Title: Asymptotic analysis of spin foam amplitude with timelike triangles

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Abstract: <p>The large j asymptotic behavior of 4-dimensional spin foam amplitude is investigated for the extended spin foam model (Conrady-Hnybida extension) on a simplicial complex. We study the most general situation in which timelike tetrahedra with timelike triangles are taken into account. The large j asymptotic behavior is determined by critical configurations of the amplitude. We identify the critical configurations that correspond to the Lorentzian simplicial geometries with timelike tetrahedra and triangles. Their contributions to the amplitude are phases asymptotically, whose exponents equal to Regge action of gravity. The amplitude also contains critical configurations corresponding to non-degenerate split signature 4-simplices and degenerate vector geometries. </p>

Asymptotic Analysis of Spin Foam Amplitude with Timelike Triangles:

Towards emerging gravity from spin foam models

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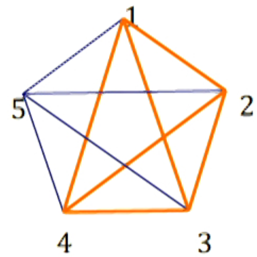
Outline

- Introduction
- Amplitude and asymptotic analysis
- Geometric interpretation
- Amplitude at critical configurations
- Conclusion

Triangulation

The building block: 4 simplex

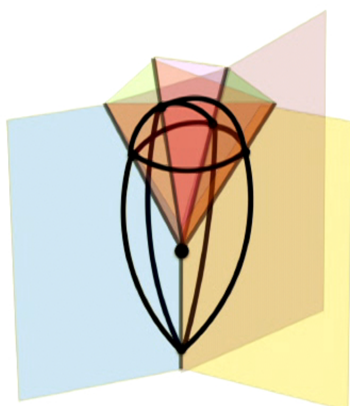


Boundary of a 4-simplex:
5 tetrahedron and 10 triangles

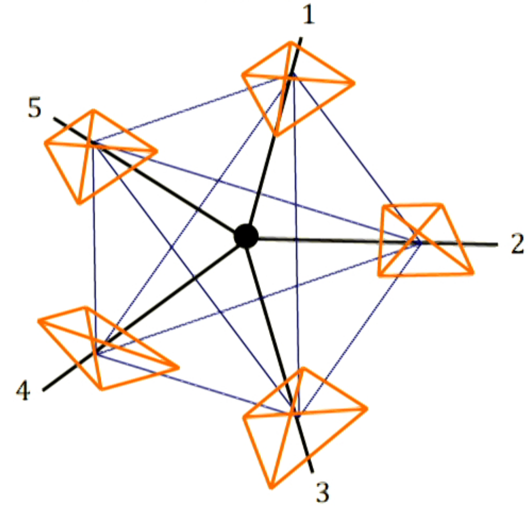
Dual graph

Boundary graph of a vertex:
5 node and 10 links

Dual Graph



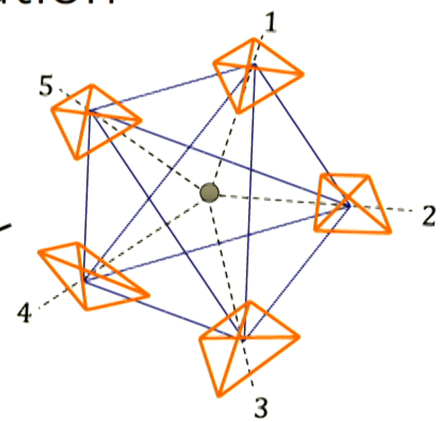
5 valent vertex and boundary

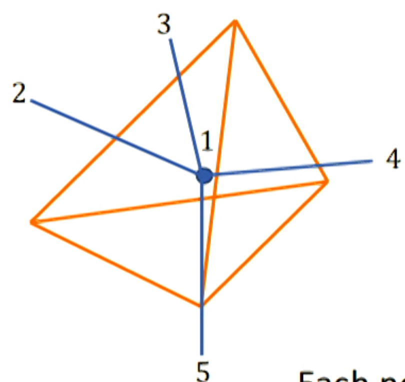


Boundary Graph:

Triangulation

Each link —— dual to a triangle :
Colored by spin!





Gluing triangles →

Spin network of boundary graph

Each node ● dual to a tetrahedron.
Gauge invariance: an intertwiner i_τ
(rank-4 invariant tensor)

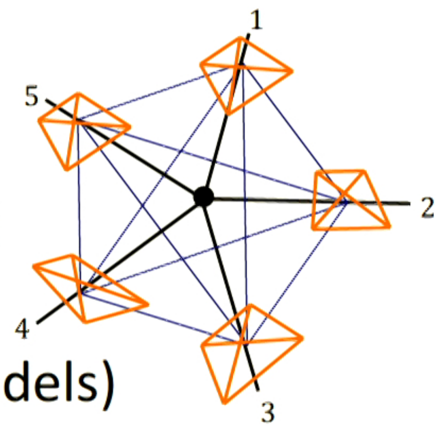
Spin foam models

- A state sum model on $\mathcal{K}$ inspired from BF action

$$Z(\mathcal{K}) = \sum_{\vec{j}} \sum_{\vec{i}} \prod_f A_f(j_f) \prod_v A_v(j_f, i_\tau)$$

- Lorentzian theory:

  Gauge group $\mathrm{SL}(2,\mathbb{C})$



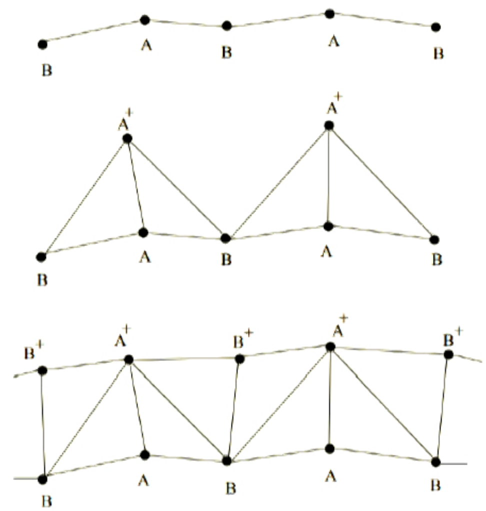
- Boundary gauge fixing: fix the normal perpendicular to the tetrahedron
- Time gauge $u = (1,0,0,0)$ (EPRL models)
- Space gauge $u = (0,0,0,1)$ (Conrady-Hybrida Extension)

Barrett, Rovelli, Dittrich, Engle, Livine, Freidel, Han, Dupris, Conrady, ...

Motivation: (3+1)-D Regge calculus Model

- A discrete formulation of General relativity.
- Discrete geometry: Sorkin triangulation
  - Initial slice triangulation (spacelike tetrahedra).
  - dragging vertices forward to construct the spacetime triangulation.

Result : spacetime triangulation with
- Every 4-simplex contains both timelike and spacelike tetrahedra
- Every timelike tetrahedron contains both timelike and spacelike triangles



(1 + 1)-dimensional analogue of the Sorkin scheme

Known predictions: Gravitational wavers, Kasner solution, etc...

Asymptotics of spin foam models:
What we have so far?

- Spacelike tetrahedron (Time gauge) [Freidel, Conrady, Barrett et al, Han, Zhang,...]
- Timelike tetrahedron with all faces spacelike [Kaminski et al]

- Asymptotics of the amplitude is dominated by critical configurations.
- Critical configurations are simplicial geometry (possibly degenerate)
- Asymptotic limit related to Regge action (discrete GR)

Are you satisfied?

Asymptotics of spin foam models:
What we have so far?

- Spacelike tetrahedron (Time gauge) [Freidel, Conrady, Barrett et al, Han, Zhang,...]
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- Asymptotics of the amplitude is dominated by critical configurations.
- Critical configurations are simplicial geometry (possibly degenerate)
- Asymptotic limit related to Regge action (discrete GR)

Non of them contains timelike triangles!!!

In all examples the Regge geometries contain timelike triangles.

Summary

- Asymptotic analysis of spin foam model with timelike triangles.
- The asymptotics of the amplitude is dominated by critical configurations.
- Critical configuration are again simplicial geometry.
- There will be no degenerate sector in the critical configuration.
- Asymptotic formula

$$A \sim N_+ e^{iS_{\mathcal{K}}} + N_- e^{-iS_{\mathcal{K}}}$$

$S_{\mathcal{K}}$ Regge action on on the simplicial complex.

Conrady-Hnybida Extension:
EPRL/FK with timelike triangles

From "time" gauge to "space" gauge

- Timelike tetrahedron with normals $u = (0,0,0,1)$
- Stabilize group SU(1,1)

Y map $\mathcal{H}^j \to \mathcal{H}^{(\rho,n)}$: physical Hilbert space

$$\rho = \begin{cases} \gamma n \\ -n/\gamma \end{cases} \qquad j = \begin{cases} \frac{n}{2}, & \text{spacelike triangle} \\ -\frac{1}{2} + \frac{i}{2}\sqrt{n^2/\gamma^2 - 1} & \text{timelike triangle} \end{cases}$$

Area spectrum:

$$A_f = \begin{cases} \frac{n_f}{2} & \text{timelike triangle} \\ \gamma\sqrt{j_f(j_f+1)} & \text{spacelike triangle} \end{cases}$$

$\rho \in \mathbb{R}, n \in \mathbb{Z}/2$ labels of SL(2, $\mathbb{C}$) irreps

$j = \begin{cases} \mathbb{Z}/2 \\ -\frac{1}{2} + i\, s \in \mathbb{R} \end{cases}$ labels of SU(1,1) irreps

SFM Amplitude

$\mathfrak{su}(1,1)$ generators: $\vec{F} = (J^3, K^1, K^2)$ $\quad$ K_1 eigenstates: $K_1|j,\lambda,\pm\rangle = \lambda\,|j,\lambda,\pm\rangle,\ \lambda \in \mathbb{R}$

$\uparrow\ t \quad \uparrow\ x^1 \quad \uparrow\ x^2$

Coherent states: on each triangle is defined as (with eigenstate of K_1)

$$\Psi_{ef} = Y\, D^j(v)|j,-s,+\rangle \in \mathcal{H}^{(\rho,n)}, \quad v \in SU(1,1)$$

The amplitude now given by

$$A(\mathcal{K}) = \sum_{j_f} \prod_f \mu(j_f) \prod_{(\nu,e)} \int_{\mathbb{SL}(2,\mathbb{C})} dg_{\nu e} \prod_{(e,f)} \int_{S^2} dN_{ef} \prod_{\nu \in f} \left\langle \Psi_{\rho_f n_f}(N_{ef}) \right| D^{(\rho_f,n_f)}(g_{ev} g_{ve'}) \left| \Psi_{\rho_f n_f}(N_{e'f}) \right\rangle$$

Amplitude appears in integration form

$$A_v(\mathcal{K}) = \int_{CP_1} \frac{\Omega_{z_{vf}}}{h_{vef-} h_{ve'f+}} \left(e^{S_{vf+}} + e^{S_{vf-}} + e^{S_{vfx+}} + e^{S_{vfx-}} \right)$$

- Actions S are pure imaginary!
- ½ order singularity appears in denominator h

Basic variables and gauge transformations

Actions

$$S_{vf\pm} = S_{ve'f\pm} - S_{vef\pm}, \quad S_{vfx\pm} = S_{ve'f\pm} - S_{vef\mp}$$

$$S_{vef\pm} = \frac{n_f}{2} \ln \frac{\langle Z_{vef}, l^{\pm}_{ef}\rangle}{\langle l^{\pm}_{ef}, Z_{vef}\rangle} \mp \mathrm{i}s \ln \langle Z_{vef}, l^{\pm}_{ef}\rangle\langle l^{\pm}_{ef}, Z_{vef}\rangle + \mathrm{i}(\rho_f \pm s) \ln \langle Z_{vef}, Z_{vef}\rangle$$

Variables

$z_{vf} \in \mathbb{C}P_1, \quad g_{ve} \in SL(2,\mathbb{C}), \quad Z_{vef} = g^{\dagger}_{ve}\bar{z}_{vf} \in \mathbb{C}^2$

$l^{\pm}_{ef} \in \mathbb{C}^2$ s.t. $\langle l^+, l^-\rangle = 1, \langle l^+, l^+\rangle = \langle l^-, l^-\rangle = 0.$ ← Null basis in $\mathbb{C}^2$

Gauge transformations

$$g_{ve} \to g_v g_{ve}, \qquad z_{vf} \to \lambda_{vf}(g_v^T)^{-1} z_{vf}$$

$$g_{ve} \to s_{ve} g_{ve}, \qquad s_{ve} = \pm 1$$

$$g_{ve} \to g_{ve} v_e, \qquad l^{\pm}_{ef} \to v_e l^{\pm}_{ef}$$

Asymptotics of the amplitude

stationary phase approximation

- Semiclassical limit of spinfoam models:
  - Area spectrum: $A_f = l_p^2 n_f / 2$
  - Keep area fixed & Take $l_p^2 \to 0$
  - Results in scaling $n_f \sim s_f \to \infty$ uniformly. Large - j asymptotics

SU(1,1) continuous series

$j = -\frac{1}{2} + i\, s$, $\mathrm{n} \sim \gamma s$

- Recall integration form of the amplitude

$$A_v(\mathcal{K}) = \int_{CP_1} \frac{\Omega_{z_{vf}}}{h_{vef-} h_{ve'f+}} \left(\mathrm{e}^{S_{vf+}} + \mathrm{e}^{S_{vf-}} + \mathrm{e}^{S_{vfx+}} + \mathrm{e}^{S_{vfx-}} \right)$$

S linear in j_f and pure imaginary

½ order singularity appears in denominator h

Stationary phase approximation with branch point!

Stationary phase analysis

with branch points

Asymptotics of integration I: $\Lambda \to \infty$

$$I = \int \mathrm{d}x \frac{1}{\sqrt{x - x_0}} g(x) \mathrm{e}^{\Lambda S(x)}$$

Critical point x_c: solutions of $\delta_x S(x) = 0$ x_0: ½ order singularity

Stationary phase approximation with branch point:

- Critical point locates at the branch point ← Our case ✔

$$I \sim g(x_c) \frac{\pi \mathrm{e}^{\mathrm{i}\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda |S_{xx}(x_c)|} \right)^{1/4} \mathrm{e}^{\Lambda S(x_c)}$$

- Critical point and branch point are separated ✘

Multivariable case: iterate evaluation

Critical point equations

for a tetrahedron with both timelike and spacelike triangles

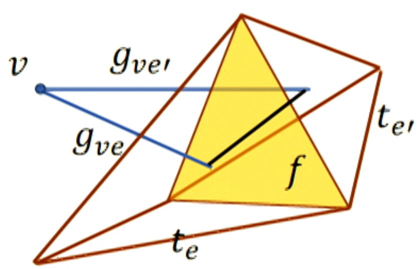
Decomposition of Z_{vef} using $l^{\pm}_{ef}$: change of variables

$$Z_{vef} \in \mathbb{C}^2 \xrightarrow{\; Z_{vef} \,=\, \zeta_{vef}\left(l^{\pm}_{ef} + \alpha_{vef}\, l^{\mp}_{ef}\right) \;} \zeta_{vef} \in \mathbb{C}, \quad \alpha_{vef} \in \mathbb{C}$$

$Z_{vef} = g^{\dagger}_{ve} \bar{z}_{vf}$ Impose:

$$g_{ve}\, J\,(l^{\pm}_{ef} + \alpha_{vef} l^{\mp}_{ef}) = \frac{\bar{\zeta}_{ve'f}}{\bar{\zeta}_{vef}} g_{ve'}\, J\,(l^{\pm}_{e'f} + \alpha_{ve'f} l^{\mp}_{ef})$$

Variation respect to z_{vf} : gluing condition on edges $e \;\to e'$



$$\delta S_{vf+} = (\gamma - \mathrm{i}) s_f \Big(\frac{g_{ve}\eta l^{+}_{ef}}{\bar{\zeta}_{vef}} - \frac{g_{ve'}\eta l^{+}_{e'f}}{\bar{\zeta}_{ve'f}}\Big) = 0 \qquad \text{with } Z = \zeta(l^{-} + \alpha l^{+})$$

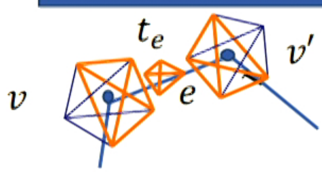
$$\delta S_{vf-} = -\mathrm{i} s_f \Big(\frac{g_{ve}\eta n_{vef}}{\mathrm{Re}(\alpha_{vef})\bar{\zeta}_{vef}} - \frac{g_{ve'}\eta n_{ve'f}}{\mathrm{Re}(\alpha_{ve'f})\bar{\zeta}_{ve'f}}\Big) = 0 \qquad \text{with } Z = \zeta(l^{+} + \alpha l^{-}$$

$$n_{vef} := l^{+}_{ef} + \mathrm{i}\,(\gamma\, \mathrm{Re}(\alpha_{vef}) + \mathrm{Im}(\alpha_{vef}))\, l^{-}_{ef}$$

Critical point equations

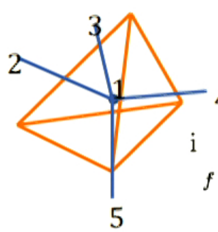
for a tetrahedron with both timelike and spacelike triangles

Variation respect to l_{ef} : gluing condition on vertices $v \to v'$



$$\gamma\,\mathrm{Re}(\alpha_{vef}) \mp \mathrm{Im}(\alpha_{vef}) = \gamma\,\mathrm{Re}(\alpha_{v'ef}) \mp \mathrm{Im}(\alpha_{v'ef})$$

Variation respect to g_{ve} : closure on faces f



null vectors

$$(1+\gamma^2)\sum_{f\ /\mathrm{w}\ S_{+(x)}} s_f\,\mathrm{Re}(\alpha_{vef})\langle l^+_{ef}, F^i l^+_{ef}\rangle + \sum_{f\ /\mathrm{w}\ S_{-(x)}} s_f \frac{\langle n_{ef}, F^i n_{ef}\rangle}{\mathrm{Re}(\alpha_{vef})} = 0$$

$$\mathrm{i}\sum_{f\ /\mathrm{w}\ S_{+(x)}} s_f\langle l^-_{ef} + \mathrm{i}\,\mathrm{Im}(\alpha_{vef})\, l^+_{ef}, F^i l^+_{ef}\rangle + \mathrm{i}\sum_{f\ /\mathrm{w}\ S_{+(x)}} s_f\langle l^-_{ef}, F^i n_{ef}\rangle + \sum_{f\ /\mathrm{w}\ S_{sp}} j_f\langle \xi^\pm_{ef}, F^i \xi^\pm_{ef}\rangle = 0$$

spacelike vectors

Timelike vectors

Check the paper for cases when all triangles are timelike

Geometric interpretation

of critical configuration

Critical point solutions $\mathbb{C}^2, \mathrm{SL}(2,\mathbb{C})$ → Geometrical interpretation $\mathbb{R}^{(1,3)}, \mathrm{SO}(1,3)$ → Simplicial geometry

- Define maps $\mathbb{C}^2 \to \mathbb{R}^{(1,3)}$, $\pi: \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$
- With gauge transformation $g \to -g$, we can always gauge fix $G = \pi(g) \in \mathrm{SO}_+(1,3)$
- Geometric vector and bivectors

$u = (0,0,0,1)$

$$V = v^i F^i = -\eta l^- \otimes (l^+)^\dagger + \frac{1}{2}\langle l^+, l^- \rangle I_2 \xrightarrow{\text{spin 1}} \begin{pmatrix} 0 & -v^1 & -v^2 & 0 \\ v^1 & 0 & v^0 & 0 \\ v^2 & -v^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = *(v \wedge u)^{IJ}$$

$$v^i = 2\langle l^+, F^i l^- \rangle \longrightarrow v^I := (v^0, -v^2, v^1, 0) = \mathrm{i}(\langle l^- | \hat{\sigma}^I | l^+ \rangle + u^I)$$

normal of triangles in a tetrahedron

Geometric solution

for a tetrahedron with both timelike and spacelike triangles

A non-degenerate tetrahedron geometry exists only when timelike triangles is with action S_{vf+}

We define a bivector

$$B_{ef} = 2A_f X_{ef} = 2A_f * (v_{ef} \wedge u)$$

with vectors and areas

$$v_{ef}^I = \begin{cases} -\mathrm{i}(\left\langle l_{ef}^+ \middle| \sigma^I \middle| l_{ef}^- \right\rangle - u^I) & \text{for timelike triangle} \\ \left\langle \xi_{ef}^\pm \middle| \sigma^I \middle| \xi_{ef}^\pm \right\rangle - \langle \xi^\pm, \xi^\pm \rangle u^I & \text{for spacelike case} \end{cases}, \quad A_f = \begin{cases} \gamma s_f = n_f/2 & \text{for timelike triangle} \\ \gamma j_f = \gamma n_f/2 & \text{for spacelike triangle} \end{cases}$$

Critical point equations

$$B_{ef}(v) = B_{e'f}(v) = B_f(v)$$

Parallel transport

$$N_e^I B_f(v)^{IJ} = 0 \qquad \sum_{f \in t_e} \epsilon_{ef}(v) B_f(v) = 0$$

Simplicity condition

Closure

Geometric Reconstruction

single non-degenerate 4 simplex

A geometrical 4-simplex

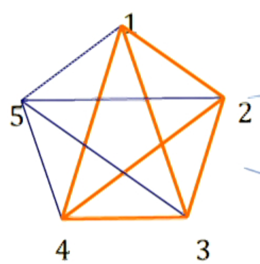
Geometrical solutions

Outpointing normals: N_i^Δ ⟷ $N_i = \pm N_i^\Delta$ ⟷ Normals N_i

Bivectors B_{ij}^Δ ⟷ $B_{ij} = \pm B_{ij}^\Delta$ ⟷ Bivectors B_{ij}

Geometric rotations: $G^\Delta \in O(1,3)$ ⟷ $G_i = G_i^\Delta I^{s_i} (IR_u)^s$ ⟷ Geometrical solution $G \in SO(1,3)$

Determine tetrahedron edge lengths uniquely



Length matching condition

Orientation matching condition

[Barrett et al, Han et al, Kaminski et al,....

Geometric Reconstruction

single 4 simplex

Reconstruction theorem

The non-degenerate geometrical solution exists if and only if the lengths and orientations matching conditions are satisfied.

There will be two gauge in-equivalent geometric solutions $\{G_{ve}\}$, such that the bivectors $B_f(v) = * \left(G_{ve} v_{ef} \wedge G_{ve} u\right)$ correspond to bivectors of a reconstructed 4 simplex as

$$B_f(v) = r(v) B_f^{\Delta}(v), \; r(v) = I^{s(v)} = \pm 1$$

And normals $$N_e(v) = I^{s_e(v)} N_e^{\Delta}(v) = \pm N_e^{\Delta}(v)$$

The two gauge equivalence classes of geometric solutions are related by

$$\tilde{G} = R_{e_\alpha} G R_{u_e} I^{s_e} \in SO_+(1,3)$$

Geometric Reconstruction

Simplicial complex with many 4 simplicies

Gluing condition $e = (v, v')$

$$N_e(v) = G_{v\,v'} N_e(v')$$
$$B_f(v) = G_{v\,v'} B_f(v') G^{-1}_{v\,v'}$$

Consistent Orientation:

$$v: [p_0, p_1, p_2, p_3, p_4]$$
$$v': -[p_0, p_1, p_2, p_3, p_4]$$

Reconstruction at v, v'

$$N_e(v) = I^{s_e(v)} N_e^{\Delta}(v)$$
$$B_f(v) = r(v) B_f^{\Delta}(v)$$

→ $$sgn\big(V(v)\big) r(v) = const$$

Reconstruction theorem

The simplicial complex $\mathcal{K}$ can be subdivided into sub-complexes $\mathcal{K}_1, \ldots, \mathcal{K}_2$, such that (1) each $\mathcal{K}_i$ is a simplicial complex with boundary, (2) within each sub-complex $\mathcal{K}_i$, $sgn\big(V(v)\big)$ is a constant. Then there exist

$$G_f^{\Delta} \in SO(1,3) = I^{\sum_{v \in f} 1} I^{\sum_{v, e \in f} s_e(v)} G_f = \pm G_f .$$

such that G_f^{Δ} are the discrete spin connection compatible with the co-frame.

For two non gauge inequivalent geometric solutions with different r:

$$\widetilde{G_f} = R_u G_f R_u$$

Degenerate Solutions?

Degenerate Condition: All normals are parallel to each other: $G_{ve} \in SO(1,2)$

if the 4-simplex contains both timelike and spacelike tetrahedra

→ **Can not be degenerate!!**

Vector geometries

$$G_{ve}\, v_{ef} = G_{ve'} v_{e'f}, \quad \sum_f \varepsilon_{ef}(v)\; v_{ef} = 0$$

Flipped signature solution

1-1 correspondence to solutions in split signature space M' with (-,+,+,-)

- pair of two non-gauge equivalent vector geometries $G^{\pm}_{ve}$,
- Geometric $SO(M')$ non-degenerate solution G'_{ve}.

The two vector geometries are obtained from $SO(M')$ solutions with map $\Phi^{\pm}$:

$$\Phi^{\pm}(G'_{ve}) = G^{\pm}_{ve}$$

Geometric $SO(M')$ solution G'_{ve} degenerate: vector geometries $G^{\pm}_{ve}$ gauge equivalent

Reconstruction → Reconstruction 4-simplex in M'

Action at critical configuration

timelike triangles

Recall stationary phase formula $I \sim g(x_c) \frac{\pi e^{i\pi(\mu-2)/8}}{\Gamma(3/4)} \left(\frac{2}{\Lambda |S_{xx}(x_c)|} \right)^{1/4} e^{\Lambda S(x_c)}$

Action after decomposition: A phase

$$S_f = \sum_{v \in \partial f} S_{vf} = -2 i s_f \left(\sum_{v \in \partial f} \theta_{e'vef} + \gamma \sum_{v \in \partial f} \phi_{e'vef} \right)$$

Asymptotics of amplitude: determine $\theta_{e'vef}$ and $\phi_{e'vef}$ at critical points

$$\theta_{e'vef} := \ln \frac{|\zeta_{ve'f}|}{|\zeta_{vef}|}, \qquad \phi_{e'vef} := \arg(\zeta_{ve'f}) - \arg(\zeta_{vef})$$

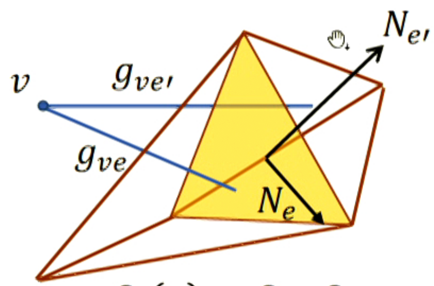
U(1) ambiguity from boundary coherent states $\Psi = D\big(v(N) e^{i\,u\,K_1}\big)|j, -s\rangle$

$$e^S \sim \langle \Psi | D(g^{-1} g') | \Psi' \rangle$$

Phase difference at two critical points $\Delta S = S(G) - S(\tilde{G})$

Phase difference

on one 4 simplex with boundary triangles



Boundary tetrahedron normals

$$\cos\theta_f = N_e^\Delta \cdot N_{e'}^\Delta, \qquad \theta_f \in (0,\pi)$$

Reconstruction theorem

Two solutions for given boundary data.

$$\tilde{G} = R_{e_\alpha} G R_{u_e} I^{s_e} \in SO_+(1,3)$$

$$G_{ve}\tilde{G}_f^{-1} G_f G_{ev} = R_{N_e} R_{N_{e'}}$$

Reflection

Rotation in a spacelike plane

From critical point equations

Reconstruction

$$g_{ve}(\tilde{g}_{e'v}\tilde{g}_{ve})^{-1} g_{e'v} = \mathrm{e}^{-2\Delta\theta_{e'vef} X_f + 2\mathrm{i}\Delta\phi_{e'vef} X_f}$$

$$\mathrm{e}^{2r\Delta\theta_{e'vef}\frac{*(N_{e'}^\Delta \wedge N_e^\Delta)}{|N_{e'}^\Delta \wedge N_e^\Delta|} + 2r\Delta\phi_{e'vef}\frac{N_{e'}^\Delta \wedge N_e^\Delta}{|N_{e'}^\Delta \wedge N_e^\Delta|}} = R_{N_e^\Delta} R_{N_{e'}^\Delta} = \mathrm{e}^{2\theta_f \frac{N_e^\Delta \wedge N_{e'}^\Delta}{|N_e^\Delta \wedge N_{e'}^\Delta|}}$$

$$\Delta\theta_{e'vef} = 0, \qquad -r\Delta\phi_{e'vef} = \theta_f \mod \pi$$

Phase difference

on simplicial complex with many 4 simplicies

Face holonomy: $G_f(v) = \prod_v G_{e'v} G_{ve}$

Reconstruction theorem: for two gauge inequivalent solutions (with different uniform orientation r)

$$\widetilde{G_f} = R_u G_f R_u \qquad G_{ve}\tilde{G}_f^{-1} G_f G_{ev} = R_{N_e} R_{N_{e'}}$$

Rotation in a spacelike plane

Now the normals become

$$N_e = G_{ve} u, \qquad N_{e'} = G_{ve}(G_f^{-1} u)$$

parallel transported vector along the face

From critical point equations

$$g_{ve}\tilde{G}_f^{-1} G_f g_{ev} = \mathrm{e}^{-2\sum_{v\in\partial f} \Delta\theta_{e'vef} X_f + 2\mathrm{i}\sum_{v\in\partial f} \Delta\phi_{e'vef} X_f}$$

Reconstruction

$$\mathrm{e}^{2r\sum_{v\in\partial f} \Delta\theta_{e'vef} \frac{*(N_{e'}^{\Delta}\wedge N_e^{\Delta})}{|N_{e'}^{\Delta}\wedge N_e^{\Delta}|} + 2r\sum_{v\in\partial f} \Delta\phi_{e'vef} \frac{N_{e'}^{\Delta}\wedge N_e^{\Delta}}{|N_{e'}^{\Delta}\wedge N_e^{\Delta}|}} = R_{N_e^{\Delta}} R_{N_{e'}^{\Delta}} = \mathrm{e}^{2\theta_f \frac{N_e^{\Delta}\wedge N_{e'}^{\Delta}}{|N_e^{\Delta}\wedge N_{e'}^{\Delta}|}}$$

$$\sum_{v\in\partial f} \Delta\theta_{e'vef} = 0, \qquad -r\sum_{v\in\partial f} \Delta\phi_{e'vef} = \theta_f \mod \pi$$

Phase difference

Now phase difference $\quad A_f = \gamma s_f = n_f/2 \in Z/2$

$$\Delta S_f = 2ir A_f \theta_f \quad \mathrm{mod} \quad i\pi$$

$i\pi$ ambiguity relates to the lift ambiguity!

→ Some of them may be absorbed to gauge transformations $g_{ve} \to -g_{ve}$

Fixed at each vertex $\sum_{v\in\partial f} \Delta\phi_{e'vef} = -\sum_{v\in\partial f} \theta_f(v) \quad \mathrm{mod}\ 2\pi$

θ_f here a rotation angle $\cos\theta_f = N_e^\Delta \cdot N_{e'}^\Delta, \qquad \theta_f \in (0,\pi)$

No. of simplicies in the bulk

→ Related to dihedral angles $\Theta_f(v) = \pi - \theta_f(v)$

deficit angles $\epsilon_f(v) = 2\pi - \sum_{v\in f} \Theta_f(v) = (1 - m_f)\pi - \theta_f(v)$

The total phase difference

$$\exp(\Delta S_f) = \exp\left\{2ir \sum_{f\ \mathrm{bulk}} A_f\left[(2-m_f)\pi - \varepsilon_f\right] + 2ir \sum_{f\ \mathrm{boundary}} A_f\left[(1-m_f)\pi - \theta_f\right]\right\}$$

When m_f even, Regge action up to a sign

Degenerate solutions

From critical point equations: two non-gauge equivalent solution $g^{\pm}_{ve}$

$$g^{\pm}_{ev} g^{\mp}_{ev} g^{\mp}_{ve'} g^{\pm}_{e'v} = \mathrm{e}^{\mp 2\Delta\theta_{e'vef} X^{\pm}_f \pm 2\mathrm{i}\Delta\phi_{e'vef} X^{\pm}_f}$$

Degenerate: $g^{\pm}_{ve} \in SU(1,1)$ → $2\Delta\phi_{e'vef} = 0 \mod 2\pi$ ← Lift ambiguity

1-1 Correspondence to flipped signature non-degenerate solutions:

$$\Phi^{\pm}(G'_{ev}\tilde{G}'_{ev}\tilde{G}'_{ve'}G'_{e'v}) = G^{\pm}_{ev}G^{\mp}_{ev}G^{\mp}_{ve'}G^{\pm}_{e'v} = \Phi^{\pm}(R_{N_e}R_{N_{e'}})$$

$$\Phi^{\pm}(\mathrm{e}^{2\Delta\theta_{e'vef} *' X_f}) = \mathrm{e}^{\mp 2\Delta\theta_{e'vef} X^{\pm}_f}$$

bivectors in flipped signature space

$$\mathrm{e}^{-r2\Delta\theta_{e'vef}\frac{N^{\Delta}_{e'}\wedge N^{\Delta}_{e}}{|*(N^{\Delta}_{e'}\wedge N^{\Delta}_{e})|}} = R_{N_e}R_{N_{e'}} = \mathrm{e}^{2\theta_f \frac{N^{\Delta}_{e}\wedge N^{\Delta}_{e'}}{|N^{\Delta}_{e}\wedge N^{\Delta}_{e'}|}},$$

$$\Delta\theta_{e'vef} = -r\theta_f \longrightarrow \exp(\Delta S_f) = \exp\left(2\mathrm{i}r\frac{1}{\gamma}A_f\theta_f\right)$$

↳ Regge action

Conclusion & Outlook

- The asymptotic analysis of spin foam model is now complete (with non degenerate boundaries)
- The asymptotics of the amplitude is dominated by critical configurations which are simplicial geometry
- The model excludes degenerate geometry sector.
- The asymptotic limit of the amplitude is related to Regge action

Outlooks

- Continuum limit
- Method and Calculations in Regge calculus: emerging gravity from spin foams (Kasner cosmology model with Han.)
- Black holes in semi-classical limit of spin foam models
- Calculation of the Propagators
- ....

Thanks for your attention