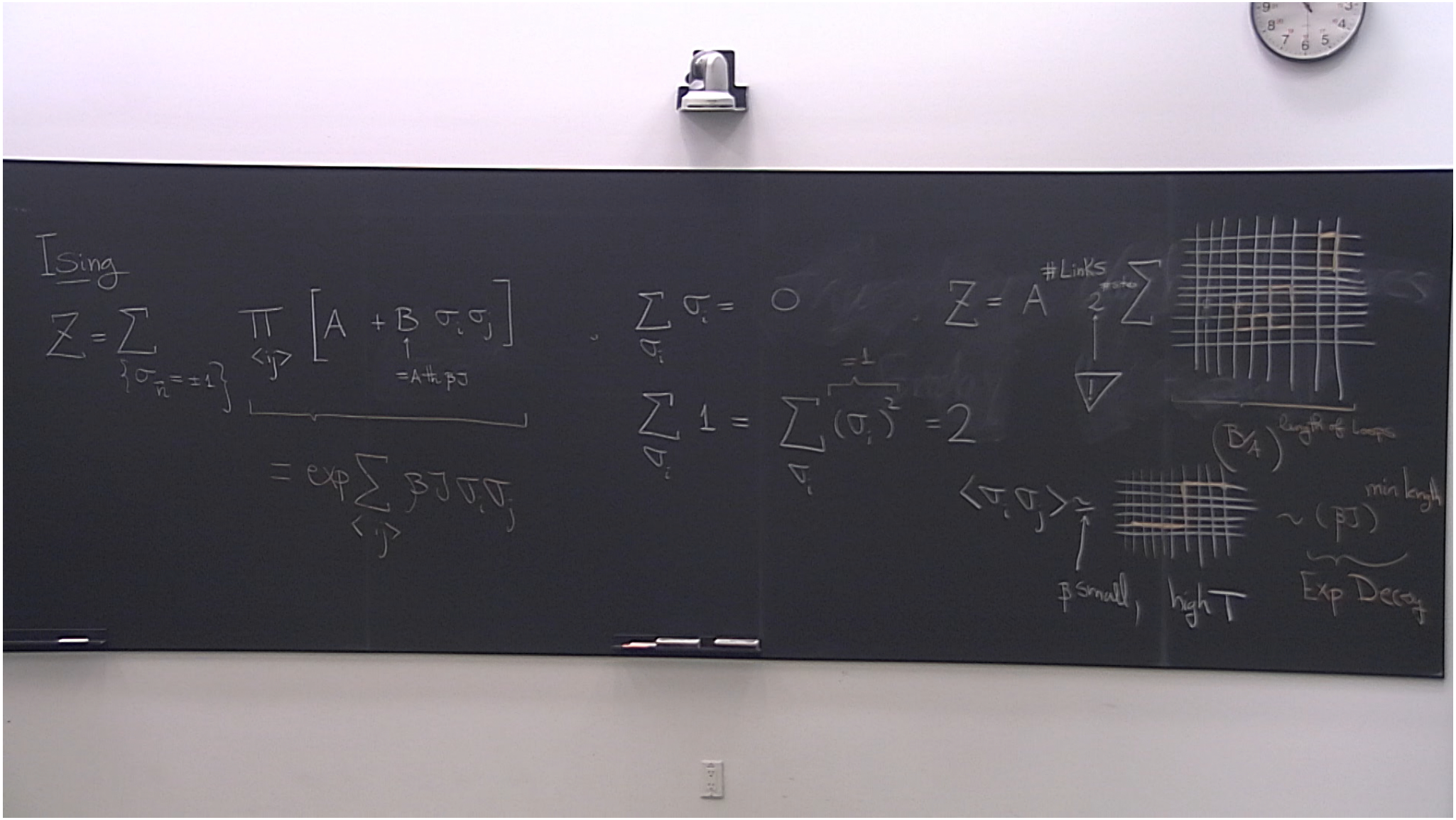
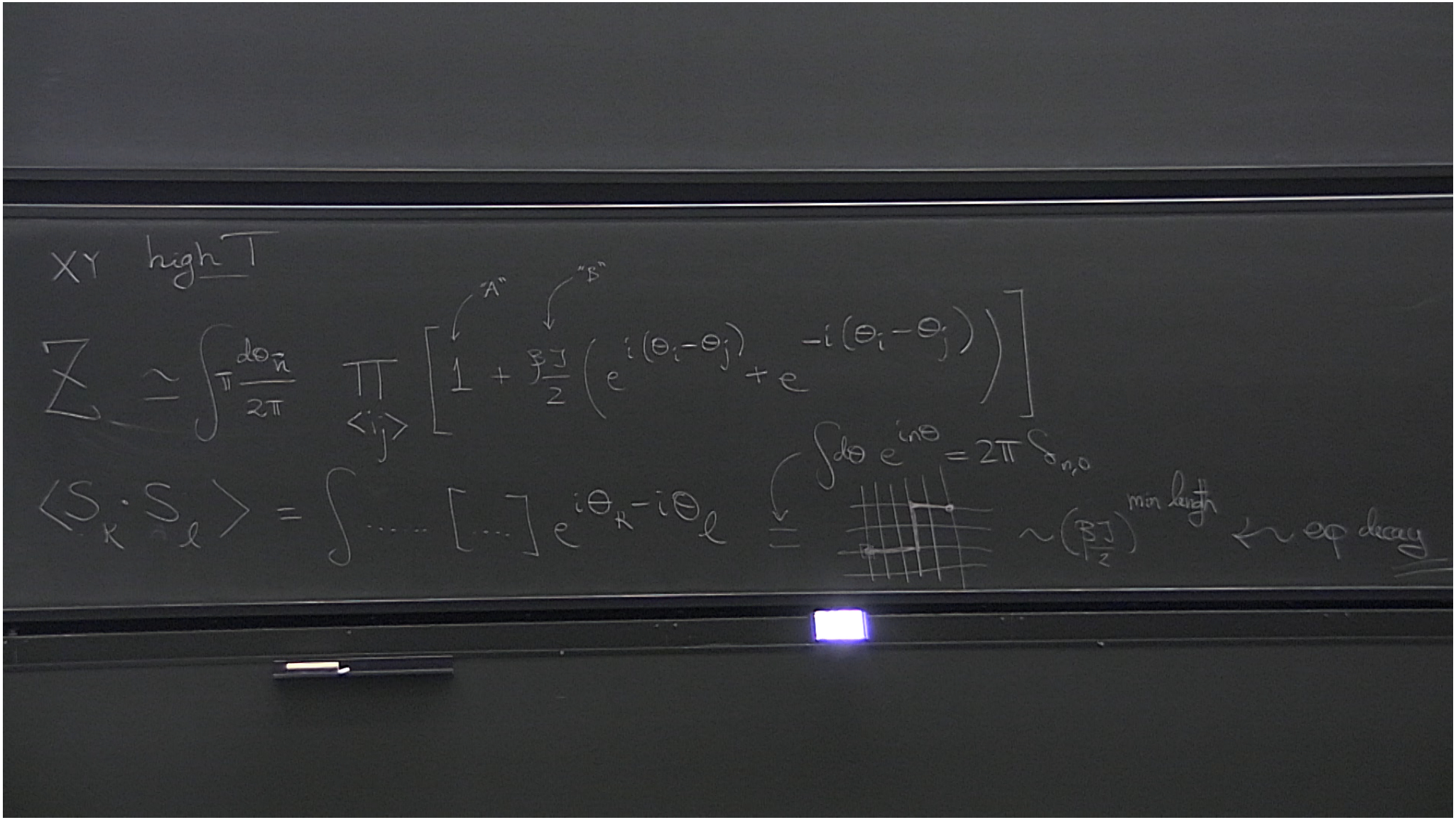
Title: PSI 2018/2019 - Statistical Mechanics - Lecture 9

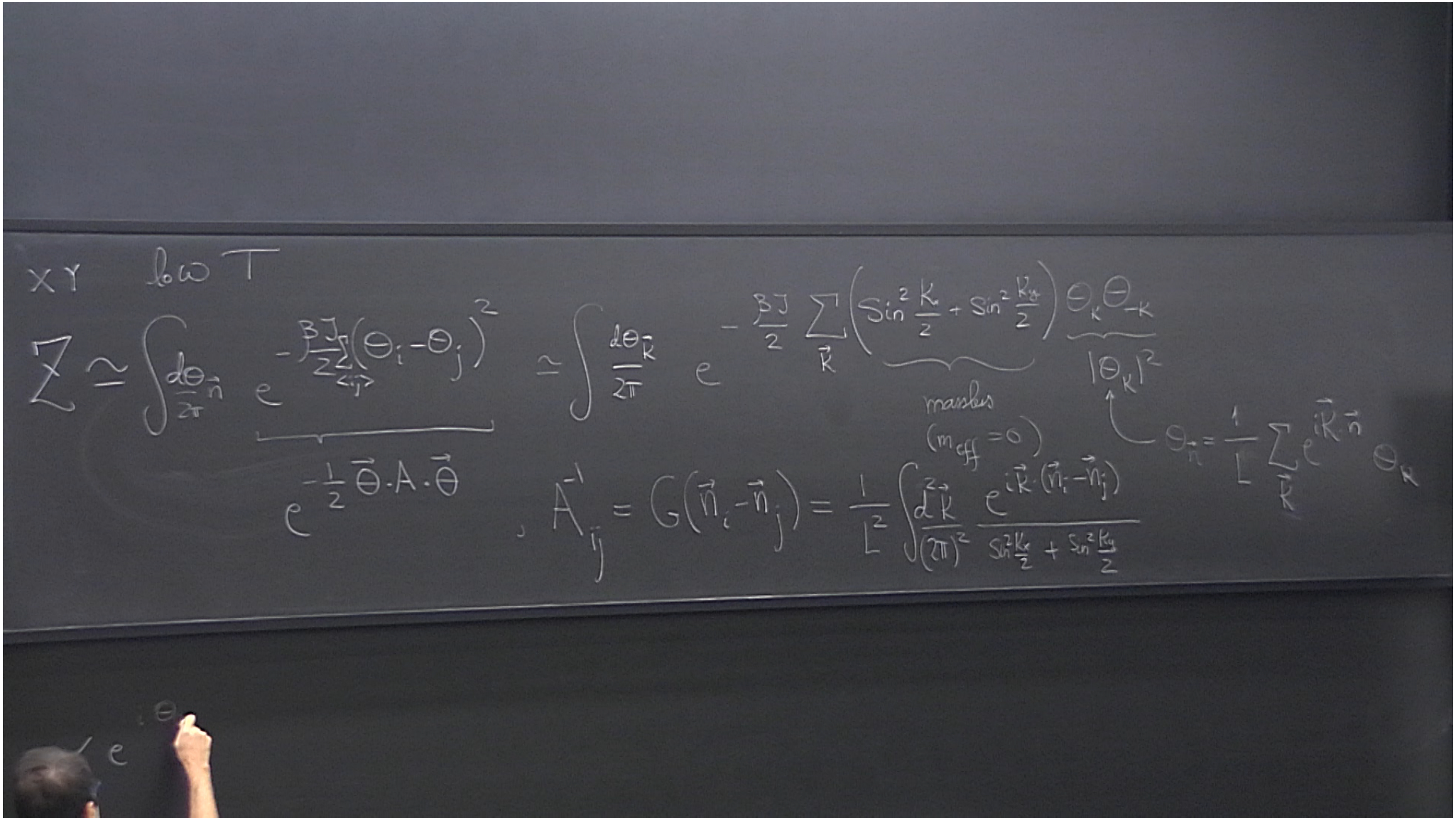
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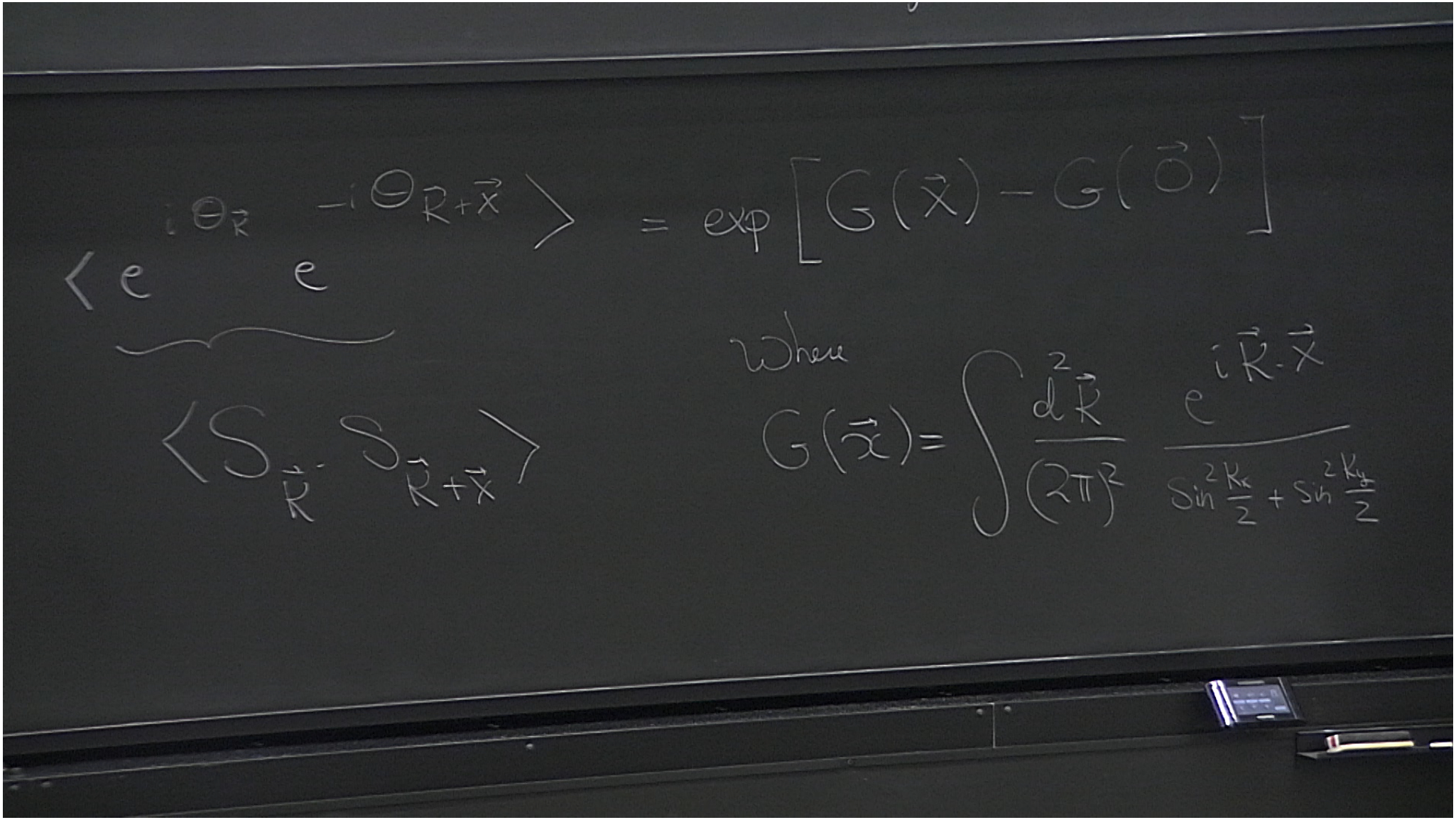
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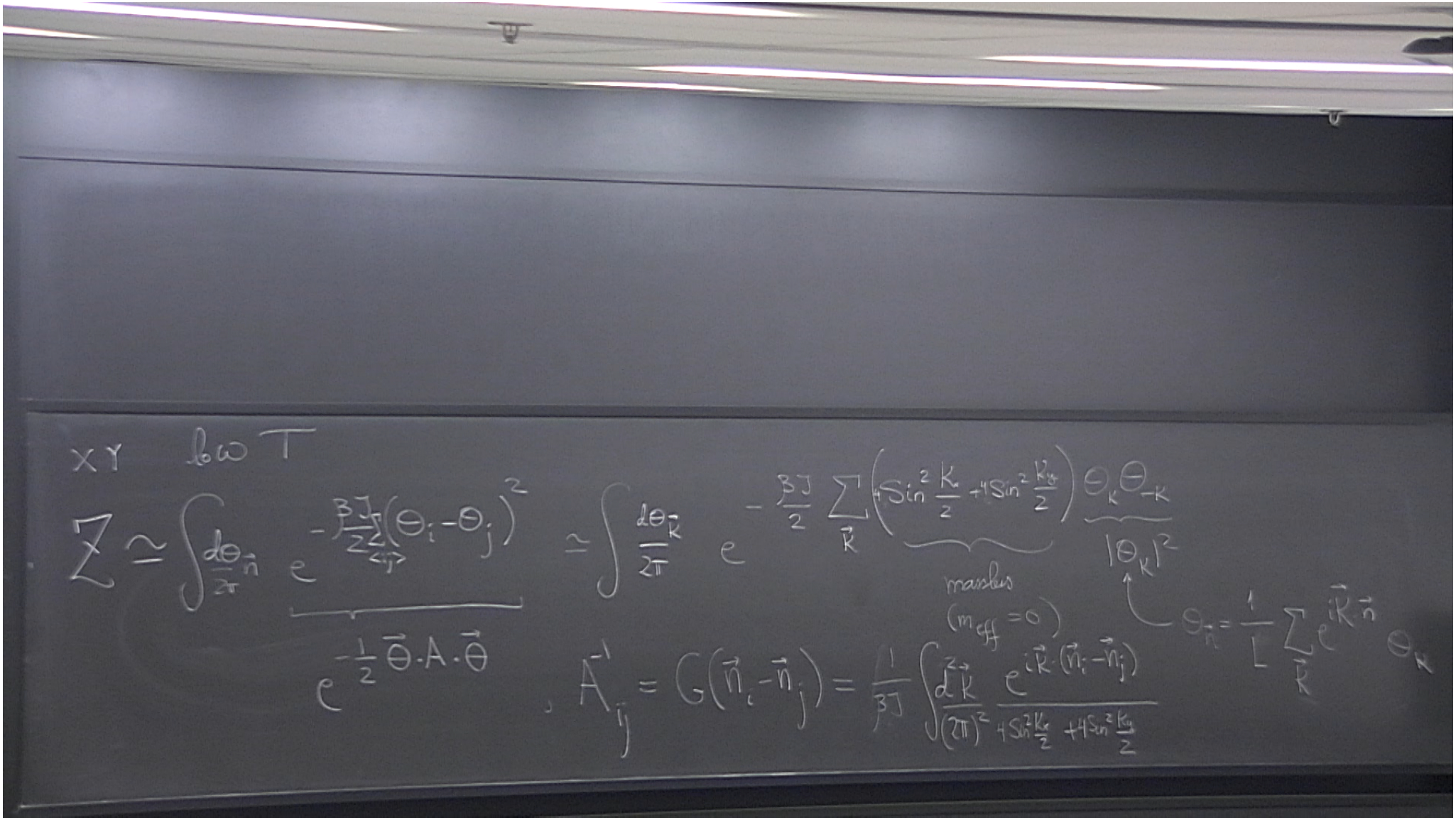
Abstract:

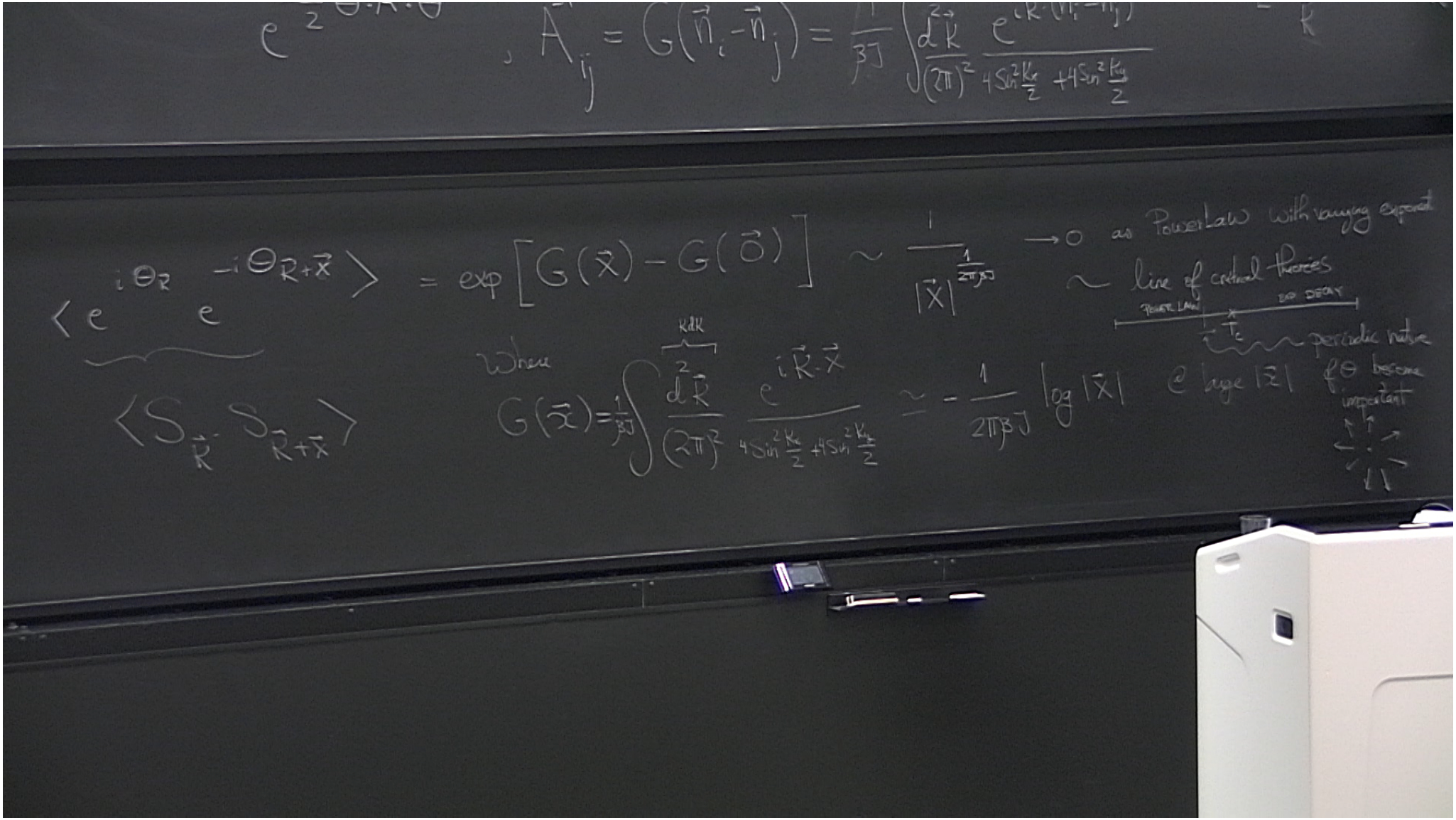

Ising
$Z = \sum_{\{\sigma_n = \pm 1\}} \prod_{\langle ij \rangle} \left[A + B \sigma_i \sigma_j \right]$
$= A + \beta J$
$= \exp \sum_{\langle ij \rangle} \beta J \sigma_i \sigma_j$
$\sum_{\sigma_i} \sigma_i = 0$
$\sum_{\sigma_i} 1 = \sum_{\sigma_i} \overbrace{(\sigma_i)^2}^{=1} = 2$
$Z = A^{\#\text{Links}} \sum_{\#\text{sites}} 2$
(B_A)
length of Loops
$\langle \sigma_i \sigma_j \rangle \approx$
$\sim (\beta J)$
min length
β small, high T
Exp Decay


XY high T
"A"
"B"
$Z \simeq \int \frac{d\theta_n}{2\pi} \prod_{\langle ij \rangle} \left[1 + \frac{\beta J}{2} \left(e^{i(\theta_i - \theta_j)} + e^{-i(\theta_i - \theta_j)} \right) \right]$
$\langle S_k \cdot S_\ell \rangle = \int \cdots [\cdots] e^{i\theta_k - i\theta_\ell}$
$\int d\theta\, e^{in\theta} = 2\pi \delta_{n,0}$
$\sim \left(\frac{\beta J}{2}\right)^{\text{min length}}$
exp decay


XY low T
$Z \simeq \int \frac{d\theta_{\vec{n}}}{2\pi} e^{-\frac{\beta J}{2}\sum_{\langle ij\rangle}(\theta_i-\theta_j)^2} \simeq \int \frac{d\theta_{\vec{k}}}{2\pi} e^{-\frac{\beta J}{2}\sum_{\vec{k}}\left(\sin^2\frac{k_x}{2}+\sin^2\frac{k_y}{2}\right)\theta_k\theta_{-k}}$
massless
$(m_{eff}=0)$
$|\theta_k|^2$
$e^{-\frac{1}{2}\vec{\theta}\cdot A\cdot\vec{\theta}}$
$\theta_{\vec{n}_i} = \frac{1}{L}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{n}_i}\theta_k$
$A^{-1}_{ij} = G(\vec{n}_i-\vec{n}_j) = \frac{1}{L^2}\int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{n}_i-\vec{n}_j)}}{\sin^2\frac{k_x}{2}+\sin^2\frac{k_y}{2}}$
$e^{i\theta}$

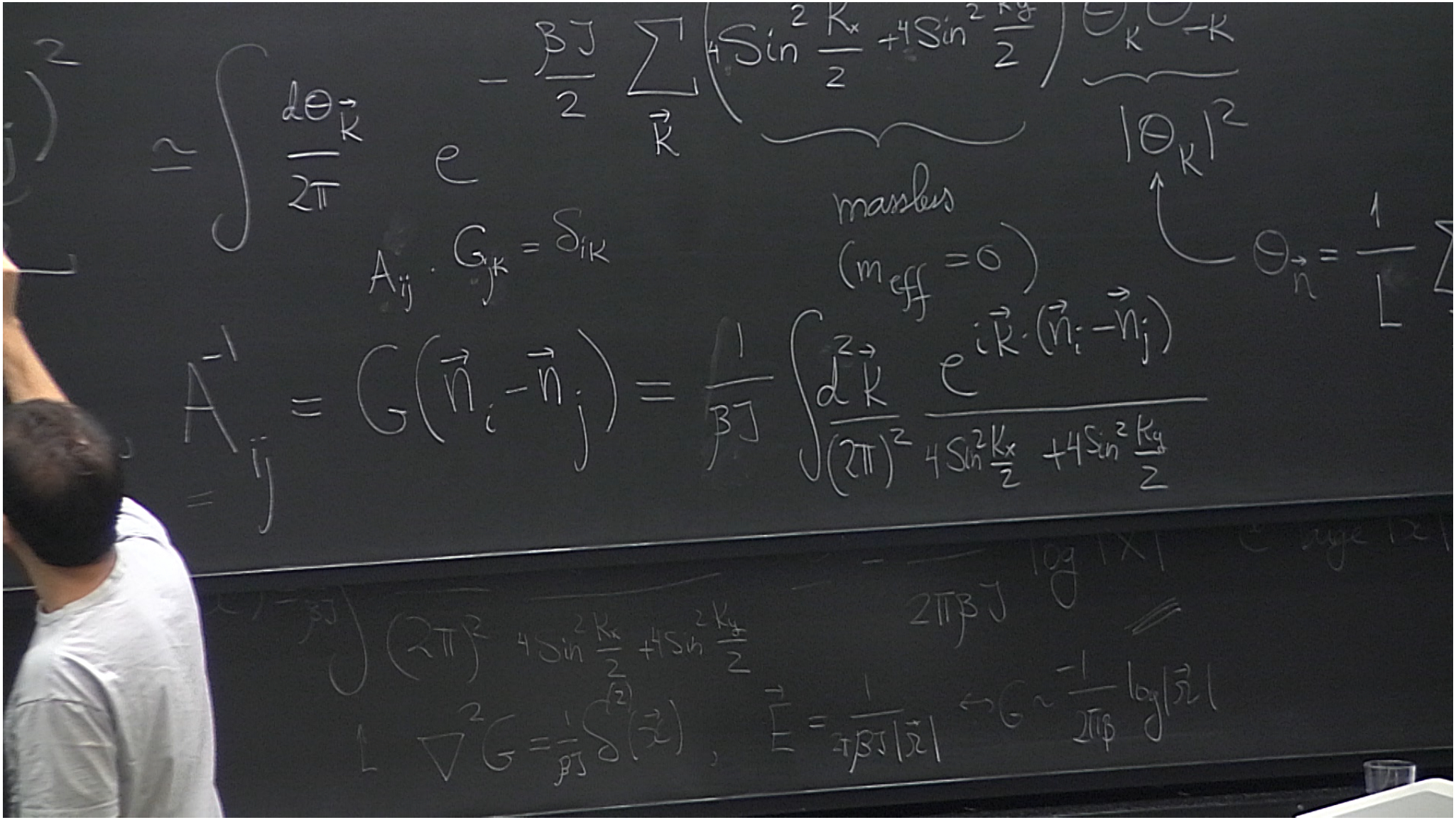

$\langle e^{i\theta_{\vec{R}}} e^{-i\theta_{\vec{R}+\vec{x}}} \rangle = \exp\left[G(\vec{x}) - G(\vec{0})\right]$
$\langle S_{\vec{R}} \cdot S_{\vec{R}+\vec{x}} \rangle$
where
$G(\vec{x}) = \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{x}}}{\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2}}$

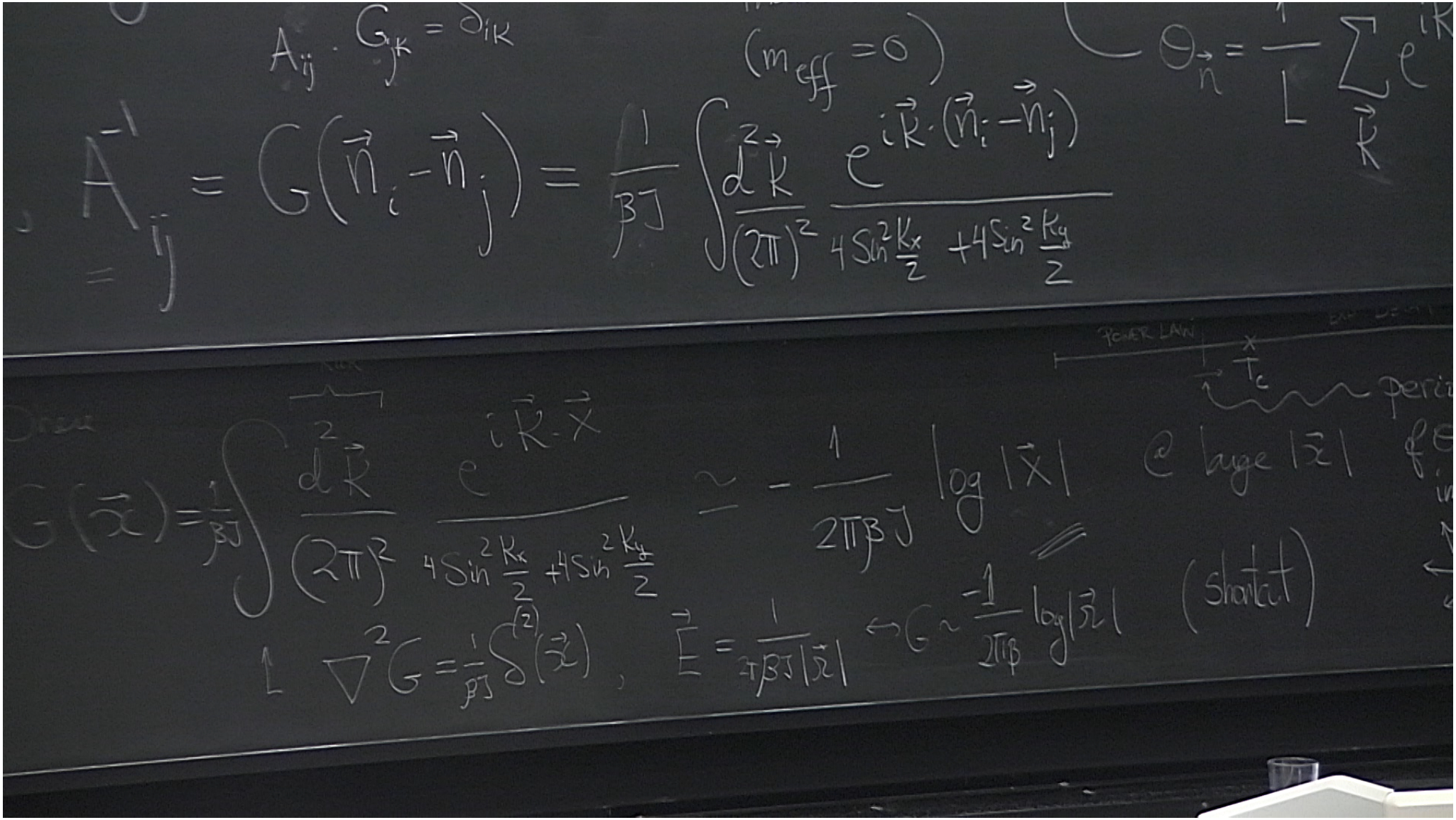

XY low T
$Z \simeq \int \frac{d\theta_{\vec{n}}}{2\pi} e^{-\frac{\beta J}{2}\sum_{\langle ij\rangle}(\theta_i-\theta_j)^2} \simeq \int \frac{d\theta_{\vec{k}}}{2\pi} e^{-\frac{\beta J}{2}\sum_{\vec{k}}\left(4\sin^2\frac{k_x}{2}+4\sin^2\frac{k_y}{2}\right)\frac{\theta_k\theta_{-k}}{|\theta_k|^2}}$
massless
$(m_{eff}=0)$
$e^{-\frac{1}{2}\vec{\theta}\cdot A\cdot\vec{\theta}}$
$A^{-1}_{ij} = G(\vec{n}_i-\vec{n}_j) = \frac{1}{\beta J}\int \frac{d^2\vec{k}}{(2\pi)^2}\frac{e^{i\vec{k}\cdot(\vec{n}_i-\vec{n}_j)}}{4\sin^2\frac{k_x}{2}+4\sin^2\frac{k_y}{2}}$
$\theta_{\vec{n}} = \frac{1}{L}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{n}}\theta_k$

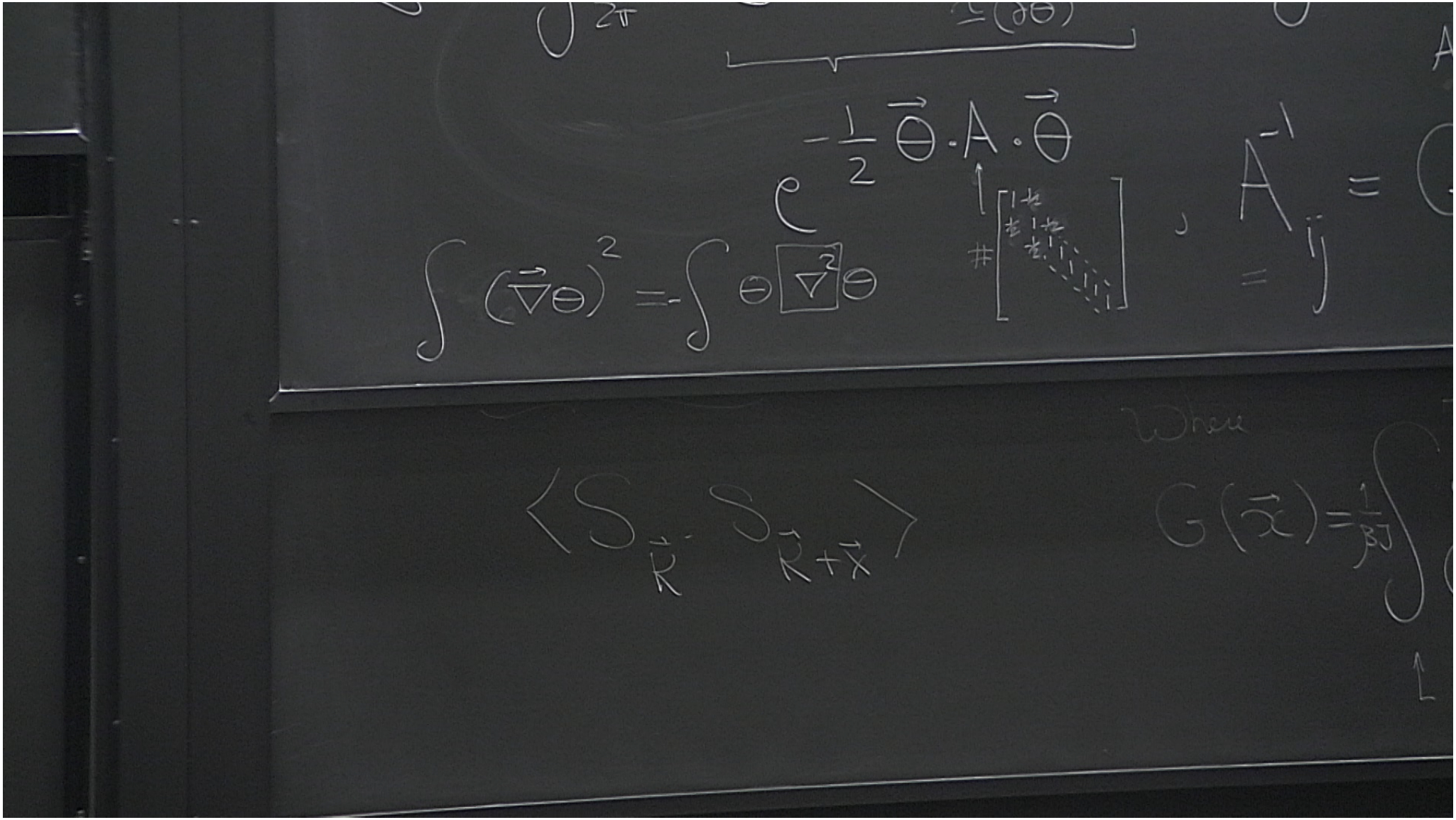

$A_{ij} = G(\vec{n}_i - \vec{n}_j) = \frac{1}{\beta J}\int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{n}_i-\vec{n}_j)}}{4\sin^2\frac{k_x}{2} + 4\sin^2\frac{k_y}{2}}$
$\langle e^{i\theta_{\vec{R}}} e^{-i\theta_{\vec{R}+\vec{x}}} \rangle = \exp\left[G(\vec{x}) - G(\vec{0})\right] \sim \frac{1}{|\vec{x}|^{\frac{1}{2\pi\beta J}}} \to 0$ as Power Law with varying exponent
$\sim$ line of critical theories
POWER LAW
EXP DECAY
T_c
periodic native if θ become important
$\langle S_{\vec{R}} S_{\vec{R}+\vec{x}} \rangle$
where
$G(\vec{x}) = \frac{1}{\beta J}\int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{x}}}{4\sin^2\frac{k_x}{2} + 4\sin^2\frac{k_y}{2}} \simeq -\frac{1}{2\pi\beta J}\log|\vec{x}|$ @ large $|\vec{x}|$

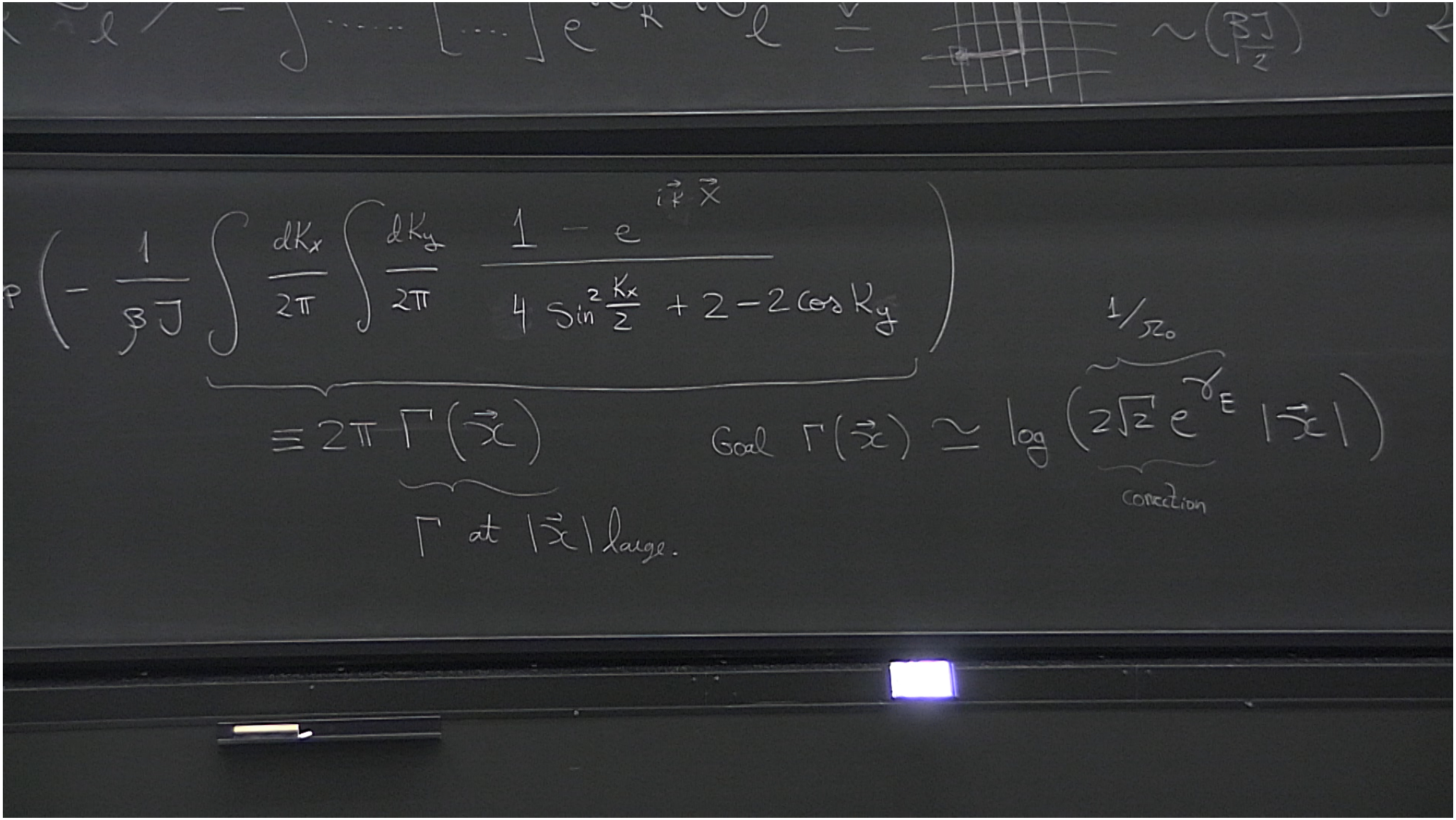
$$\exp\left(-\frac{1}{\beta J}\underbrace{\int\frac{dK_x}{2\pi}\int\frac{dK_y}{2\pi}\,\frac{1-e^{i\vec{k}\,\vec{x}}}{4\sin^2\frac{K_x}{2}+2-2\cos K_y}}_{\equiv 2\pi\,\Gamma(\vec{x})}\right)$$

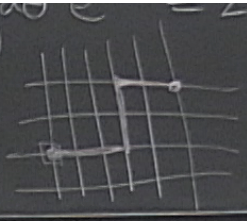
Γ at $|\vec{x}|$ large.


massless
$(m_{eff} = 0)$
$A_{ij} \cdot G_{jk} = \delta_{ik}$
$A^{-1}_{ij} = G(\vec{n}_i - \vec{n}_j) = \frac{1}{\beta J} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{n}_i - \vec{n}_j)}}{4\sin^2\frac{k_x}{2} + 4\sin^2\frac{k_y}{2}}$
$\nabla^2 G = \frac{1}{\beta J}\delta^{(2)}(\vec{x})$, $\vec{E} = \frac{1}{2\pi\beta J}\frac{\vec{x}}{|\vec{x}|}$
$G \sim \frac{-1}{2\pi\beta}\log|\vec{x}|$
$\theta_{\vec{n}} = \frac{1}{L}$


$A_{ij} \cdot G_{jk} = \delta_{ik}$
$(m_{eff} = 0)$
$\theta_{\vec{n}} = \frac{1}{L} \sum_{\vec{k}} e^{i\vec{k}}$
$A^{-1}_{ij} = G(\vec{n}_i - \vec{n}_j) = \frac{1}{\beta J} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{n}_i - \vec{n}_j)}}{4\sin^2\frac{k_x}{2} + 4\sin^2\frac{k_y}{2}}$
POWER LAW
T_c
$G(\vec{x}) = \frac{1}{\beta J} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{x}}}{4\sin^2\frac{k_x}{2} + 4\sin^2\frac{k_y}{2}} \simeq -\frac{1}{2\pi\beta J} \log|\vec{x}|$
@ large $|\vec{x}|$
$\nabla^2 G = \frac{1}{\beta J} \delta^{(2)}(\vec{x})$, $\vec{E} = \frac{1}{2\pi\beta J |\vec{x}|}$ $\leftrightarrow G \sim \frac{-1}{2\pi\beta} \log|\vec{x}|$ (Shortcut)


$\int_0^{2\pi}$
$\frac{1}{2}(\delta\theta)$
$-\frac{1}{2}\vec{\theta}\cdot A\cdot\vec{\theta}$
e
$\int(\vec{\nabla}\theta)^2 = -\int\theta\,\nabla^2\theta$
A^{-1}
$A_{ij} =$
$\langle S_{\vec{R}} \cdot S_{\vec{R}+\vec{x}} \rangle$
Where
$G(\vec{x}) = \frac{1}{\beta J}$


$\sim \left(\frac{\beta J}{2}\right)$
$P\left(-\frac{1}{\beta J}\int \frac{dK_x}{2\pi}\int \frac{dK_y}{2\pi}\ \frac{1-e^{i\vec{k}\vec{x}}}{4\sin^2\frac{K_x}{2}+2-2\cos K_y}\right)$
$\equiv 2\pi\,\Gamma(\vec{x})$
Γ at $|\vec{x}|$ large.
Goal $\Gamma(\vec{x}) \simeq \log\left(2\sqrt{2}\,e^{\gamma_E}\,|\vec{x}|\right)$
$1/z_0$
correction

$\langle S_k \cdot S_l \rangle = \int \cdots [\cdots] e^{i\theta_k - i\theta_l} \cong$  $\sim \left(\frac{\beta J}{z}\right)^{\text{min length}}$ ← eq dec

$$\exp\left(-\frac{1}{\beta J}\int \frac{dK_x}{2\pi}\int \frac{dK_y}{2\pi}\ \frac{1-e^{i\vec{k}\cdot\vec{x}}}{4\sin^2\frac{K_x}{2}+2-2\cos K_y}\right)$$

$\equiv 2\pi\,\Gamma(\vec{x})$ ← large x

Γ at $|\vec{x}|$ large.

$\equiv$

$x=0$ ↓

$1 = \langle \vec{S}\vec{S} \rangle$

1

Goal $\Gamma(\vec{x}) \simeq \log\left(2\sqrt{2}\, e^{\gamma_E} |\vec{x}|\right)$ ← $1/z_0$ correction

$\| \leftarrow |\vec{x}|=1$

1.5708

$= 1.6169$ ← $|\vec{x}|=1$

$\Gamma(\vec{x}=0) = 0$

Γ at $|\vec{x}|$ large.

$\Leftarrow$

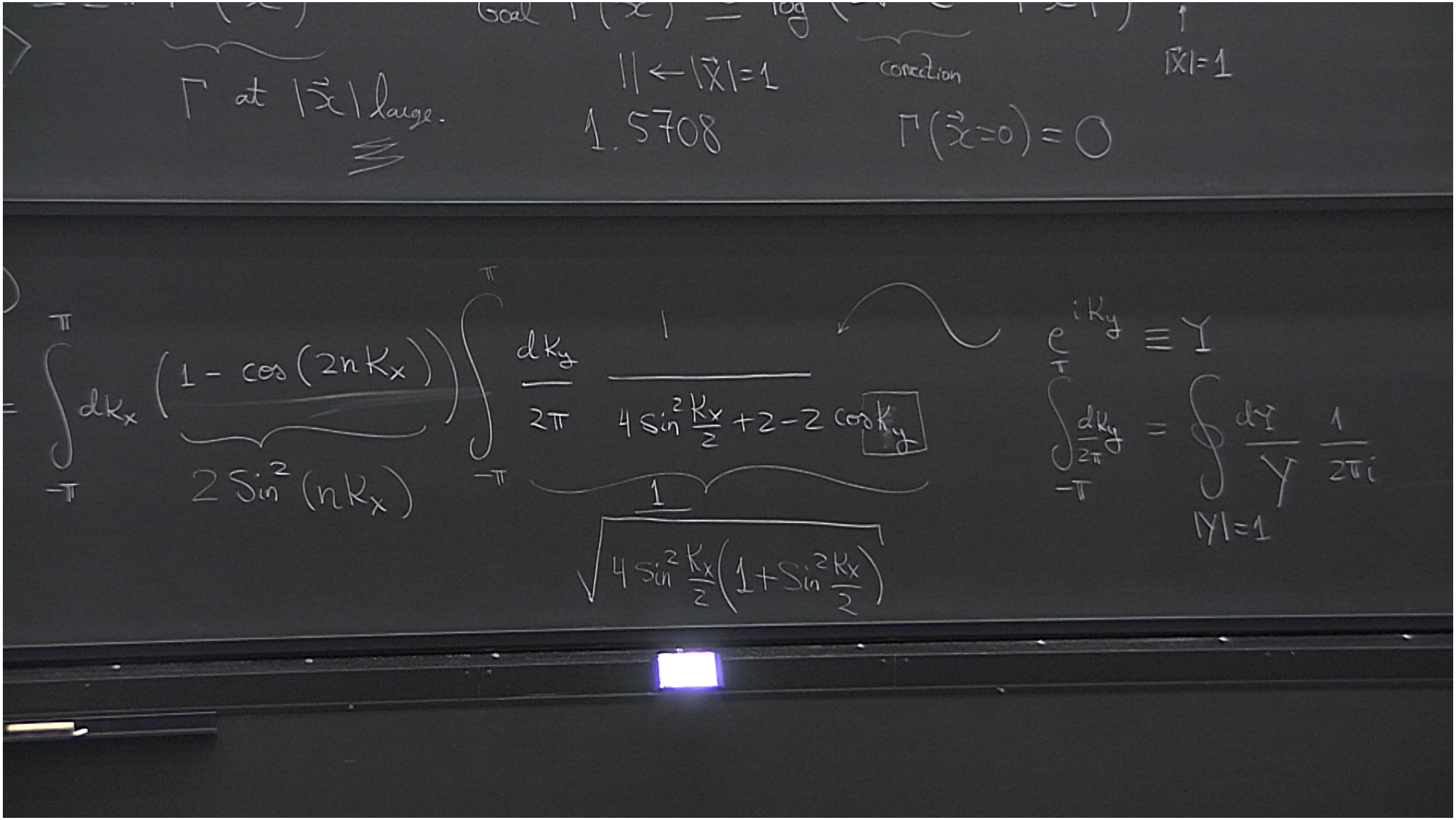
$|\vec{x}|=1$

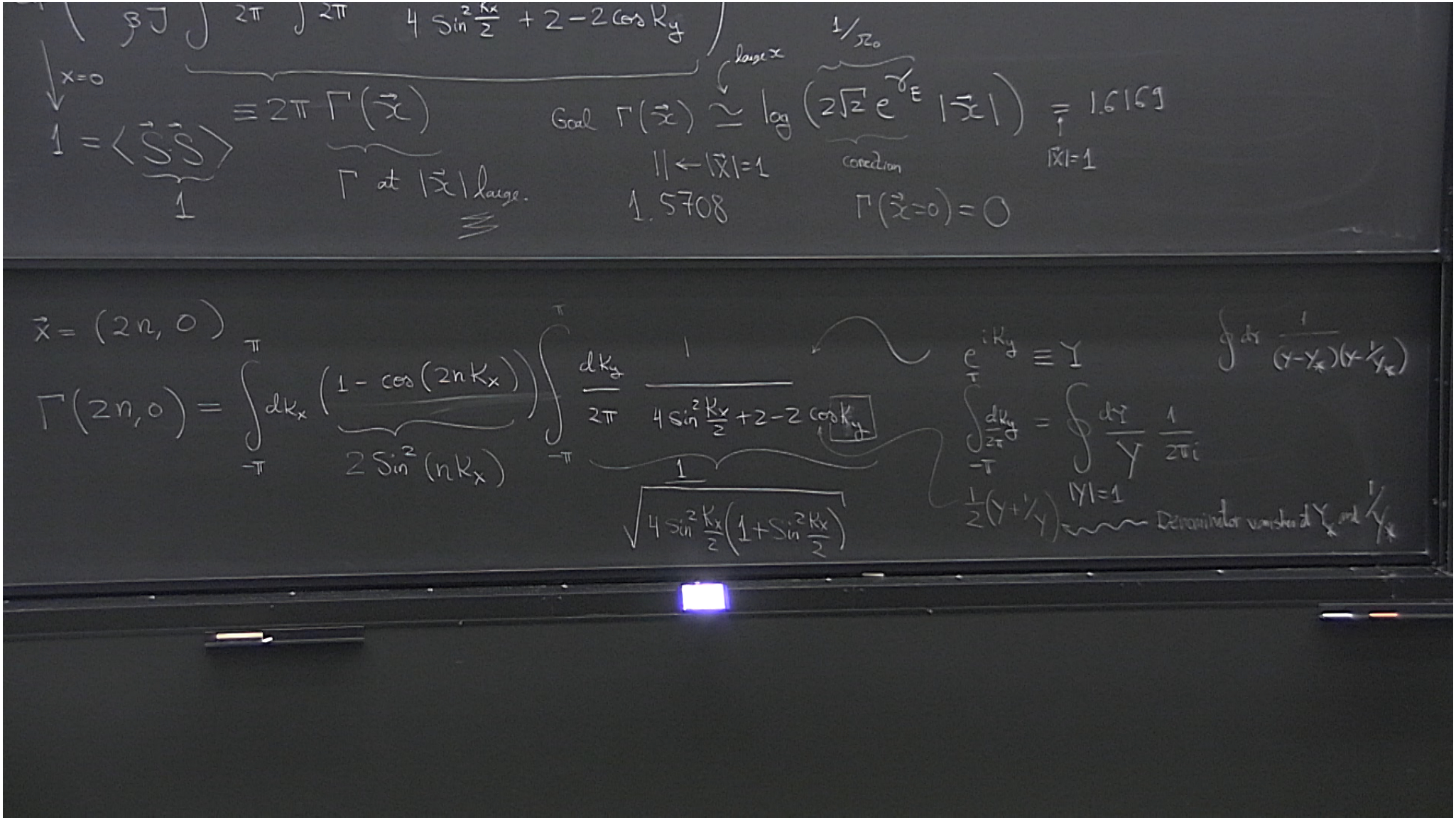
1

1.5708

$\vec{x} = (2n, 0)$

$$\Gamma(2n,0) = \int_{-\pi}^{\pi} dk_x \underbrace{\left(1-\cos(2nk_x)\right)}_{2\sin^2(nk_x)} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \underbrace{\frac{1}{4\sin^2\frac{k_x}{2}+2-2\cos k_y}}_{\frac{1}{\sqrt{4\sin^2\frac{k_x}{2}\left(1+\sin^2\frac{k_x}{2}\right)}}}$$


Goal
Γ at |x⃗| large.
≡
|| ← |x⃗|=1
1.5708
correction
Γ(x⃗=0)=0
|x⃗|=1
$\int_{-\pi}^{\pi} dk_x \left(\underbrace{1-\cos(2nk_x)}_{2\sin^2(nk_x)}\right) \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \frac{1}{4\sin^2\frac{k_x}{2}+2-2\cos k_y}$
$\underbrace{\frac{1}{\sqrt{4\sin^2\frac{k_x}{2}\left(1+\sin^2\frac{k_x}{2}\right)}}}$
$e^{ik_y} \equiv Y$
$\int_{-\pi}^{\pi} \frac{dk_y}{2\pi} = \oint_{|Y|=1} \frac{dY}{Y} \frac{1}{2\pi i}$


$\frac{\beta J}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{4\sin^2\frac{k_x}{2}+2-2\cos k_y}$
$x=0$
$1=\langle \vec{S}\vec{S}\rangle$
1
$\equiv 2\pi\,\Gamma(\vec{x})$
Γ at $|\vec{x}|$ large.
large x
$1/\pi_0$
Goal $\Gamma(\vec{x}) \simeq \log\left(2\sqrt{2}\,e^{\gamma_E}|\vec{x}|\right) = 1.6169$
$\| \leftarrow |\vec{x}|=1$
correction
$|\vec{x}|=1$
1.5708
$\Gamma(\vec{x}=0)=0$
$\vec{x}=(2n,0)$
$\Gamma(2n,0)=\int_{-\pi}^{\pi} dk_x \underbrace{\left(1-\cos(2nk_x)\right)}_{2\sin^2(nk_x)} \int_{-\pi}^{\pi}\frac{dk_y}{2\pi}\frac{1}{4\sin^2\frac{k_x}{2}+2-2\cos k_y}$
$\frac{1}{\sqrt{4\sin^2\frac{k_x}{2}\left(1+\sin^2\frac{k_x}{2}\right)}}$
$e^{ik_y}\equiv Y$
$\int_{-\pi}^{\pi}\frac{dk_y}{2\pi} = \oint_{|Y|=1}\frac{dY}{Y}\frac{1}{2\pi i}$
$\frac{1}{2}(Y+1/Y)$ Denominator vanishes at Y_x and $1/Y_x$
$\oint dY \frac{1}{(Y-Y_x)(Y-1/Y_x)}$

$$\left(\beta J \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{}{4\sin^2\frac{k_x}{2} + 2 - 2\cos k_y} \right)$$

$x=0$

$$1 = \langle \vec{S}\vec{S} \rangle \equiv 2\pi\, \Gamma(\vec{x})$$

Γ at $|\vec{x}|$ large.

large x

$1/\pi_0$

$$\text{Goal } \Gamma(\vec{x}) \simeq \log\left(2\sqrt{2}\, e^{\gamma_E} |\vec{x}|\right) = 1.6169$$

$\Gamma(\vec{x}=0) = 0$

$\vec{x} = (2n, 0)$

$$\Gamma(2n,0) = \int_{-\pi}^{\pi} dk_x \underbrace{\left(1 - \cos(2nk_x)\right)}_{2\sin^2(nk_x)} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \frac{1}{4\sin^2\frac{k_x}{2} + 2 - 2\cos k_y}$$

$$\frac{1}{\sqrt{4\sin^2\frac{k_x}{2}\left(1 + \sin^2\frac{k_x}{2}\right)}}$$

$$e^{ik_y} \equiv Y$$

$$\int_{-\pi}^{\pi} \frac{dk_y}{2\pi} = \oint_{|Y|=1} \frac{dY}{Y} \frac{1}{2\pi i}$$

$$\frac{1}{2}\left(Y + \frac{1}{Y}\right)$$

Denominator vanishes at Y_x and $1/Y_x$

$$\oint_{|Y|=1} dY \frac{1}{(Y - Y_x)(Y - 1/Y_x)}$$

$$\sim \frac{1}{Y_x - 1/Y_x}$$

2 residues

1.5708 $\qquad \Gamma(\vec{x}=0) = 0$

$$\frac{1}{4\sin^2\frac{k_x}{2} + 2 - 2\cos k_y}$$

$$\frac{1}{\sqrt{4\sin^2\frac{k_x}{2}\left(1+\sin^2\frac{k_x}{2}\right)}}$$

$$e^{ik_y} \equiv Y$$

$$\int_{-\pi}^{\pi} \frac{dk_y}{2\pi} = \oint_{|Y|=1} \frac{dY}{Y}\,\frac{1}{2\pi i}$$

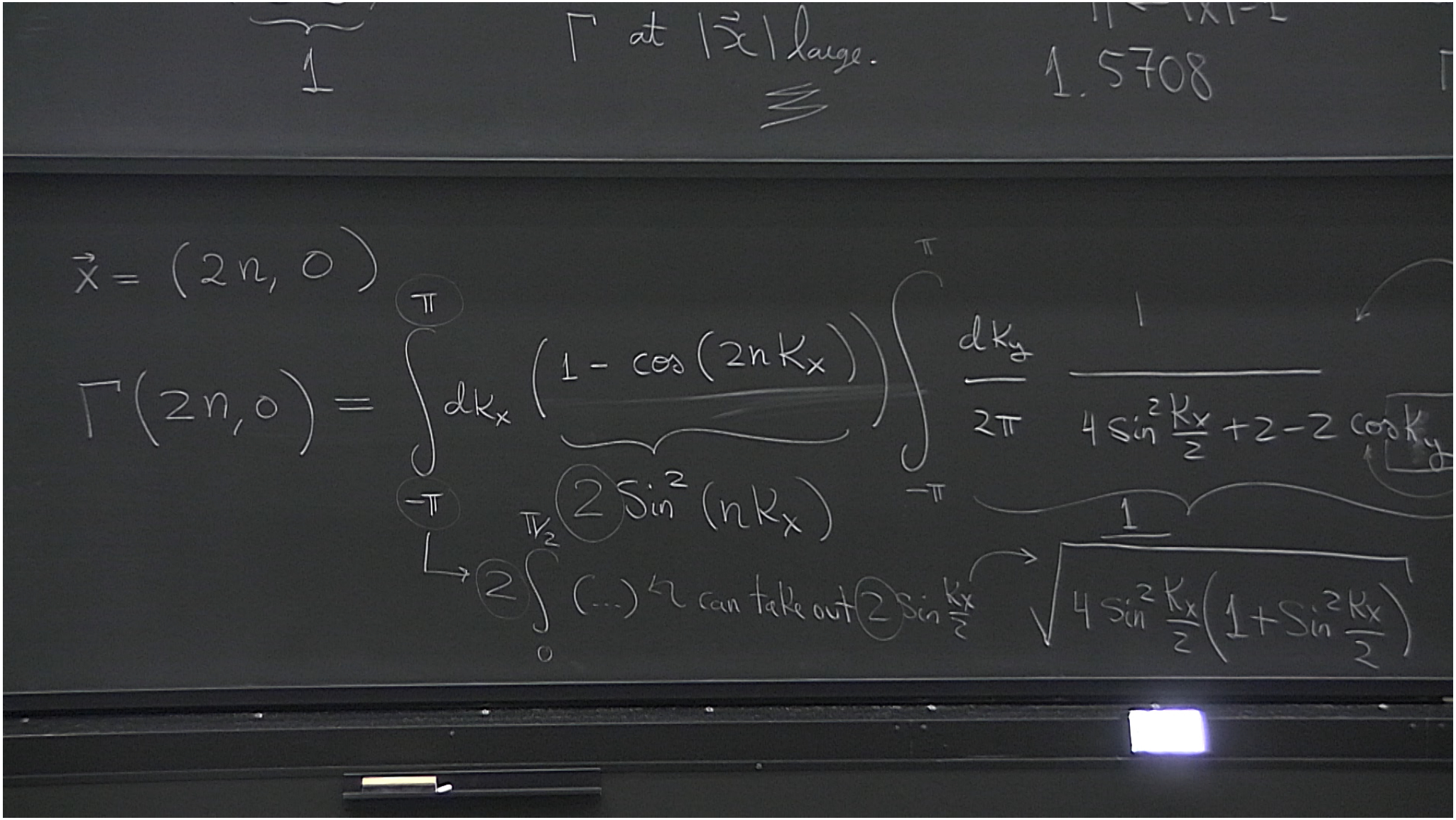
$$\frac{1}{2}\left(Y + \frac{1}{Y}\right)$$

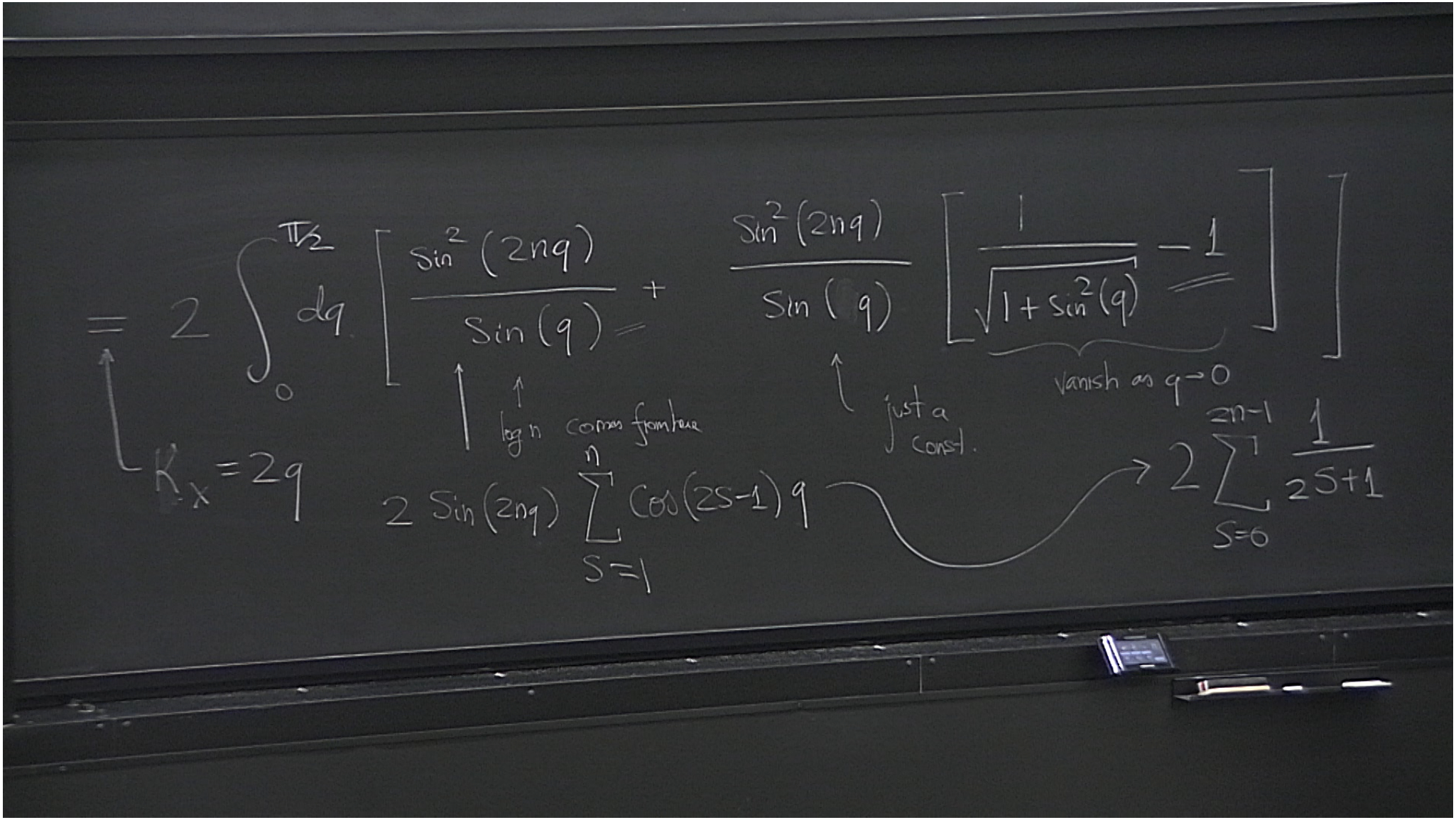
$$\oint_{|X|=1} dY\, \frac{1}{(Y - Y_*)\left(Y - \frac{1}{Y_*}\right)}$$

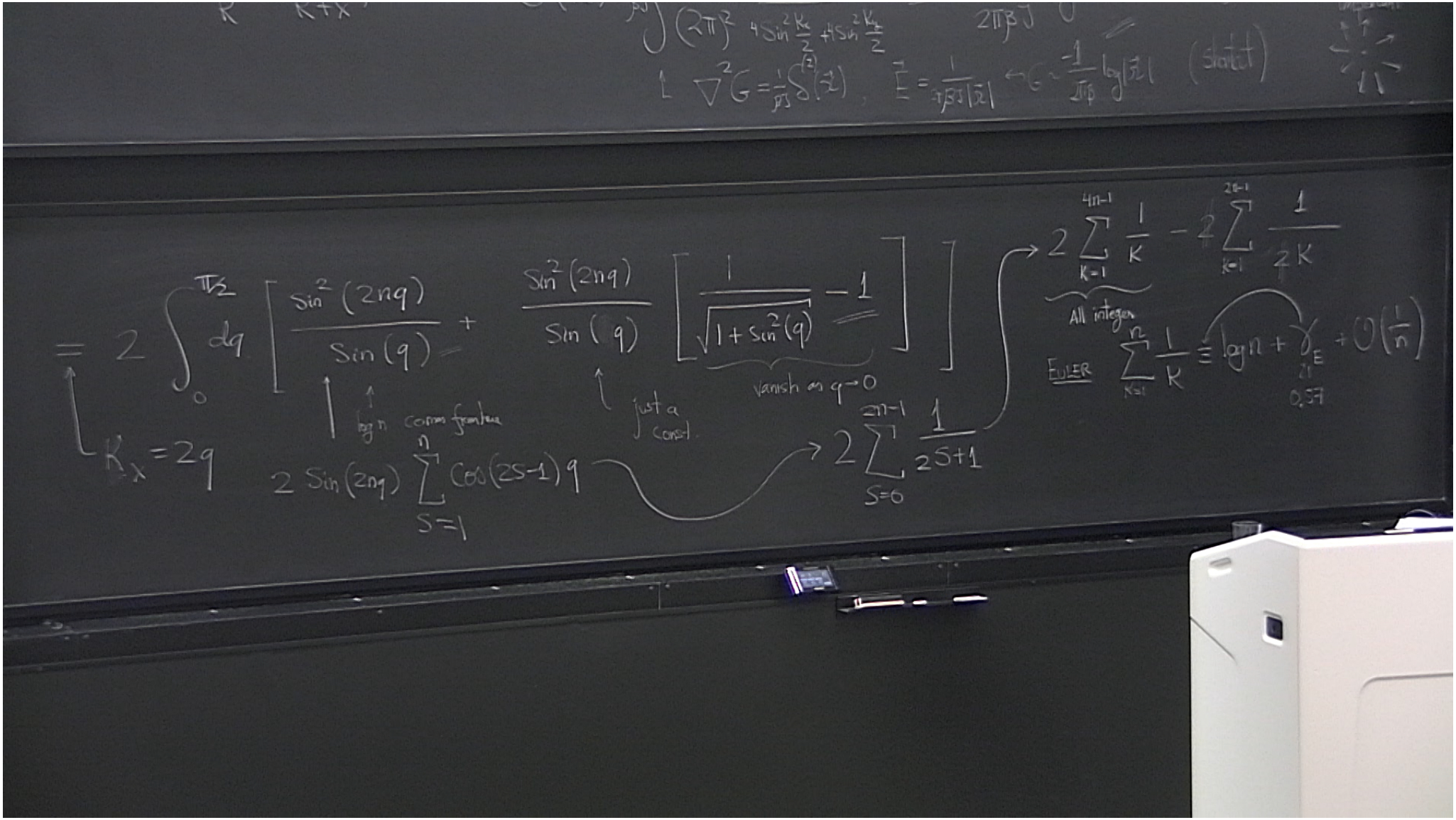
$$\sim \frac{1}{Y_* - \frac{1}{Y_*}}$$

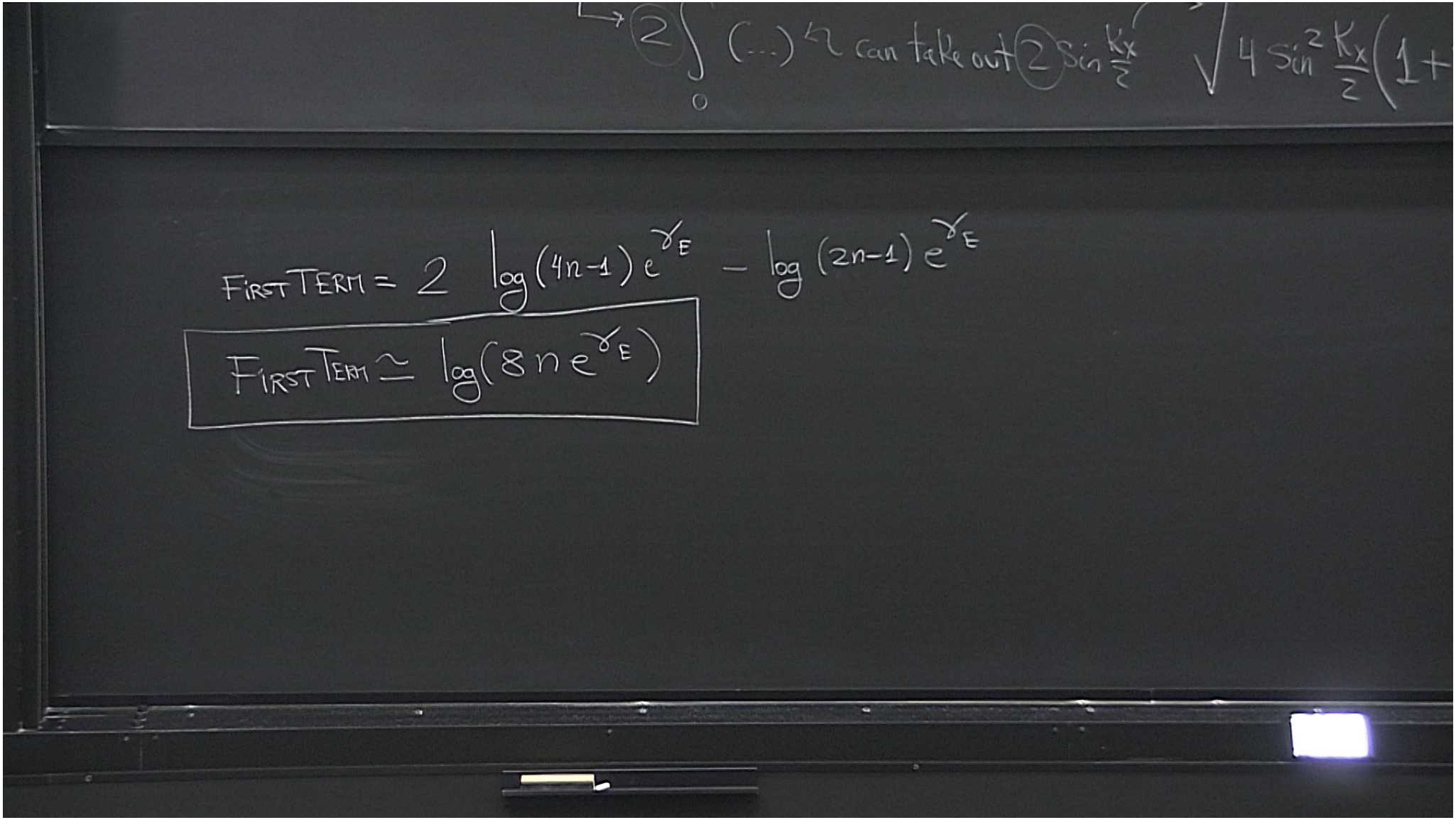
one residue

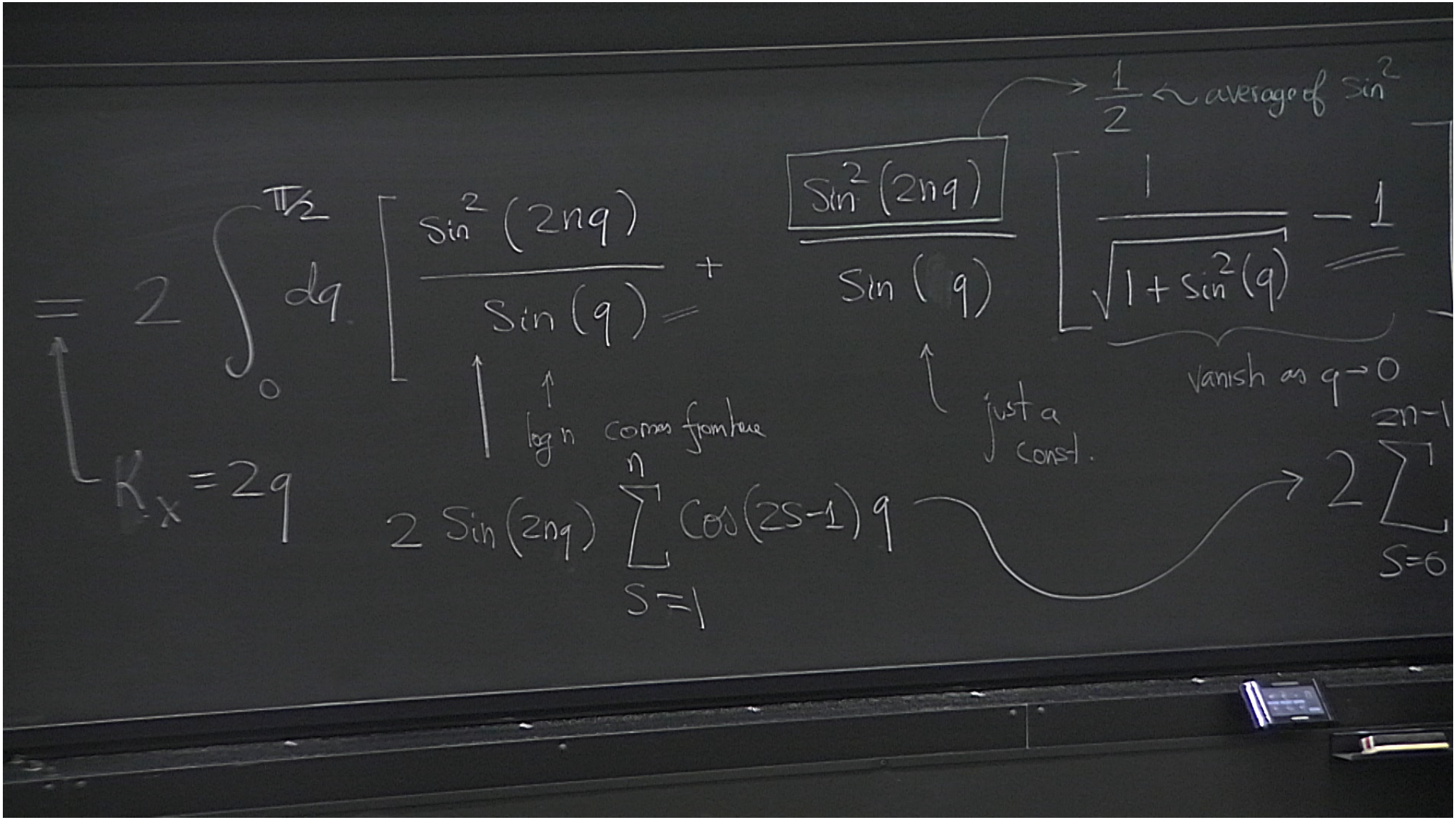
Denominator vanishes at Y_* and $\frac{1}{Y_*}$

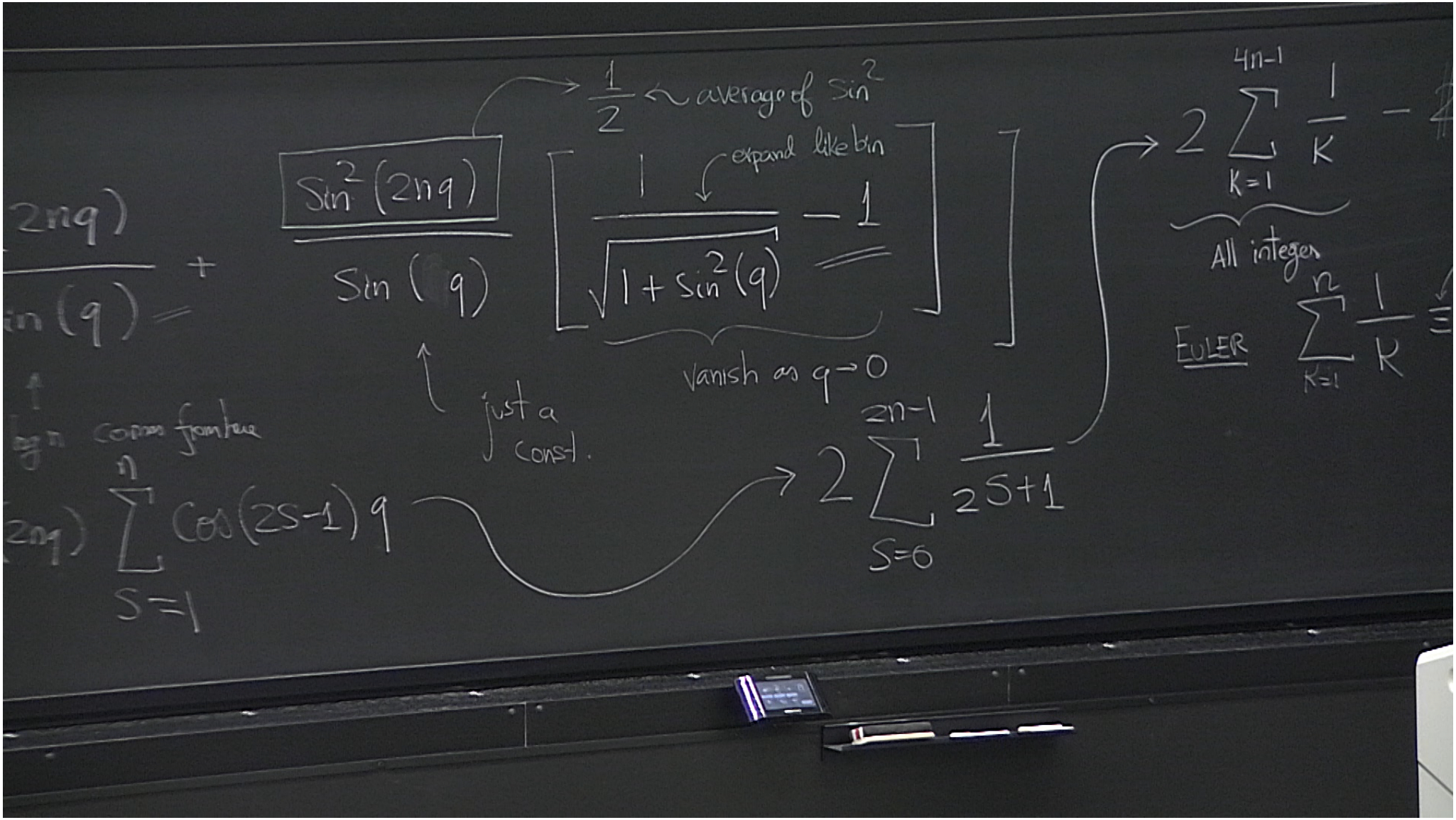

1
Γ at |x⃗| large.
≡
1.5708
x⃗ = (2n, 0)
Γ(2n,0) = ∫_{-π}^{π} dk_x (1 - cos(2n k_x)) ∫_{-π}^{π} dk_y/2π · 1/(4 sin² k_x/2 + 2 - 2 cos k_y)
2 sin²(n k_x)
2 ∫_0^{π/2} (...) can take out 2 sin k_x/2
1/√(4 sin² k_x/2 (1 + sin² k_x/2))

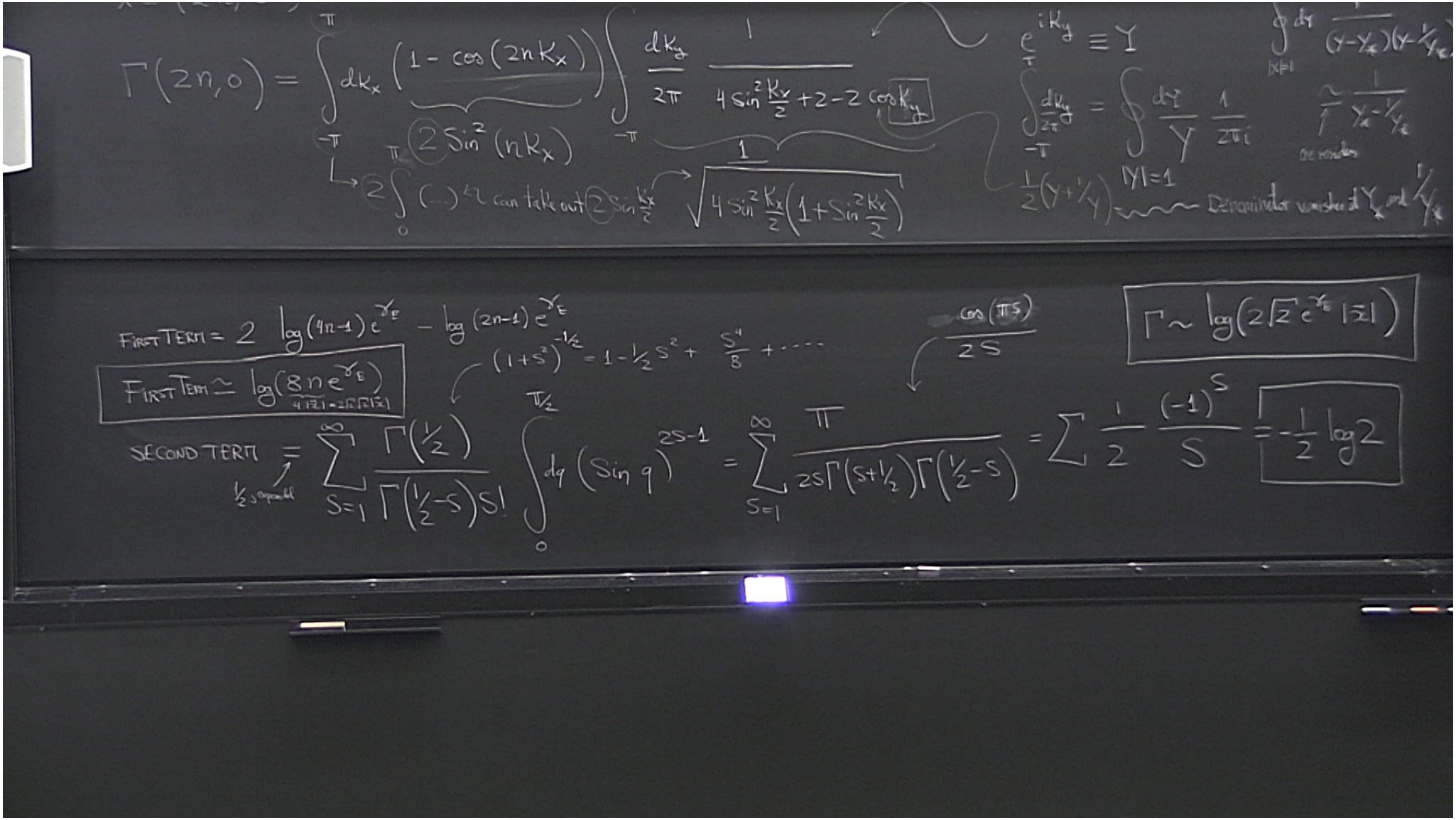

$= 2\int_0^{\pi/2} dq \left[\frac{\sin^2(2nq)}{\sin(q)} + \frac{\sin^2(2nq)}{\sin(q)} \left[\frac{1}{\sqrt{1+\sin^2(q)}} - 1 \right] \right]$
$K_x = 2q$
$2\sin(2nq)\sum_{s=1}^{n}\cos(2s-1)q \rightarrow 2\sum_{s=0}^{2n-1}\frac{1}{2s+1}$
log n comes from here
just a const.
vanish as $q \to 0$

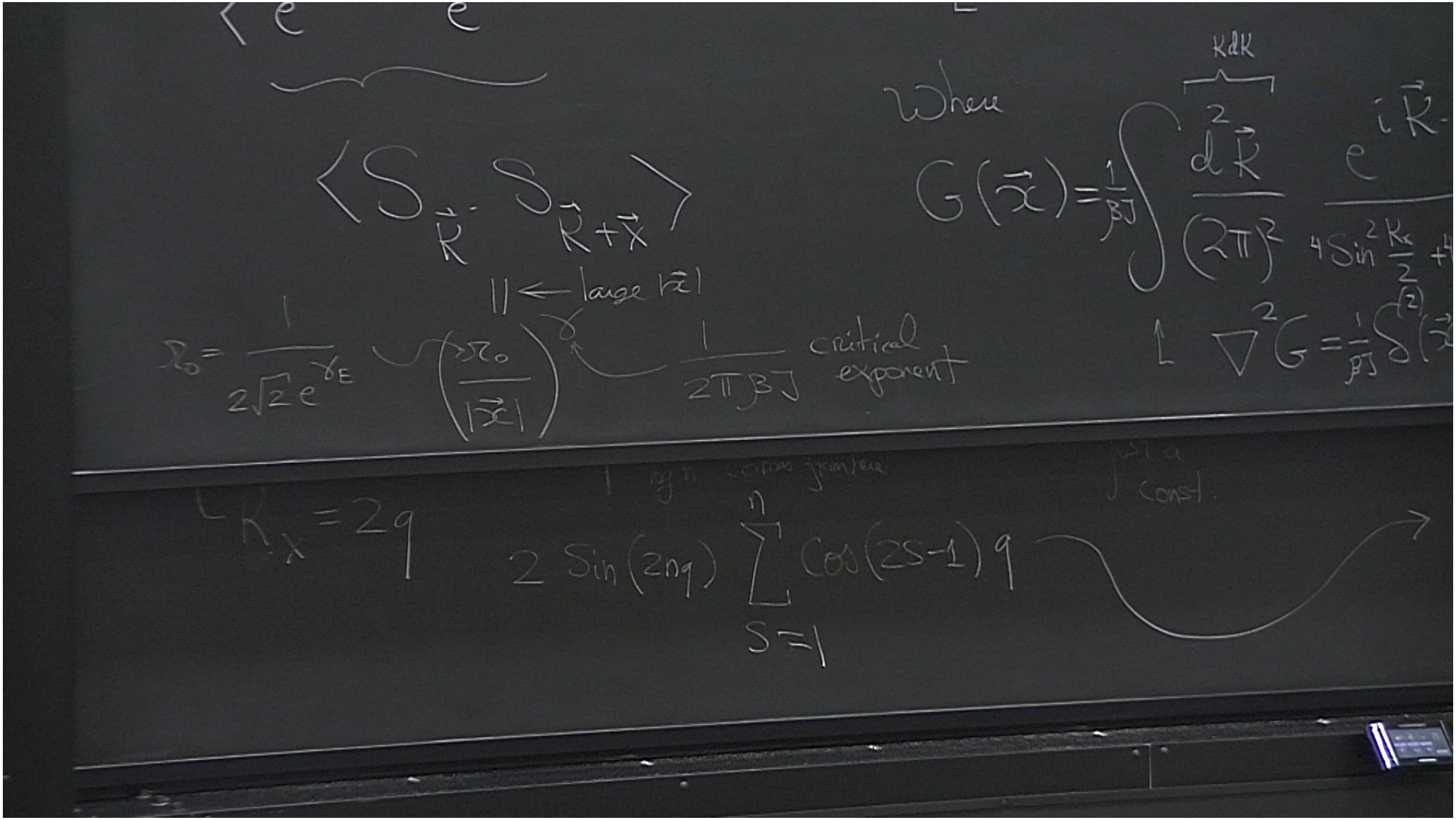

$\nabla^2 G = \frac{1}{\rho_0}\delta^{(2)}(\vec{x})$
$\vec{E} = \frac{1}{2\pi\rho_0}\frac{\vec{x}}{|\vec{x}|}$
$G \sim \frac{-1}{2\pi\rho_0}\log|\vec{x}|$
$= 2\int_0^{\pi/2} dq \left[\frac{\sin^2(2nq)}{\sin(q)} + \frac{\sin^2(2nq)}{\sin(q)}\left[\frac{1}{\sqrt{1+\sin^2(q)}} - 1\right]\right]$
$K_x = 2q$
$2\sin(2nq)\sum_{s=1}^{n}\cos(2s-1)q$
$2\sum_{s=0}^{2n-1}\frac{1}{2s+1}$
$2\sum_{K=1}^{4n-1}\frac{1}{K} - 2\sum_{K=1}^{2n-1}\frac{1}{2K}$
$\sum_{K=1}^{n}\frac{1}{K} = \log n + \gamma_E + O\left(\frac{1}{n}\right)$
$\gamma_E \approx 0.57$
log n comes from here
just a const.
vanish as $q \to 0$
All integer
EULER


↳ 2 ∫ (...) ↳ can take out 2 sin kx/2 √4 sin² kx/2 (1+
0
FIRST TERM = 2 log(4n-1) e^γE − log(2n-1) e^γE
FIRST TERM ≃ log(8n e^γE)


$= 2\int_0^{\pi/2} dq \left[\frac{\sin^2(2nq)}{\sin(q)} + \frac{\sin^2(2nq)}{\sin(q)} \left[\frac{1}{\sqrt{1+\sin^2(q)}} - 1 \right] \right.$
$\frac{1}{2}$ ~ average of $\sin^2$
vanish as $q \to 0$
log n comes from here
just a const.
$K_x = 2q$
$2\sin(2nq) \sum_{s=1}^{n} \cos(2s-1)q$
$2\sum_{s=0}^{2n-1}$


$\frac{1}{2}$ ~ average of $\sin^2$
expand like bin
$\frac{\sin^2(2nq)}{\sin(q)}\left[\frac{1}{\sqrt{1+\sin^2(q)}}-1\right]$
vanish as $q\to 0$
just a const.
$\sin(q)$ =
$\sin(2nq)$
$\sum_{S=1}^{n}\cos(2S-1)q$
$2\sum_{S=0}^{2n-1}\frac{1}{2S+1}$
$2\sum_{K=1}^{4n-1}\frac{1}{K}$ -
All integers
EULER
$\sum_{K=1}^{n}\frac{1}{K}$


$\Gamma(2n,0) = \int_{-\pi}^{\pi} dK_x \left(1-\cos(2nK_x)\right) \int_{-\pi}^{\pi} \frac{dK_y}{2\pi} \frac{1}{4\sin^2\frac{K_x}{2}+2-2\cos K_y}$
$1-\cos(2nK_x) = 2\sin^2(nK_x)$
$2\int_0^{\pi}$ (...) can take out $2\sin\frac{K_x}{2}$
$\frac{1}{\sqrt{4\sin^2\frac{K_x}{2}\left(1+\sin^2\frac{K_x}{2}\right)}}$
$e^{iK_y} \equiv Y$
$\int_{-\pi}^{\pi} \frac{dK_y}{2\pi} = \oint_{|Y|=1} \frac{dY}{Y} \frac{1}{2\pi i}$
$\frac{1}{2}\left(Y+\frac{1}{Y}\right)$ ~~~ Denominator vanishes at Y_x and $\frac{1}{Y_x}$
$\oint_{|X|=1} dX \frac{1}{(X-Y_x)(X-\frac{1}{Y_x})}$
one residue $\frac{1}{Y_x-\frac{1}{Y_x}}$
FIRST TERM $= 2\log(4n-1)e^{\gamma_E} - \log(2n-1)e^{\gamma_E}$
FIRST TERM $\simeq \log\left(\frac{8ne^{\gamma_E}}{4\sqrt{2}\cdot 2\sqrt{2}}\right)$
$(1+s^2)^{-1/2} = 1-\frac{1}{2}s^2+\frac{s^4}{8}+\cdots$
SECOND TERM $= \sum_{s=1}^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-s)s!} \int_0^{\pi/2} dq\,(\sin q)^{2s-1}$
$\frac{1}{2}$ s appeal
$\frac{\cos(\pi s)}{2s}$
$= \sum_{s=1}^{\infty} \frac{\pi}{2s\Gamma(s+\frac{1}{2})\Gamma(\frac{1}{2}-s)} = \sum \frac{1}{2}\frac{(-1)^s}{s} = -\frac{1}{2}\log 2$
$\Gamma \sim \log\left(2\sqrt{2}e^{\gamma_E}|z|\right)$

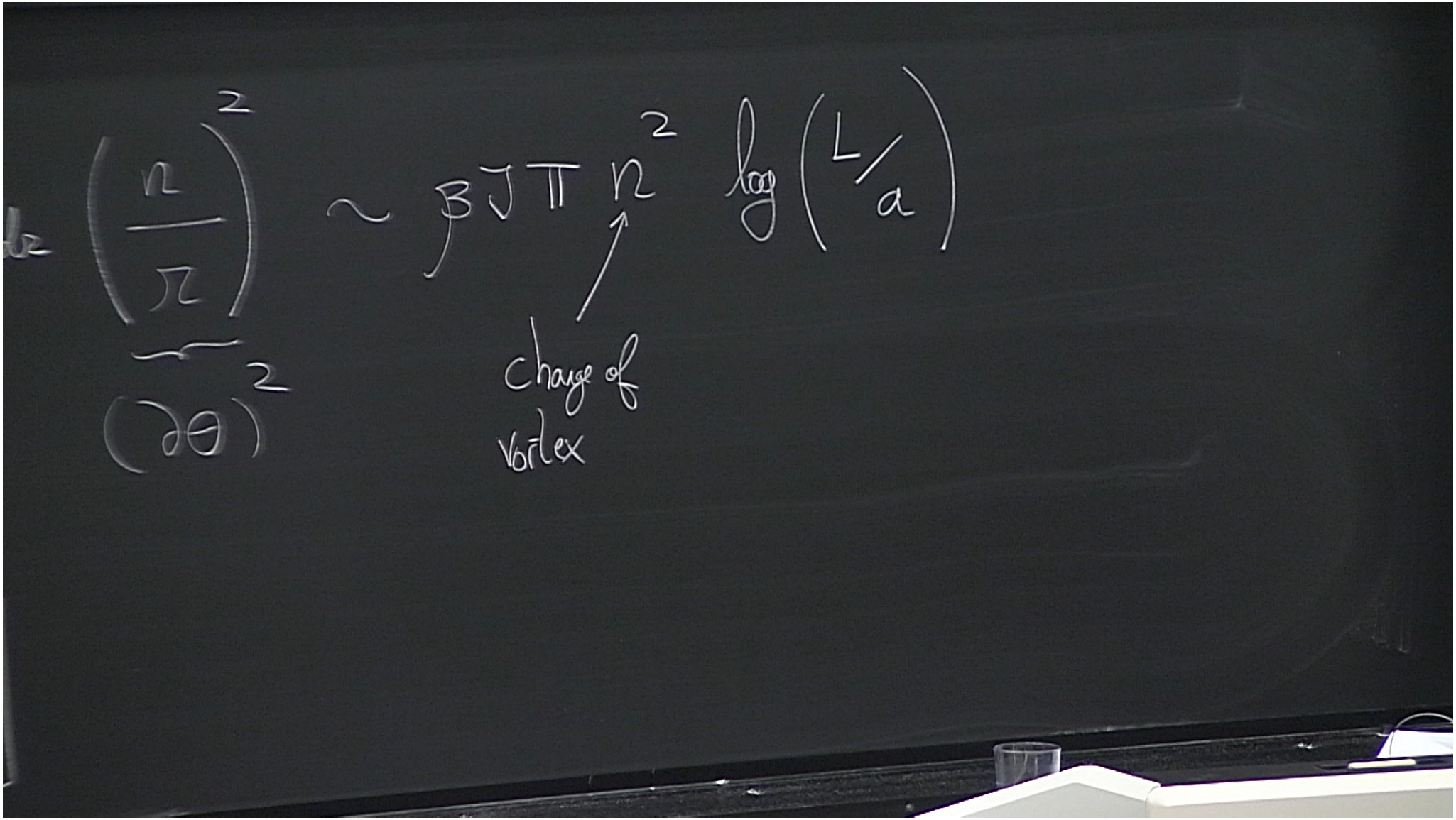

$\langle e \quad e \rangle$
$\langle S_{\vec{R}} \cdot S_{\vec{R}+\vec{x}} \rangle$
$\| \leftarrow$ large $|\vec{x}|$
$\xi_0 = \frac{1}{2\sqrt{2}e^{\gamma_E}}$
$\left(\frac{\xi_0}{|\vec{x}|}\right)^{\gamma}$
$\frac{1}{2\pi\beta J}$ critical exponent
Where
$G(\vec{x}) = \frac{1}{\beta J}\int \frac{d^2\vec{k}}{(2\pi)^2} \frac{e^{i\vec{k}\cdot}}{4\sin^2\frac{k_x}{2} +}$
$\nabla^2 G = -\frac{1}{\beta J}\delta^{(2)}(\vec{x}$
$n_x = 2q$
$2\sin(2nq)\sum_{s=1}^{n}\cos(2s-1)q$
const.

$$E_{\text{vortex}} \sim \frac{\beta J}{2} \int_{a \gtrsim 1}^{L} 2\pi r \, dz \underbrace{\left(\frac{n}{r}\right)^2}_{(\partial\theta)^2}$$

[rigorously, $\nabla^2 \theta = 0$

where $\theta \sim n\phi$ at large R

"$\theta = n \arctan\left(\frac{x - x_i}{y - y_i}\right)$"]


$\left(\frac{n}{r}\right)^2 \sim \beta J \pi n^2 \log\left(\frac{L}{a}\right)$
$(\partial\theta)^2$
charge of vortex

$$\left(\frac{n}{r}\right)^2 \sim \beta J \pi n^2 \log\left(\frac{L}{a}\right) \sim \beta J \pi n^2 \log L$$

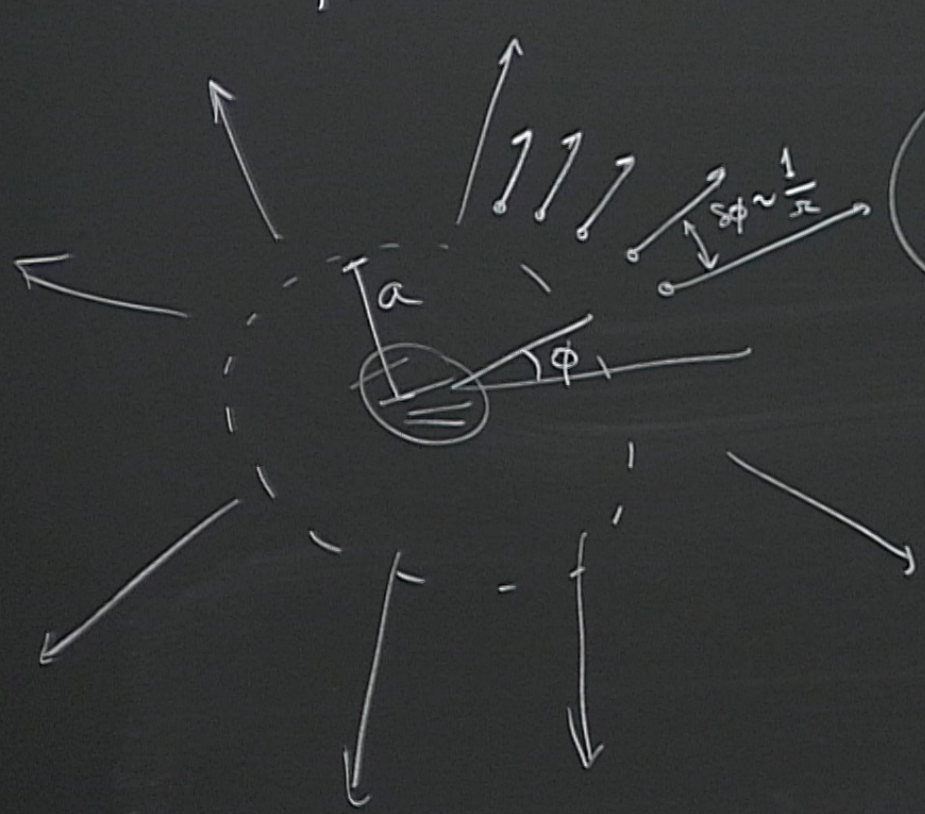
$(\partial\theta)^2$

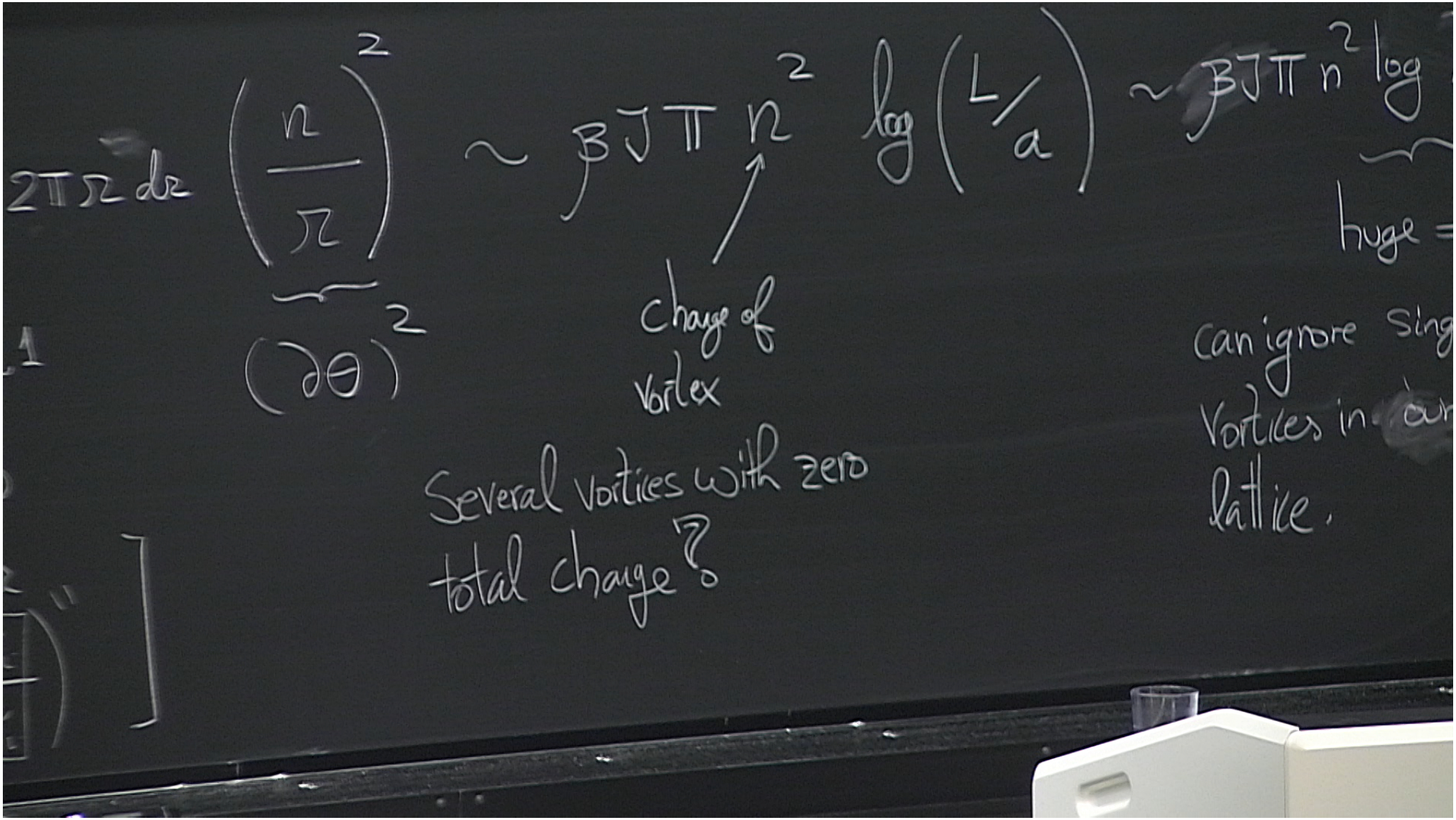
charge of vortex

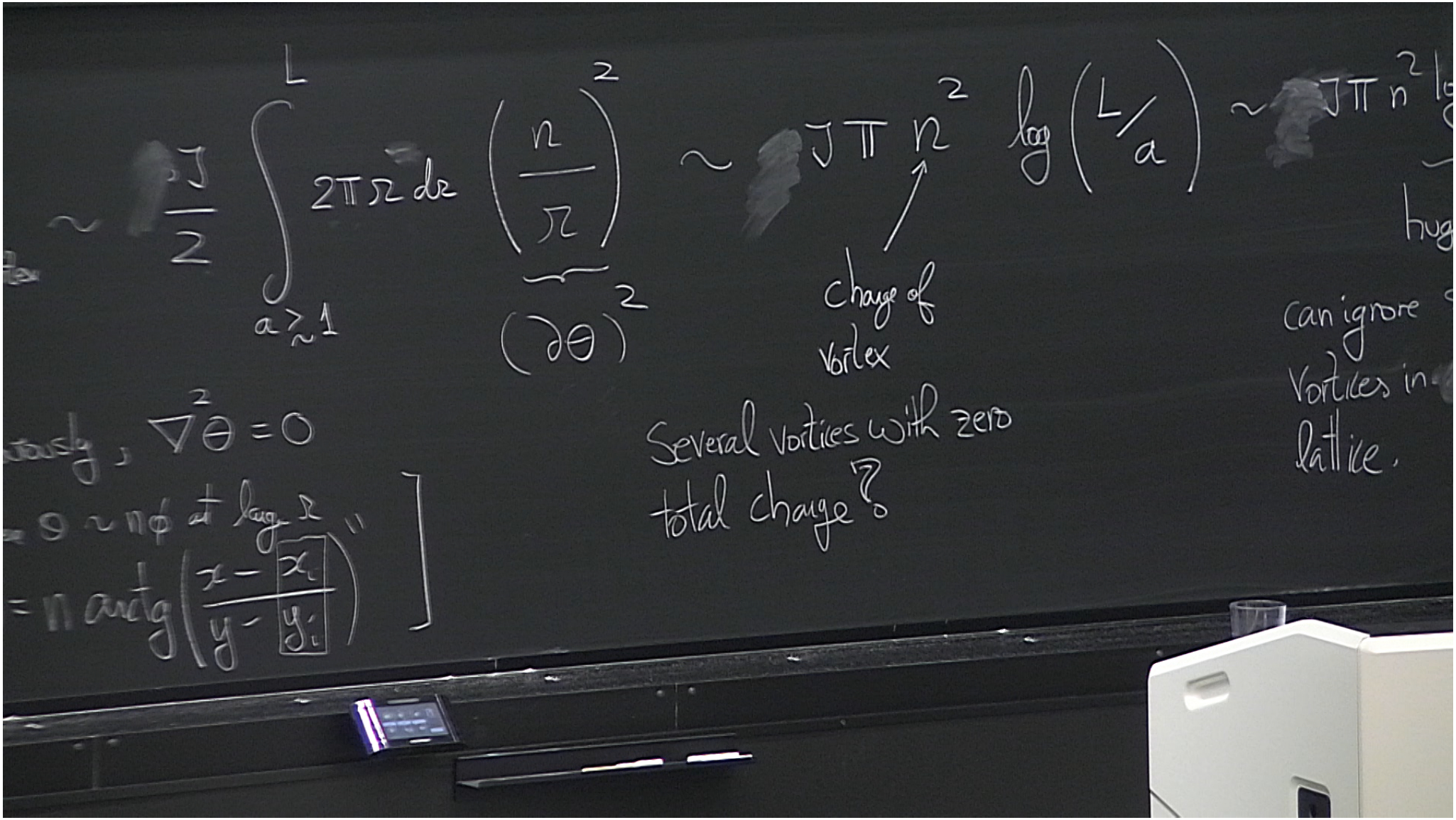
huge $\Rightarrow$

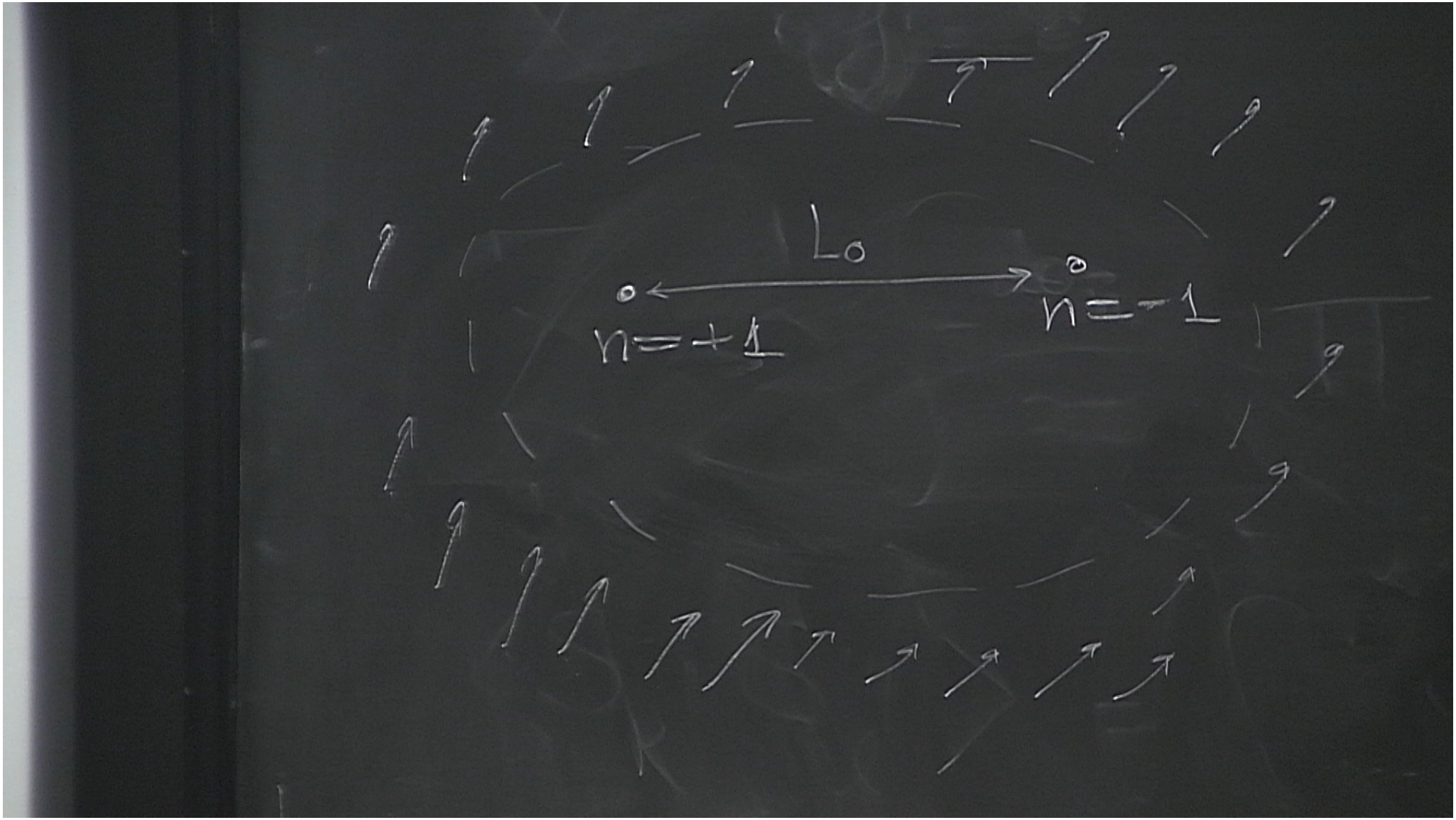
can ignore single vortices in our lattice.

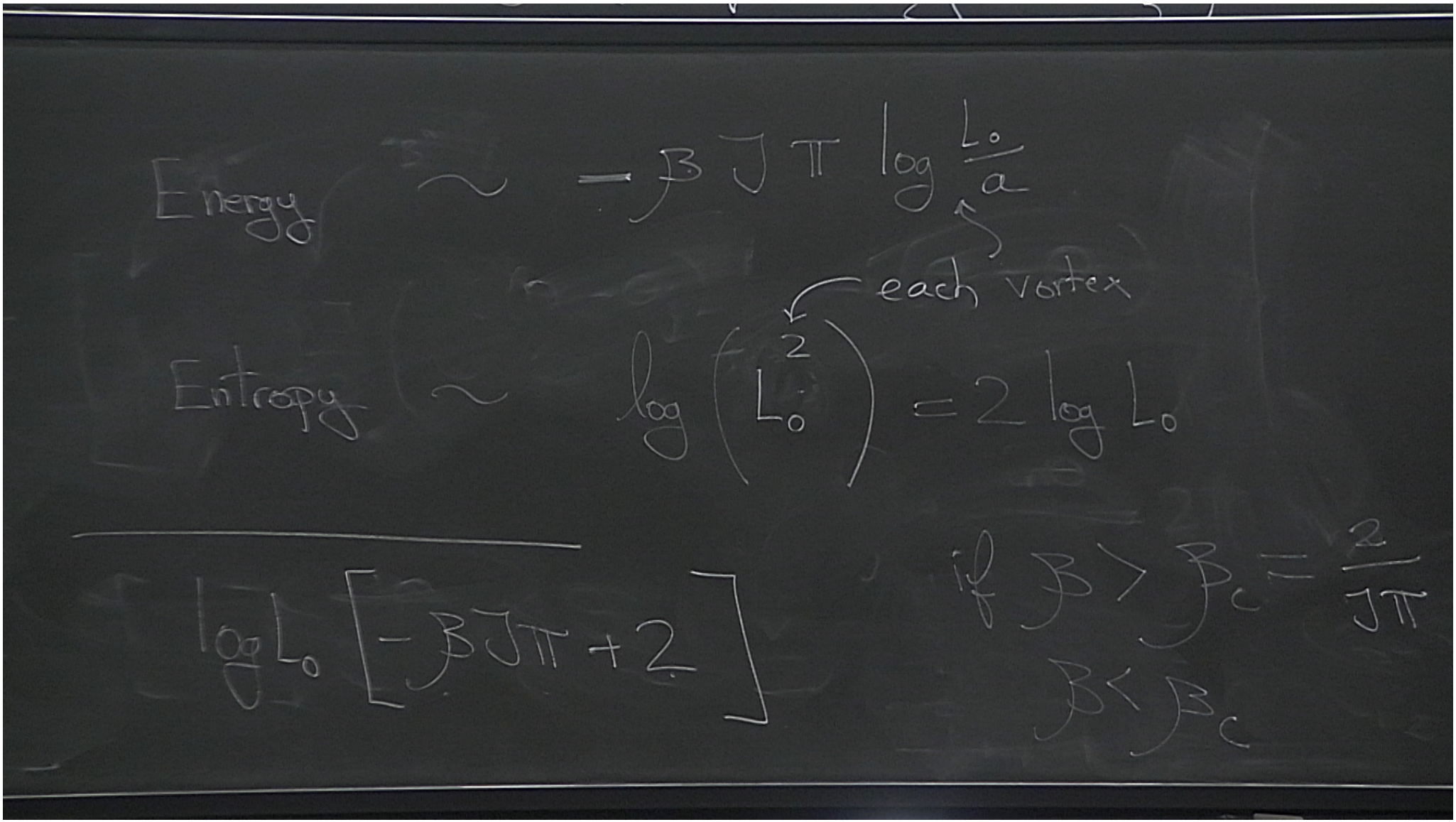
Importance of Vortices

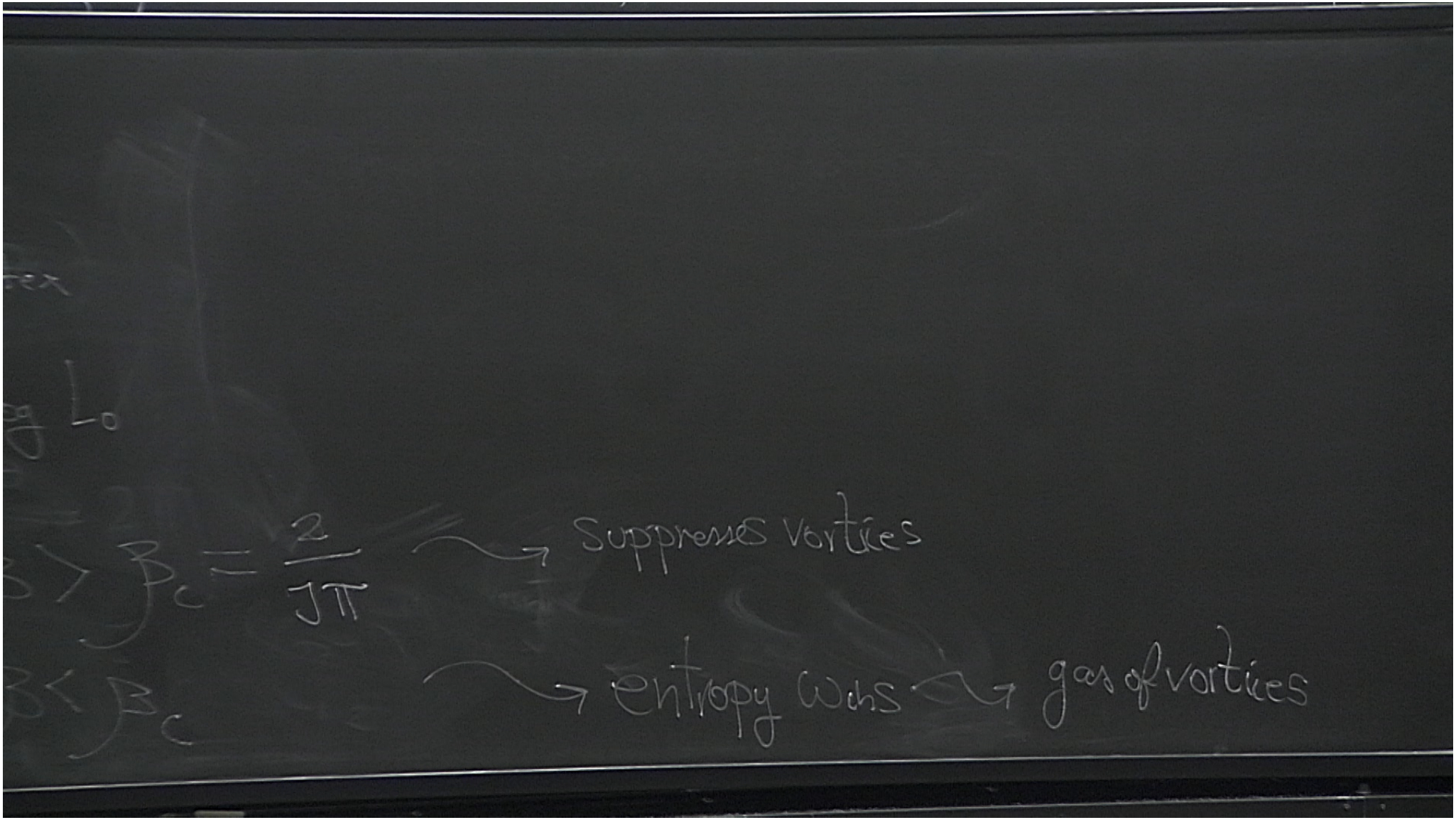


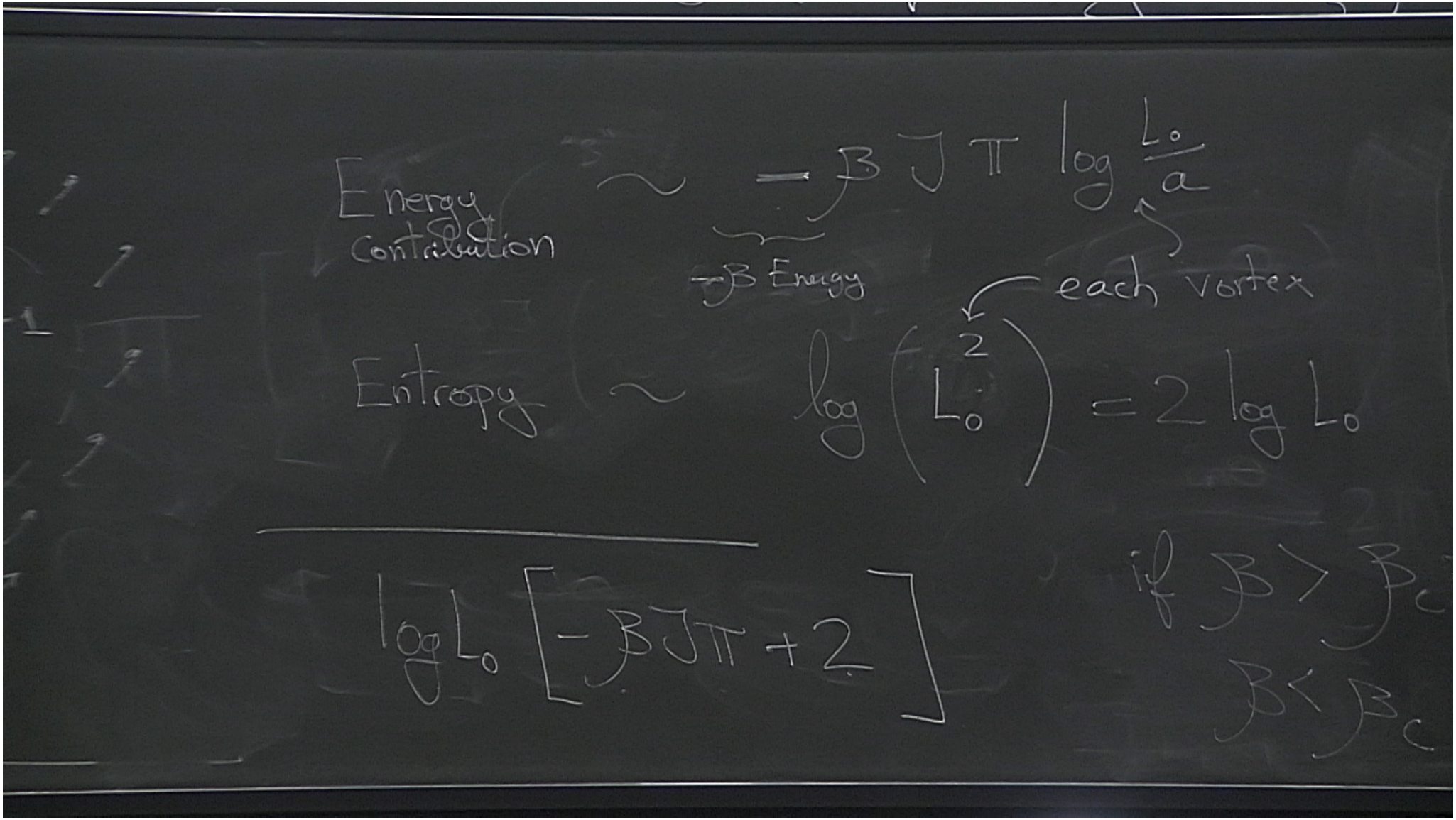

$2\pi r\,dr \left(\frac{n}{r}\right)^2 \sim \beta J \pi n^2 \log\left(\frac{L}{a}\right) \sim \beta J \pi n^2 \log$
$(\partial\theta)^2$
charge of vortex
huge
Several vorties with zero total charge?
Can ignore sing
Vortices in
lattice.

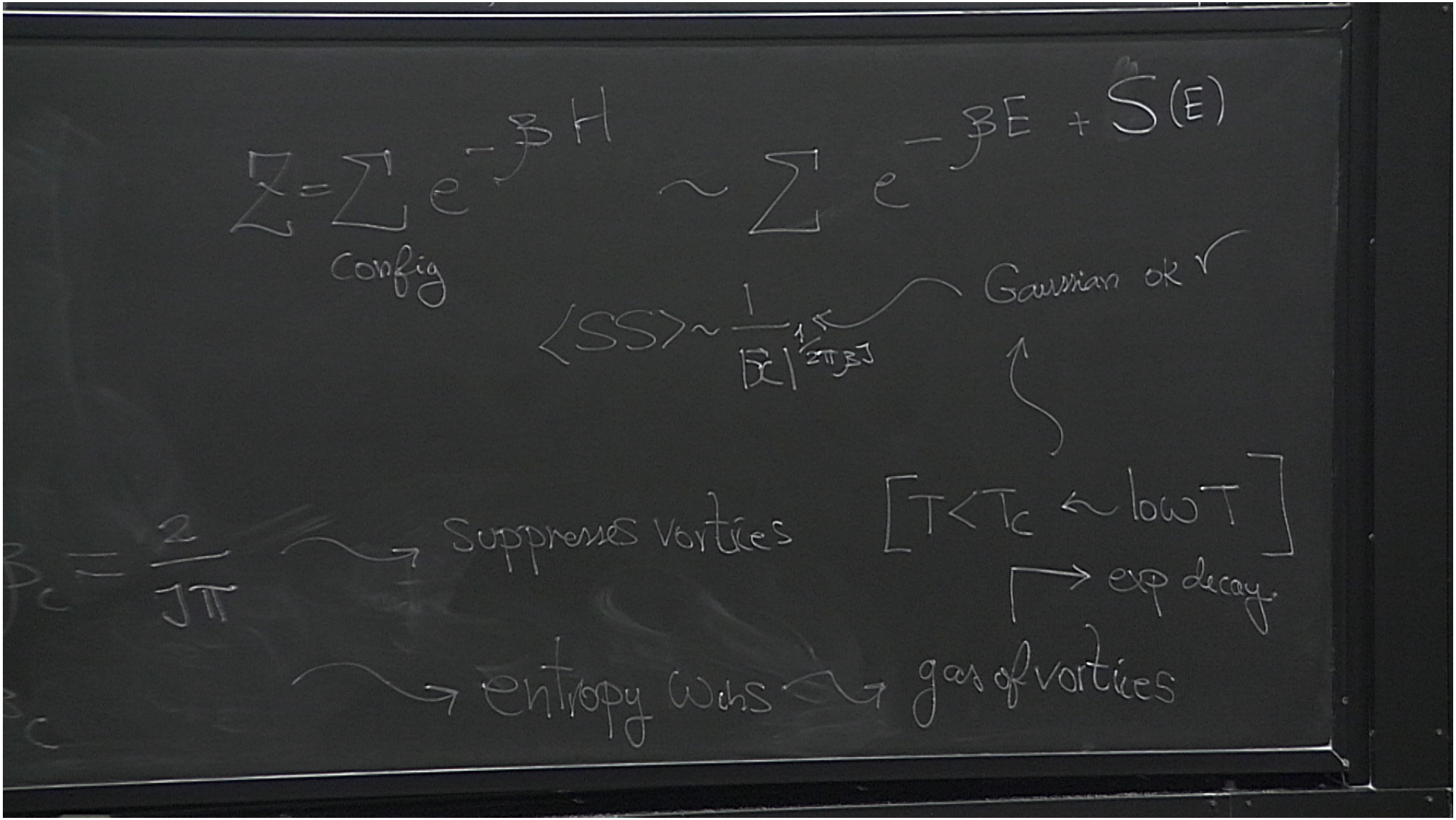

$\sim \frac{J}{2} \int_{a \gtrsim 1}^{L} 2\pi r\, dr \left(\frac{n}{r}\right)^2 \sim J\pi n^2 \log\left(\frac{L}{a}\right) \sim J\pi n^2$
$(\partial\theta)^2$
charge of vortex
Several vorties with zero total charge?
Can ignore vortices in lattice.
$\nabla^2\theta = 0$
$\theta \sim n\phi$ at large r
$= n \arctg\left(\frac{x - x_i}{y - y_i}\right)$

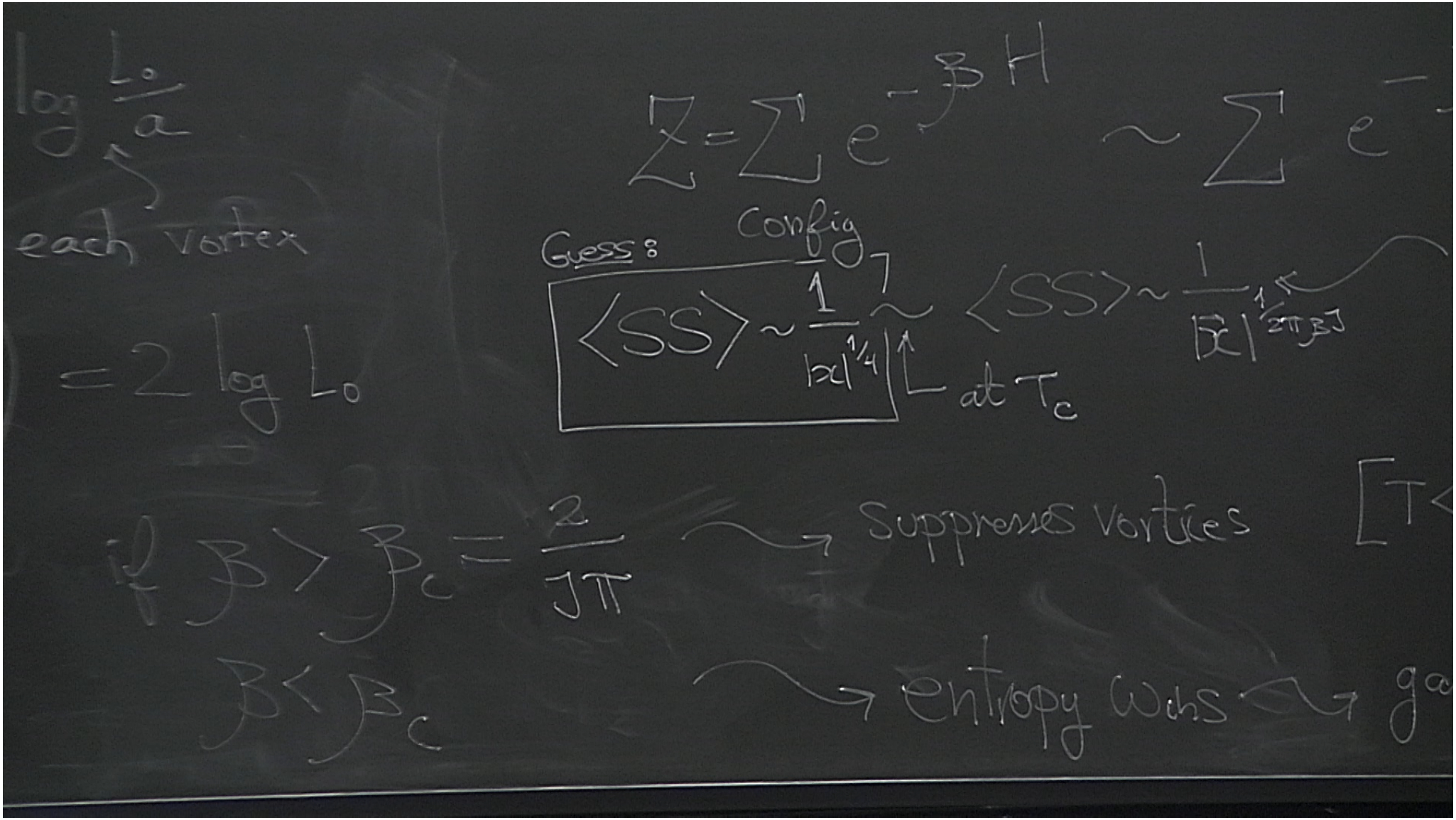

L_0
$n = +1$
$n = -1$

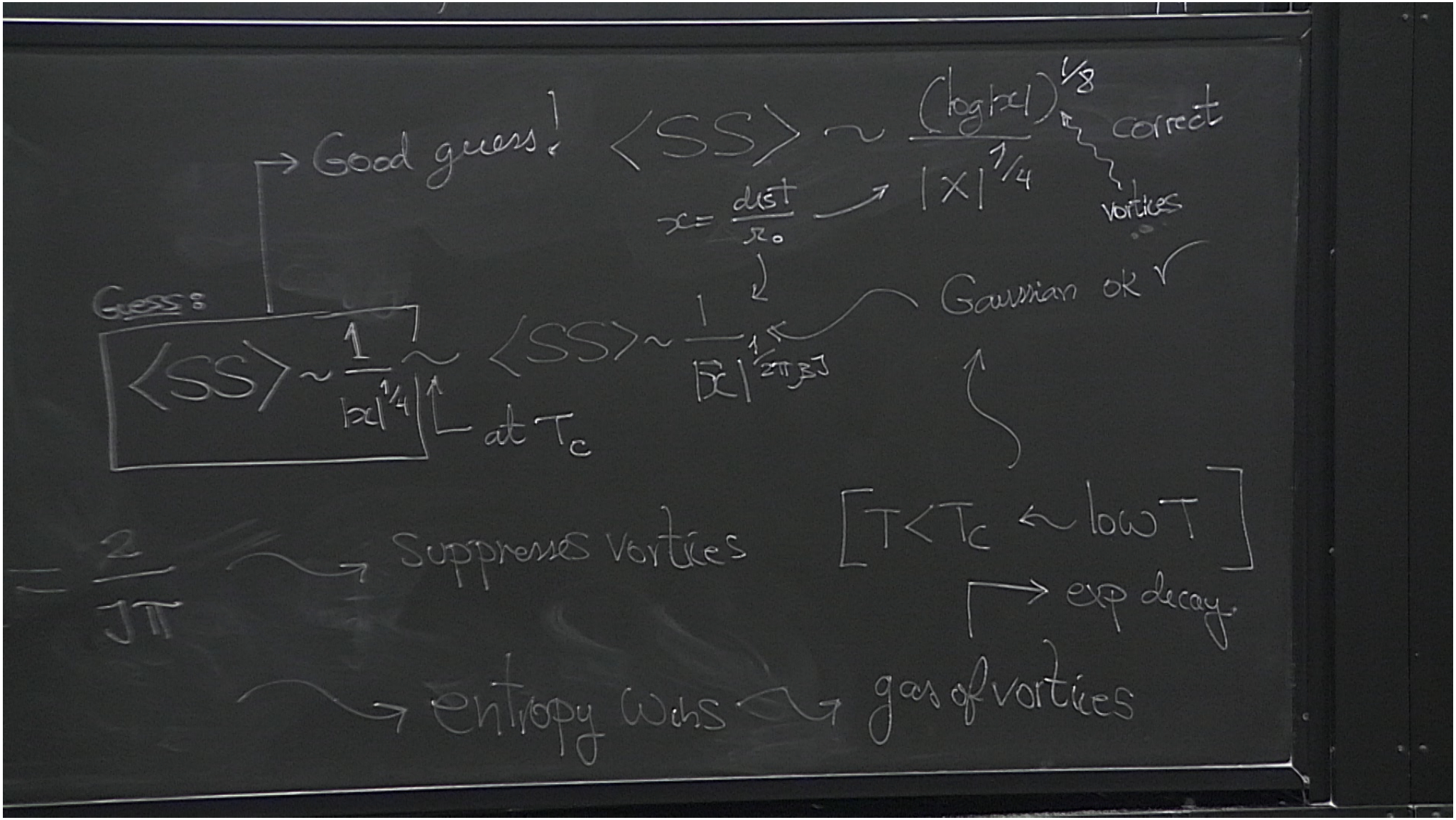

Energy $\sim -\beta J \pi \log \frac{L_0}{a}$
each vortex
Entropy $\sim \log\left(L_0^2\right) = 2 \log L_0$
$\log L_0 \left[-\beta J \pi + 2\right]$
if $\beta > \beta_c = \frac{2}{J\pi}$
$\beta < \beta_c$


L_0
$\beta > \beta_c = \frac{2}{J\pi}$
Suppresses Vortices
$\beta < \beta_c$
entropy wins
gas of vortices


Energy Contribution
$\sim -\beta J \pi \log \frac{L_0}{a}$
$-\beta$ Energy
each vortex
Entropy $\sim \log\left(L_0^2\right) = 2 \log L_0$
$\log L_0 \left[-\beta J \pi + 2\right]$
if $\beta > \beta_c$
$\beta < \beta_c$


$Z = \sum_{\text{Config}} e^{-\beta H} \sim \sum e^{-\beta E + S(E)}$
$\langle SS \rangle \sim \frac{1}{|x|^{\frac{1}{2\pi\beta J}}}$
Gaussian ok
$\beta_c = \frac{2}{J\pi}$
Suppresses Vortices
$[T < T_c \leftarrow \text{low } T]$
exp decay.
entropy wins
gas of vortices


log L/a
each Vortex
= 2 log L₀
Z = Σ e^{-βH} ~ Σ e^{-
Config
Guess:
⟨SS⟩ ~ 1/|x|^{1/4}
L at T_c
⟨SS⟩ ~ 1/|x|^{1/2πβJ}
if β > β_c = 2/Jπ
Suppresses Vortices
[T<
β < β_c
entropy wins
g∞


Good guess!
$\langle SS \rangle \sim \frac{(\log|x|)^{1/8}}{|x|^{1/4}}$
correct
vortices
$x = \frac{dist}{r_0}$
Guess:
$\langle SS \rangle \sim \frac{1}{|x|^{1/4}}$
at T_c
$\langle SS \rangle \sim \frac{1}{|x|^{\frac{1}{2\pi\beta J}}}$
Gaussian ok
$= \frac{2}{J\pi}$
Suppresses vorties
$[T < T_c \sim$ low $T]$
exp decay
entropy wins
gas of vorties