Title: Topological invariants and entanglement negativity in SPT phases of fermions

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Abstract: <p>Searching for a proper set of order parameters which distinguishes different phases of matter sits in the heart of condensed matter physics. In this talk, I discuss topological invariants as (non-local) order parameters for symmetry protected topological (SPT) phases of fermions in the presence of time-reversal symmetry. It turns out that topological invariants provide a natural definition for the partial transpose of density matrices. The partial transpose can then be used to define an entanglement measure (analog of entanglement negativity) for mixed states of fermions. I will show that this quantity captures the mixed state entanglement in fermionic SPTs as well as in a system of free fermions with a Fermi surface.</p>

THE UNIVERSITY OF CHICAGO

Topological invariants and entanglement negativity in SPT phases of fermions

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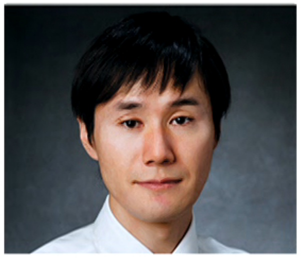
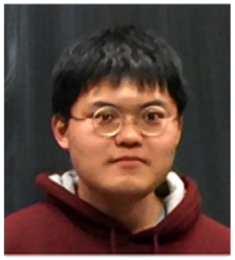
Perimeter Institute
10/04/2018

SPT= symmetry protected topological

Acknowledgement





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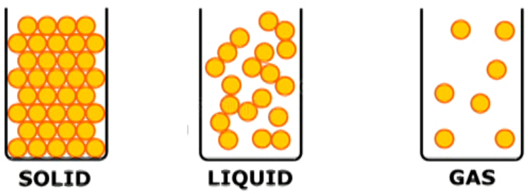
HS, K. Shiozaki, S. Ryu, PRL **118**, 216402 (2017)
K. Shiozaki, **HS (Equal contribution)**, K. Gomi, S. Ryu, PRB **98**, 035151 (2018)
HS, K. Shiozaki, S. Ryu, PRB **95**, 165101 (2017)
HS, S. Ryu, arXiv: 1804.08637

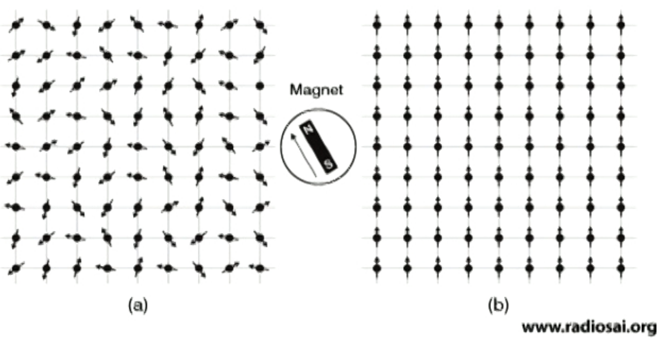
Outline

- Topological phases of fermions
- Many-body topological invariants for topological superconductors
- Applications
    - Fermionic Quantum Information: Entanglement negativity
    - Statistical Mechanics: Finite-temperature Fermi surfaces

Phases of matter

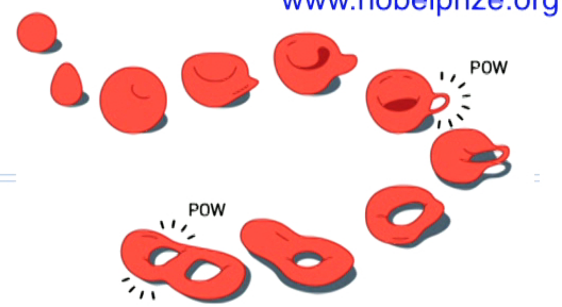
- Conventional phases
  - Characterized by broken symmetries and order parameters



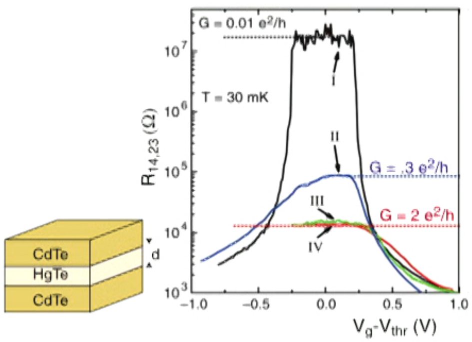


Topological phases of matter

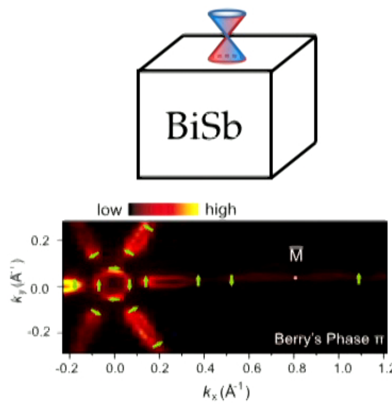
www.nobelprize.org



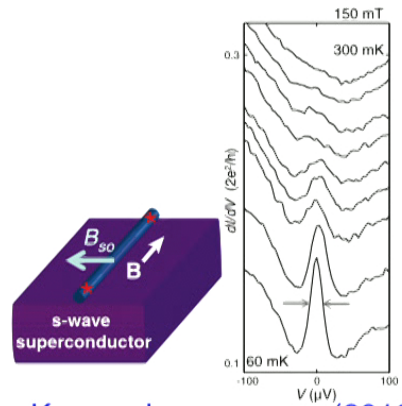
- Exotic boundary modes



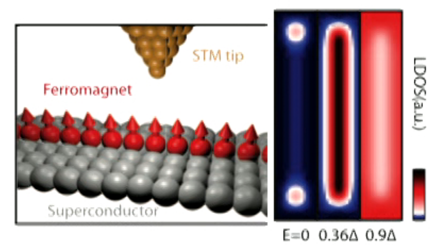
Molenkamp group (2007)



Hasan and Cava (2009)



Kouwenhoven group (2012)



Yazdani group (2014)

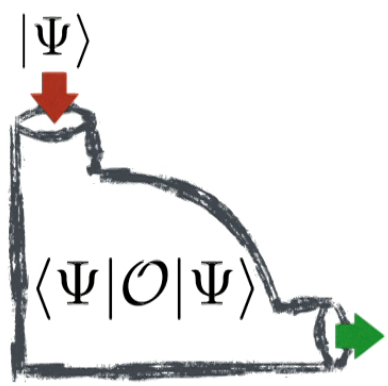
✓Helical edge modes

✓Dirac surface states

✓Majorana zero modes

Many-body topological invariant

- (non-local) **Order parameter** to distinguish **many-body** fermionic SPT phases
  - **Bulk** quantity
  - Does **not** require **single particle** description.



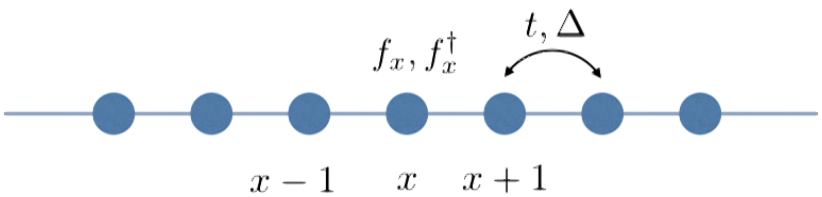
ν : **Many-body topological index**

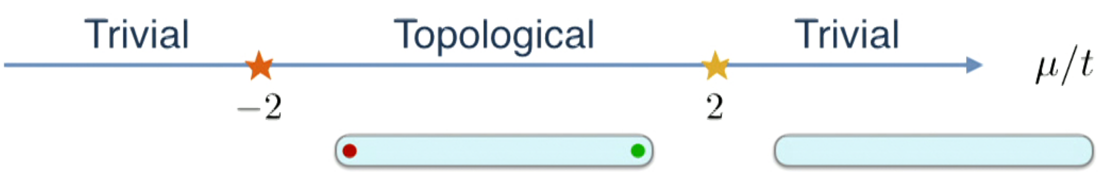
The Kitaev chain

Kitaev (2000)

- p-wave superconductivity on a chain

$$H = -\sum_x \left[t f_{x+1}^\dagger f_x - \Delta f_{x+1}^\dagger f_x^\dagger + \mathrm{H.c.} \right] - \mu \sum_x f_x^\dagger f_x ,$$





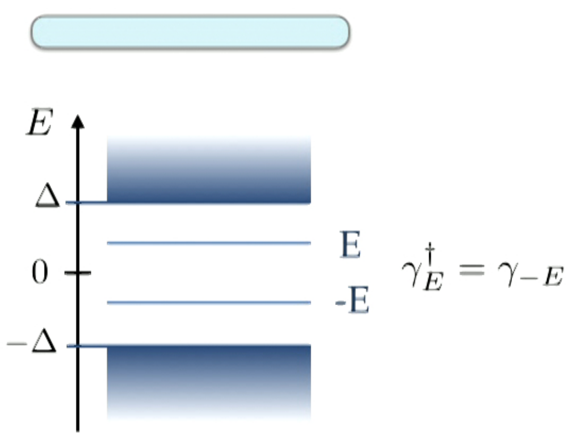
Read Green (2000)

The Kitaev chain

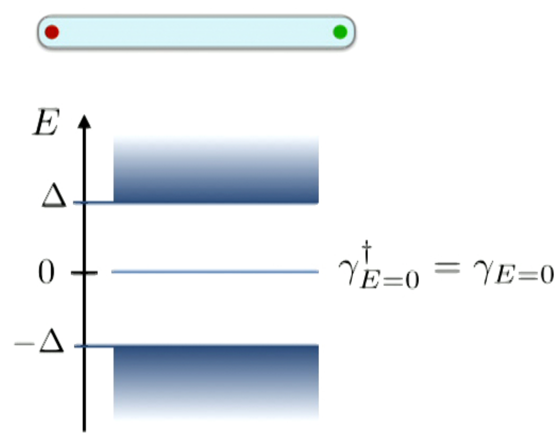
Kitaev (2000)

- p-wave superconductivity on a chain $$H=-\sum_x\left[tf^\dagger_{x+1}f_x-\Delta f^\dagger_{x+1}f^\dagger_x+\text{H.c.}\right]-\mu\sum_x f^\dagger_x f_x,$$

Trivial ($|\mu|>2t$)



Topological ($|\mu|<2t$)



Topological superconductors

- Class BDI
  - **Time-reversal:** $\mathcal{T}\Psi(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \Psi^*(\mathbf{r}_1, \mathbf{r}_2, \cdots)$

    $\mathcal{T}^2 = 1$

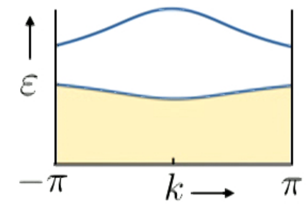
- Non-interacting classification is $\mathbb{Z}$.

Classification of non-interacting fermions

- Boundary states
  - Gappability criterion

Classification of non-interacting fermions

- Boundary states
  - Gappability criterion
- Band structure related topological invariants
  - E.g., winding numbers $w = \frac{i}{2\pi}\int_{-\pi}^{\pi} dk \, \langle u_k|\partial_k|u_k\rangle$
    Schnyder, Ryu et al(2008)



$$H(\mathbf{k})|u_n(\mathbf{k})\rangle = \varepsilon_n(\mathbf{k})|u_n(\mathbf{k})\rangle$$

Classification of non-interacting fermions

- Boundary states
  - Gappability criterion
- Band structure related topological invariants
  - E.g., winding numbers $w = \frac{i}{2\pi} \int_{-\pi}^{\pi} dk \, \langle u_k | \partial_k | u_k \rangle$
- Structure of the space of Hamiltonians $H(\mathbf{k})$

Schnyder, Ryu et al(2008), Kitaev (2009)

class\\δ	0	1	2	3
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$\mathbb{Z}$	0	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	0	$2\mathbb{Z}$	0
CI	0	0	0	$2\mathbb{Z}$

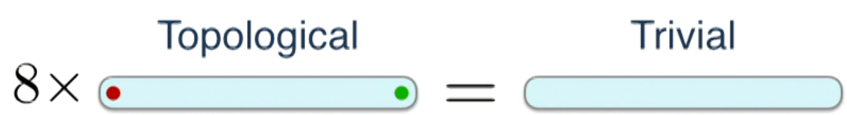
Interacting classification

- Class BDI
  - **Time-reversal:** $\mathcal{T}\Psi(\mathbf{r}_1, \mathbf{r}_2, \cdots) = \Psi^*(\mathbf{r}_1, \mathbf{r}_2, \cdots)$

    $\mathcal{T}^2 = 1$

- Non-interacting classification is $\mathbb{Z}$.

- **Interacting** classification is $\mathbb{Z}_8$.
  Fidkowski and Kitaev (2010)



What is the bulk distinction of 8 possible phases?

Recap: TKNN invariant

- Chern number in 2D band structures
  Thouless et al (1982)

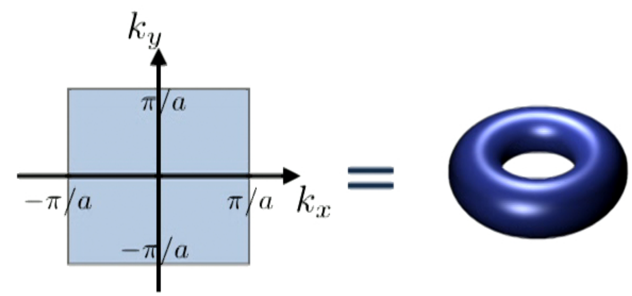
Bloch function: $H(\mathbf{k})|u_n(\mathbf{k})\rangle = \varepsilon_n(\mathbf{k})|u_n(\mathbf{k})\rangle$

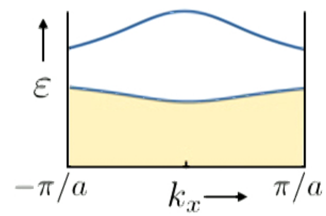
Berry connection: $\mathbf{A} = -i\langle u_n(\mathbf{k})|\nabla_{\mathbf{k}}|u_n(\mathbf{k})\rangle$

Berry curvature: $\mathbf{F} = \nabla_{\mathbf{k}} \times \mathbf{A}$

Chern number: $n = \frac{1}{2\pi} \int_{\mathrm{BZ}} d^2k \; \mathbf{F}(\mathbf{k})$

Hall conductance: $\sigma_{xy} = n\frac{e^2}{h}$



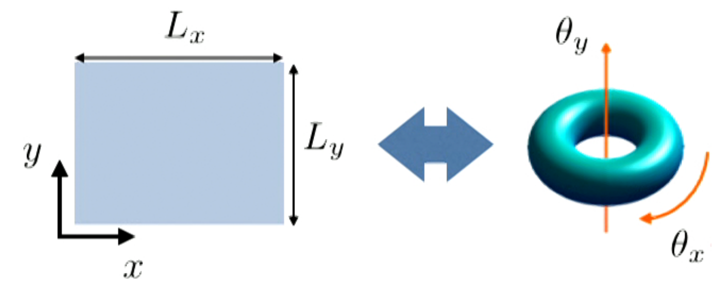


Many-body Chern number

- Twist boundary conditions by U(1) symmetry

Niu, Thouless, and Wu (1985)
Avron and Seiler (1985)

$$\Psi_{(\theta_x,\theta_y)}(\mathbf{r}_1,\cdots,\mathbf{r}_k+L_i\hat{\mathbf{x}}_i,\cdots,\mathbf{r}_N)$$
$$=e^{i\theta_i}\Psi_{(\theta_x,\theta_y)}(\mathbf{r}_1,\cdots,\mathbf{r}_k,\cdots,\mathbf{r}_N)$$



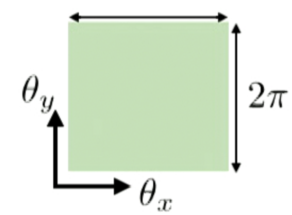
- Define a family of **many-body** states

$$|\Psi_{(\theta_x,\theta_y)}\rangle \qquad 0\le\theta_x,\theta_y\le 2\pi \quad \text{(torus)}$$

- Compute Berry phase

$$\mathcal{A}=-i\langle\Psi_{(\theta_x,\theta_y)}|\nabla_\theta|\Psi_{(\theta_x,\theta_y)}\rangle$$
$$\mathcal{B}=\nabla_\theta\times\mathcal{A}$$

$$\mathrm{Ch}_1=\frac{1}{2\pi}\int_{T^2}d^2\theta\ \mathcal{B}(\theta_x,\theta_y)$$

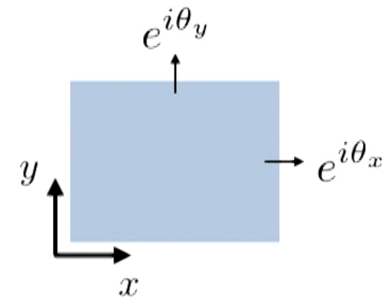


Many-body Chern number

- **Many-body** Chern number in two dimensions

Niu, Thouless, and Wu (1985)
Avron and Seiler (1985)

$$\Psi_{(\theta_x,\theta_y)}(\mathbf{r}_1,\cdots,\mathbf{r}_k+L_i\hat{\mathbf{x}}_i,\cdots,\mathbf{r}_N)$$
$$=e^{i\theta_i}\Psi_{(\theta_x,\theta_y)}(\mathbf{r}_1,\cdots,\mathbf{r}_k,\cdots,\mathbf{r}_N)$$



Intuitive message: **twist** boundary conditions by symmetry

Let us experiment this idea

- Topological invariant ➡ **partition function** with **twisted** boundary conditions
- **Canonical** formalism in terms of GS wave function

SPT phases and TQFT

- Partition function of a gapped quantum system

$$Z(X, \eta, A) = \int \mathcal{D}[\phi_i] \; e^{-S(X,\eta,A,\phi_i)}$$

X spacetime manifold

A background gauge field [on-site symmetry group G]

η spin structures ((anti-)periodic boundary conditions)

- For topological phases, the action may have a **complex phase**

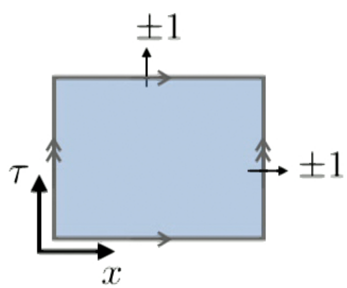
$$Z(X, \eta, A) \sim e^{iS_{\text{top}}(X,\eta,A)}$$

→ Topological quantum field theory (TQFT)

Kapustin, Freed, Hopkins, Witten, …

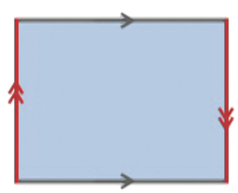
Time-reversal symmetric superconductors

- Class BDI: $\mathbb{Z}_2^f \times \mathcal{T}$
- Twist spacetime boundary conditions of partition function
  - Twist by fermion-number parity
  - Twist by time-reversal



$Z(T^2)$

Torus



$Z(KB)$

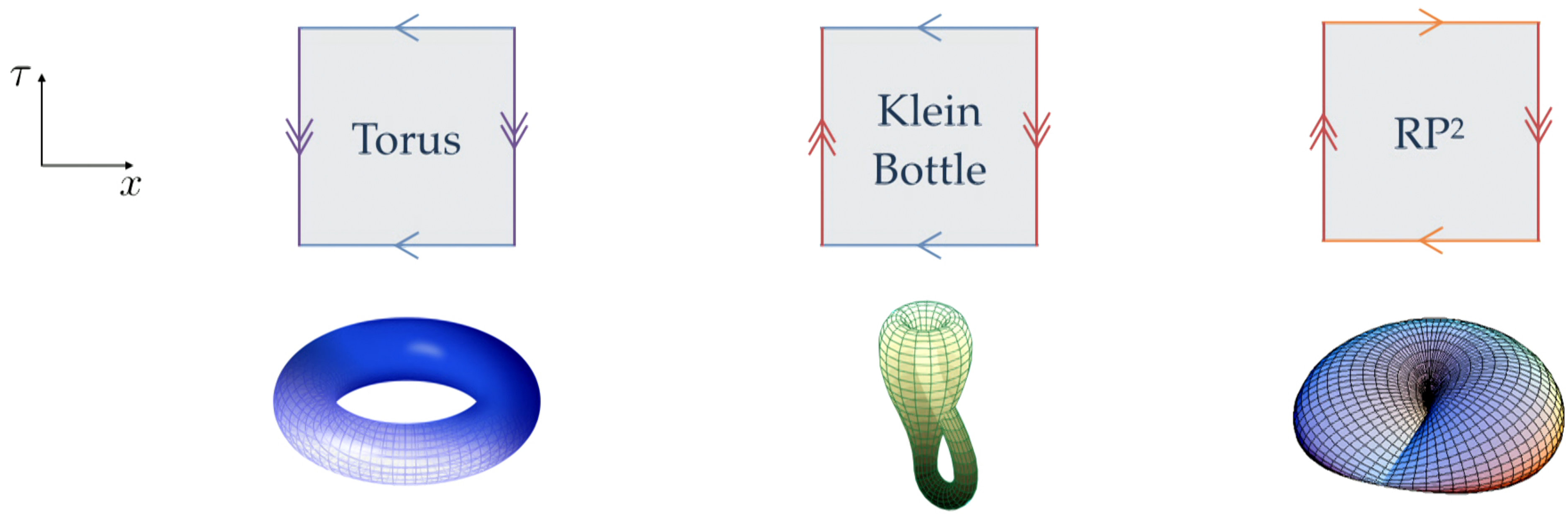
Klein bottle

Non-orientable spacetime

Hsieh and Ryu (2014)
Cho and Ryu (2015)
HS, Shiozaki, Ryu (2016)
Barkeshli and Cheng (2016)
Wang and Levin (2016)
Tachikawa and Yonekura (2016)

Twisting by time-reversal

	Torus	Klein Bottle	RP^2
Temporal BC	$\psi(x,\tau+\beta) = \pm\psi(x,\tau)$	$\psi(x,\tau+\beta) = \pm\psi(x,\tau)$	$\psi(x,\tau+\beta) = \pm\psi(L-x,\tau)$
Spatial BC	$\psi(x+L,\tau) = \pm\psi(x,\tau)$	$\psi(x+L,\tau) = \pm\psi(x,\beta-\tau)$	$\psi(x+L,\tau) = \pm\psi(x,\beta-\tau)$



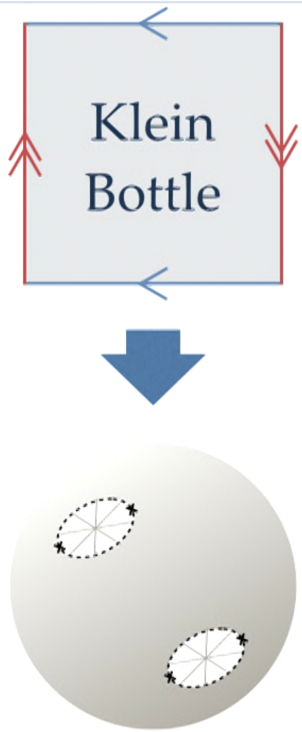
Partition function of class BDI

Spacetime	Partition function	Detected Classification
Torus	$Z(T^2)_{r,r} = -1$	$\mathbb{Z}_2$
Klein bottle	$Z(KB)_r = e^{i\frac{2\pi}{4}}$	$\mathbb{Z}_4$
Real projective plane	$Z(\mathbb{R}P^2) = e^{i\frac{2\pi}{8}}$	$\mathbb{Z}_8$

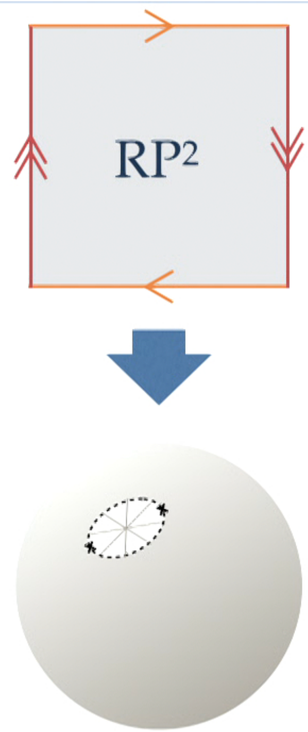
Canonical formalism?

Spacetime	Partition function	Detected Classification	Canonical formalism
Torus	$Z(T^2)_{r,r} = -1$	$\mathbb{Z}_2$	$Z^{T^2}_{(r,X)} = \langle \mathrm{GS}_X \| (-1)^{\hat{n}} \| \mathrm{GS}_X \rangle$
Klein bottle	$Z(KB)_r = e^{i\frac{2\pi}{4}}$	$\mathbb{Z}_4$	$Z^{KB} = ?$
Real projective plane	$Z(\mathbb{R}P^2) = e^{i\frac{2\pi}{8}}$	$\mathbb{Z}_8$	$Z^{\mathbb{R}P^2} = ?$

Cross-cap



Sphere+ 2 cross-caps



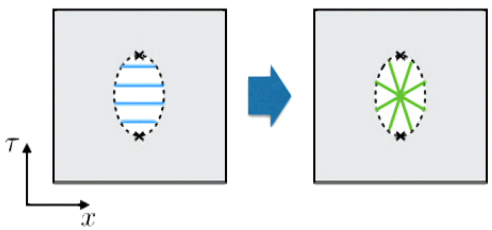
Sphere+ cross-cap

Hsieh and Ryu (2014)
Cho and Ryu (2015)
HS, Shiozaki, Ryu (2016)
Barkeshli and Cheng (2016)

Cross-cap in canonical formalism

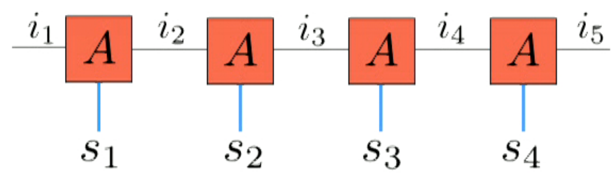
- $\mathbb{R}P^2$ in canonical formalism

Create a cross-cap



Matrix product state (MPS)

Perez-Garcia, et al (2007)



$s_i = \uparrow, \downarrow$

$$\Psi(s_1, s_2, \cdots) = \sum_{\{i_n = 1, \cdots\}} A^{s_1}_{i_1 i_2} A^{s_2}_{i_2 i_3} A^{s_3}_{i_3 i_4} \cdots$$

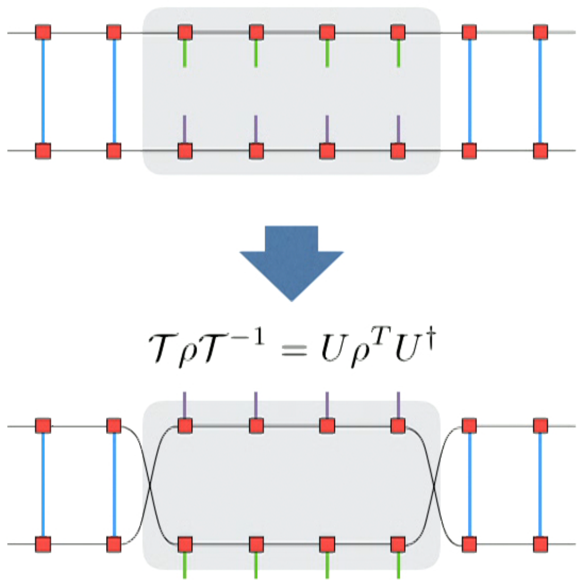
Fermions and chiral phases:

Wahl et al (2013)
Dubail and Read (2015)
Williamson et al (2016)
Bultinck et al (2017)

How to create a cross-cap?

- Time-reversal symmetry $\mathcal{T} = U\mathcal{K}$

$$\psi(x,\tau) \xrightarrow{\mathcal{T}} \psi(x,-\tau)$$



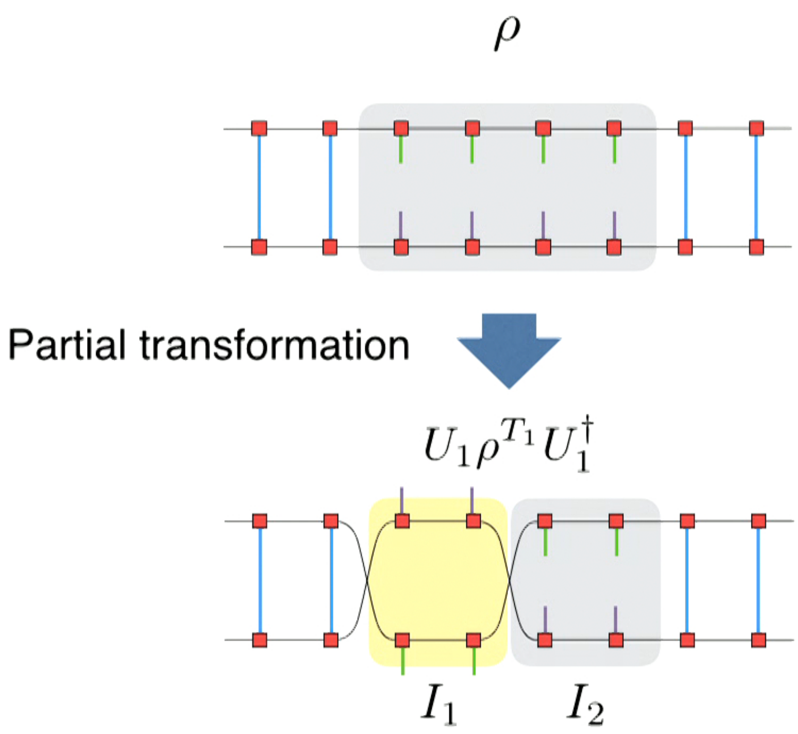
ρ

$\mathcal{T}\rho\mathcal{T}^{-1} = U\rho^T U^\dagger$

cross-cap ⟺ partial time-reversal

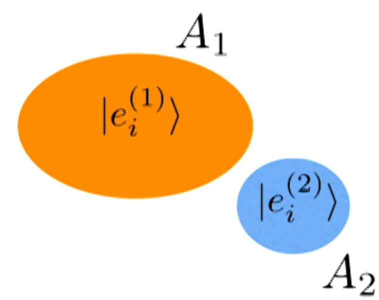
- Time-reversal symmetry

$$\psi(x,\tau) \xrightarrow{\mathcal{T}} \psi(x,-\tau)$$



Partial transpose (bosons vs. fermions)

- Full transpose $\langle e_i^{(1)}, e_j^{(2)} | \rho^T | e_k^{(1)}, e_l^{(2)} \rangle = \langle e_k^{(1)}, e_l^{(2)} | \rho | e_i^{(1)}, e_j^{(2)} \rangle$



- Partial transpose $\langle e_i^{(1)}, e_j^{(2)} | \rho^{T_1} | e_k^{(1)}, e_l^{(2)} \rangle = \langle e_k^{(1)}, e_j^{(2)} | \rho | e_i^{(1)}, e_l^{(2)} \rangle$

- **Fermions**: Partial transpose must be consistent with **anti-commutation** relation $f_j f_k^\dagger + f_k^\dagger f_j = \delta_{jk}$
  Eisler, Zimboras (2015)

- Fock space basis $|\{n_j\}_{j\in A_1}, \{n_j\}_{j\in A_2}\rangle$

$$\langle \{n_j\}_{A_1}, \{n_j\}_{A_2} | \rho^{T_1} | \{m_j\}_{A_1}, \{m_j\}_{A_2} \rangle = (-1)^{\phi(\{n_j\},\{m_j\})} \langle \{m_j\}_{A_1}, \{n_j\}_{A_2} | \rho | \{n_j\}_{A_1}, \{m_j\}_{A_2} \rangle$$

Phase factor

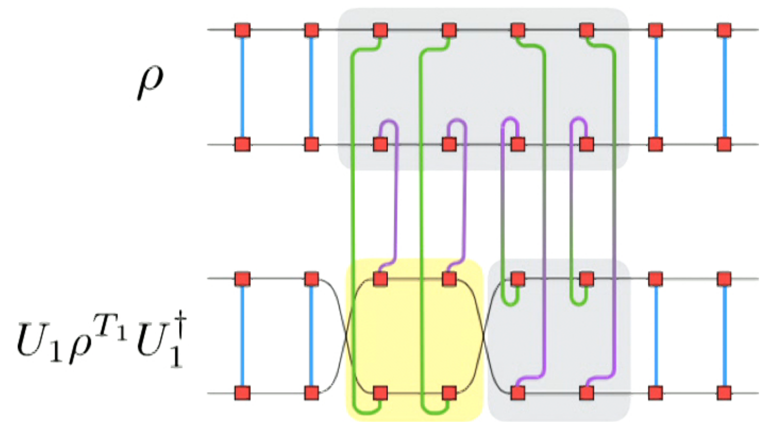
$\mathbb{R}P^2$ from partial time-reversal

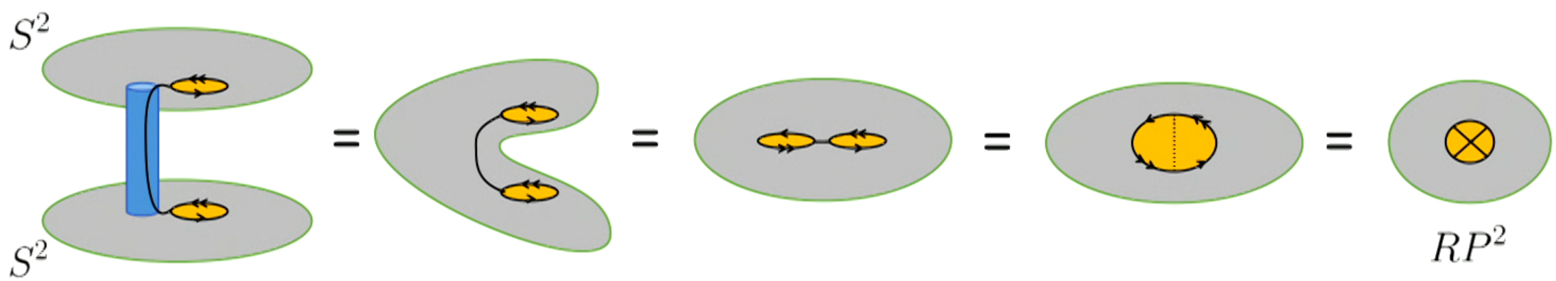
$$Z = \mathrm{Tr}(\rho U_1 \rho^{T_1} U_1^\dagger)$$

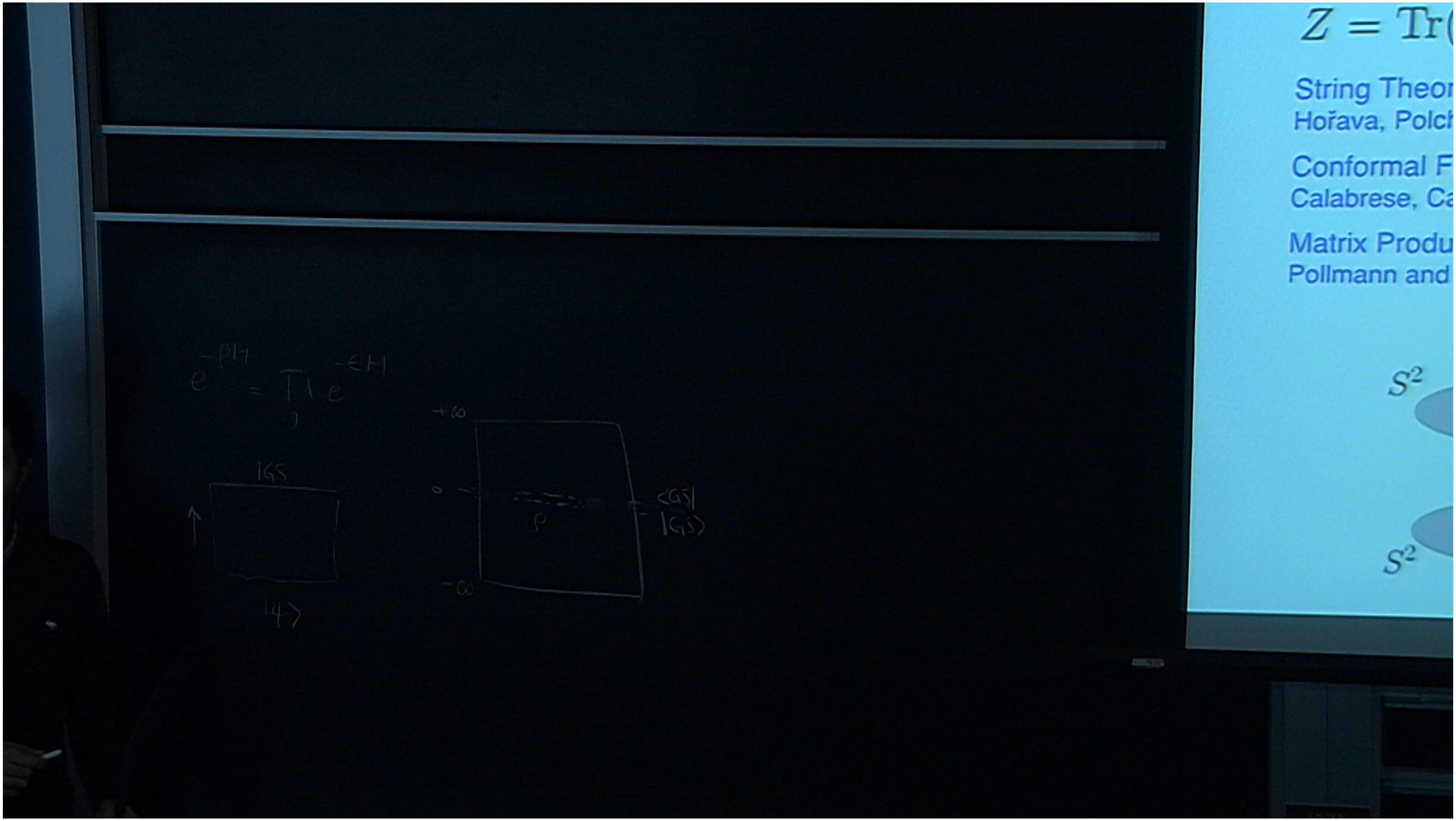
String Theory
Hořava, Polchinski, Sagnotti,…

Conformal Field Theories
Calabrese, Cardy, Tonni (2010)

Matrix Product States
Pollmann and Turner (2012)



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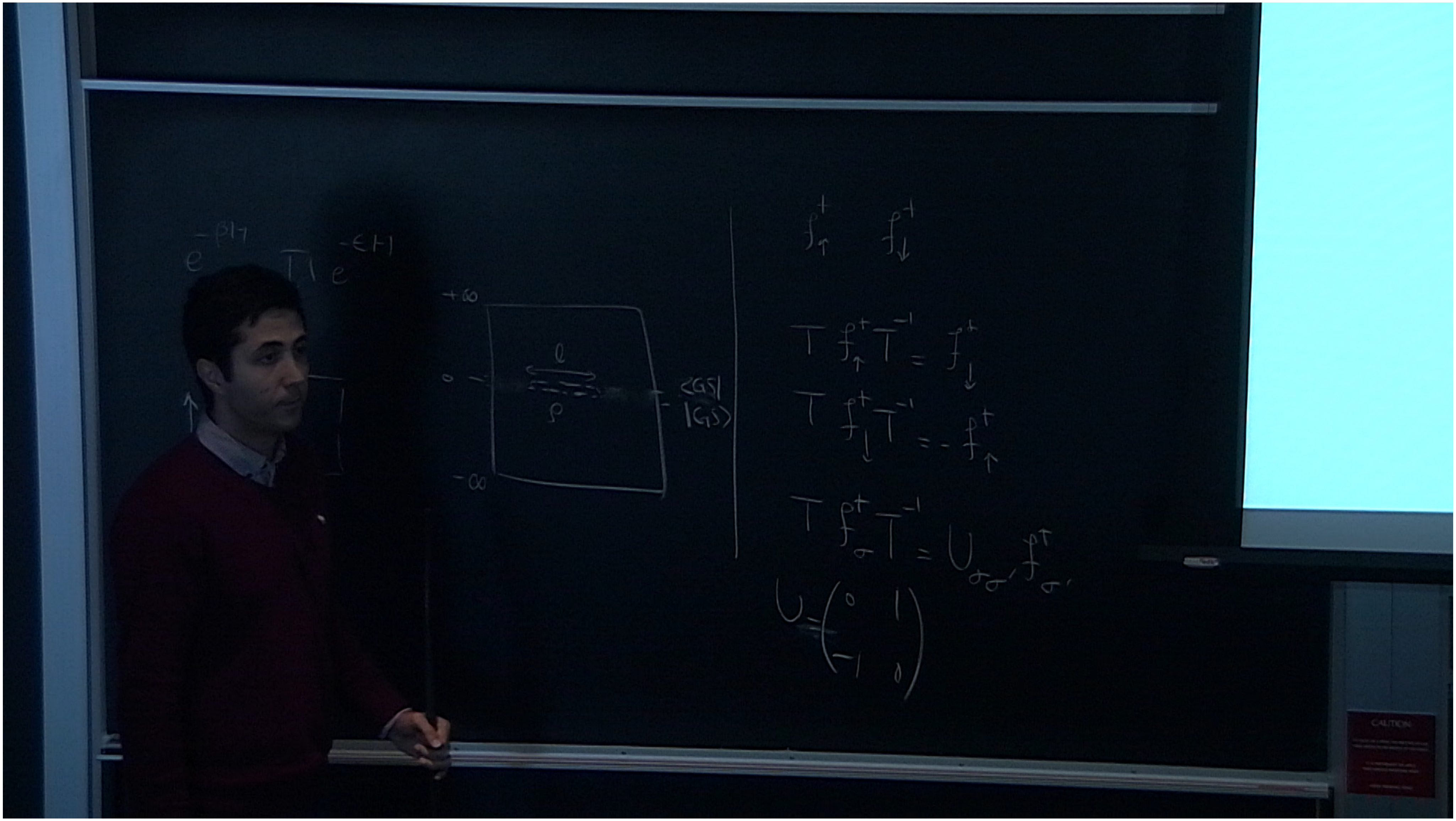
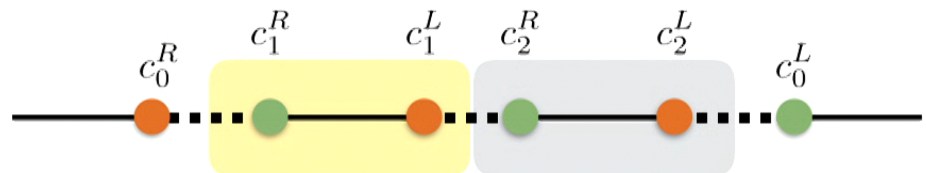


(1+1)d class BDI

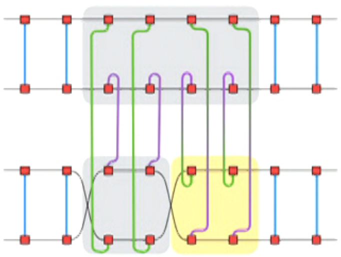
- Topological invariant: **Partial** time-reversal

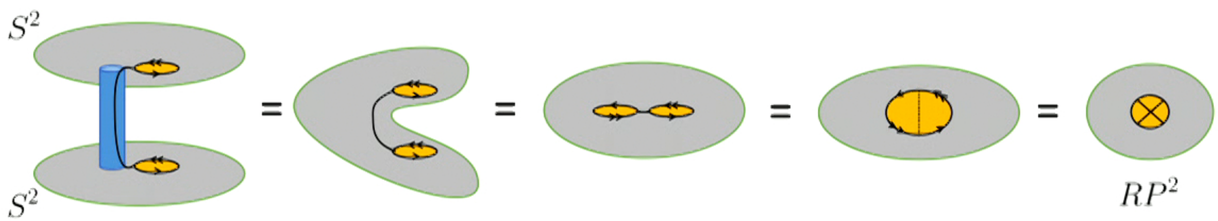
$$Z = \mathrm{Tr}(\rho\rho^{T_1}) = \begin{cases} \dfrac{e^{i\frac{2\pi}{8}}}{2\sqrt{2}} & \text{Topological} \\ 1 & \text{Trivial} \end{cases}$$



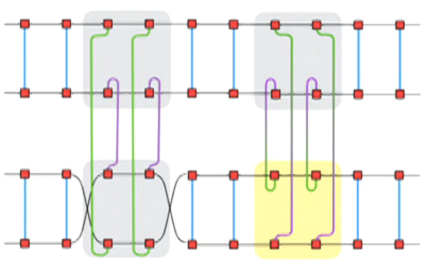
Different geometries of subsystem

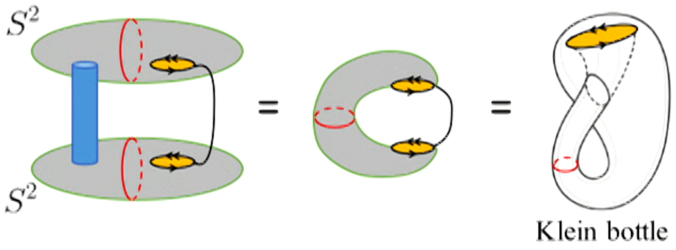
- Adjacent intervals ➡ Real projective plane $Z = \mathrm{Tr}(\rho U_1 \rho^{T_1} U_1^\dagger)$





- Disjoint intervals ➡ Klein bottle

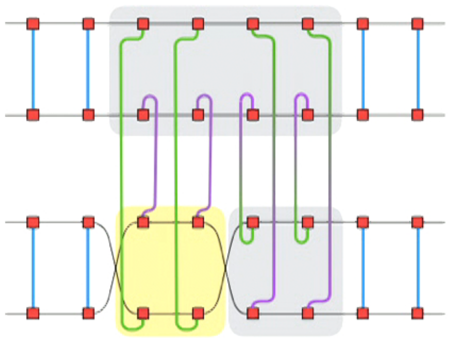


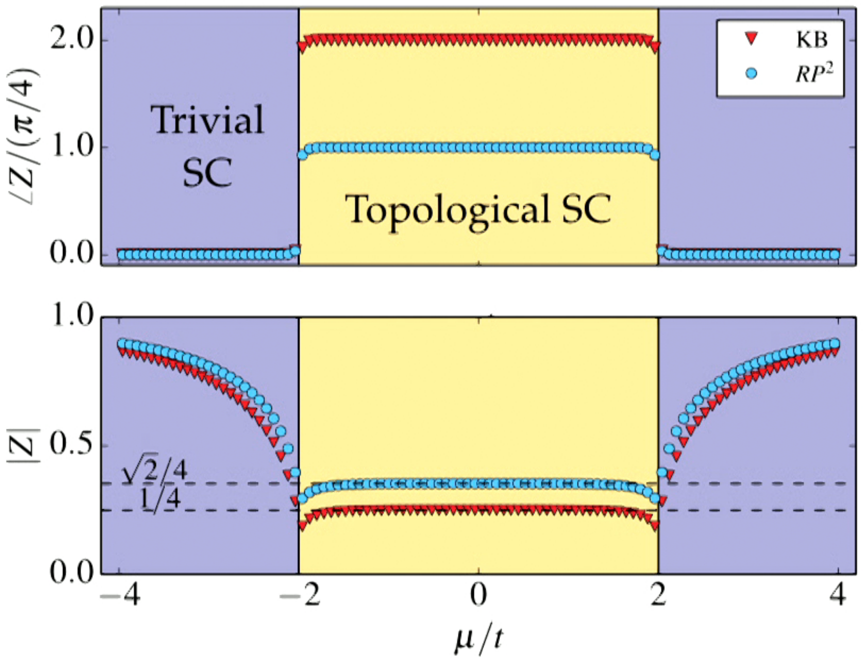


The Kitaev chain

Class BDI

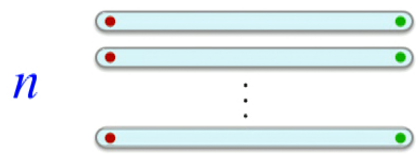
$$Z = \mathrm{Tr}(\rho\rho^{T_1})$$



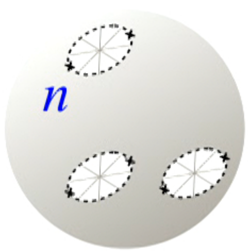


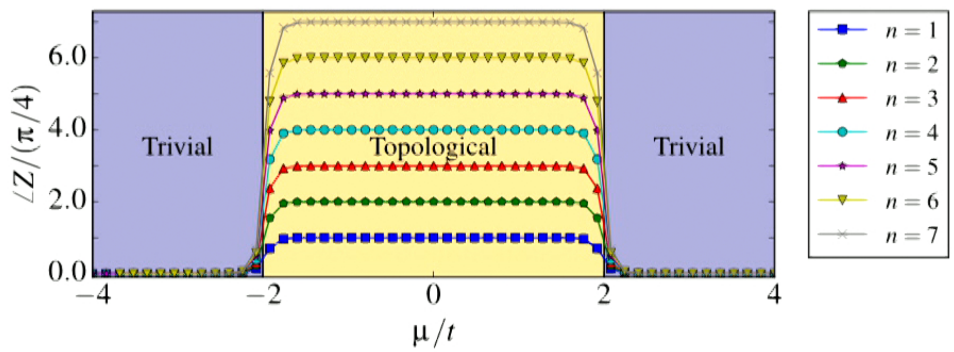
Additivity of topological invariant

- Multiple chains $[Z^{\mathbb{R}P^2}]^n \sim e^{i\frac{2\pi n}{8}}$



- Multiple cross-caps $Z(\otimes^n) \sim e^{i\frac{2\pi n}{8}}$





- Show $\mathbb{Z}_8$ cyclic group!

Cobordism Theory

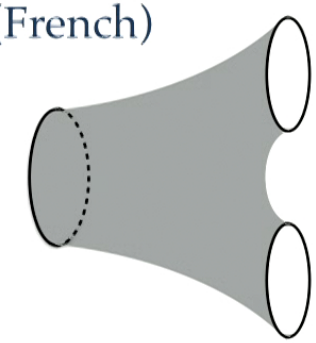
- SPT phases of fermions are characterized by spin-TQFT

  Kapustin, Freed, Hopkins, Witten, Thorngren …

$$Z(X,\eta,A) \sim e^{iS_{\text{top}}(X,\eta,A)}$$

$$Z : (X,\eta,A) \to U(1)$$

bord = boundary
(French)



- Manifolds are classified into co**bordism group** $\Omega_d^{\text{str}}(BG)$

- Two manifolds from **same** cobordism class give **same** U(1) phase
- Two manifolds from **different** cobordism classes give **different** U(1) phases

SPTs protected by anti-unitary symmetries

AZ class and space dim.	Cobordism	Generating space-time manifold	Topological invariant
BDI in $(1+1)d$	$\Omega_2^{\text{pin}_-} = \mathbb{Z}_8$	$\mathbb{R}P^2$	$\text{tr}\,[\rho_I C_T^{I_1} \rho_I^{\mathsf{T}_1} [C_T^{I_1}]^\dagger]$
DIII in $(1+1)d$	$\Omega_2^{\text{pin}_+} = \mathbb{Z}_2$	KB, (R,R) sector	$\text{tr}\,[\rho_{I_1\cup I_3}((-1)^{F_2})C_T^{I_1} \rho_{I_1\cup I_3}^{\mathsf{T}_1}((-1)^{F_2})[C_T^{I_1}]^\dagger]$
AIII in $(1+1)d$	$\Omega_2^{\text{pin}^c} = \mathbb{Z}_4$	$\mathbb{R}P^2$, the flux threading $\mathbb{R}P^2$ is quantized to $\pm i$	$\text{tr}\,[\rho_I U_S^{I_1} \rho_I^{\mathsf{T}_1} [U_S^{I_1}]^\dagger]$
AI in $(1+1)d$	$\Omega_2^{\text{pin}_-^{\tilde{c}}} = \mathbb{Z} \times \mathbf{Z}_2$	$\mathbb{R}P^2$ for $\mathbf{Z}_2$ and a two-manifold with a unit magnetic flux for $\mathbb{Z}$	$\text{tr}\,[\rho_I C_T^{I_1} \rho_I^{\mathsf{T}_1} [C_T^{I_1}]^\dagger]$ for $\mathbf{Z}_2$
AII in $(1+1)d$	$\Omega_2^{\text{pin}_+^{\tilde{c}}} = \mathbb{Z}$	$\mathbb{R}P^2$ with a half magnetic flux $\int_{\mathbb{R}P^2} F = \pi$	$\text{tr}\,[\rho_I \prod_{x\in I_1} e^{\frac{\pi i x \hat{n}(x)}{2\|I_1\|}} C_T^{I_1} \rho_I^{\mathsf{T}_1} [C_T^{I_1}]^\dagger \prod_{x\in I_1} e^{\frac{-\pi i x \hat{n}(x)}{2\|I_1\|}}]$

Conclusions and outlook

- Many-body topological invariants of SPT phases protected by anti-unitary symmetries

- Higher dimensions, point-group symmetries

  K. Shiozaki, **HS**, S. Ryu, PRB **95**, 205139 (2017)

$$\langle \Psi | g_D | \Psi \rangle \sim \exp \left[i\theta + \gamma - \alpha \frac{\text{Area}(\partial D)}{\xi^{d-1}} \right]$$

- **Outlook:**
  - Numerical applications (Quantum Monte Carlo, Tensor Network Models, etc)
  - Possibilities in symmetry enriched topological order?
  - Fermionic partial transpose ρ^{T_1} → anyons?
  - Entanglement distillation?