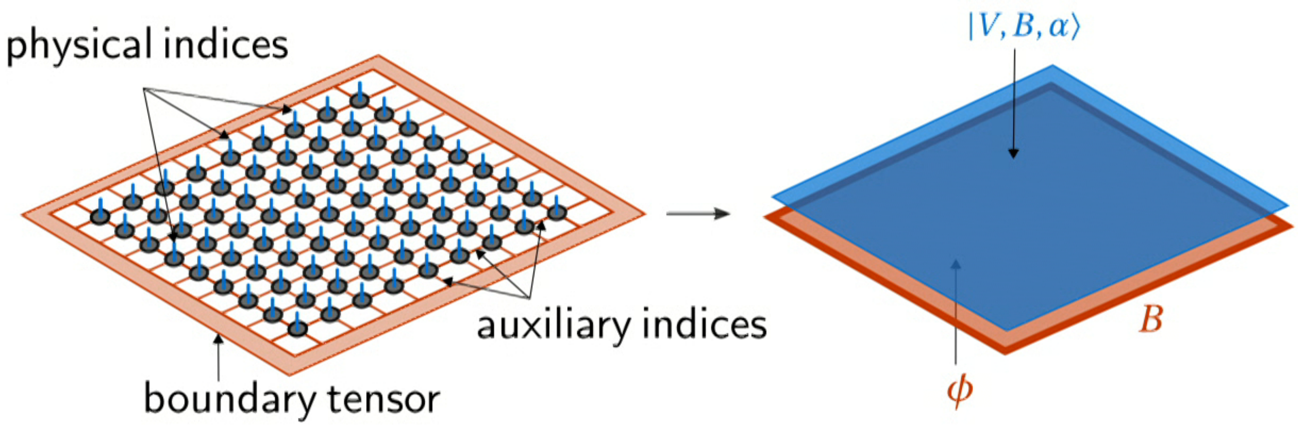
$$\exp\left\{ -\varepsilon^2 V\left[\phi(1), \cdots, \phi(4)\right] + \varepsilon^2 \alpha\left[\phi(1), \cdots, \phi(4)\right] \psi^\dagger(x) \right\}$$

Functional integral definition



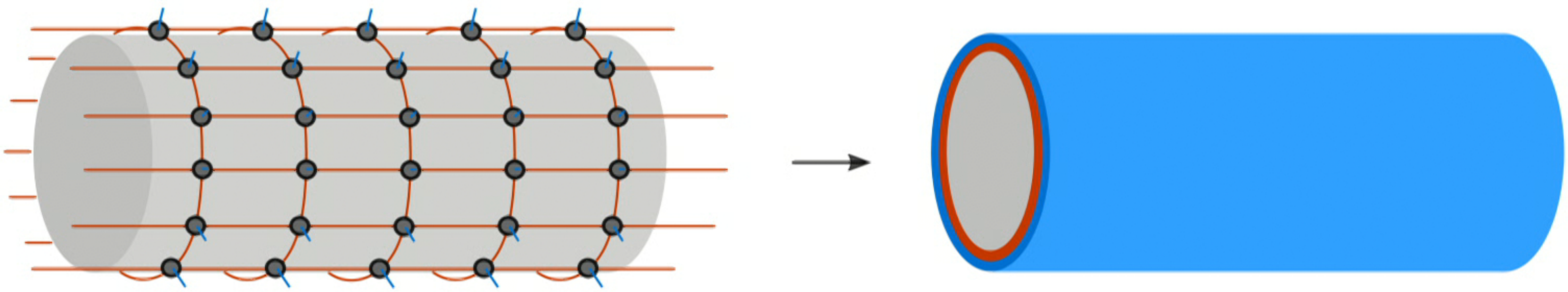
$$|V,\alpha\rangle = \int \mathcal{D}\phi\ \exp\left\{-\int_\Omega \mathrm{d}^d x\ \frac{1}{2}\sum_{k=1}^{D}[\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)]\,\psi^\dagger(x)\right\}|0\rangle$$

Functional integral definition



$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{ -\int_\Omega \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

Operator definition



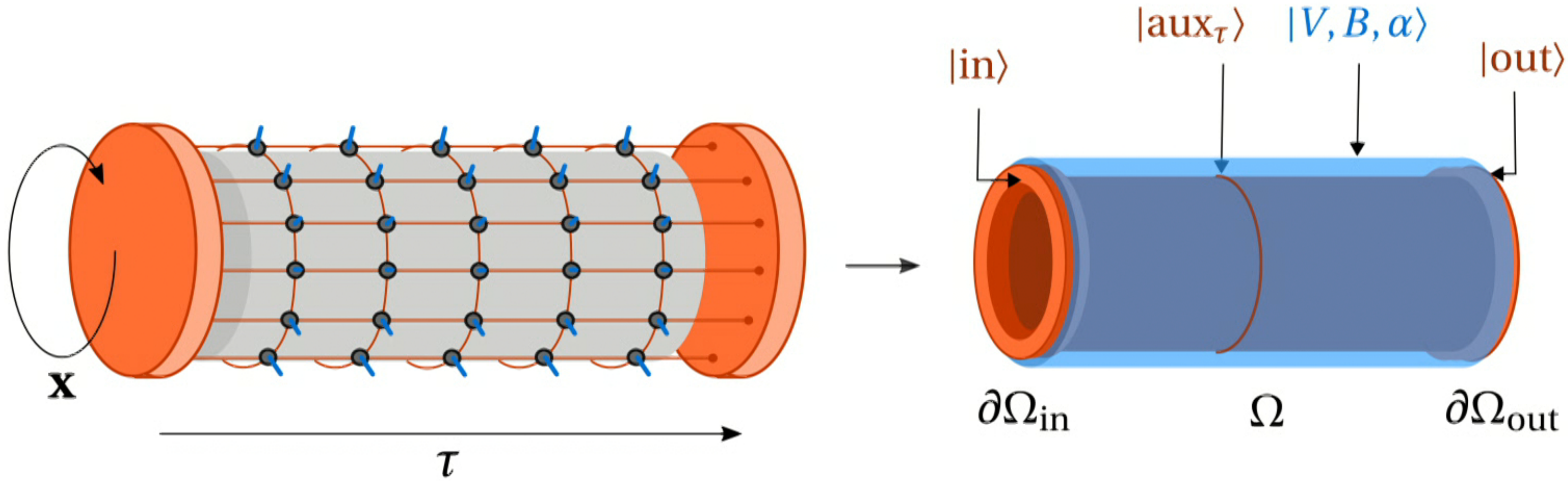
$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T d\tau \int_S dx\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are k independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x),\hat{\phi}_l(y)] = 0$, $[\hat{\pi}(x)_k,\hat{\pi}_l(y)] = 0$, and $[\hat{\phi}_k(x),\hat{\pi}_l(y)] = i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

Operator definition



$|V,B,\alpha\rangle =$

$$\mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T d\tau\int_S dx\,\frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2}+\frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2}+V[\hat{\phi}(x)]-\alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

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Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) |0\rangle$$

where ϕ_n is a symmetric n-particle wave-function

Functional integral representation

$$\varphi_n = \int d\mu(\phi) \mathcal{A}_V(\phi)\, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)]$$

with:

- $d\mu(\phi) =$
  $\mathcal{D}\phi \exp\left[-\frac{1}{2}\int_\Omega d^d x\, [\nabla\phi_k(x)]^2\right]$
- $\mathcal{A}_V(\phi) =$
  $B(\phi|_{\partial\Omega}) \exp\left\{-\int_\Omega d^d x\; V[\phi(x)]\right\}$

Operator representation

$\varphi_n =$

$$\text{tr}\left[\hat{B}\, \hat{G}_{T,\tau_n}\, \hat{\alpha}(x_n)\, \hat{G}_{\tau_n,\tau_{n-1}}\, \hat{\alpha}(x_{n-1}) \cdots \hat{\alpha}(x_1)\, \hat{G}_{\tau_1,0}\right]$$

with:

- $\hat{G}_{u,v} = \mathcal{T} \exp[-\int_v^u d\tau \int_S dx \mathcal{H}(x)]$

♡ Extension of Moore-Read

Expressivity and stability

How big are cTNS?

Stability

The sum of two cTNS of bond field dimension D_1 and D_2 is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

Expressiveness

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 0$
- Stability allows to get all sums of field coherent states

Note: expressiveness can also be obtained with $D = 1$ but it is less natural. Flexibility in D makes the expressivity higher for restricted classes of V and α.

Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V,\alpha|V,\alpha\rangle} \langle V,\alpha| \exp\left(\int \mathrm{d}x\, j'(x)\psi^\dagger(x)\right) \exp\left(\int \mathrm{d}x\, j(x)\psi(x)\right) |V,\alpha\rangle$$

Functional integral representation

- Use formula for overlap of field coherent states

$$\langle\beta|\alpha\rangle = \exp\left(\int \mathrm{d}x\, \beta^*(x)\,\alpha(x)\right)$$

- Compute with Gaussian integration + Feynman diagrams or Monte Carlo

Operator representation

Similar to cMPS

- Transfer matrix

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \mathrm{tr}\left(\Phi_{\mathcal{O}} \cdot e^{-(y-x)T}\Phi_{\mathcal{O}} \cdot \rho_{\mathrm{stat}}\right)$$

with $T = Q \otimes \mathbb{1} + \mathbb{1} \otimes \bar{Q} + R \otimes \bar{R}$ with

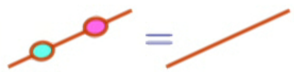
$$Q = -\int \frac{\hat{\pi}_k(x)^2 + [\nabla\hat{\phi}_k(x)]^2}{2} + V(\hat{\phi}(x))$$

and $R \otimes \bar{R} = \int V(\hat{\phi}(x)) \otimes V(\hat{\phi}(x))^\dagger$

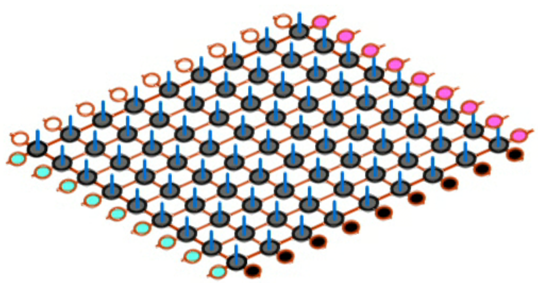
Redundancies

Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and 

up to **boundary** terms:



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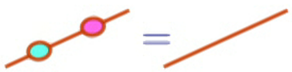
when  and 

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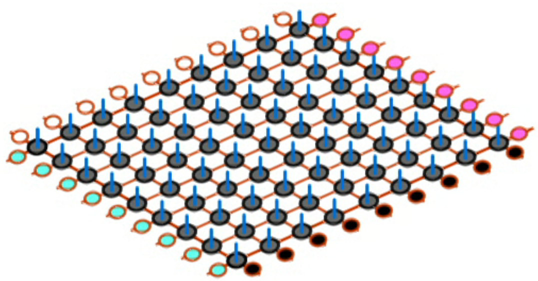
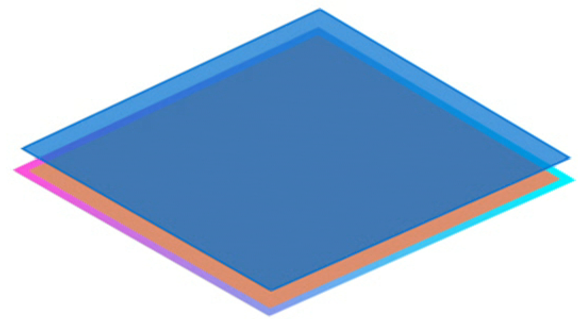


Continuum redundancy

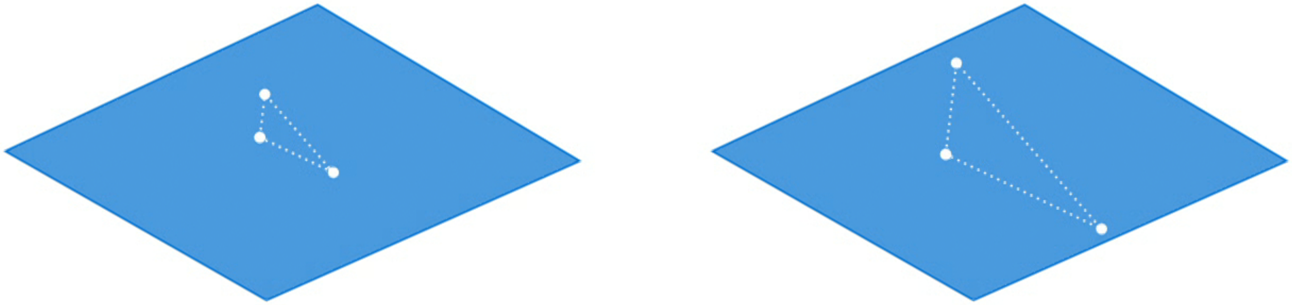
$$V(\phi) \to V(\phi) + \nabla \cdot \mathscr{F}[x, \phi(x)]$$

Just Stokes' theorem. If Ω has a boundary $\partial\Omega$:

$$\mathcal{D}[\phi] \to \mathcal{D}[\phi] \exp\left\{ \oint_{\partial\Omega} \mathrm{d}^{d-1}x \, \mathscr{F}[x, \phi(x)] \cdot \mathbf{n}(x) \right\}$$



Renormalization



$$C(x_1, \cdots, x_n) = \langle T(1) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(1) \rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1, \cdots, \lambda x_n) \propto \langle T(\lambda) | \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) | T(\lambda) \rangle.$$

Doable exactly:

$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

- $d = 2$, All powers of the field in V and α yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in V yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in V. For α, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other p are irrelevant.

Getting back cMPS

One can get back cMPS with finite bond dimension by:

1. **Compactification** Take $d-1$ dimensions out of d to be very small

$$|V,B,\alpha\rangle \simeq \mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \sum_{k=1}^{D} \frac{\hat{P}_k^2}{2} + V[\hat{X}] - \alpha[\hat{X}]\,\psi^\dagger(\tau)\right)\right]|0\rangle$$

$\Longrightarrow$ Hilbert space of a quantum particle in D space dimensions.

2. **Quantization** Take V with D deep minima to force the auxiliary field to take only D possibilities

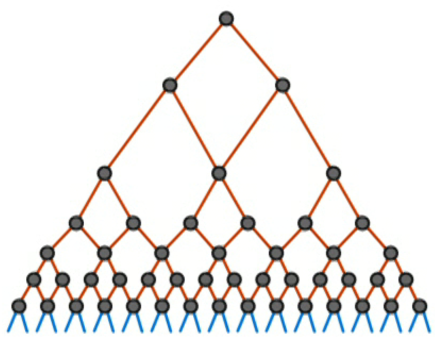
Generalization

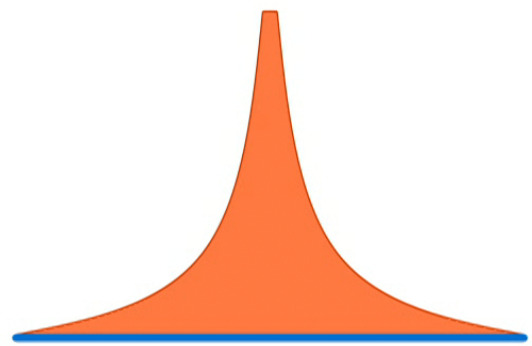
For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi\, B(\phi|_{\partial\mathcal{M}}) \exp\left\{-\int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g}\left(\frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\,\psi^\dagger\right)\right\}|0\rangle$$

i.e. add curvature and possible anisotropies in V and α

Example: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix g:





$\rightarrow$ **cMERA** in $d-1$ dimensions

Future

Limitations and work for the future

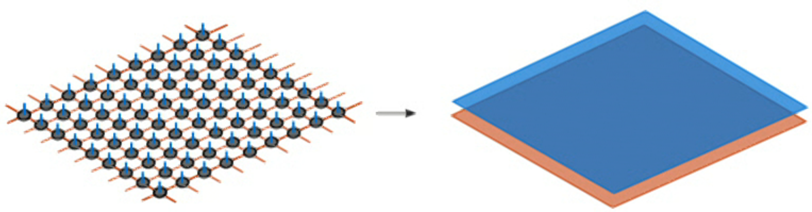
- Quite formal out of the Gaussian regime (back to perturbative)
- Limited to bosonic field theories (so far)
- Parent Hamiltonian?
- Gauge invariant states
- Topology?

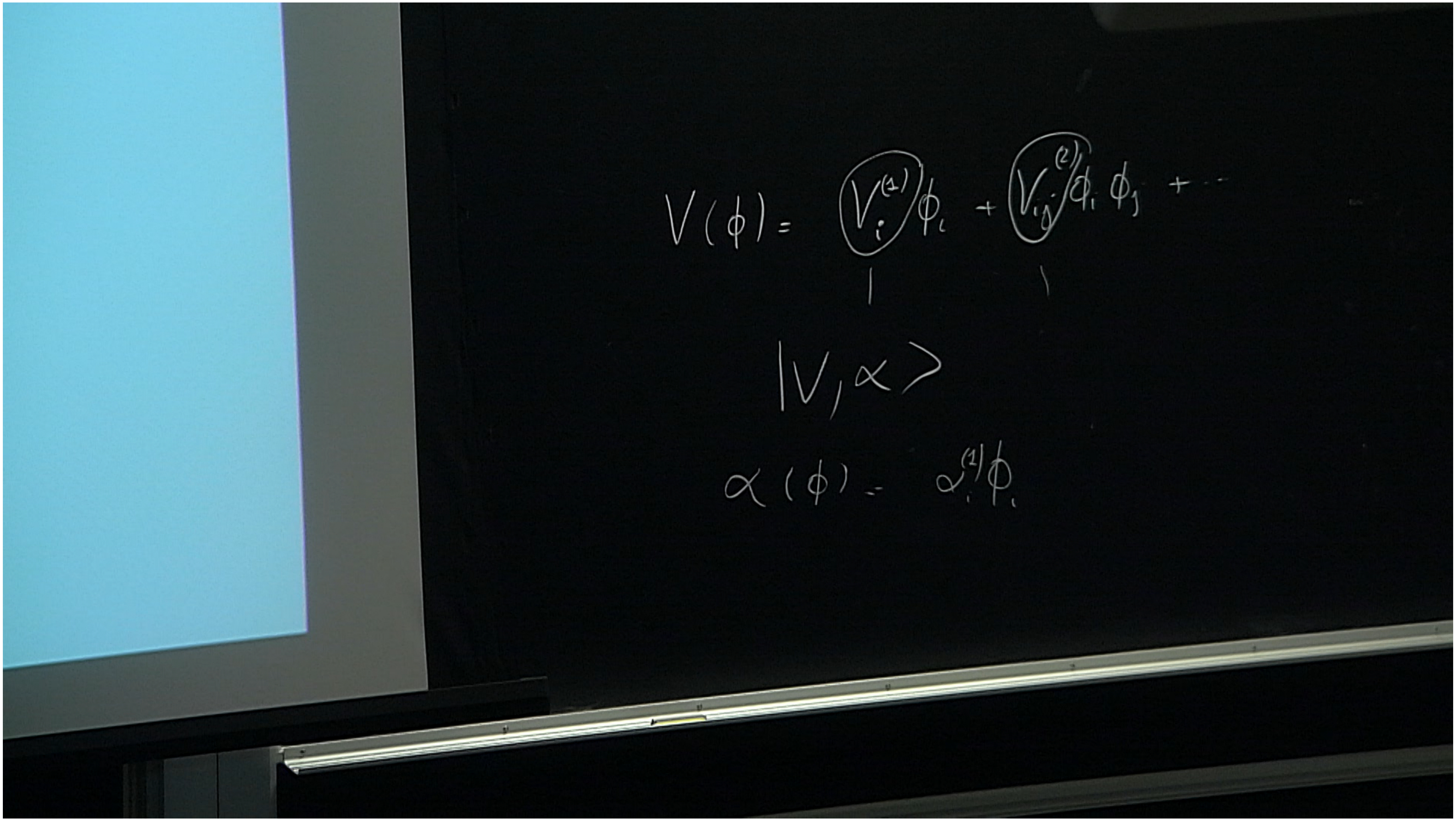
Summary

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher d extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. Have functional and operator representations
4. Have a geometrical equivalent of the discrete gauge redundancies
5. Have an exact and explicit "renormalization" flow




$V(\phi) = V_i^{(1)} \phi_i + V_{ij}^{(2)} \phi_i \phi_j + \cdots$
$|V, \alpha\rangle$
$\alpha(\phi) = \alpha_i^{(1)} \phi_i$