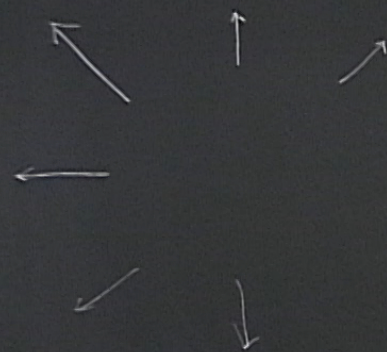


Title: PSI 2018/2019 - Statistical Mechanics - Lecture 10

Date: Oct 25, 2018 10:45 AM

URL: <http://pirsa.org/18100033>

Abstract:



$$\Theta \simeq n \phi, \quad -\beta E = \frac{-3J}{1} 2\pi \int_{r_0}^L r dr \frac{1}{r^2} \simeq -2$$

$\uparrow$
 charge

$$\vec{\nabla} \Theta = \left(\frac{y}{r^2}, -\frac{x}{r^2} \right), \quad (\vec{\nabla} \Theta)^2 = \frac{x^2 + y^2}{r^4} = \frac{1}{r^2}$$

$$\int_{\pi_0}^L \pi dr \frac{1}{r^2} \simeq -2\pi\beta J \log L, \quad S \sim \log \frac{L^2}{\text{where to put the vortex}}$$

$\beta_c = \frac{1}{\pi}$

$\sim @ T = T_c$

$$\frac{dy^2}{r^4} = \frac{1}{r^2}$$

$$\langle SS \rangle \sim \frac{1}{|X|^{1/4}} \sim \exp\left(-\frac{1}{4} \log r\right)$$

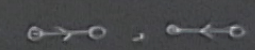
XY model

$$Z = \int \prod_{\vec{r}} \pi d\theta_{\vec{r}} \exp\left(\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right) \stackrel{\sim}{\sim}_{\text{low } T} \int \prod_{\vec{r}} \pi d\vec{\theta}_{\vec{r}} \exp\left(-\frac{\beta J}{2} \sum_{\langle ij \rangle} (\vec{\theta}_i - \vec{\theta}_j)^2\right)$$

$$\langle S_{\vec{y}} S_{\vec{y}+\vec{x}} \rangle = \langle e^{i\theta_{\vec{y}} - i\theta_{\vec{y}+\vec{x}}} \rangle = e^{G(\vec{x}) - G(0)} = e^{-\frac{1}{4\pi\beta J} \log|\vec{x}|} \approx \frac{1}{|\vec{x}|^{\frac{1}{4\pi\beta J}}} \approx \exp\left[-\frac{1}{4\pi\beta J} \log|\vec{x}|\right]$$


 $\theta \sim n\phi \quad -\beta E = \frac{-\beta J}{4} 2\pi \int_0^L dx \frac{1}{2} \approx -2\pi\beta J$

$$Z \approx \exp\left(-\frac{\beta J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2\right) \underset{\substack{\text{LARGE} \\ \text{DISTANCES}}}{\approx} \int \mathcal{D}\vec{\Theta}(\vec{x}) \exp\left(-\frac{\beta J}{2} \int (\vec{\nabla}\vec{\Theta})^2 d\vec{x}\right)$$



$$G(0) = e^{-\frac{1}{4\pi\beta J} \Gamma(\vec{x})} \int \frac{d^2\vec{k}}{2\pi} \frac{e^{i\vec{k}\vec{x}}}{4\sin^2\frac{k_x}{2} + (y)} \approx \log \frac{\pi}{x_0} \frac{1}{2\sqrt{2}e^{\gamma_E}}$$

$$\approx \exp\left[-\frac{1}{4\pi\beta J} \log|\vec{x}| \right]$$

$$\frac{1}{2} \approx -2\pi\beta J \log L, \quad S \sim \log L^2$$

when to put the vortex

$$|\vec{x}|^{4\pi\beta J} = \exp[-4\pi\beta J \phi]$$

$\theta \simeq n\phi$, $-\beta E = \frac{-\beta J}{1} 2\pi \int_{r_0}^L r dr \frac{1}{r^2} \simeq -2\pi\beta J \log L$, S

↑
 Charge, $n = \pm 1$
 ↓

$\vec{\nabla}\theta = \left(\frac{y}{r^2}, -\frac{x}{r^2}\right)$, $(\vec{\nabla}\theta)^2 = \frac{x^2+y^2}{r^4} = \frac{1}{r^2}$

$J\beta_c = \frac{1}{\pi}$

$$\Theta(\vec{x}) = \underbrace{\Theta_{\text{vortices@positions } \vec{r}_j}} + \Theta_{\text{spin waves}}$$

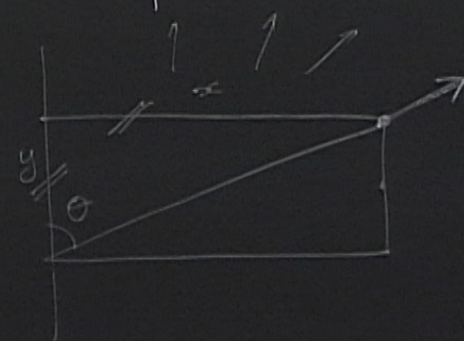
pin waves

$$Z = \sum_{\text{vortices}} \times \sum_{\substack{\text{Gaussian} \\ \text{S.W.}}}$$

$$\Theta(\vec{x}) = \underbrace{\Theta_{\text{vortices @ positions } \vec{x}_j}} + \Theta_{\text{spin waves}}$$

$$\Theta = \sum_j n_j \arctan \frac{x - x_j}{y - y_j}$$

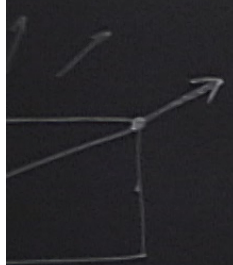
min. action, $\nabla^2 \Theta = 0$



$$Z = 1$$

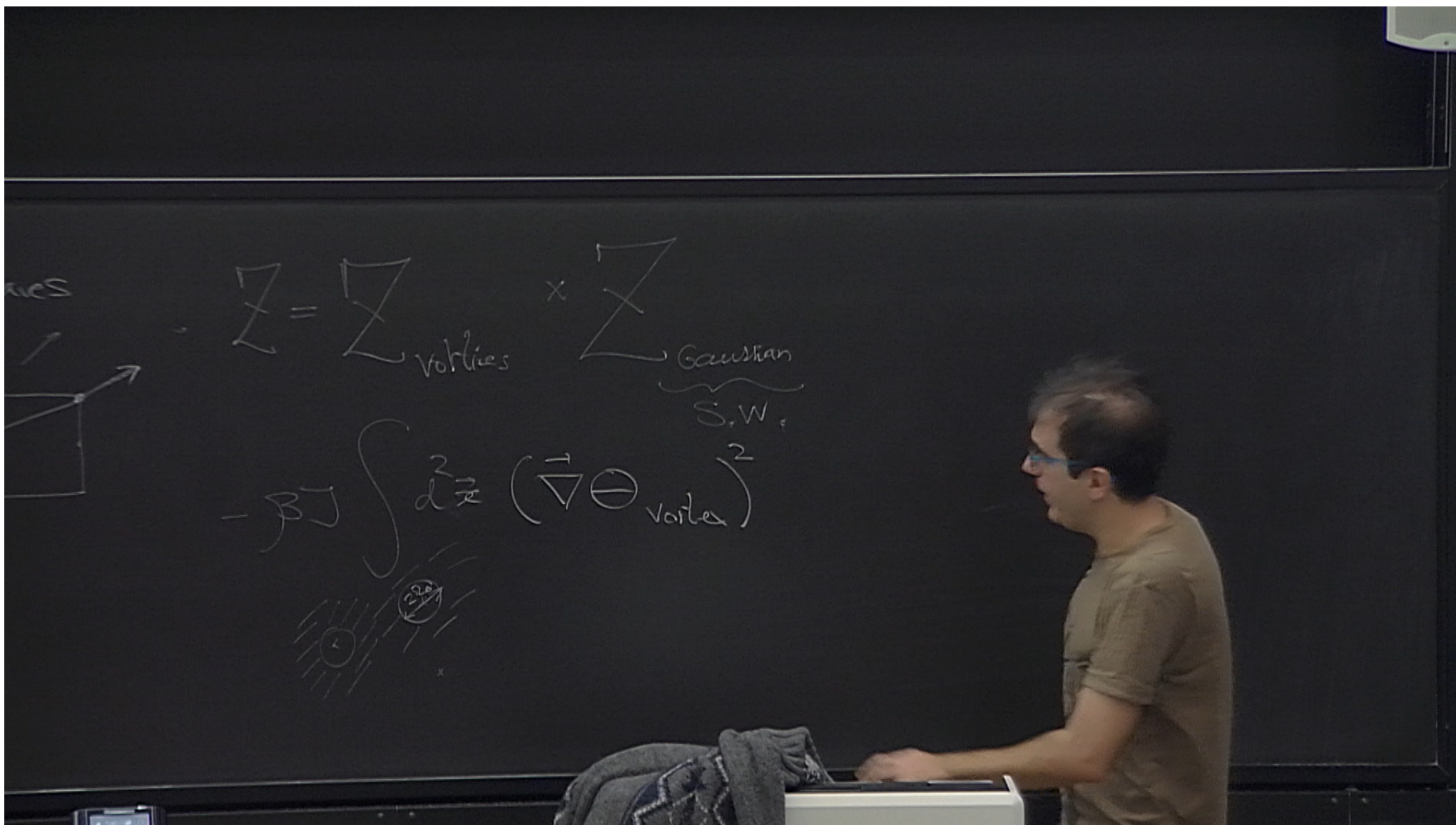
$$- \beta J$$

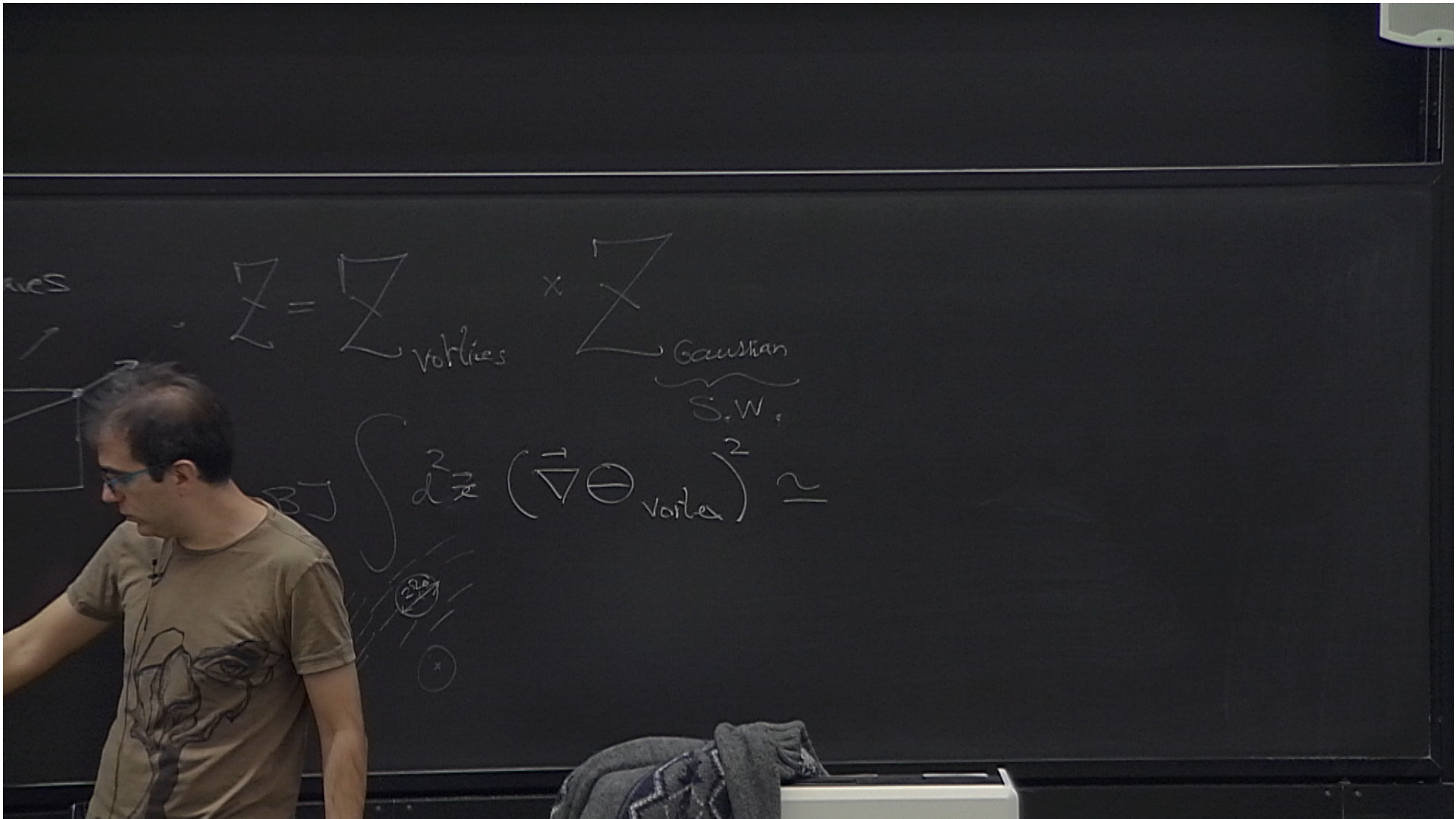
naives



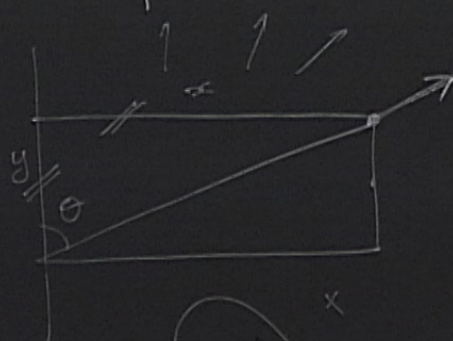
$$Z = \sum_{\text{vortices}} \times \underbrace{\sum_{\text{Gaussian}}}_{\text{S.W.}}$$

$$- \beta J \int d^2 \vec{z} \left(\vec{\nabla} \Theta_{\text{vortex}} \right)^2$$





$\vec{r}_j$ spin waves



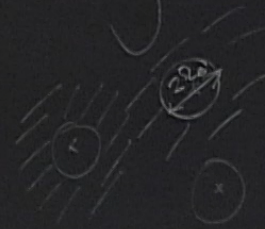
$$i_j \arctan \frac{x-x_j}{y-y_j}$$

min. action, $\nabla^2 \Theta = 0$

$$2\pi(n_1+n_2) = \oint \vec{dl} \cdot \vec{\nabla} \Theta$$

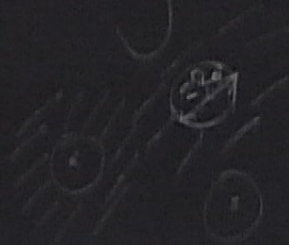
$$Z = \sum_{\text{vortices}} \times \sum_{\text{Gaussian S.W.}}$$

$$- \beta J \int d^2 \vec{x} (\vec{\nabla} \Theta_{\text{vortex}})^2 \approx$$



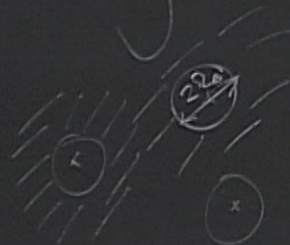
$$Z = Z_{\text{vortices}} \times Z_{\text{Gaussian}} \\ \text{S.W.}$$

$$-\beta J \int d^2x (\vec{\nabla} \Theta_{\text{vortex}})^2 \simeq S_{\text{vortex}}(n_j, \vec{r}_j)$$



$$Z = \sum_{\text{vortices}} \times \underbrace{\sum_{\text{Gaussian}}}_{S, W,}$$

$$-\beta J \int d^2 \vec{x} \left(\vec{\nabla} \Theta_{\text{vortex}} \right)^2 \simeq - \sum_{\text{vortex}} \left(n_j, \vec{r}_j \right)$$



$$\Theta = \sum_j n_j \arctan \frac{y - y_j}{x - x_j}$$

min. action, $\nabla^2 \Theta = 0$

$2\pi(n_1 + n_2) = \oint \vec{\nabla} \Theta \cdot d\vec{l}$

$-\beta J \int d\vec{z} (\nabla \Theta_{\text{vortex}})^2 \approx$

$$S_{\text{vortex}} = 2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{z}_i - \vec{z}_j|}{a} - 2\pi\beta J \sum_i n_i^2 \log \frac{\xi_0}{a}$$

$\uparrow$ lattice spacing $= 1$

$$\sum n_i^2 \log \frac{\pi_0}{a} \quad [\text{actually a ind.}]$$

$$\begin{aligned}
 & \frac{|\vec{r}_i - \vec{r}_j|}{a} = 2\pi \beta \sum_i n_i^2 \log \frac{r_0}{a} \quad \left[\text{actually a ind.} \right. \\
 & \quad \uparrow \text{lattice spacing} \\
 & \quad = 1 \\
 & \quad \sum_j n_j = 0, \quad \sum_i n_i \sum_{j \neq i} n_j = - \sum_i n_i^2
 \end{aligned}$$

min. action,

$$\vec{\nabla}^2 \Theta = 0$$

$$-S_{\text{vortex}} = +2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{x}_i - \vec{x}_j|}{a} + 2\pi\beta J$$

$\uparrow$
lattice
spacing
 $= 1$

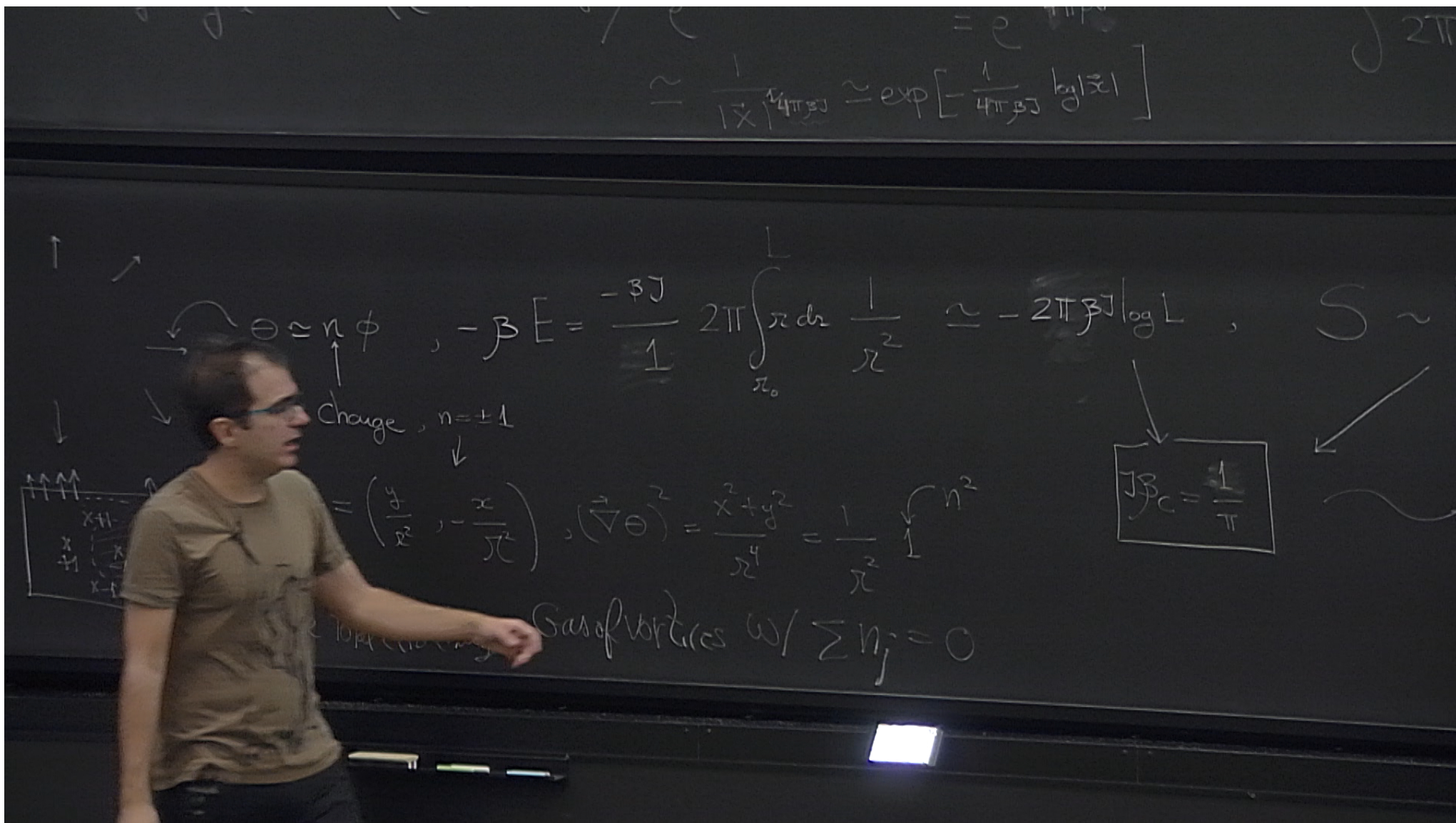
min. action, $\vec{\nabla}^2 \Theta = 0$

$$-S_{\text{vortex}} = +2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{z}_i - \vec{z}_j|}{a}$$

$n_i = +1$ vortices
 $n_i = -1$ antivortices

$$Z_{\text{vortices}} = \sum_{n=0}^{\infty}$$

a
lattice spacing
 $= 1$



min. action,

$$\vec{\nabla}^2 \Theta = 0$$

$$2\pi(n_1+n_2) = \oint \vec{\nabla} \Theta \cdot \vec{dl}$$

z_i

$$-S_{\text{vortex}} = +2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{z}_i - \vec{z}_j|}{a} + 2\pi\beta J$$

$n_i = +1$ vortices:
 $n_i = -1$ antivortices

a
lattice spacing
 $= 1$

$$Z_{\text{vortices}} = \sum_{n=0}^{\infty} \frac{K^n}{(n!)^2}$$

$$\text{min. action, } \vec{\nabla}^2 \Theta = 0 \quad 2\pi(n_1+n_2) = \oint d\vec{l} \cdot \vec{\nabla} \Theta$$

$$-S_{\text{vortex}} = +2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{r}_i - \vec{r}_j|}{a} + 2\pi\beta J \sum_i n_i$$

$n_i = +1$ vortex;
 $n_i = -1$ antivortex

a
 lattice spacing = 1

$$Z_{\text{vortices}} = \sum_{n=0}^{\infty} \frac{K^n}{(n!)^2} \int_{|\vec{r}_i - \vec{r}_j| > a} d\vec{r}_1 \dots d\vec{r}_{2n} \exp\left(2\pi\beta J \sum_{\langle ij \rangle} n_i n_j \log |\vec{r}_i - \vec{r}_j|/a\right)$$

$$+ 2\pi\beta J \sum_i n_i^2 \log \frac{z_0}{a} \quad [\text{actually a ind.}]$$

$$\sum_j n_j = 0, \quad \sum_i n_i \sum_{j \neq i} n_j = - \sum_i n_i^2$$

$$\log |\vec{r}_i - \vec{r}_j|/a$$

where $K = \exp(2\pi\beta J \log(z/a))$

$$X^{1/4} \sim \exp\left(-\frac{1}{4} \log \tilde{x}\right)$$

$$Z_{\text{vortices}} = \sum_{n=0}^{\infty} \frac{K^{2n}}{(n!)^2} \int \frac{d\vec{r}_1 \dots d\vec{r}_{2n}}{|\vec{r}_i - \vec{r}_j| > a} \exp\left(2\pi\beta J \sum_{\langle i,j \rangle} \dots\right)$$

↑
Grand canonical Z
for a gas of
Coulomb charged
particles is

$$X^{1/4} \sim \exp\left(-\frac{1}{4} \log \bar{z}\right)$$

$$Z_{\text{vortices}} = \sum_{n=0}^{\infty} \frac{K^{2n}}{(n!)^2} \int \frac{d\vec{r}_1 \dots d\vec{r}_{2n}}{|\vec{r}_i - \vec{r}_j| > a} \exp\left(2\pi\beta J \sum_{\langle i,j \rangle} \dots\right)$$

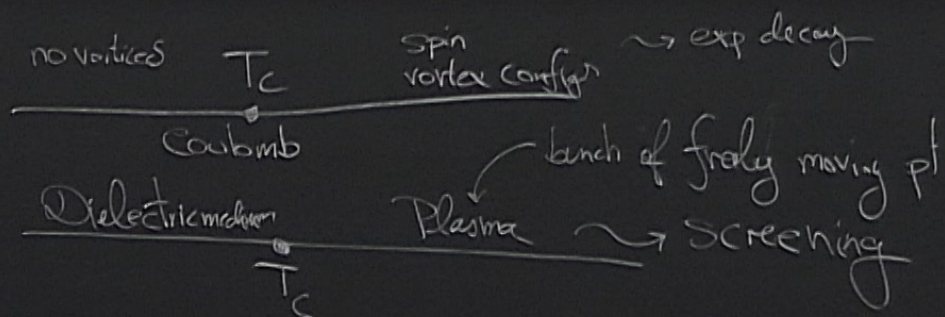
↑
Grand canonical Z
for a gas of
Coulomb charged
particles in 2D

Vortices

$$= \sum_{n=0}^{\infty} \frac{h}{(n!)^2} \int_{|\vec{r}_i - \vec{r}_j| > a} d\vec{r}_1 \dots d\vec{r}_{2n} \exp\left(2\pi\beta J \sum_{\langle i,j \rangle} n_i n_j \log |\vec{r}_i - \vec{r}_j|/a\right)$$

XY model

↑
Grand canonical $\mathcal{Z}$
for gas of
charged
particles in 2D

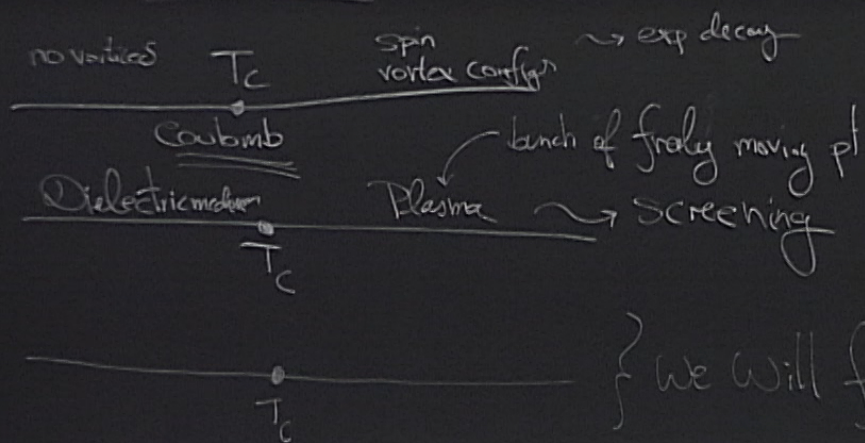


$$\int_{|\vec{r}_i - \vec{r}_j| > a} d\vec{r}_1 \dots d\vec{r}_n \exp\left(2\pi\beta J \sum_{i \neq j} n_i n_j \log |\vec{r}_i - \vec{r}_j|/a\right) \quad \text{where } K = \exp\left(2\pi\beta J \log(r/a)\right)$$

Spacing = 1

$\sum_j n_j = 0, \quad \sum_i n_i = \sum_{j \neq i} n_j$

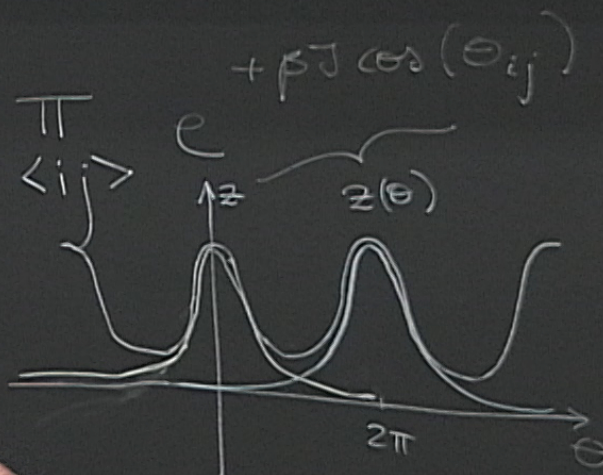
XY model



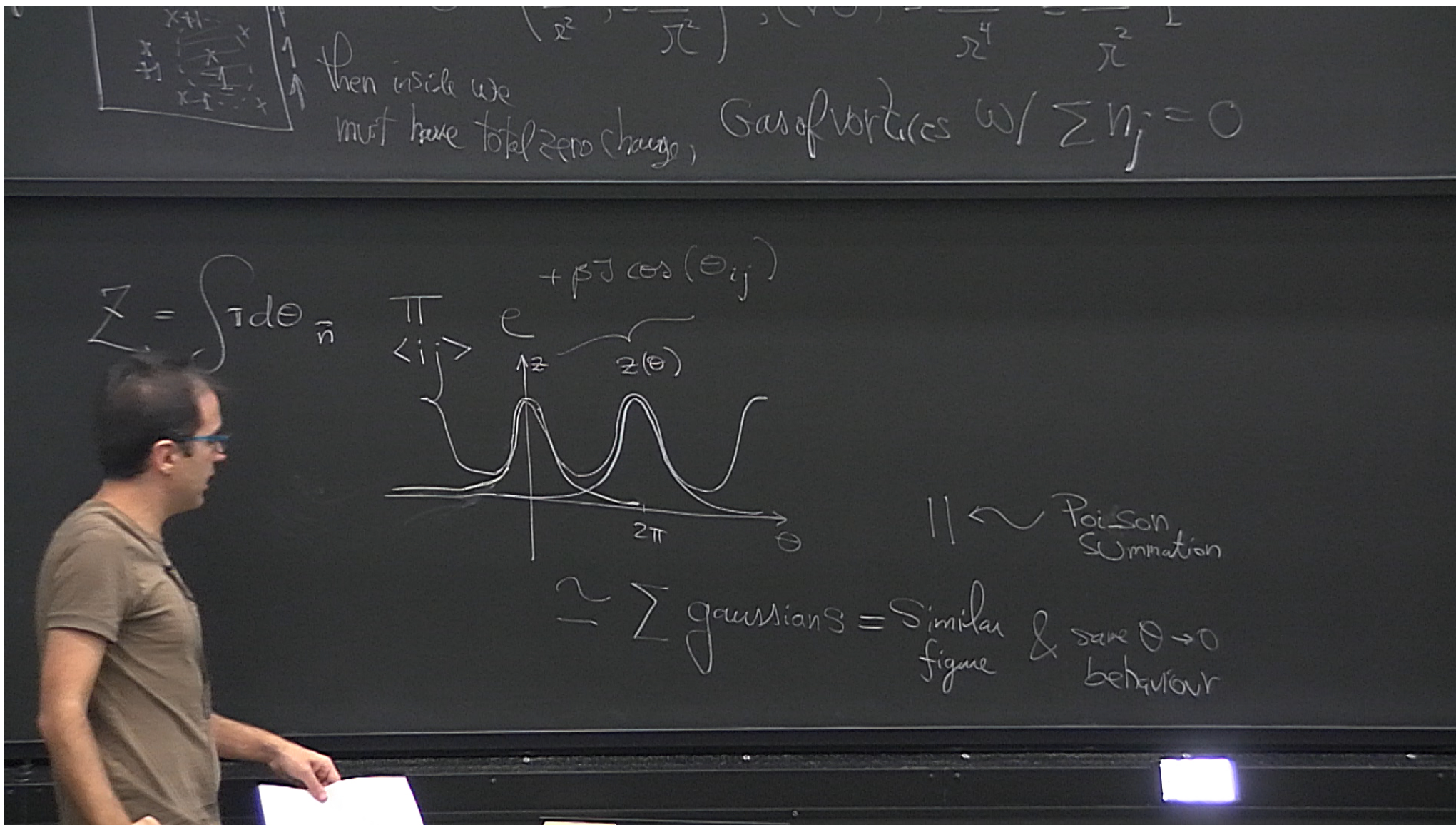


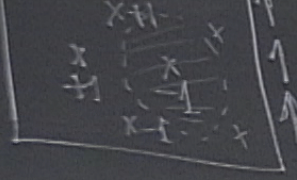
then inside we must have total zero charge, Gas of vortices w/ $\sum \eta_j = 0$

$$Z = \int \prod d\theta \, \bar{n}$$



$\approx \sum \text{gaussians} = \text{similar figure \& same } \theta \rightarrow 0 \text{ behaviour}$

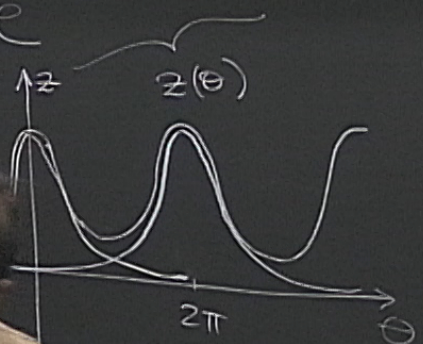




then inside we must have total zero charge, Gas of vortices w/ $\sum n_j = 0$

$$Z = \int \prod d\theta_{\vec{n}}$$

$\pi \langle ij \rangle$

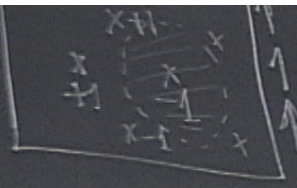


$$+ \beta J \cos(\theta_{ij})$$

$$\sum_n e^{-\frac{n^2}{2\beta J} + i n \theta}$$

$\parallel \sim$ Poisson summation

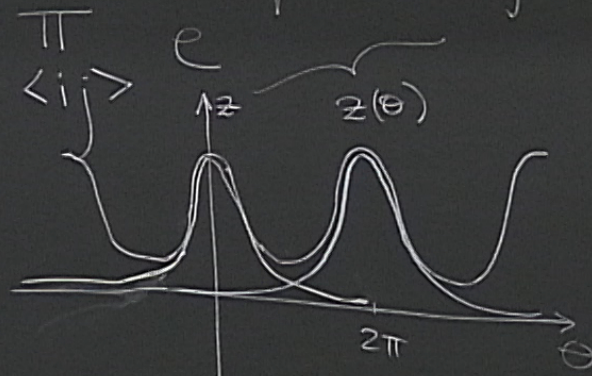
$$\approx \sum \text{gaussians} = \text{similar figure \& same } \theta \rightarrow 0 \text{ behaviour}$$



then inside we must have total zero charge, Gas of vortices w/ $\sum n_j = 0$

$$Z = \int d\theta \bar{n}$$

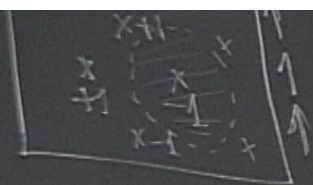
$$+ \beta J \cos(\theta_{ij})$$



$$\approx \sum_n e^{-\frac{n^2}{2\beta J}} + i n \theta$$

$\parallel \leftarrow$ Poisson summation

$\approx \sum \text{gaussians} = \text{similar figure \& same } \theta \rightarrow 0 \text{ behaviour}$



then inside we
must have total zero charge,

Gas of vortices w/ $\sum n_j = 0$

$$Z = \int \prod d\theta_{\vec{n}} e^{+\beta J \cos(\theta_{ij})}$$

$\langle ij \rangle$ π z $z(\theta)$ 2π θ

same universality class

$$\approx \sum_n e^{-\frac{n^2}{2\beta J} + i n \theta}$$

Poisson summation

$\approx \sum \text{gaussians} = \text{similar figure \& same } \theta \rightarrow 0 \text{ behaviour}$

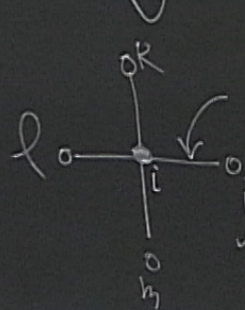
$+in\Theta$

$\swarrow$ bad : new n_{ij} per link !
 $\searrow$ good : integrate over Θ !

$+in\Theta$

bad : new n_{ij} per link !

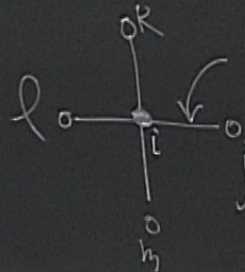
good : integrate over Θ !


$$e^{i\Theta_i} (n_{ij} + n_{ik} + n_{il} + n_{im})$$
$$= 0 \leftarrow \int d\Theta$$

$+in\Theta$

bad : new n_{ij} per link!

good : integrate over Θ ! $\leadsto$ just $\sum n_{ij}$ with

$$e^{i\Theta_i} (n_{ij} + n_{ik} + n_{il} + n_{im})$$


$$= 0 \leftarrow \int d\Theta$$

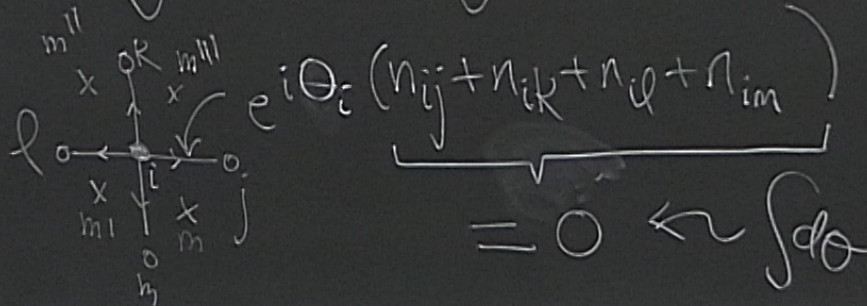
$$\sum_{\langle j,i \rangle} n_{ij} = 0$$

$+in\Theta$

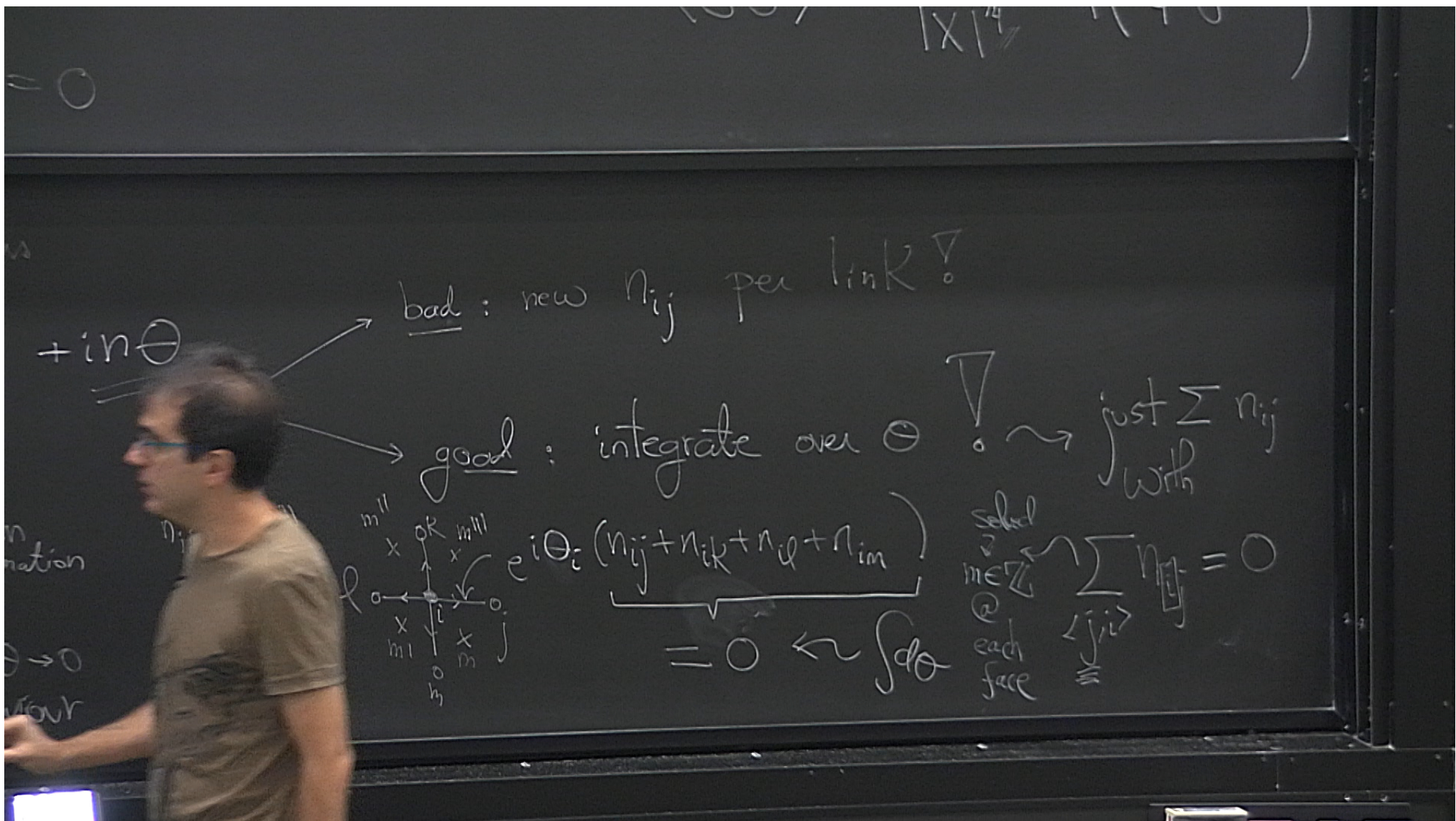
bad : new n_{ij} per link!

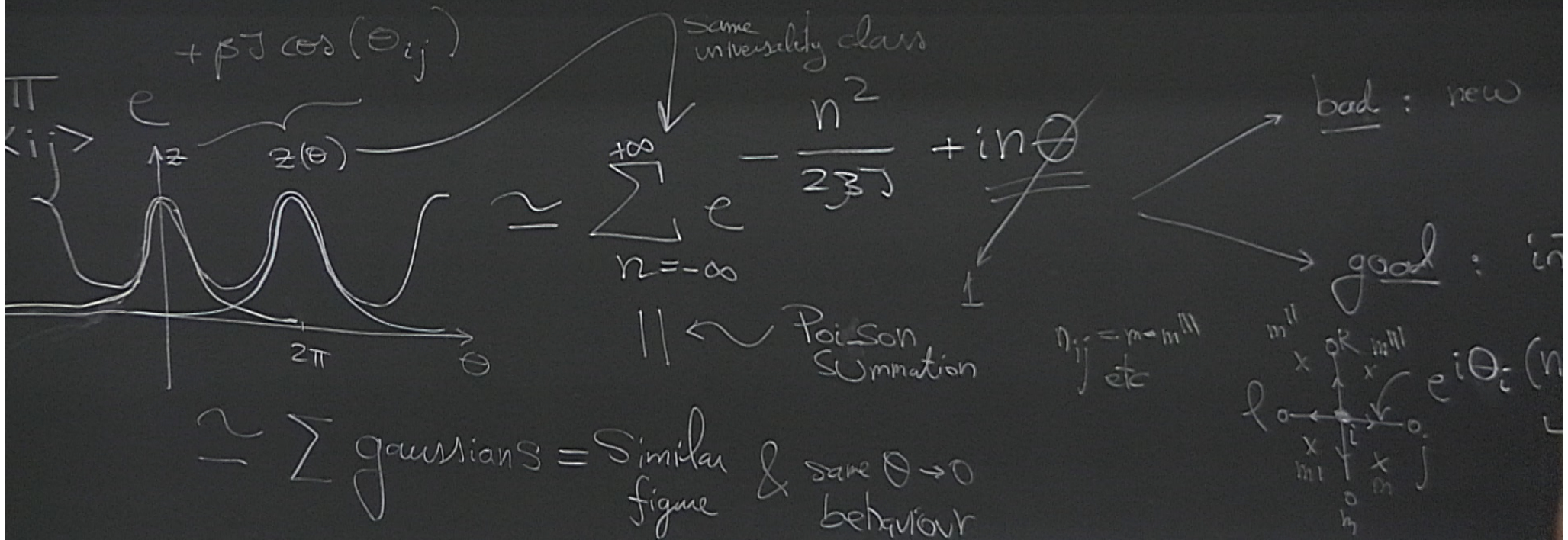
good : integrate over Θ ! $\leadsto$ just $\sum n_{ij}$ with

$n_{ij} = n - m^{III}$
etc



$$\sum_{\langle j|i \rangle} n_{ij} = 0$$

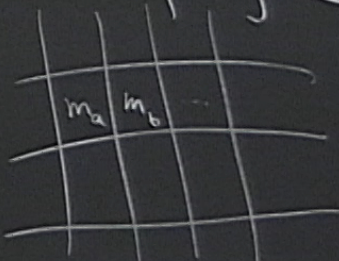




$\approx \sum \text{gaussians} = \text{Similar figure} \& \text{ same } \theta \rightarrow 0 \text{ behaviour}$

$$Z_{xy} \approx \sum_{z(\mathbf{x}) = \sum \text{gaussians}} \quad \text{Same Universality class}$$

$$= \sum \text{SOS or Fluctuating Surface model}$$

$$= \sum_{m_a \in \mathbb{Z}} \quad \text{per face}$$


$\theta \rightarrow 0$
 uniform

$$\sum_i \frac{x_i}{m_i} = 0$$

$$= 0 \leftarrow \int d\phi \text{ @ each face } \langle j_i \rangle$$

$$\exp \left(- \frac{1}{2\beta J} \sum_{\langle ab \rangle} (m_a - m_b)^2 \right)$$

$m_a \in \mathbb{Z}$

↑ per face

m_a	m_b	...

& same $0 \rightarrow 0$
behaviour

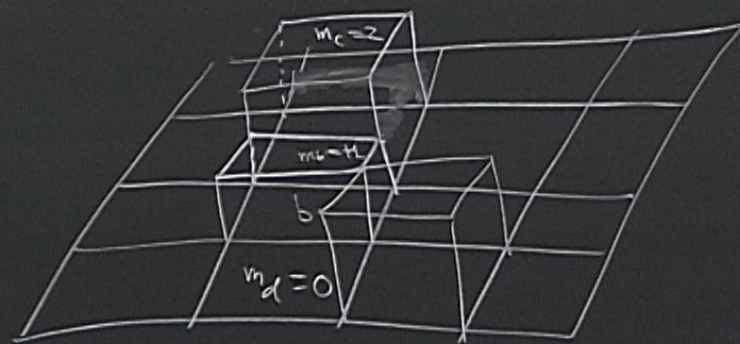
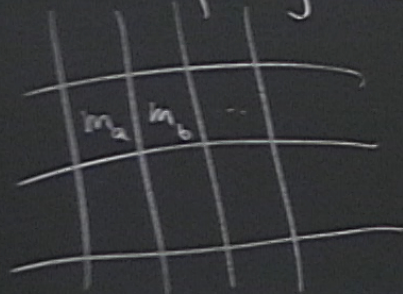
$$\frac{x_{m1}}{h} \frac{1}{h} \frac{x_m}{h}$$

$$= 0 \leftarrow \int d\theta \text{ @ each face } \langle \vec{j}_i \rangle$$

$$= \sum_{m_a \in \mathbb{Z}} \exp \left(- \frac{1}{2\beta J} \sum_{\langle ab \rangle} (m_a - m_b)^2 \right)$$

all

$m_a \in \mathbb{Z}$
↑ per face



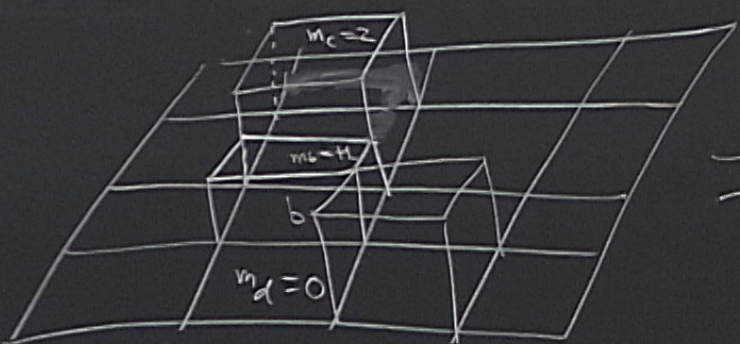
$\theta \rightarrow 0$
 flat

$$\sum_{m_i} \frac{x_i}{h} \frac{y_i}{h} \frac{z_i}{h}$$

$$= 0 \leftarrow \int d\Omega \text{ @ each face } \langle \vec{j}_i \rangle$$

$$\exp \left(- \frac{1}{2\beta J} \sum_{\langle ab \rangle} (m_a - m_b)^2 \right)$$

$m_a \in \mathbb{Z}$
 ↑ per face



Smooth $\neq m$'s suppressed

Rough $\leftarrow$ large fluctuations

$\theta \rightarrow 0$
 flat

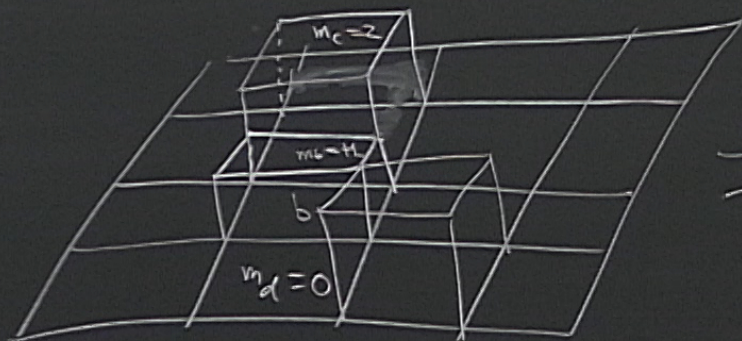
$$\sum_{m_i} \frac{x_i}{h} \frac{y_i}{h} \frac{z_i}{h}$$

$$= 0 \leftarrow \int d\Omega \text{ @ each face } \langle \mathbf{j}_i \rangle$$

$$\exp \left(- \frac{1}{2\beta J} \sum_{\langle ab \rangle} (m_a - m_b)^2 \right)$$

$m_a \in \mathbb{Z}$

per face

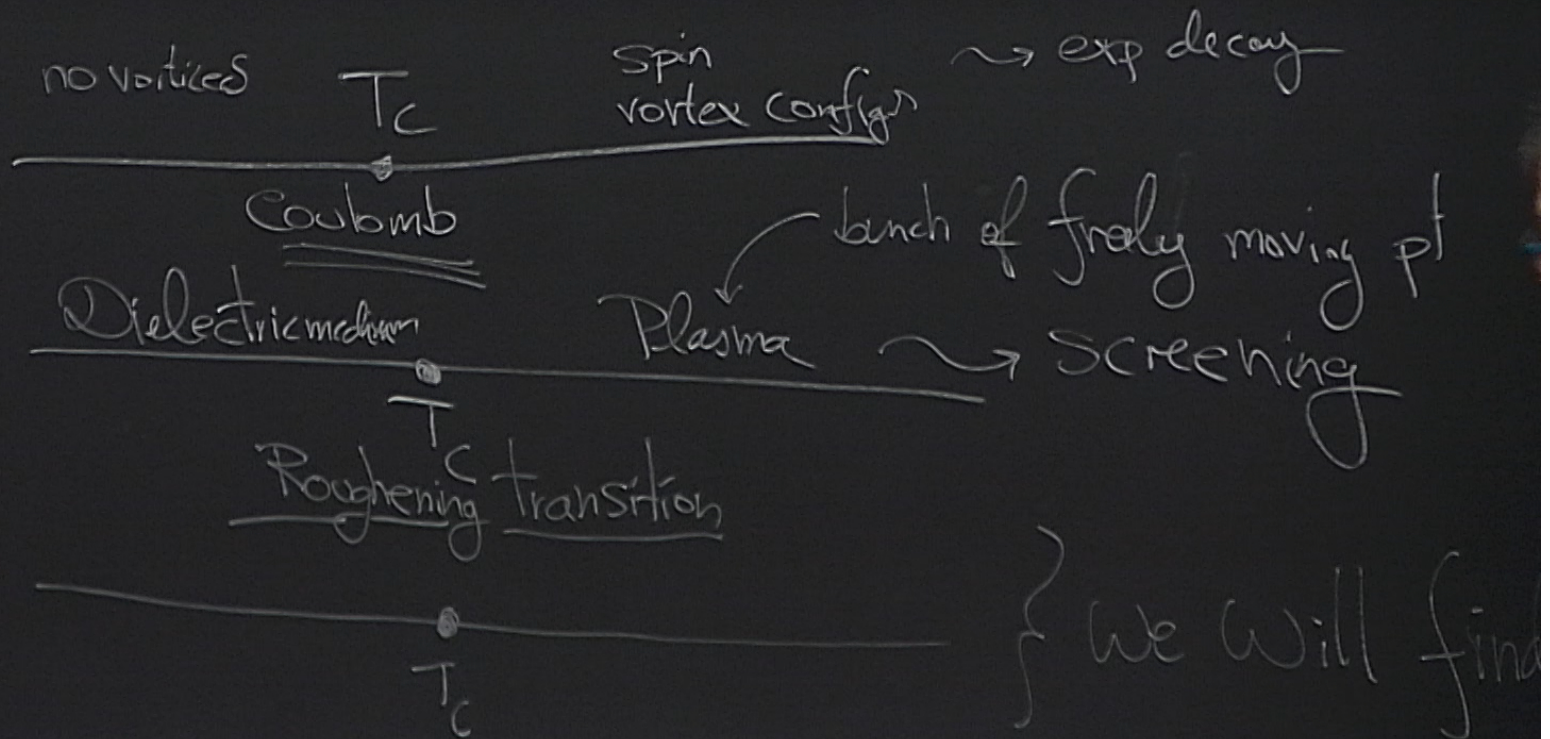


Smooth $\neq m$'s suppressed
 β small (T large)

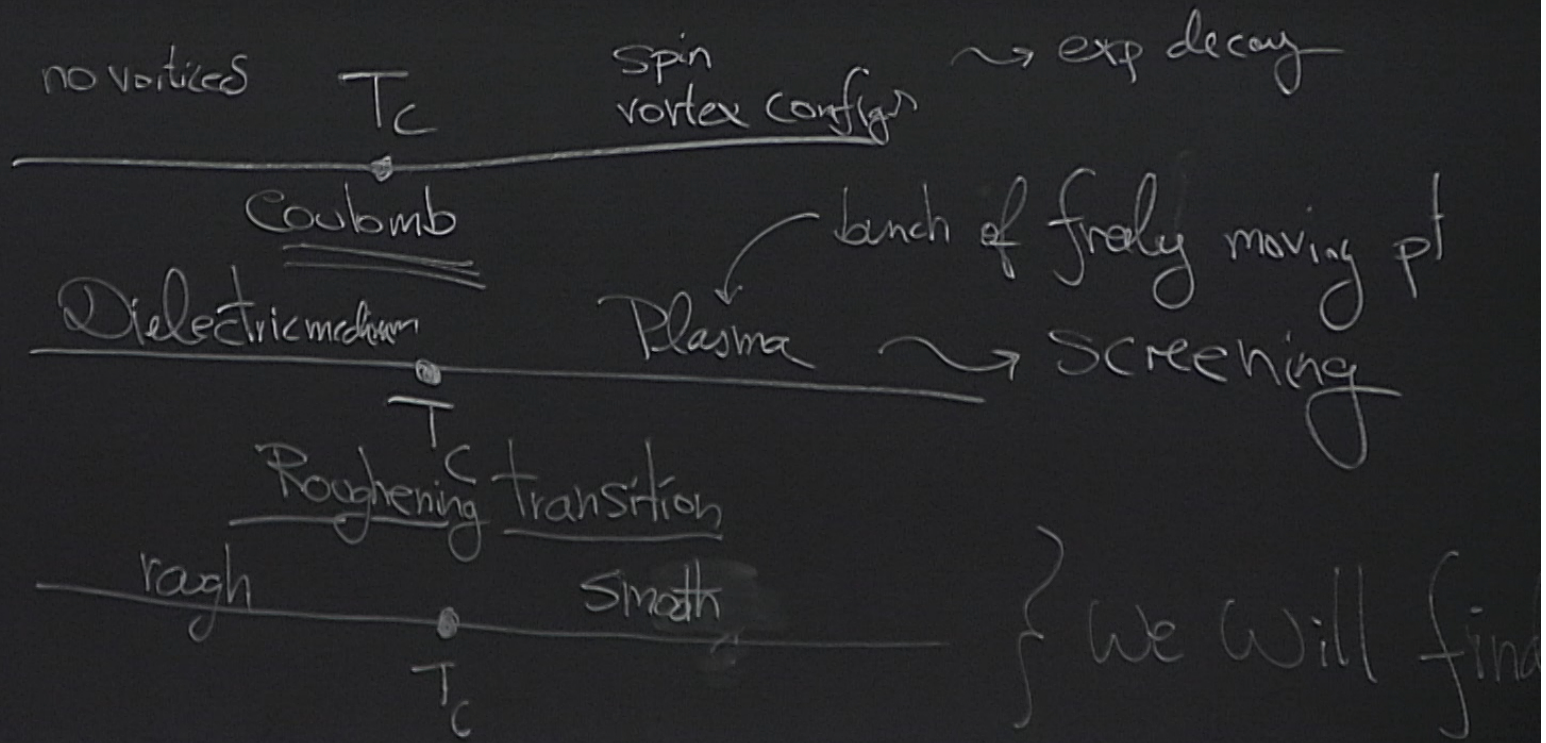
β large (T low)

Rough $\leftarrow$ large fluctuations

XY model

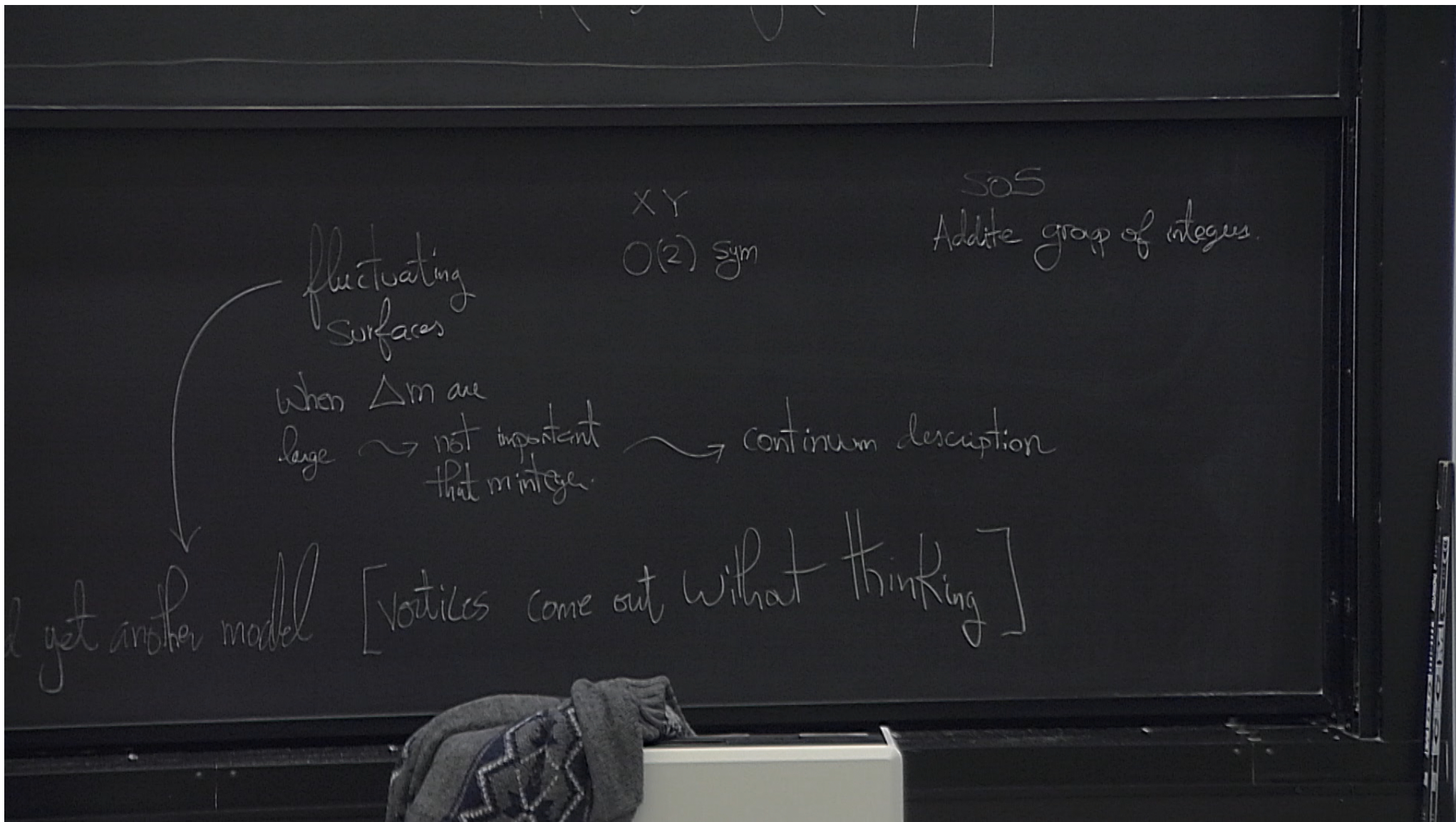


XY model



fluctuating
surfaces
when Δm are
large

just another model [vortices come out without thinking]



fluctuating
surfaces

when Δm are
large $\leadsto$ not important
that m integer.

XY
* $O(2)$ sym
* Lattice (cusps)

SOS
Additive group of integers.
Dual Lattice (faces)

$\leadsto$ continuum description

get another model [vortices come out without thinking]

fluctuating
surfaces

when Δm are
large $\leadsto$ not important
that m integer.

- XY
- * $O(2)$ sym
 - * Lattice (asps)
 - * βJ

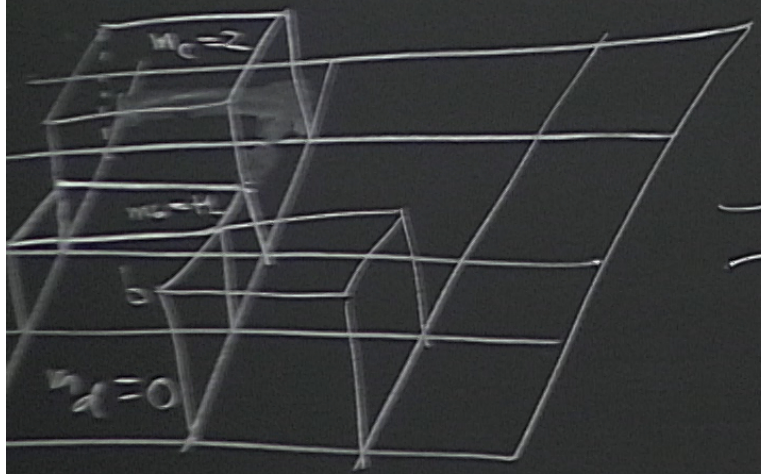
SOS
Additive group of integers.
Dual Lattice (faces)
 $\frac{1}{\beta J}$

$\leadsto$ continuum description

get another model [vortices come out without thinking]

$$\sum_{\langle ab \rangle} (m_a - m_b)^2$$

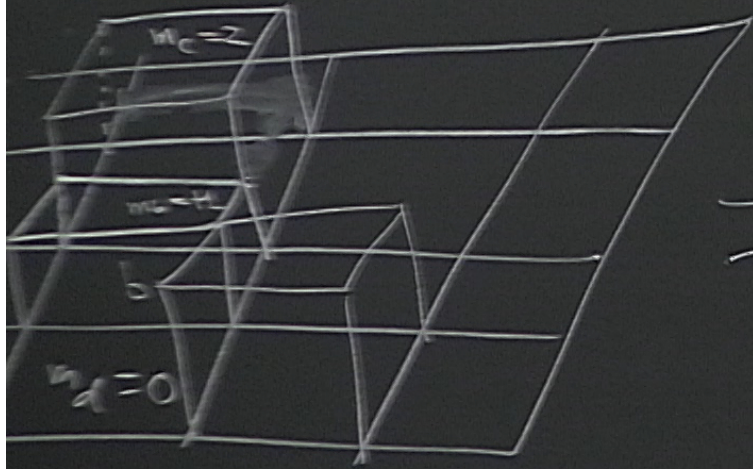
NEXT: Take the SOS model



Smooth $\neq m$'s sup
 β small (T large)
 β large (T low)
 Rough $\leftarrow$ large fluct

$$\sum_{\langle ab \rangle} (m_a - m_b)^2$$

NEXT: Take the SOS model
 $\leadsto$ Vortices



Smooth $\neq$ m 's suppressed
 β small (T large)

β large (T low)

Rough $\leftarrow$ large fluctuations

$$\sum_{m \in \mathbb{Z}} f(m) = \int d\varphi \, f(\varphi) \underbrace{\sum_m \delta(\varphi - m)}_{\text{cont}}$$

$$\sum_{n \in \mathbb{Z}} e^{2\pi i n \varphi}$$

will find get another model [variables come out of the box]

$$Z_{\text{SOS}} = \int \pi d\psi_{\vec{a}} \sum_{q_{\vec{a}}} \exp \left(-\frac{1}{2\beta J} \sum_{\vec{a}, \vec{b}} (\psi_{\vec{a}} - \psi_{\vec{b}})^2 + 2\pi i \sum_{\vec{a}} q_{\vec{a}} \psi_{\vec{a}} \right) \frac{Z}{Z}$$

$\uparrow$ face $\uparrow$ face $\uparrow$ Gaussian! $\log \frac{Z}{Z_0}$

$$= Z_{\text{SW}} \sum_{q_{\vec{a}}} \exp \left(2\pi \beta \sum_{\vec{a}, \vec{b}} \bar{q}_{\vec{a}} \cdot \Gamma(\vec{\kappa}_{\vec{a}} - \vec{\kappa}_{\vec{b}}) \cdot \vec{q}_{\vec{b}} \right)$$

$$Z_{\text{SOS}} = \int \pi d\psi_{\vec{a}} \sum_{\substack{q_{\vec{a}} \\ \text{face}}} \exp \left(-\frac{1}{2\beta J} \sum_{\vec{a}, \vec{b}} (\psi_{\vec{a}} - \psi_{\vec{b}})^2 + 2\pi i \sum_{\vec{a}} q_{\vec{a}} \psi_{\vec{a}} \right) \frac{Z}{Z}$$

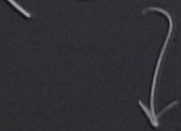
$$= Z_{\text{SW}} \sum_{q_{\vec{a}}} \exp \left(2\pi\beta \sum_{\vec{a}, \vec{b}} \vec{q}_{\vec{a}} \cdot \Gamma(\vec{\kappa}_{\vec{a}} - \vec{\kappa}_{\vec{b}}) \cdot \vec{q}_{\vec{b}} \right)$$

Annotations:
 - $\psi_{\vec{a}}$ is labeled "face" with an arrow.
 - $q_{\vec{a}}$ is labeled "face" with an arrow.
 - "with $\sum q_{\vec{a}} = 0$ " is written near the summation.
 - "zero mode $m \rightarrow m+n$ " is written near the summation.
 - "Gaussian!" is written with an arrow pointing to the quadratic term.
 - " $\log \frac{Z}{Z_0}$ " is written with an arrow pointing to the ratio of partition functions.

$$-S_{\text{vortex}} = +2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{x}_i - \vec{x}_j|}{a} +$$

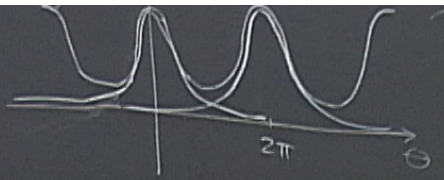
$$Z_{\text{vortices}} = \sum_{n=0}^{\infty} \frac{K^{2n}}{(n!)^2} \int_{|\vec{x}_i - \vec{x}_j| > a} d\vec{x}_1 \dots d\vec{x}_{2n} \exp\left(2\pi\beta J \sum_{i \neq j} n_i n_j \log \frac{|\vec{x}_i - \vec{x}_j|}{a}\right)$$

$$\langle e^{J \cdot x} \rangle = e^{\frac{1}{2} J \cdot A^{-1} \cdot J}$$



$$\langle e^{i\theta} e^{-i\theta} \rangle \sim e$$

$$1(-1) \Gamma(x)$$


 $\approx \sum_{n=-\infty}^{\infty} e^{i n \theta}$

Poisson Summation

goal: integrate over θ

$\approx \sum \text{Gaussians} = \text{Similar to sum } \theta \rightarrow 0$

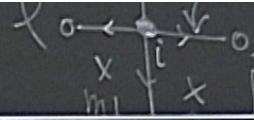
$\rho_{ij} = \rho_{i-m, j}$

$e^{i \theta_c (n_{ij} + n_{ik} + n_{il} + n_{im})}$

$$\sum_{\text{SOS}} = \sum_{\text{SW}} \sum_{\vec{q}_a \text{ surface}} \exp \left(2\pi i \beta \sum_{\vec{a}, \vec{b}} q_a \log \frac{|\vec{x}_a - \vec{x}_b|}{a} q_b + \pi \beta \sum q_a^2 \log \frac{r_0}{a} \right)$$

$\sum q_a = 0$

ans = Similar & same $\theta \rightarrow 0$



$$\beta J \sum_{\vec{a}, \vec{b}} q_{\vec{a}} \log \frac{|x_{\vec{a}} - x_{\vec{b}}|}{a} q_{\vec{b}} + \pi \beta J \sum q_{\vec{a}}^2 \log \frac{r_0}{a}$$

$$+ \pi \beta J \sum \vec{q}_a^2 \log \frac{R_0}{a}$$

$q=2, n=2$ vertex here

$q=1, n=1$ vertex here

$$n \rightarrow 0$$

$$\vec{q}_a = 0 \quad \forall \text{ faces}$$

$$n=1$$

$$q=0 \quad \text{except}$$


etc



$q=1$

$n = -\infty$
 $\leftarrow$ Poisson Summation
 $\sum \text{gaussians} = \text{similar} \text{ as } \theta \rightarrow 0$
 $\eta_{ij} = m - m^{\text{all}}$ etc
 $\rightarrow \text{goal: integrate over } \theta$
 $e^{i\theta_i (n_{ij} + n_{ik} + n_{il} + n_{im})}$
 GAS OF VORTICES !

$$\sum \exp \left(2\pi i \beta \sum_{\vec{a}, \vec{b}} q_{\vec{a}} \log \frac{|\vec{x}_{\vec{a}} - \vec{x}_{\vec{b}}|}{a} q_{\vec{b}} + \pi \beta \sum q_{\vec{a}}^2 \log \frac{r_0}{a} \right)$$

 $q_{\vec{a}} \leftarrow \text{surface}$
 $\sum q_{\vec{a}} = 0$
 $q=2, n=2 \text{ vertex here}$
 $q=1, n=1 \text{ vertex here}$
 $\text{another vertex here}$

$$\sum_{m \in \mathbb{Z}} f(m) = \int d\varphi \, f(\varphi) \underbrace{\sum_m \delta(\varphi - m)}_{\text{cont}}, \quad \sum_{\text{SOS}} =$$

$\leftarrow$ wants to be $\simeq$ integer

$$S = \int (\partial \varphi)^2$$

$$\sum_{q \in \mathbb{Z}} e^{2\pi i q \varphi}$$

$$\sum_{m \in \mathbb{Z}} f(m) = \int d\varphi \, f(\varphi) \underbrace{\sum_m \delta(\varphi - m)}_{\text{cont}}, \quad \sum_s =$$

$$S = \underbrace{\int (\partial\varphi)^2 - 2\cos(2\pi\varphi)}_{\text{Sine Gordon model}}$$

wants to be
 $\simeq$ integer
 $\square\varphi = \sin\varphi$ vs $m\varphi$

$$\sum_{q \in \mathbb{Z}} e^{2\pi i q \varphi}$$