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Abstract: <p>Abstract TBD.</p>

# Continuous Tensor Network States of Quantum Fields

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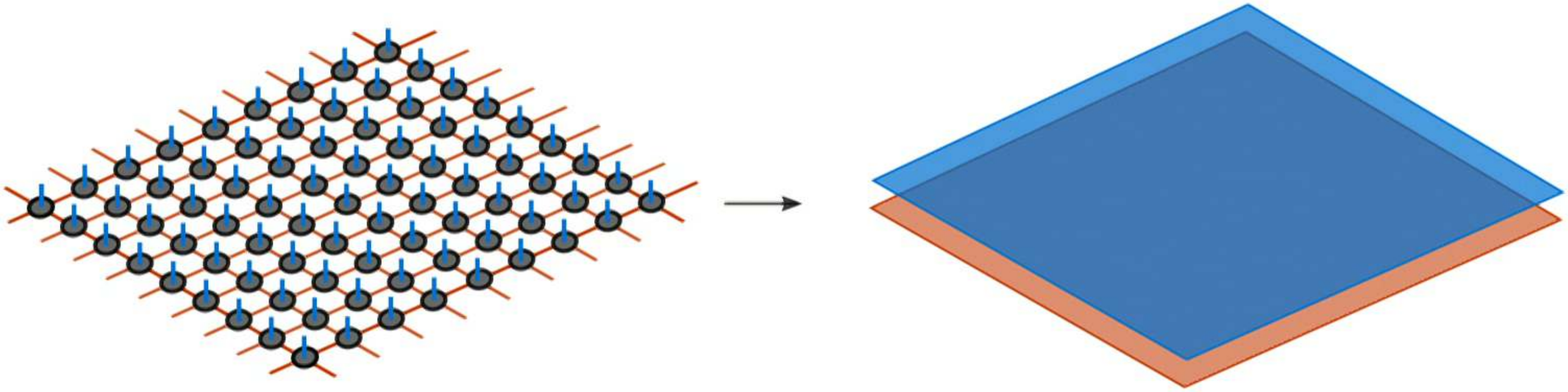

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Stiftung/Foundation

MPQ

# Problem

Many-body states are complicated.

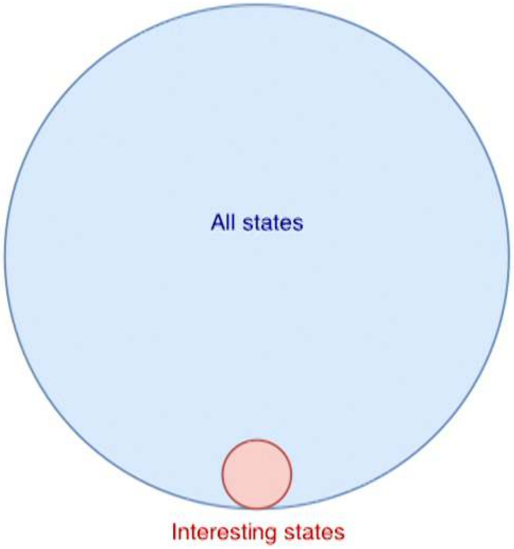
$$|\psi\rangle = \sum_{i_1, i_2, \cdots, i_n} c_{i_1, i_2, \cdots, i_n} |i_1, \cdots, i_n\rangle$$

$2^n$ parameters $c_{i_1, i_2, \cdots, i_n}$.

Typical many-body Hamiltonians are simple.

$$H = \sum_{k=1}^{n} h_k$$

$\sim$ const $\times$ $n$ parameters.



**Variational optimization**

To find the ground state:

$$|\text{ground}\rangle = \min_{|\psi\rangle \in \mathscr{S}} \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

Can we find a subspace $\mathscr{S}$ s. t.:

- $|\mathscr{S}| \propto n^k \ll e^n$
- $\mathscr{S}$ approximates well interesting states
- *bonus* $\langle\psi|\mathcal{O}(x)|\psi\rangle$ is computable

# An idea popular in many fields

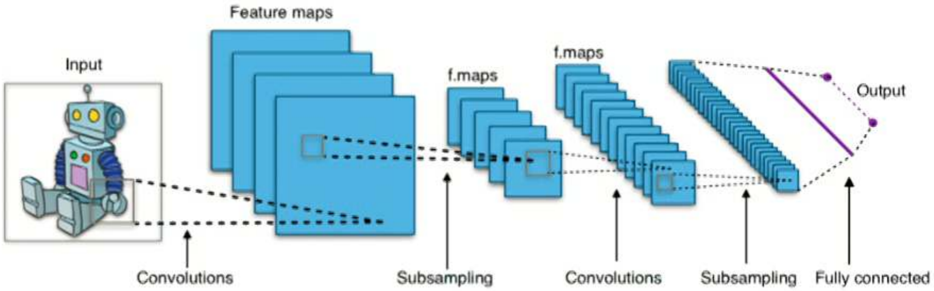
- **Mean field** approximation (of which TNS are an extension)

$$\psi(x_1, x_2, \cdots, x_n) = \psi_1(x_1)\,\psi_2(x_2)\cdots\psi_n(x_n)$$

- Special variational wave functions in **Quantum chemistry** (whole industry of ansatz)
- **Moore-Read wavefunctions** in the study of the quantum Hall effect

$$\psi(x_1, x_2, \cdots, x_n) = \left\langle \hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\hat{\phi}(x_n) \right\rangle_{\text{CFT}}$$

- Fully connected and convolutional **neural networks** used in machine learning



# Matrix product states

$$|\psi\rangle = \sum_{i_1,i_2,\cdots,i_n} c_{i_1,i_2,\cdots,i_n} \, |i_1,\cdots,i_n\rangle$$

**Matrix Product States (MPS)**

$$|A,L,R\rangle = \sum_{i_1,i_2,\cdots,i_n} \langle \mathrm{L}|A_{i_1}(1)A_{i_2}(2)\cdots A_{i_n}(n)|\mathrm{R}\rangle \, |i_1,\cdots,i_n\rangle$$

- $A_i$ are $D \times D$ complex matrices
- $A$ is a $2 \times D \times D$ tensor $[A_i]_{k,l}$
- $|L\rangle$ and $|R\rangle$ are $D$-vectors.

**Remark**: actually equivalent with the density matrix renormalization group (DMRG)

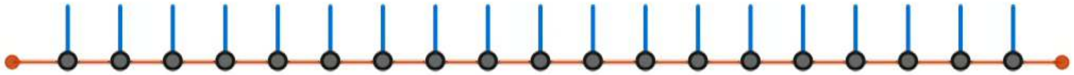
- ♢ $n \times 2 \times D^2$ parameters instead of $2^n$
- ♢ $D$ is the **bond dimension** and encodes the size of the variational class

# Graphical notation

$$|A, L, R\rangle = \sum_{i_1, i_2, \cdots, i_n} \langle \mathrm{L} | A_{i_1}(1) A_{i_2}(2) \cdots A_{i_n}(n) | \mathrm{R} \rangle \; |i_1, \cdots, i_n\rangle$$

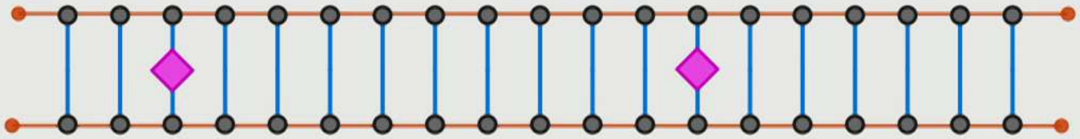
**Notation:** $[A_i]_{\mathbf{k},\mathbf{l}} =$  and $k$  $l = \sum \delta_{k,l}$ gives:

$|A, L, R\rangle =$

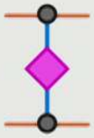


**Example: computation of correlations**

$\langle A | \mathcal{O}(i_k) \mathcal{O}(i_\ell) | A \rangle =$



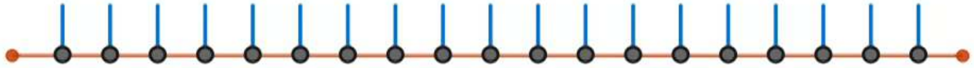
can be done by iteration 2 maps:

$\Phi =$  and $\Phi_{\mathcal{O}} =$ 

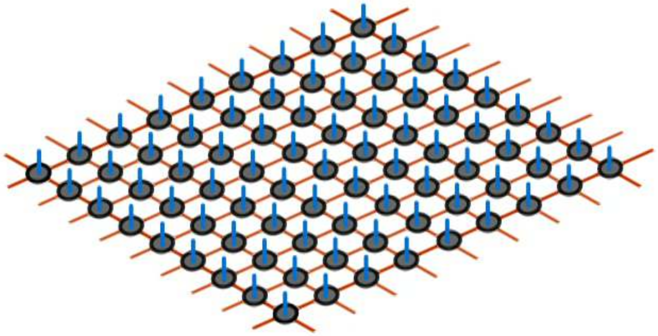
The contraction for a $d = 1$ system, can be seen as an open-system dynamics in $d = 0$.

# Generalizations: different tensor networks

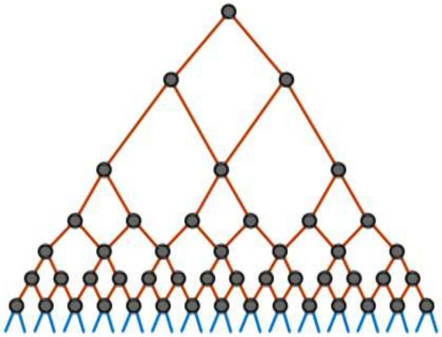
**Matrix Product States** (MPS)



**Projected Entangled Pair States** (PEPS)



**Multi-scale Entanglement Renormalization Ansatz** (MERA)



# Some facts

A list of theorems [very colloquially]:

- **Expressiveness** [trivial] Tensor Network States cover $\mathscr{H}$ when $D \propto 2^n$
- **Area law** The entanglement of a subregion of space scales as its area for a TNS
- **Efficiency** [gapped] Matrix Product States approximate well the ground states of gapped systems in $1$ spatial dimension
- **Efficiency** [critical] Multi-scale Entanglement Renormalization Ansatz (MERA) approximate well the ground states of critical systems in $1$ spatial dimension.
- **Symmetries** Physical symmetries can be implemented locally on the bond space
- **Inverse problem** TNS are the ground state of a local parent Hamiltonian

# Successes and limits

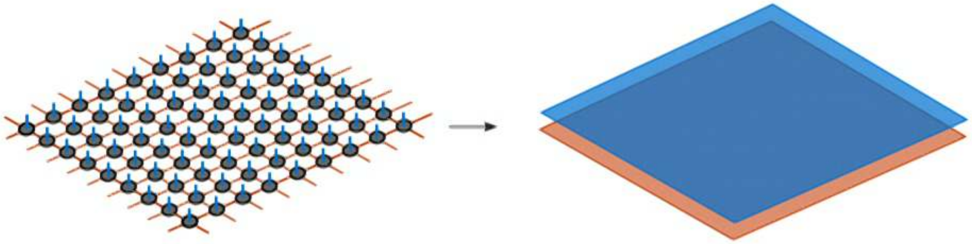
**Successes**

- ♡ Arbitrary precision for $1d$ quantum systems
- ♡ Classification of topological phases in $1d$ and $2d$
- ♡ Progress on non-Abelian lattice Gauge theories
- ♡ AdS/CFT toy models

**Limits**

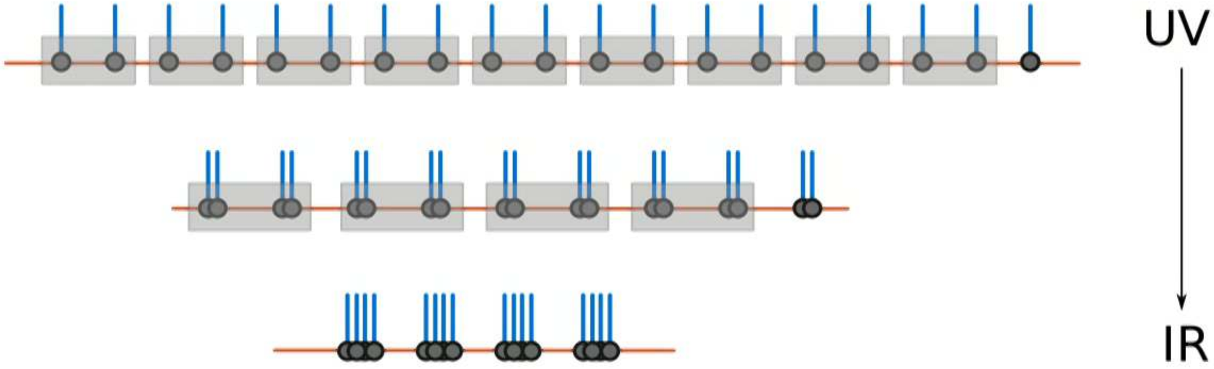
- ♠ Hard to contract in $d \geqslant 2$
- ♠ No continuum limit in $d \geqslant 2$
- ♠ Lack of analytic techniques

***Can one apply tensor network techniques directly in the continuum, to QFT?***



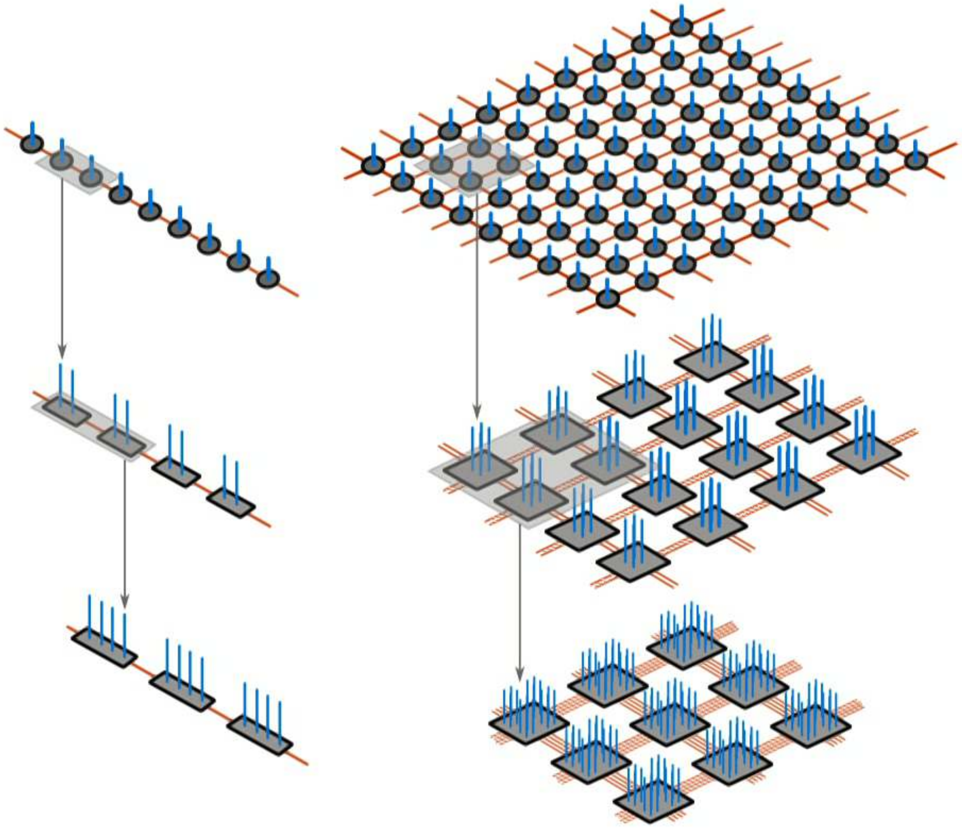
# Continuous Matrix Product States (cMPS)

Taking the continuum limit of a MPS



- the bond dimension $D$ stays fixed
- the local physical dimension explodes $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \longrightarrow \mathscr{F}(L^2([x, x+dx]))$.
  $\Longrightarrow$ **Spins** become **fields** – ($\simeq$ central limit theorem $\simeq$ quantum noises $d\xi$, $d\xi^\dagger$)
- A cMPS is a quantum field state parameterized by finite dimensional matrices:
  $|Q, R, \omega\rangle = \langle \omega_L | \mathcal{P} \exp \left\{ \int_0^L dx \; Q(x) \otimes \mathbb{1} + R(x) \otimes \psi^\dagger(x) \right\} |\omega_R\rangle \; |0\rangle$

# Continuous Tensor Networks: blocking



Upon blocking:

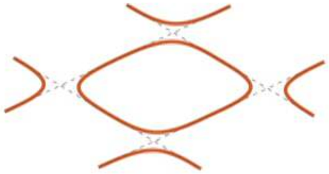
- ♣ The **physical** Hilbert space dimension $d$ increases (idem cMPS $\Longrightarrow$ physical field)
- ♣ The **bond** dimension $D$ increases too

# Choice of trivial tensor

For **MPS**, not much choice:

For **TNS** in $d \geqslant 2$, many options:

1. Take a $\delta$ between all legs ~ GHZ state $T^{(0)} =$ 
   $\Longrightarrow$ trivial geometry
2. Take two identities $T^{(0)} =$ 
   $\Longrightarrow$ breakdown of Euclidean invariance
3. Take the sum of pairs of identities in both directions $T^{(0)} =$ 



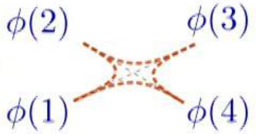
We will consider a softer modification of the first version:

$T^{(0)} \sim$ 

# Ansatz

**1** – Take a “Trivial” tensor:

$$T^{(0)}_{\phi(1),\phi(2),\phi(3),\phi(4)} =$$



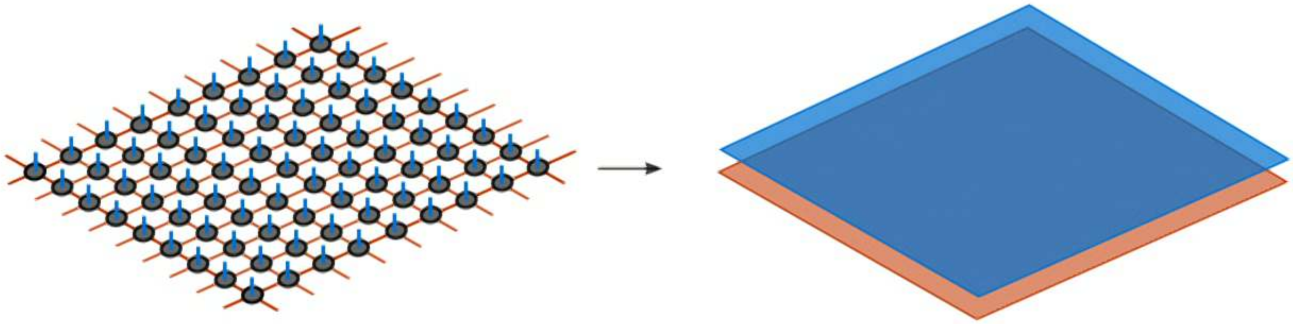
$$\sim \exp\left\{ \frac{-1}{2} \sum_{k=1}^{D} [\phi_k(1) - \phi_k(2)]^2 + [\phi_k(2) - \phi_k(3)]^2 + [\phi_k(3) - \phi_k(4)]^2 + [\phi_k(4) - \phi_k(1)]^2 \right\}$$

The indices $\phi$ are in $\mathbb{R}^D$ (and **not** $1, \cdots, D$)

**2** – And add a “correction”:

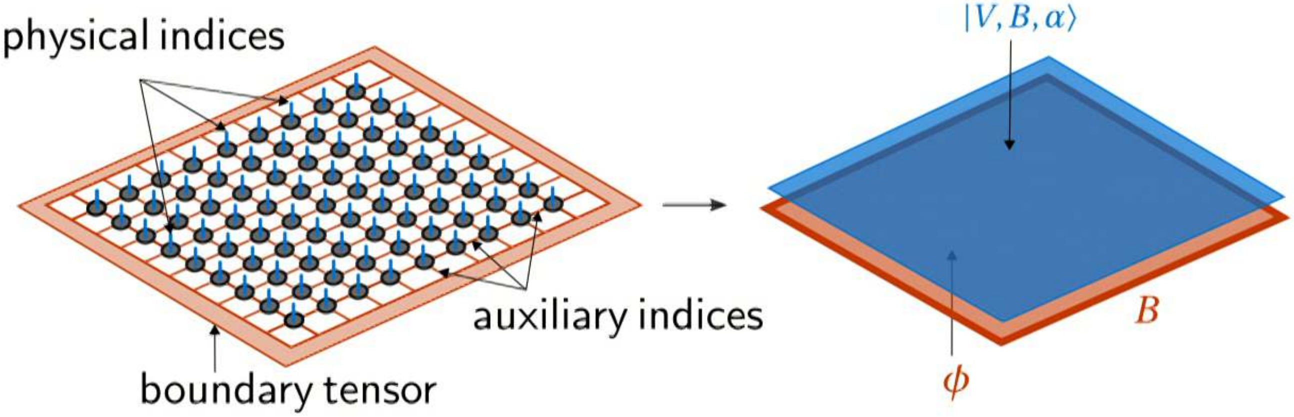
$$\exp\left\{ -\varepsilon^2 V\left[\phi(1), \cdots, \phi(4)\right] + \varepsilon^2 \alpha\left[\phi(1), \cdots, \phi(4)\right] \psi^\dagger(x) \right\}$$

# Functional integral definition



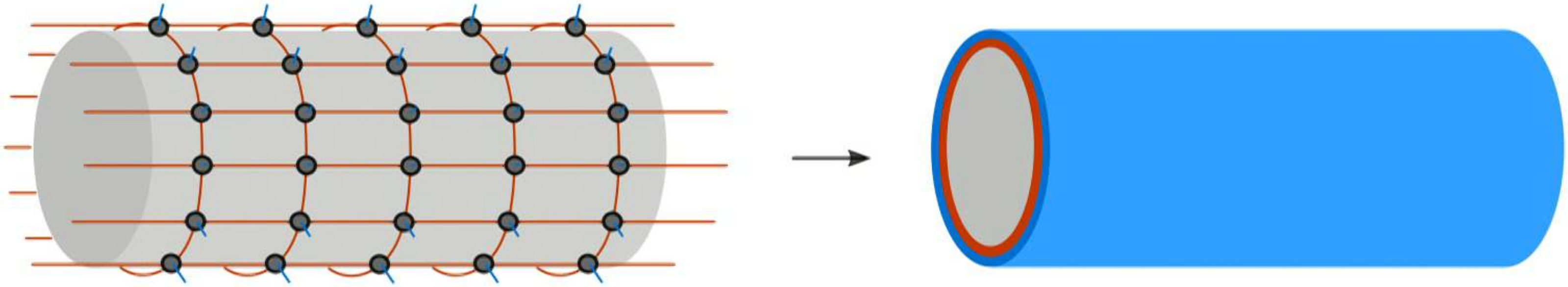
$$|V,\alpha\rangle = \int \mathcal{D}\phi \, \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x) \right\} |0\rangle$$

# Functional integral definition



$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{-\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla \phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^\dagger(x)\right\} |0\rangle$$

# Operator definition



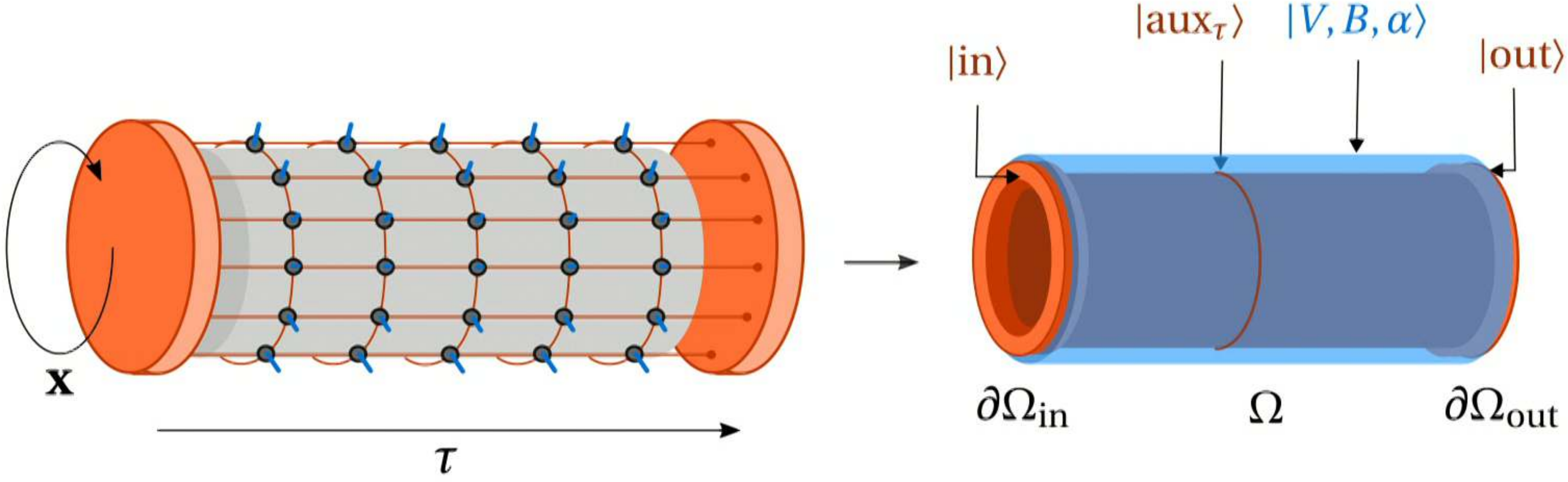
$$|V,\alpha\rangle =$$

$$\mathrm{tr}\left[\mathcal{T}\exp\left(-\int_0^T d\tau\int_S dx\, \frac{\hat{\pi}_k(x)\hat{\pi}_k(x)}{2} + \frac{\nabla\hat{\phi}_k(x)\nabla\hat{\phi}_k(x)}{2} + V[\hat{\phi}(x)] - \alpha[\hat{\phi}(x)]\,\psi^\dagger(\tau,x)\right)\right]|0\rangle$$

where:

- $\hat{\phi}_k(x)$ and $\hat{\pi}_k(x)$ are $k$ independent canonically conjugated pairs of (auxiliary) field *operators*: $[\hat{\phi}_k(x),\hat{\phi}_l(y)]=0$, $[\hat{\pi}(x)_k,\hat{\pi}_l(y)]=0$, and $[\hat{\phi}_k(x),\hat{\pi}_l(y)]=i\delta_{k,l}\,\delta(x-y)$ acting on a space of $d-1$ dimensions.

# Operator definition



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# Wave-function definition

A generic state $|\Psi\rangle$ in Fock space can be written:

$$|\Psi\rangle = \sum_{n=0}^{+\infty} \int_{\Omega^n} \frac{\varphi_n(x_1, \cdots, x_n)}{n!} \psi^\dagger(x_1) \cdots \psi^\dagger(x_n) \, |0\rangle$$

where $\phi_n$ is a symmetric $n$-particle wave-function

**Functional integral representation**

$$\varphi_n = \int d\mu(\phi) \mathcal{A}_V(\phi) \, \alpha[\phi(x_1)] \cdots \alpha[\phi(x_n)]$$

with:

- $d\mu(\phi) =$
  $\mathcal{D}\phi \exp\left[-\frac{1}{2}\int_\Omega d^d x \, [\nabla \phi_k(x)]^2\right]$
- $\mathcal{A}_V(\phi) =$
  $B(\phi|_{\partial\Omega}) \exp\left\{-\int_\Omega d^d x \; V[\phi(x)]\right\}$

**Operator representation**

$$\varphi_n =$$
$$\mathrm{tr}\left[\hat{B} \, \hat{G}_{T,\tau_n} \, \hat{\alpha}(x_n) \, \hat{G}_{\tau_n,\tau_{n-1}} \, \hat{\alpha}(x_{n-1}) \cdots \hat{\alpha}(x_1) \, \hat{G}_{\tau_1,0}\right]$$

with:

- $\hat{G}_{u,v} = \mathcal{T} \exp[-\int_v^u d\tau \int_S dx \mathcal{H}(x)]$

♡ Extension of Moore-Read

# Expressivity and stability

How big are cTNS?

**Stability**

The sum of two cTNS of bond field dimension $D_1$ and $D_2$ is a cTNS with bond field dimension $D \leqslant D_1 + D_2 + 1$:

$$|V_1, \alpha_1\rangle + |V_2, \alpha_2\rangle = |W, \beta\rangle$$

**Expressiveness**

All states in the Fock space can be approximated by cTNS:

- A field coherent state is a cTNS with $D = 0$
- Stability allows to get all sums of field coherent states

**Note**: expressiveness can also be obtained with $D = 1$ but it is less natural. Flexibility in $D$ makes the expressivity higher for restricted classes of $V$ and $\alpha$.

# Computations

Define generating functional for normal ordered correlation functions

$$\mathcal{Z}_{j',j} = \frac{1}{\langle V,\alpha|V,\alpha\rangle}\langle V,\alpha|\exp\left(\int \mathrm{d}x\, j'(x)\psi^\dagger(x)\right)\exp\left(\int \mathrm{d}x\, j(x)\psi(x)\right)|V,\alpha\rangle$$

**Functional integral representation**

- Use formula for overlap of field coherent states

$$\langle\beta|\alpha\rangle = \exp\left(\int \mathrm{d}x\, \beta^*(x)\,\alpha(x)\right)$$

- Compute with Gaussian integration + Feynman diagrams or Monte Carlo

**Operator representation**

Similar to cMPS

- Transfer matrix

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle = \mathrm{tr}\left(\Phi_{\mathcal{O}}\cdot e^{-(y-x)T}\Phi_{\mathcal{O}}\cdot\rho_{\mathrm{stat}}\right)$$

with $T = Q\otimes\mathbb{1} + \mathbb{1}\otimes\bar{Q} + R\otimes\bar{R}$ with

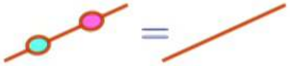
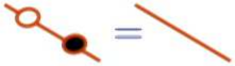
$$Q = -\int\frac{\hat{\pi}_k(x)^2 + [\nabla\hat{\phi}_k(x)]^2}{2} + V(\hat{\phi}(x))$$

and $R\otimes\bar{R} = \int V(\hat{\phi}(x))\otimes V(\hat{\phi}(x))^\dagger$

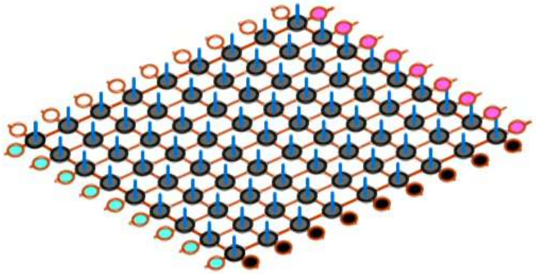
# Redundancies

## Discrete redundancy

Different elementary tensors are **equivalent**, they give the same state:

when  and 

up to **boundary** terms:



# Redundancies

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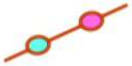
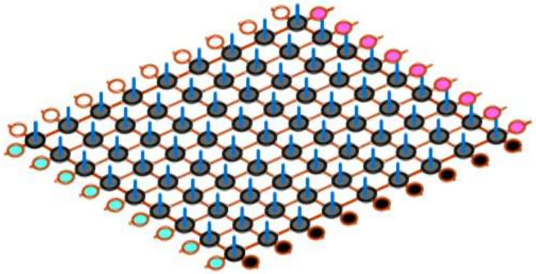
when  =  and  = 

up to **boundary** terms:

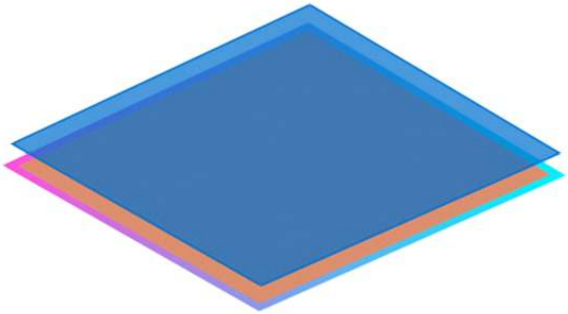


### Continuum redundancy

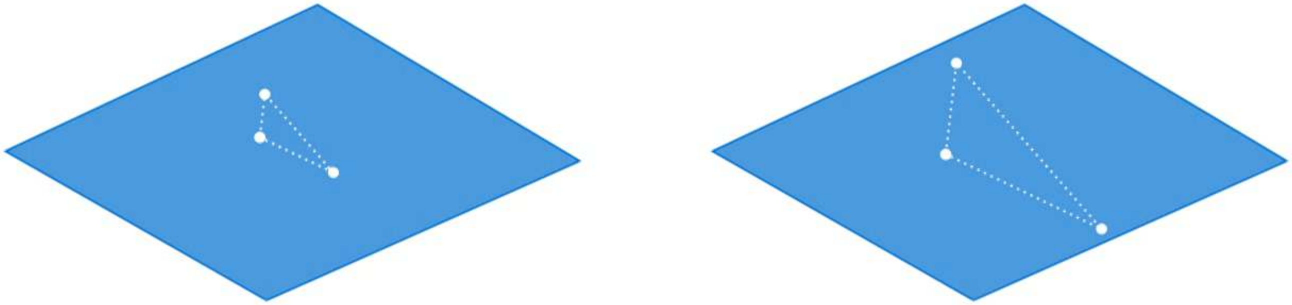
$$V(\phi) \to V(\phi) + \nabla \cdot \mathscr{F}[x, \phi(x)]$$

Just Stokes' theorem. If $\Omega$ has a boundary $\partial\Omega$:

$$\mathcal{D}[\phi] \to \mathcal{D}[\phi] \exp\left\{ \oint_{\partial\Omega} \mathrm{d}^{d-1}x \, \mathscr{F}[x, \phi(x)] \cdot \mathbf{n}(x) \right\}$$



# Renormalization



$$C(x_1,\cdots,x_n) = \langle T(1)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(1)\rangle,$$

the objective is to find a tensor $T(\lambda)$ of new parameters such that:

$$C(\lambda x_1,\cdots,\lambda x_n) \propto \langle T(\lambda)|\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|T(\lambda)\rangle.$$

Doable exactly:

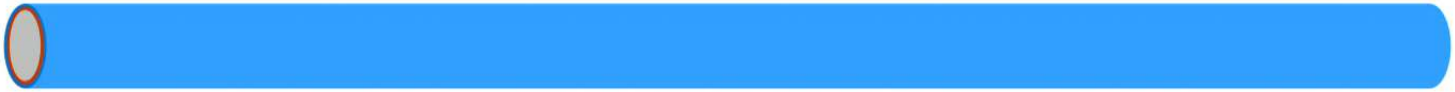
$$V \to \lambda^d V \circ \lambda^{\frac{2-d}{2}} \quad \text{and} \quad \alpha \to \lambda^{\frac{d}{2}} \alpha \circ \lambda^{\frac{2-d}{2}}$$

- $d = 2$, All powers of the field in $V$ and $\alpha$ yield relevant couplings
- $d = 3$, The powers $p = 1, 2, 3, 4, 5$ of the field in $V$ yield relevant $\Delta > 0$ couplings. $p = 6$ is marginal in $V$. For $\alpha$, $p = 1, 2$ are relevant and $p = 3$ is marginal. All other $p$ are irrelevant.

# Getting back cMPS

One can get back cMPS with finite bond dimension by:

1. **Compactification** Take $d-1$ dimensions out of $d$ to be very small



$$|V, B, \alpha\rangle \simeq \mathrm{tr}\left[\hat{B}\,\mathcal{T}\exp\left(-\int_0^T \mathrm{d}\tau \sum_{k=1}^{D} \frac{\hat{P}_k^2}{2} + V[\hat{X}] - \alpha[\hat{X}]\,\psi^\dagger(\tau)\right)\right]|0\rangle$$

$\Longrightarrow$ Hilbert space of a quantum particle in $D$ space dimensions.

2. **Quantization** Take $V$ with $D$ deep minima to force the auxiliary field to take only $D$ possibilities

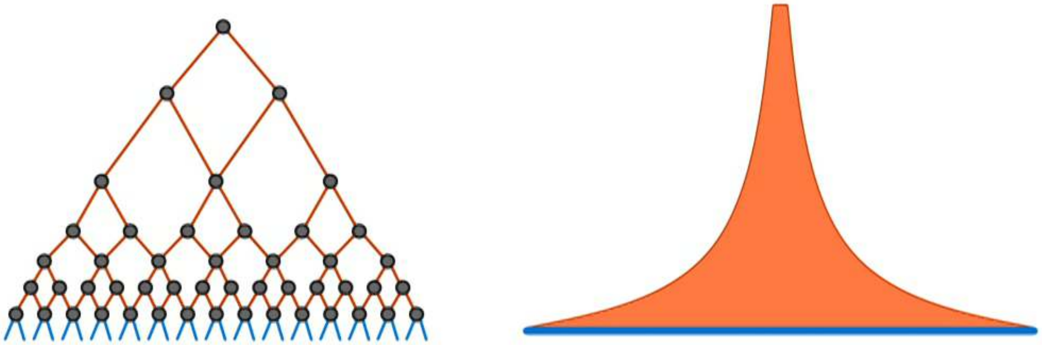
# Generalization

For a general Riemanian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, define:

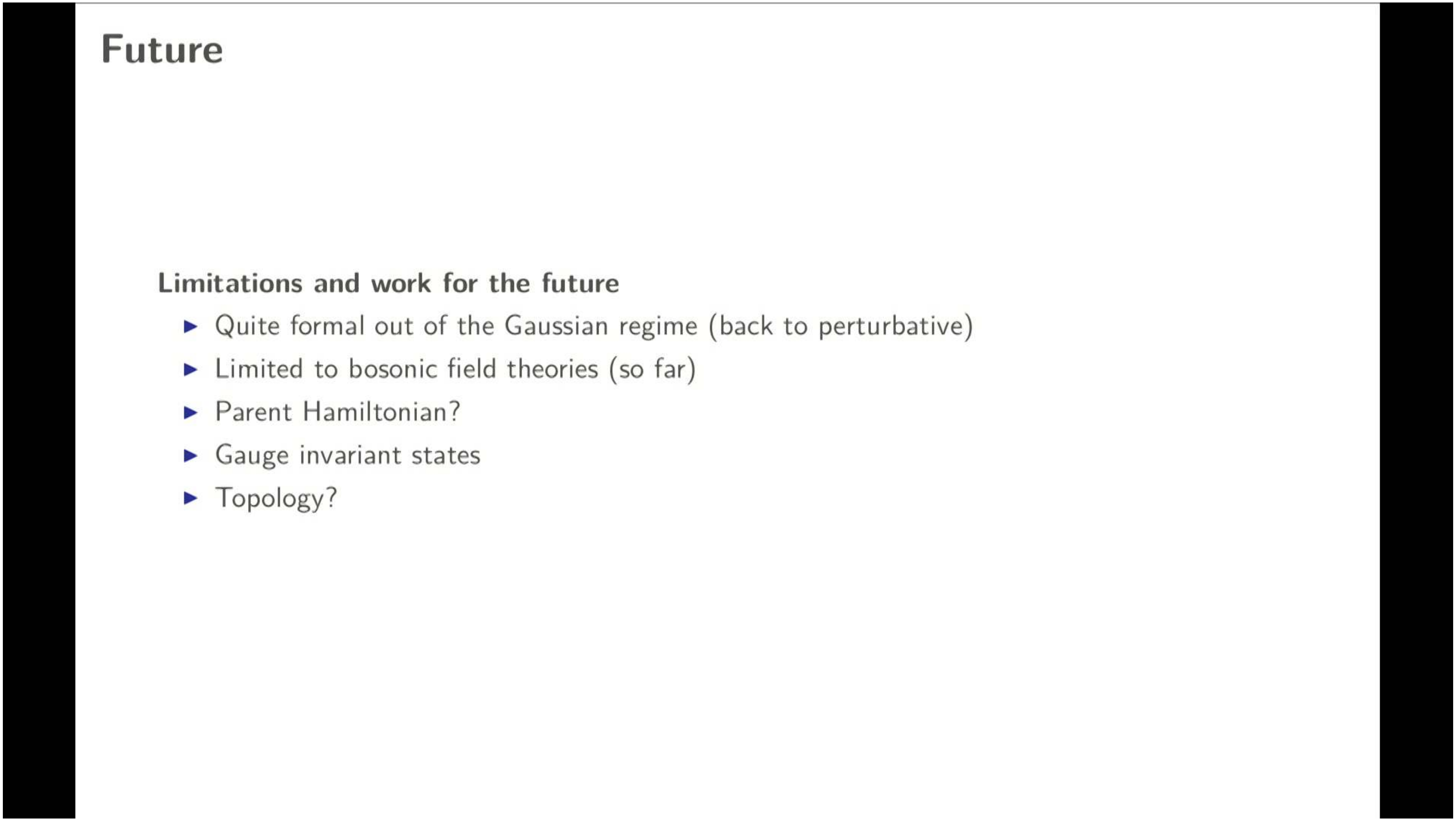
$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\mathcal{M}}) \exp\left\{ - \int_{\mathcal{M}} \mathrm{d}^d x \sqrt{g} \left( \frac{g^{\mu\nu}\partial_\mu\phi_k\partial_\nu\phi_k}{2} + V[\phi, \nabla\phi] - \alpha[\phi, \nabla\phi]\, \psi^\dagger \right) \right\} |0\rangle$$

i.e. add curvature and possible anisotropies in $V$ and $\alpha$

**Example**: $\alpha[x, \phi, \nabla\phi]$ localized on the boundary and hyberbolic metrix $g$:



→ **cMERA** in $d-1$ dimensions



# Future

**Limitations and work for the future**

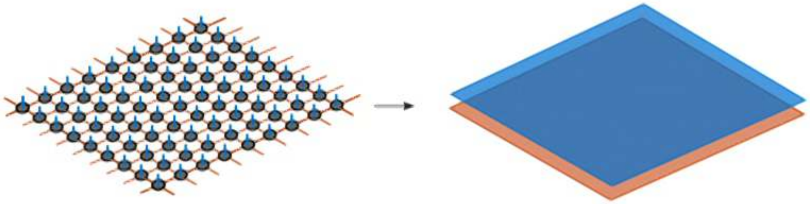
- Quite formal out of the Gaussian regime (back to perturbative)
- Limited to bosonic field theories (so far)
- Parent Hamiltonian?
- Gauge invariant states
- Topology?

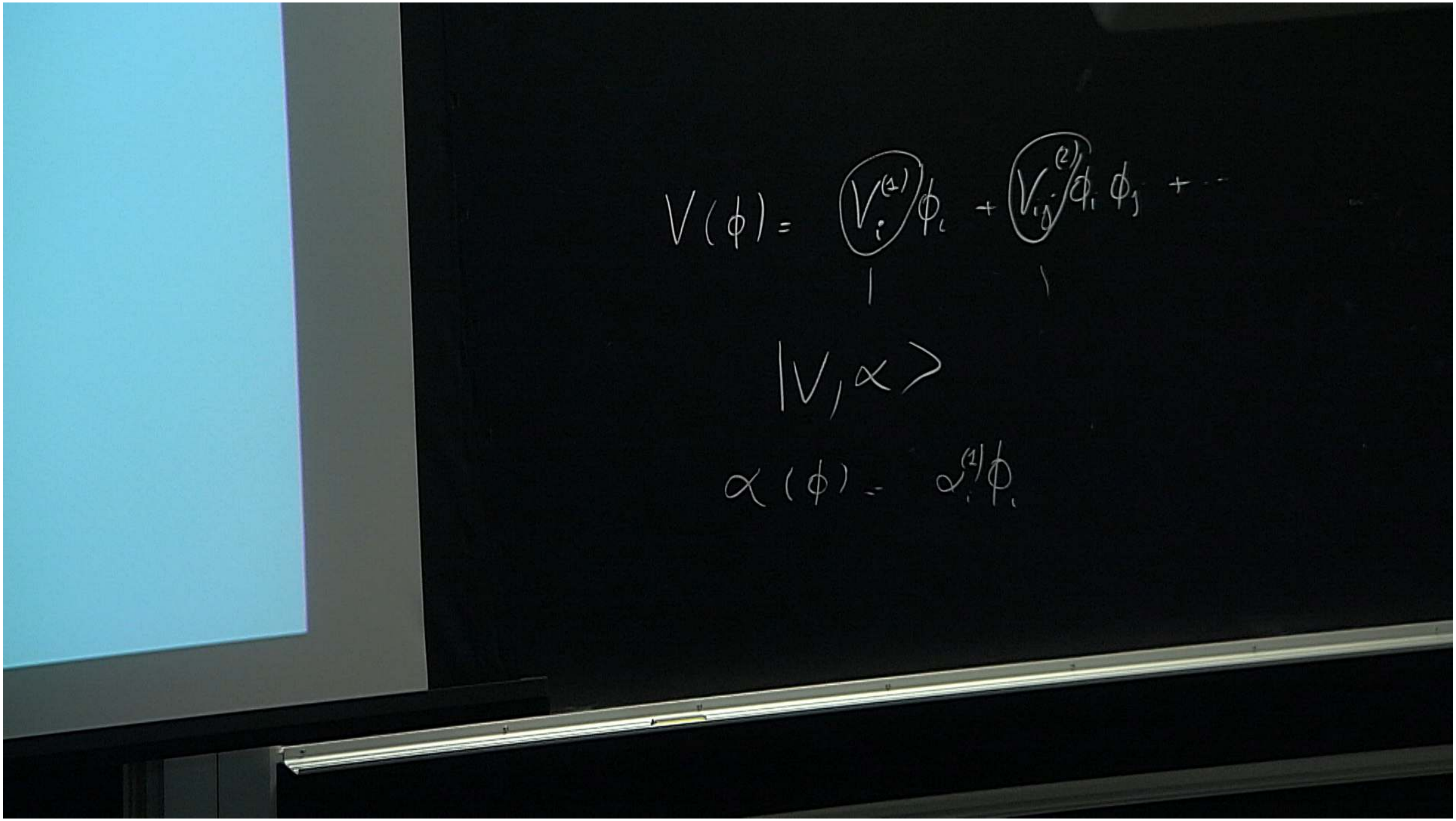
# Summary

$$|V, B, \alpha\rangle = \int \mathcal{D}\phi \, B(\phi|_{\partial\Omega}) \exp\left\{ -\int_{\Omega} \mathrm{d}^d x \, \frac{1}{2} \sum_{k=1}^{D} [\nabla\phi_k(x)]^2 + V[\phi(x)] - \alpha[\phi(x)] \, \psi^{\dagger}(x) \right\} |0\rangle$$

Continuous tensor network states are natural continuum limits of tensor network states and natural higher $d$ extensions of continuous matrix product states.

1. Obtained from discrete tensor networks
2. Can be made Euclidean invariant
3. Have functional and operator representations
4. Have a geometrical equivalent of the discrete gauge redundancies
5. Have an exact and explicit "renormalization" flow





$$V(\phi) = \boxed{V_i^{(1)}}\phi_i + \boxed{V_{ij}^{(2)}}\phi_i\phi_j + \cdots$$

$$|V, \alpha\rangle$$

$$\alpha(\phi) = \alpha_i^{(1)}\phi_i$$