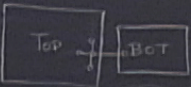


Title: PSI 2018/2019 - Statistical Mechanics - Lecture 8

Date: Oct 22, 2018 10:45 AM

URL: <http://pirsa.org/18100030>

Abstract:



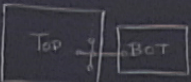
RESERVOIR
WITH A FEW (5 in the class / paper)
HARD CORE BOSONS INITIALLY IN THE VACUUM

$$\mathcal{H} = \{ b_{\vec{n}_1}^+ \dots b_{\vec{n}_K}^+ | \text{empty lattice} \rangle \quad \text{with} \quad \overbrace{(b_{\vec{n}}^+)^2 = 0}^{\text{Hardcore}}, \quad b_{\vec{n}}^2 = 0, \quad [b_{\vec{n}_1}, b_{\vec{n}_2}] = 0 \}$$

$$|\Psi(z)\rangle_{\text{Full System}} = \sum C_n e^{-iE_n t} \quad |\Psi_n\rangle \xrightarrow{t=0} |\text{Ground State of Reservoir}\rangle \quad \text{Same as } \delta H = U \sum_i \underbrace{b_i^+ b_i}_{N_i}$$

without $(b^+)^2 = b^2 = 0$. Bosons

$$H = J \sum_{\langle ij \rangle} \underbrace{b_i^+ b_j + b_j^+ b_i}_{\text{Hopping}} + U \sum_{\langle ij \rangle} \underbrace{N_i N_j}_{\text{repulsion}}$$



RESERVOIR WITH A FEW (5 in the class / paper)
HARD CORE BEGINS INITIALLY IN THE VACUUM

Handcore

$$\mathcal{H} = \{ b_{\vec{n}_1}^+ \dots b_{\vec{n}_K}^+ | \text{empty lattice} \rangle \quad \text{with} \quad (b_{\vec{n}}^+)^2 = 0, \quad b_{\vec{n}}^2 = 0, \quad [b_{\vec{n}_1}, b_{\vec{n}_2}] = 0 \}$$

$$|\Psi(t)\rangle_{\text{Full System}} = \sum C_n e^{-iE_n t} |\Psi_n\rangle \quad \xrightarrow{t=0} \quad |\Psi_n\rangle = |\text{Ground State of Reservoir}\rangle$$

Same as $\delta H = U \sum_i b_i^+ b_i$ with $(b^+)^2 = b^2 = 0$. Bosons without

$$H = J \sum_{\langle ij \rangle} b_i^+ b_j + b_j^+ b_i + U \sum_{\langle ij \rangle} N_i N_j$$

Hopping repulsion $|\Psi_0^{\text{BOT}}\rangle$

lattice $\rangle$ with $\overbrace{(b_{\vec{n}}^+)^2 = 0, b_{\vec{n}}^2 = 0, [b_{\vec{n}_1}, b_{\vec{n}_2}] = 0}^{\text{Hardcore}}$

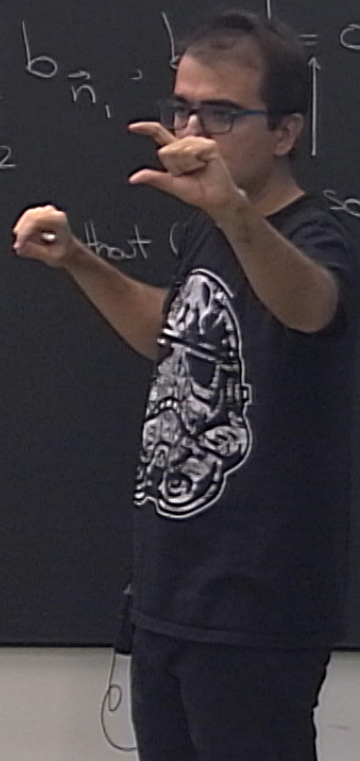
$\downarrow t=0$
 $|\Psi_n\rangle = |\text{Ground State of Reservoir}\rangle$

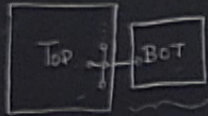
$U \sum_{\langle ij \rangle} N_i N_j$ repulsion

$\downarrow |\Psi_0^{\text{BOT}}\rangle$

Same as $\delta H = U \sum_i \underbrace{[b_i^+ b_i]^2}_{N_i}$ that's

sons





RESERVOIR
WITH A FEW (5 in the class/paper)
HARD CORE BOSONS INITIALLY IN THE VACUUM

$$\mathcal{H} = \{ b_{\vec{n}_1}^+ \dots b_{\vec{n}_K}^+ | \text{empty lattice} \rangle$$

$$|\Psi(t)\rangle_{\text{Full System}} = \sum c_n e^{-iE_n t} |\Psi_n\rangle \xrightarrow{t=0} |\text{Ground State of Reservoir}\rangle$$

$$H = J \sum_{\langle ij \rangle} \underbrace{b_i^+ b_j + b_j^+ b_i}_{\text{Hopping}} + U \sum_{\langle ij \rangle} \overbrace{N_i N_j}^{\text{repulsion}}$$

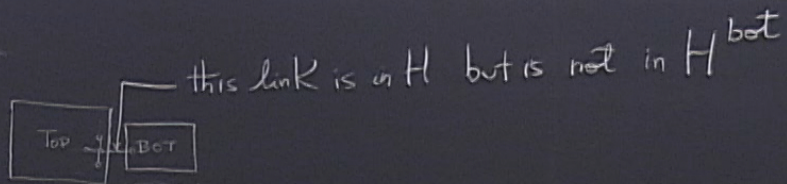
with $(b_{\vec{n}}^+)^2 = 0$, $b_n^2 = 0$, $[b_{\vec{n}_1}, b_{\vec{n}_2}] = 0$ } Hardcore
Same as $\delta H = U \sum_i \underbrace{[b_i^+ b_i]^2}_{N_i}$ } Bosons
without $(b^+)^2 = b^2 = 0$.

$$H_{\text{bottom}} |\psi_0^{\text{BOT}}\rangle = E_0 |\psi_0^{\text{BOT}}\rangle$$

$$H |\psi_0^{\text{BOT}}\rangle = E_0 |\psi_0^{\text{BOT}}\rangle + \left| \begin{array}{c} \square \\ \text{one boson in!} \end{array} \right\rangle$$

$$\langle \psi_0^{\text{BOT}} | H | \psi_0^{\text{BOT}} \rangle = E_0 \text{ still.}$$

$$= E_0 @ \text{any } t$$



RESERVOIR
WITH A FEW ($\leq$ in the class / paper)
HARD CORE BOSONS INITIALLY IN THE VACUUM

$$\mathcal{H} = \left\{ b_{\vec{n}_1}^+ \dots b_{\vec{n}_K}^+ \mid \text{empty lattice} \right\} \quad \text{with} \quad \left. \begin{array}{l} \text{Handcore} \\ (b_{\vec{n}}^+)^2 = 0, \quad b_{\vec{n}}^2 = 0, \quad [b_{\vec{n}_1}, b_{\vec{n}_2}] = 0 \end{array} \right\}$$

$$|\Psi(z)\rangle_{\text{Full System}} = \sum c_n e^{-iE_n t} |\Psi_n\rangle \quad \xrightarrow{t=0} \quad |\Psi_n\rangle = |\text{Ground State of Reservoir}\rangle$$

Same as $\delta H = U \sum_i \underbrace{[b_i^+ b_i]^2}_{N_i}$ with $(b^+)^2 = b^2 = 0$. Bosons

↑ huge

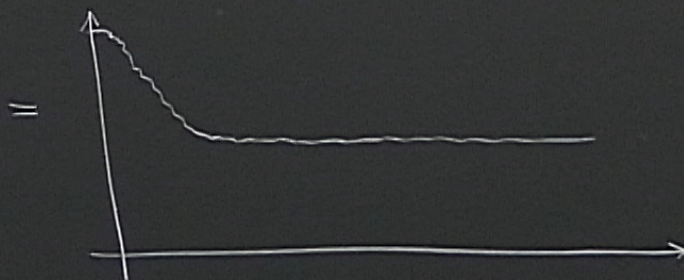
↓ $|\Psi_0^{\text{Bot}}\rangle$

$$H = J \sum_{\langle ij \rangle} \underbrace{b_i^+ b_j + b_j^+ b_i}_{\text{Hopping}} + U \sum_{\langle ij \rangle} \underbrace{N_i N_j}_{\text{repulsion}}$$

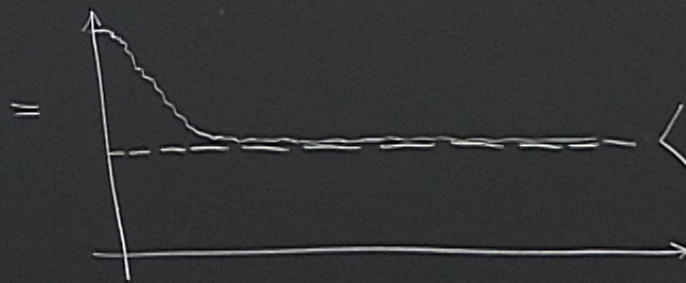
all links

$$\langle \mathcal{O} \rangle_t = \sum_n \langle n | \mathcal{O} | n \rangle |C_n|^2 + \sum_{m \neq n} \langle n | \mathcal{O} | m \rangle C_n^* C_m e^{-i(E_m - E_n)t}$$

$$\langle \mathcal{O} \rangle_t = \sum_n \langle n | \mathcal{O} | n \rangle |C_n|^2 + \sum_{m \neq n} \langle n | \mathcal{O} | m \rangle C_n^* C_m e^{-i(E_m - E_n)t}$$



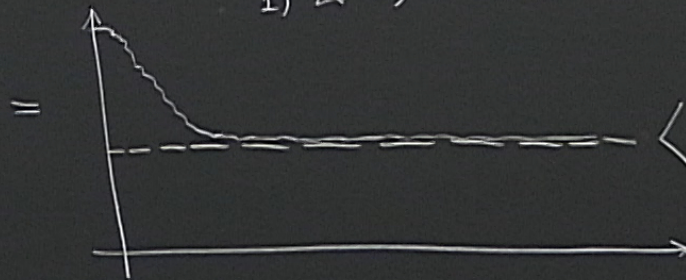
$$\langle \mathcal{O} \rangle_t = \sum_n \langle n | \mathcal{O} | n \rangle |C_n|^2 + \sum_{m \neq n} \langle n | \mathcal{O} | m \rangle C_n^* C_m e^{-i(E_m - E_n)t}$$



$$E_0 - \delta < E_n < E_0 + \delta$$

$$\langle \mathcal{O} \rangle_{\text{microcanonical @ fixed } E \simeq E_0} = \frac{\sum \langle n | \mathcal{O} | n \rangle}{\sum \langle n | n \rangle}$$

$$\langle \mathcal{O} \rangle_t = \sum_n \underbrace{\langle n | \mathcal{O} | n \rangle}_{\substack{\simeq \langle \mathcal{O} \rangle_{\text{micro}}(E_n) \\ 1) \& 2) \iff ETH}} |C_n|^2 + \sum_{m \neq n} \underbrace{\langle n | \mathcal{O} | m \rangle}_{\substack{2) \text{ small}}} C_n^* C_m e^{-i(E_m - E_n)t}$$



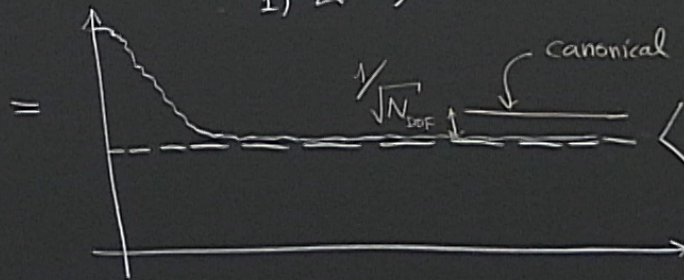
$$E_0 - \delta < E_n < E_0 + \delta$$

$$= \frac{\sum \langle n | \mathcal{O} | n \rangle}{\sum \langle n | n \rangle}$$

microcanonical
@ fixed $E \simeq E_0$

$$\langle \mathcal{O} \rangle_t = \sum_n \underbrace{\langle n | \mathcal{O} | n \rangle}_{\approx \langle \mathcal{O} \rangle_{\text{micro}}(E_n)} |C_n|^2 + \sum_{m \neq n} \underbrace{\langle n | \mathcal{O} | m \rangle}_{\approx \text{small}} \frac{C_n^* C_m}{e^{-i(E_m - E_n)t}}$$

1) & 2) $\leftrightarrow$ ETH



canonical $\langle \mathcal{O} \rangle_{\text{CAN}, \beta}$

$$E_0 - \delta < E_n < E_0 + \delta$$

$$\langle \mathcal{O} \rangle_{\text{microcanonical}} \stackrel{E_0 = \langle E \rangle_{\beta}}{=} \sum_n \frac{\langle n | \mathcal{O} | n \rangle}{\sum_n \langle n | n \rangle}$$

@ fixed $E \approx E_0$

$$\langle \mathcal{O} \rangle_t = \sum_n \underbrace{\langle n | \mathcal{O} | n \rangle}_{\approx \langle \mathcal{O} \rangle_{\text{micro}}(E_n)} |C_n|^2 + \sum_{m \neq n} \underbrace{\langle n | \mathcal{O} | m \rangle}_{\approx \text{small}} \frac{C_n^* C_m}{e^{-i(E_m - E_n)t}}$$

$\langle \mathcal{O} \rangle_{\text{extensive}} / N_{\text{DOF}} \quad 1) \& 2) \iff \text{ETH}$

$E_0 - \delta < E_n < E_0 + \delta$

$$\langle \mathcal{O} \rangle_{\text{microcanonical}} @ \text{fixed } E \approx E_0 = \frac{\sum \langle n | \mathcal{O} | n \rangle}{\sum \langle n | n \rangle}$$

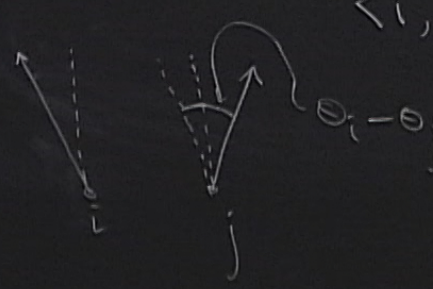
XY model

$$H = -J \sum_{\langle i, j \rangle} \vec{S}_i \cdot \vec{S}_j$$

unit vectors
with 2
components

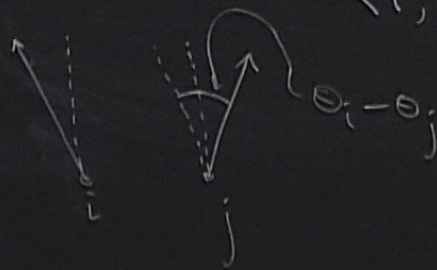
XY model

$$H = -J \sum_{\langle i, j \rangle} \vec{S}_i \cdot \vec{S}_j$$

 unit vectors with 2 components

XY model

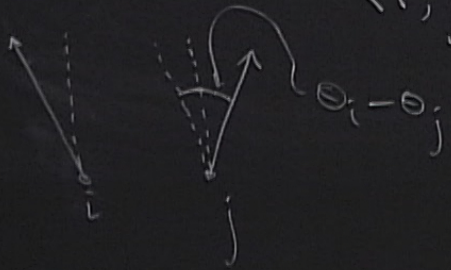
$$H = -J \sum_{\langle i, j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum \cos(\theta_i - \theta_j)$$



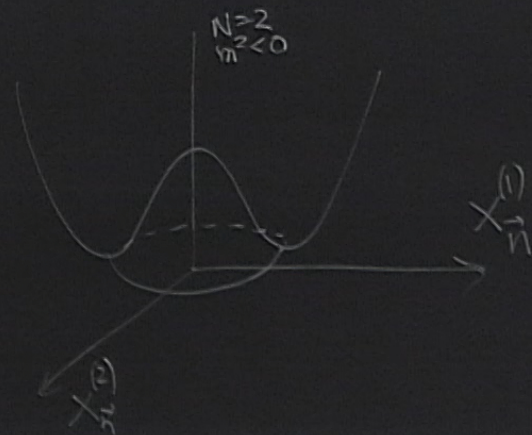
unit vectors
with 2
components

XY model

$$H = -J \sum_{\langle i, j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum \cos(\theta_i - \theta_j)$$

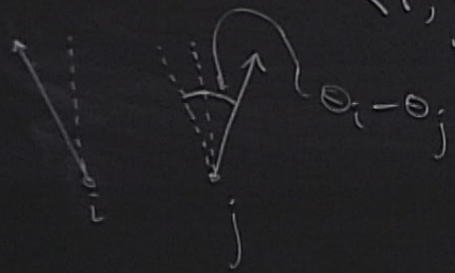


unit vectors
with 2
components



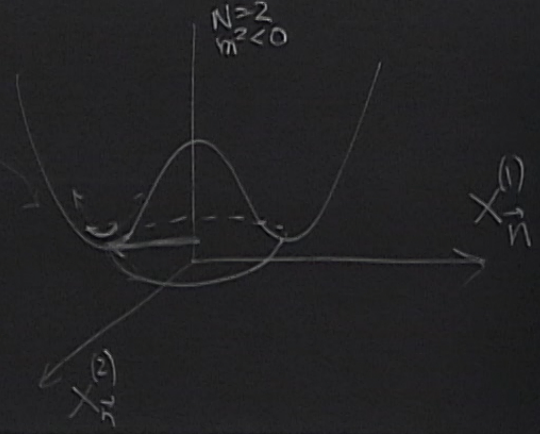
XY model

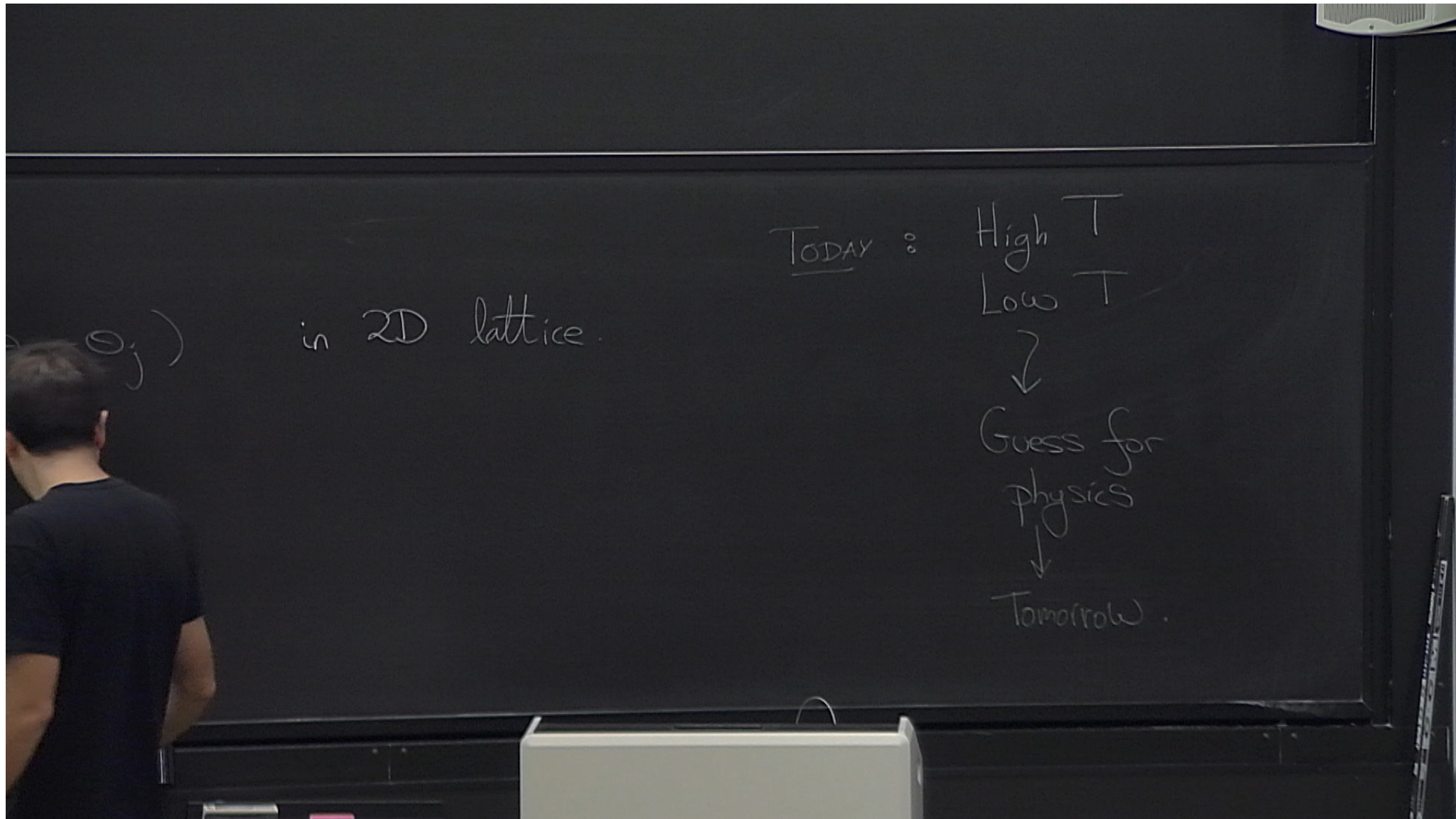
$$H = -J \sum_{\langle i, j \rangle} \vec{S}_i \cdot \vec{S}_j = -J \sum \cos(\theta_i - \theta_j)$$

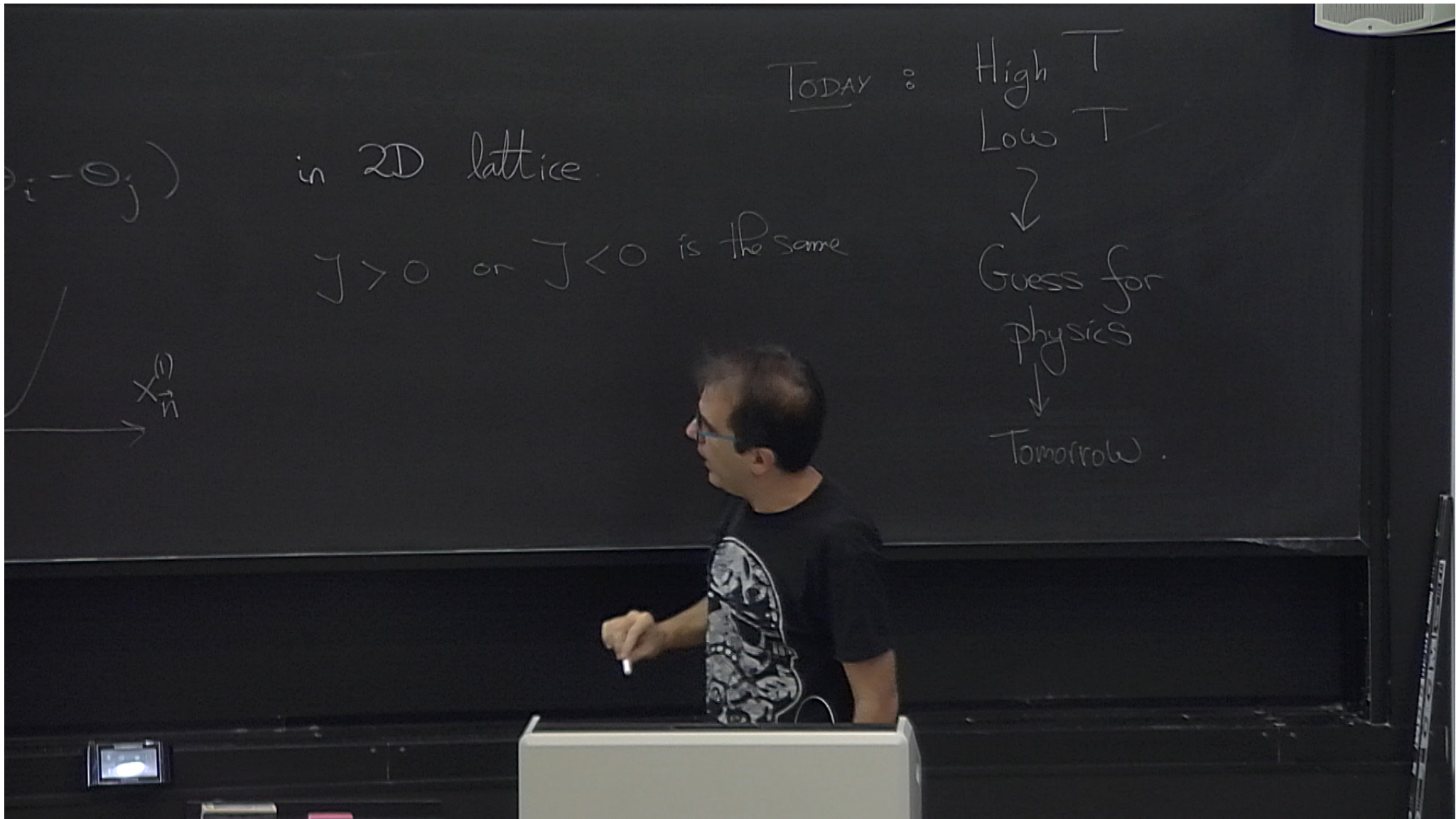


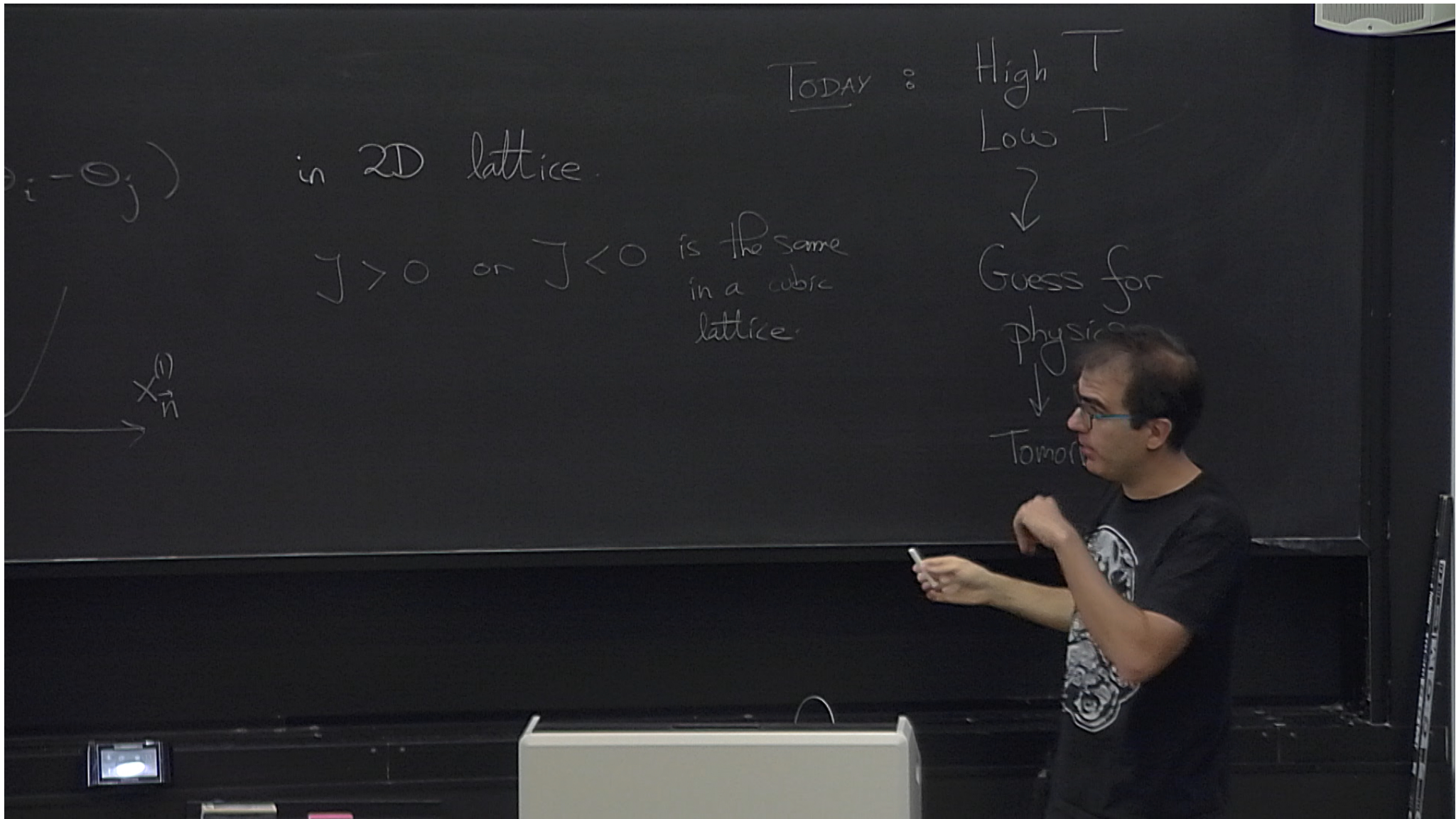
unit vectors
with 2
components

same at
low E

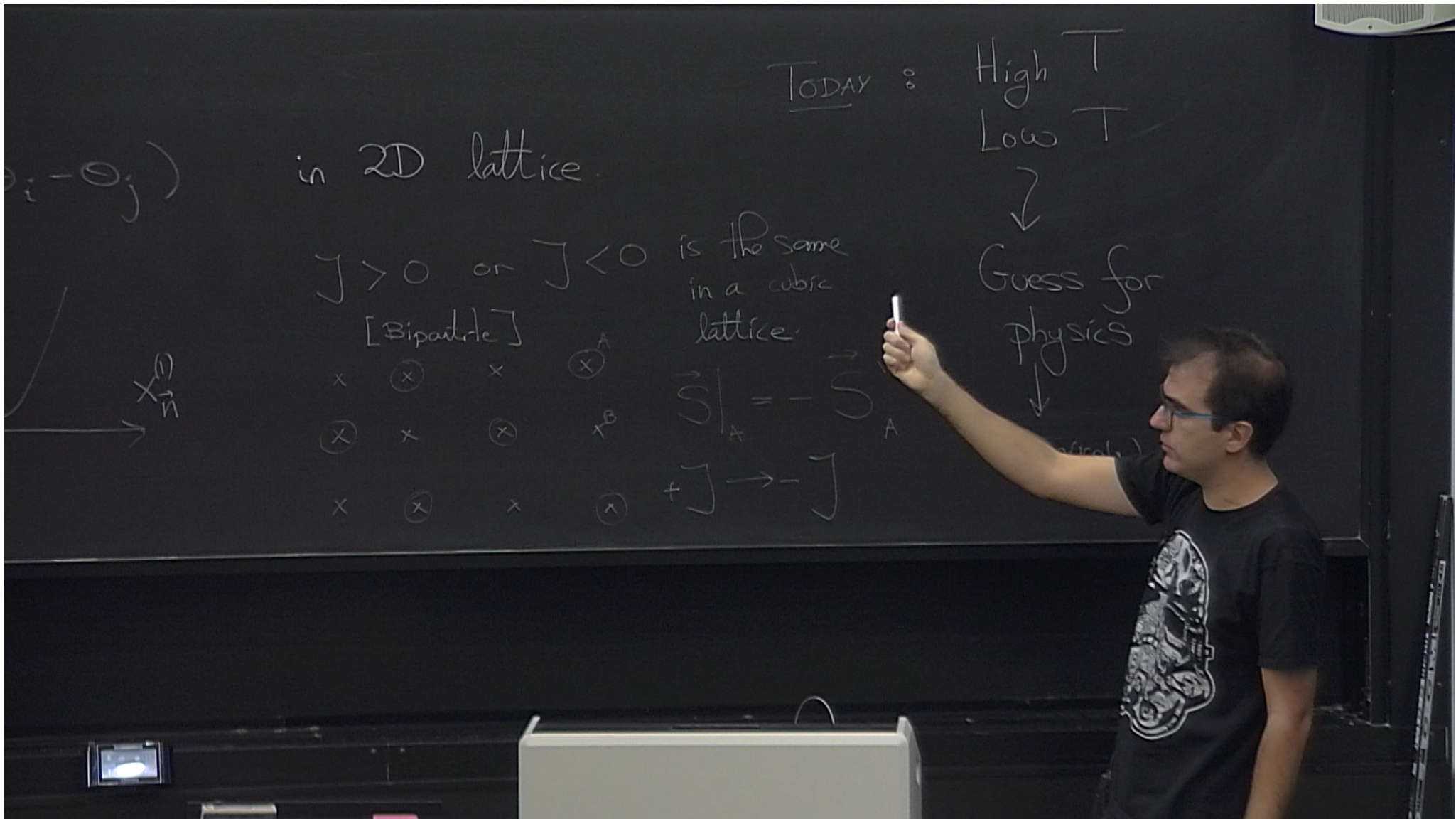












$$Z = \int_0^{2\pi} \prod_{\vec{n}} d\theta_{\vec{n}} \exp \left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right]$$

$$Z = \int_0^{2\pi} \prod_{\vec{n}} \frac{d\theta_{\vec{n}}}{(2\pi)} \exp \left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right]$$

β small (high T)

lets postpone a bit

$$\langle \vec{S}_{\vec{j}} \cdot \vec{S}_{\vec{j}+\vec{n}} \rangle = \langle \cos(\theta_{\vec{j}} - \theta_{\vec{j}+\vec{n}}) \rangle$$

$$Z = \int_0^{2\pi} \prod_{\vec{n}} \frac{d\theta_{\vec{n}}}{(2\pi)} \exp \left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right]$$

β small (high T)

lets
postpone
a bit

$\theta \rightarrow -\theta$ Sym

$$\langle \vec{S}_{\vec{j}} \cdot \vec{S}_{\vec{j}+\vec{n}} \rangle = \langle \cos(\theta_{\vec{j}} - \theta_{\vec{j}+\vec{n}}) \rangle = \langle e^{i\theta_{\vec{j}} - i\theta_{\vec{j}+\vec{n}}} \rangle$$

$$(\theta_i - \theta_j)$$

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0}$$

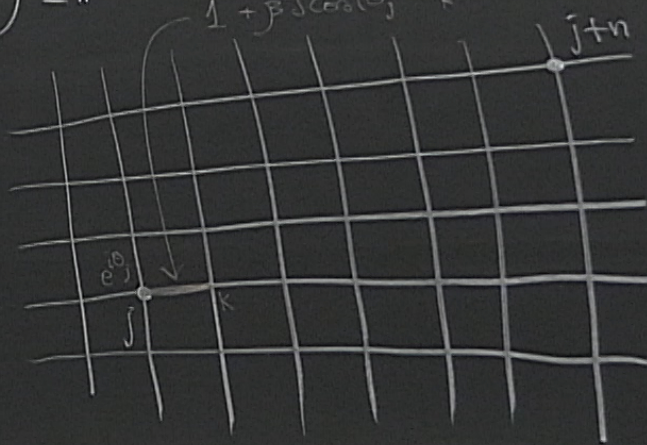
$\theta \rightarrow -\theta$ sym

$$= \langle i\theta_j - i\theta_{j+n} \rangle$$

$$s(\theta_i - \theta_j)$$

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0}$$

$$1 + \beta J \cos(\theta_j - \theta_k)$$



$\theta \rightarrow -\theta$ sym

$$\langle e^{i\theta_j - i\theta_{j+n}} \rangle$$

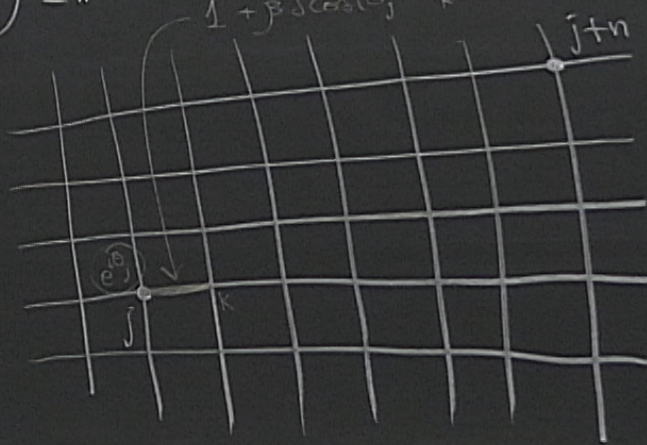
$$(\theta_i - \theta_j)$$

$$\theta \rightarrow -\theta \text{ sym}$$

$$= \langle e^{i\theta_j - i\theta_{j+n}} \rangle$$

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0}$$

$$1 + \beta J \cos(\theta_j - \theta_k) + \dots = 1 + \frac{\beta J}{2} (e^{-i\theta_j + i\theta_k} + e^{i\theta_j - i\theta_k})$$

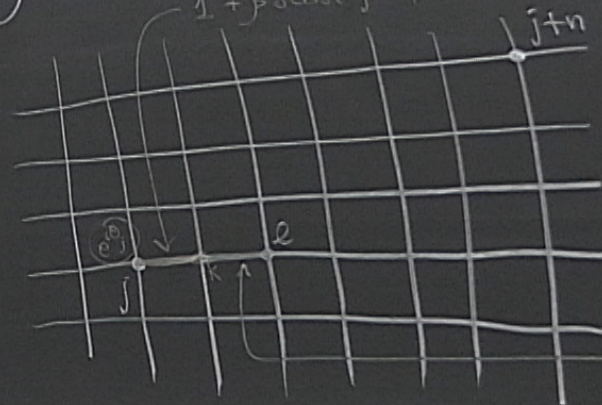


$$s(\theta_i - \theta_j)$$

$$\theta \rightarrow -\theta \text{ sym}$$

$$= \langle e^{i\theta_j - i\theta_{j+n}} \rangle$$

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0} = 1 + \sum_{j \neq k} (e^{-i\theta_j + i\theta_k} + e^{i\theta_j - i\theta_k})$$



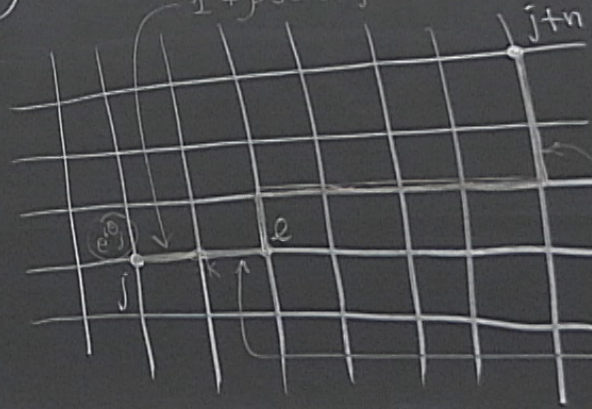
$$1 + \sum_{j \neq k} e^{-i\theta_k + i\theta_j}$$

$$(\Theta_i - \Theta_j)$$

$\Theta \rightarrow -\Theta$ sym

$$= \langle e^{i\Theta_j - i\Theta_{j+n}} \rangle$$

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0} = 1 + \sum_{k \neq 0} (e^{-i\theta_j + i\theta_k} + e^{i\theta_j - i\theta_k})$$



path connecting j and $j+n$.

$$1 + \sum_k e^{-i\theta_k + i\theta_l}$$

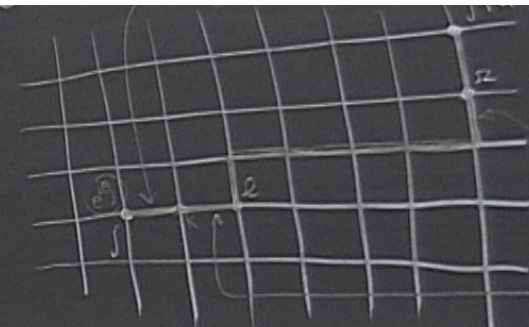
$$\langle \vec{S}_j \cdot \vec{S}_{j+\vec{n}} \rangle = \langle \cos(\theta_j - \theta_{j+\vec{n}}) \rangle = \langle e^{i\theta_j - i\theta_{j+\vec{n}}} \rangle$$

$$\begin{aligned} \langle e^{i\theta_j - i\theta_{j+\vec{n}}} \rangle &= \int \frac{\pi d\vec{\theta}_{\vec{n}}}{(2\pi)^{L^2}} e^{i\theta_j} e^{-i\theta_{j+\vec{n}}} e^{-\beta H} / Z \\ &= \sum_{\text{path that connect } j \text{ and } j+\vec{n}} \left(\frac{\beta J}{2} \right)^{\text{length of path}} \int \frac{d\theta_j}{2\pi} \frac{d\theta_k}{2\pi} \frac{d\theta_l}{2\pi} \dots e^{i\theta_j} e^{-i\theta_{j+\vec{n}}} \end{aligned}$$

let's post-pose abd

β small (high T)

$\Theta \rightarrow -\Theta$ sym

$$\langle \cos(\Theta_j - \Theta_{j+n}) \rangle = \langle e^{i\Theta_j - i\Theta_{j+n}} \rangle$$


path connecting j and $j+n$.

$1 + \frac{\beta J}{Z} e^{-i\Theta_k + i\Theta_l}$

$$\begin{aligned} &= \int \frac{\pi d\Theta_{\vec{n}}}{(2\pi)^{L^2}} e^{i\Theta_j} e^{-i\Theta_{j+n}} e^{-\beta H} \\ &= \sum_{\text{path that connect } j \text{ and } j+n} \left(\frac{\beta J}{Z} \right)^{\text{length of path}} \int \frac{d\Theta_j}{2\pi} \frac{d\Theta_k}{2\pi} \frac{d\Theta_l}{2\pi} \dots e^{i\Theta_j} e^{-i\Theta_j + i\Theta_l} e^{-i\Theta_l + i\Theta_k} \dots e^{-i\Theta_k + i\Theta_{j+n}} e^{-i\Theta_{j+n}} \end{aligned}$$

$$\langle \Theta_j - \Theta_{j+\vec{n}} \rangle = \langle e^{i\Theta_j - i\Theta_{j+\vec{n}}} \rangle$$

$$\frac{\pi d\Theta_{\vec{n}}}{(2\pi)^{L^2}} e^{i\Theta_j} e^{-i\Theta_{j+\vec{n}}} e^{-\beta H}$$

$$\sum_{\text{path that connect } j \text{ and } j+\vec{n}} \left(\frac{\beta J}{2} \right)^{\text{length of path}}$$

$$= \left(\frac{\beta J}{2} \right)^{\text{Min. Length Between } j \text{ and } j+\vec{n}} N(\text{paths with that length})$$

$$\int \frac{d\theta_j}{2\pi} \frac{d\theta_k}{2\pi} \frac{d\theta_l}{2\pi} \dots e^{i\Theta_j} e^{-i\Theta_j + i\Theta_l} e^{-i\Theta_l + \dots}$$

Z

$$\cos(\theta_j - \theta_{j+n}) = \langle e^{i\theta_j - i\theta_{j+n}} \rangle$$



$$\int \frac{\pi d\vec{\theta}}{(2\pi)^L} e^{i\theta_j} e^{-i\theta_{j+n}} e^{-\beta H} / \mathcal{Z}$$

length of path

$$\sum_{\text{path that connect } j \text{ and } j+n} \left(\frac{\beta J}{2} \right)^{\text{length of path}} \int \frac{d\theta_j}{2\pi} \frac{d\theta_k}{2\pi} \frac{d\theta_l}{2\pi} \dots e^{i\theta_j} e^{-i\theta_j + i\theta_l} e^{-i\theta_l + i\theta_k} \dots$$

Min Length Between j and $j+n$

$$\langle S_j \cdot S_{j+n} \rangle = \left(\frac{\beta J}{2} \right)^{\text{Min Length Between } j \text{ and } j+n} N(\text{paths with that length})$$

$\langle e^{i\theta_j - i\theta_{j+n}} \rangle$

$1 + \frac{\beta J}{2} e^{-i\theta_k + i\theta_l}$

$-i\theta_{j+n} e^{-\beta H}$

$\sum$

$\int \frac{d\theta_j}{2\pi} \frac{d\theta_k}{2\pi} \frac{d\theta_l}{2\pi} \dots e^{i\theta_j} e^{-i\theta_j + i\theta_l} e^{-i\theta_l + i\theta_k} \dots e^{-i\theta_{j+n} + i\theta_{j+n}} e^{-i\theta_{j+n}}$

$\frac{\beta J}{2}$ Min Length Between $\vec{j}$ and $\vec{j+n}$

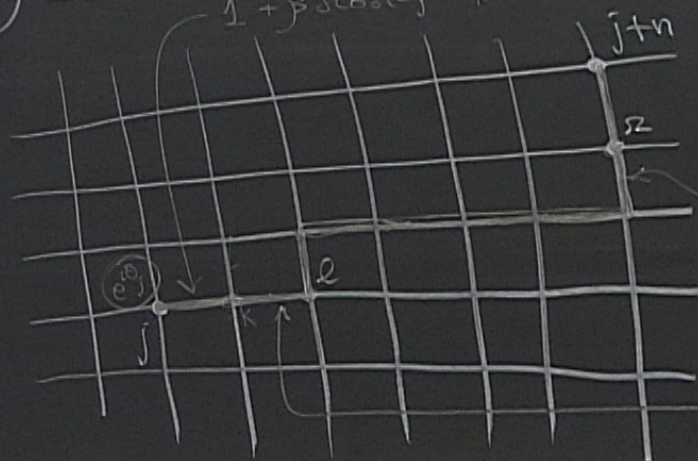
$N(\text{paths with that length})$

$\leftarrow$ exponentially suppressed @ large dist.

$$\int \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0} = 1 + \sum_{j \neq 0} \left(e^{-i\theta_j + i\theta_k} + e^{i\theta_j - i\theta_k} \right)$$

$1 + \sum_{j \neq 0} \cos(\theta_j - \theta_k) + \dots$

invariant



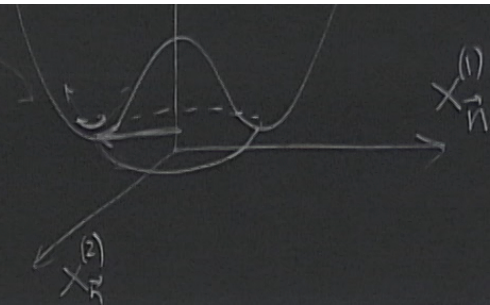
path connecting $\vec{j}$ and $\vec{j+n}$.

$$1 + \sum_{j \neq 0} e^{-i\theta_k + i\theta_l}$$



Unit vectors
with 2
components

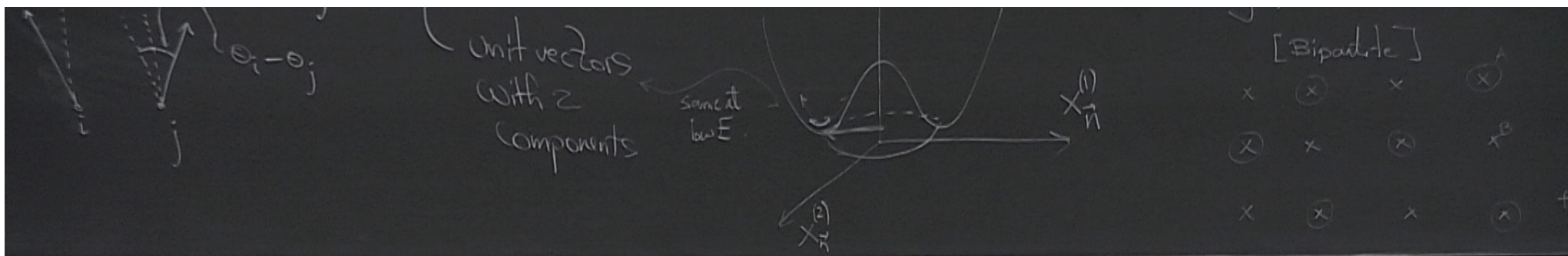
same as
low E



$$Z = \int \pi \frac{d\theta}{2\pi} \bar{n}$$

$$\prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)}$$

$$1 + \frac{\beta J}{2} \left(e^{i\theta_i - i\theta_j} + e^{-i\theta_i + i\theta_j} \right)$$



$$Z = \int \prod \frac{d\theta_n}{2\pi} \prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)} \approx 1 + \sum_{\text{loops}} \left(\frac{\beta J}{2} \right)^{\text{length}} + \dots$$

orientation

length of loop

2

Some at low E.

[Biparticle]

x	(x)	x	(x) ^A
(x)	x	(x)	x ^B
x	(x)	x	(x)

lattice.

$\vec{S}_A = -\vec{S}_A$

$+J \rightarrow -J$

physics
↓
Tomorrow.

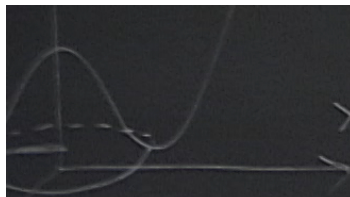
$\cos(\theta_i - \theta_j) \simeq 1 + \sum_{\text{loops}} \left(\frac{\beta J}{2} \right)^{\text{length}} + \dots$

$\frac{\beta J}{2} \left(e^{i\theta_i - i\theta_j} + e^{-i\theta_i + i\theta_j} \right) + \dots$

orientation

length of loop

Gas of Loops.



$x_n^{(0)}$

[Biparticle]

x	(x)	x	(x) ^A
(x)	x	(x)	x ^B
x	(x)	x	(x)

lattice.

$\vec{S}_A = -\vec{S}_A$

$+] \rightarrow -]$

physics

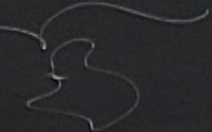
↓

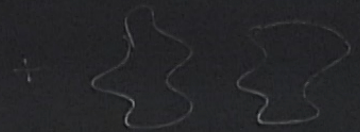
Tomorrow.

$$\approx 1 + \sum_{\text{loops}} \left(\frac{\beta J}{2} \right)^{\text{length}}$$

length

loops





+ ... = 1 + [] + more suppressed

≈ 1

$i\theta_j + e^{-i\theta_j + i\theta_j}$

length of loop

$\sim \left(\frac{\beta J}{2} \right)^{\text{length of loop}}$

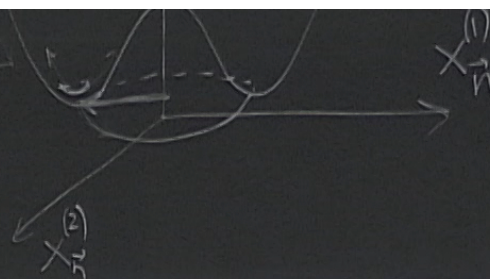
station

Gas of Loops.

What about ... ? Indeed

With 2 components

same at low E



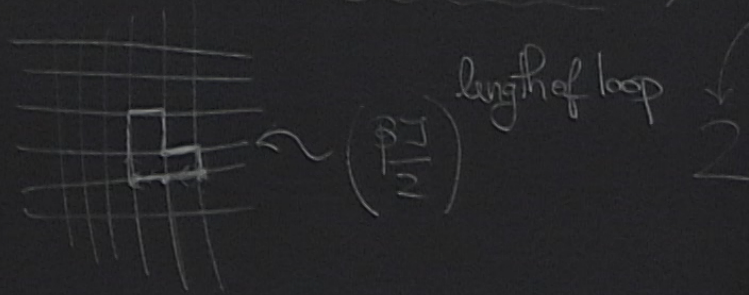
$$Z = \int \prod \frac{d\theta_n}{2\pi} \prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)} \approx 1 + \sum_{\text{loops}}$$

modified Bessel

$$e^{\beta J \cos \theta} = I_0(\beta) \left[1 + \sum_{n \neq 0} \frac{I_n(\beta)}{n! I_0(\beta)} e^{in\theta} \right]$$

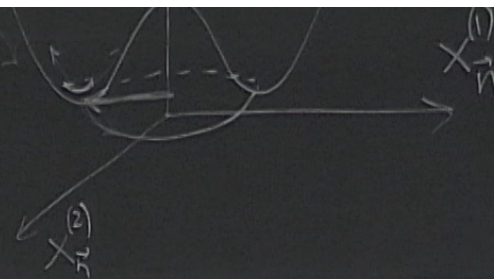
where @ $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$1 + \frac{\beta J}{2} (e^{i\theta_i - i\theta_j} + e^{-i\theta_i + i\theta_j}) + \dots$$



With 2 components

same at low E



$$Z = \int \prod \frac{d\vec{n}}{2\pi}$$

$$\prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)}$$

$$\approx 1 + \sum_{\text{loops}}$$

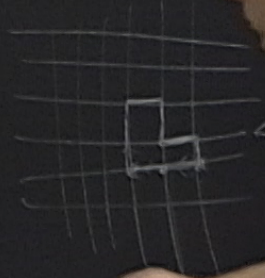
modified Bessel

$$e^{\beta J \cos \theta} = I_0(\beta J) \left[1 + \sum_n \frac{I_n(\beta J)}{n! I_0(\beta J)} e^{in\theta} \right]$$

where @ $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$\approx 1 + (\beta J) (e^{i\theta} + e^{-i\theta}) \quad n = \pm 1$$

$$1 + \frac{\beta J}{2} (e^{i\theta} + e^{-i\theta}) + \dots$$



length of loop

$$2$$

With 2 components

same at low E

$$Z = \int \prod_n \frac{d\theta_n}{2\pi} \prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)}$$

modified Bessel

$$e^{\beta J \cos \theta} = I_0(\beta J) \left[1 + \sum_{n \neq 0} \frac{I_n(\beta J)}{n! I_0(\beta J)} e^{in\theta} \right]$$

where @ $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$\approx 1 + (\beta J)(e^{i\theta} + e^{-i\theta}) + \dots$$

$\wedge n = \pm 1$

$$\approx 1 + \sum_{\text{loops}}$$

Ref loop

$$Z = \int \prod_n \frac{d\theta_n}{2\pi}$$

$$\prod_{\langle ij \rangle} e^{\beta J \cos(\theta_i - \theta_j)}$$

$$= 1 + \sum_{\text{loops}}$$

$$e^{\beta J \cos \theta} = I_0(\beta J) \left[1 + \sum_{n \neq 0} \frac{I_n(\beta J)}{n! I_0(\beta J)} e^{in\theta} \right] \sim \text{Loop Gas}$$

where @ $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$\approx 1 + (\beta J) (e^{i\theta} + e^{-i\theta}) + \dots$$

$\sim n = \pm 1$

$$1 + \frac{\beta J}{2} (e^{i\theta_i - i\theta_j} + e^{i\theta_j - i\theta_i}) + \dots$$

of loop $\downarrow$
2

$$= I_0(\beta J) \left[1 + \sum_{n \neq 0} \frac{I_n(\beta J)}{n I_0(\beta J)} e^{in\theta} \right] \xrightarrow{\text{Loop Gas}}$$

When $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$\approx 1 + (\beta J) (e^{i\theta} + e^{-i\theta}) + \dots$$

$\uparrow$
 $n = \pm 1$

length of loop $\sim \left(\frac{\beta J}{2}\right)$

orientation $\downarrow$ 2

$Z = \text{Loop Gas}$



Parentheses

2D Ising

$$e^{\beta J}, \sigma_i \sigma_j = +1 \quad \propto e^{-\beta J} \text{ for } \sigma_i \sigma_j = -1$$

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\langle ij \rangle} \left[e^{\beta J \sigma_i \sigma_j} = A + B \sigma_i \sigma_j \right]$$

$$A + B = e^{\beta J}$$

$$A - B = e^{-\beta J}$$

$$= I_0(\beta J) \left[1 + \sum_{n \neq 0} \frac{I_n(\beta J)}{n I_0(\beta J)} e^{in\theta} \right] \xrightarrow{\text{Loop Gas}}$$

when $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

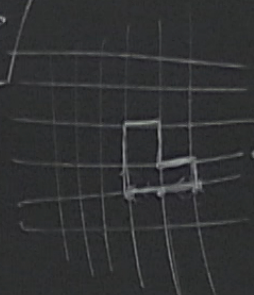
$$\approx 1 + (\beta J) (e^{i\theta} + e^{-i\theta}) + \dots$$

$\uparrow$
 $n = \pm 1$

length of loop $\sim \left(\frac{\beta J}{2}\right)$

orientation $\downarrow$

$Z = \text{Loop Gas}$



Parentheses

2D Ising

$$e^{\beta J}, \sigma_i \sigma_j = +1 \quad \text{or} \quad e^{-\beta J}, \sigma_i \sigma_j = -1$$

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\langle ij \rangle} \left[e^{\beta J \sigma_i \sigma_j} = A + B \sigma_i \sigma_j \right]$$

$$A + B = e^{\beta J}$$

$$A - B = e^{-\beta J}$$

$$= I_0(\beta J) \left[1 + \sum_n \frac{I_n(\beta J)}{n I_0(\beta J)} e^{in\theta} \right] \xrightarrow{\text{Loop Gas}}$$

when $\beta \rightarrow 0$
 $I_n(\beta) \sim \beta^n$

$$\approx 1 + (\beta J) (e^{i\theta} + e^{-i\theta}) + \dots$$

$\downarrow$
 $n = \pm 1$

length of loop $\sim \left(\frac{\beta J}{2}\right)$

orientation $\downarrow$ 2

$Z = \text{Loop Gas}$

Parentheses

2D Ising

$$e^{\beta J}, \sigma_i \sigma_j = +1 \quad \propto e^{-\beta J} \text{ for } \sigma_i \sigma_j = -1$$

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\langle ij \rangle} \left[e^{\beta J \sigma_i \sigma_j} = A + B \sigma_i \sigma_j \right]$$

$$= A \left[1 + \frac{B}{A} \sigma_i \sigma_j \right]$$

$$\begin{pmatrix} A+B = e^{\beta J} \\ A-B = e^{-\beta J} \end{pmatrix} =$$

$e^{-\beta J}$ for $\sigma_i \sigma_j = -1$

$\sum_{\sigma_i = \pm 1} \sigma_i \sigma_j = 0 = \sigma_j - \sigma_j$

α small in high T , $\beta \rightarrow 0$

$\left(\begin{array}{l} A+B=e^{\beta J} \\ A-B=e^{-\beta J} \end{array} \right) = A^{\# \text{ links}} \sum_{\{\sigma = \pm 1\}} \prod_{\langle ij \rangle} (1 + \alpha \sigma_i \sigma_j) = A^{\#} (1 + \dots)$

$\dots = \sum_{\text{loops}} \left(\frac{\text{length}}{\text{interval sum}} \right) \left(\underbrace{\sigma_i \sigma_j \sigma_k \dots}_{+1} \right)$

α

$e^{-\beta J}$ for $\sigma_i \sigma_j = -1$

$\sum_{\sigma_i = \pm 1} \sigma_i \sigma_j = 0 = \sigma_j - \sigma_j$

α small in high T , $\beta \rightarrow 0$

$$\begin{pmatrix} A+B=e^{\beta J} \\ A-B=e^{-\beta J} \end{pmatrix} = A^{\# \text{ links}} \sum_{\{\sigma = \pm 1\}} \prod_{\langle ij \rangle} (1 + \alpha \sigma_i \sigma_j) = A^{\#} (1 + \dots)$$

$$\dots = \sum_{\text{loops}} \left(\frac{\beta J}{2} \right)^{\text{Length}} \underbrace{\sum_{\sigma_i, \sigma_j} \sigma_i \sigma_j \dots}_{\text{interval sum} + 1}$$

$$\boxed{2D \text{ ISing} = \text{Loop Gas}}$$

$$-i\theta_j + e^{-i\theta_j + i\theta_j} + \dots$$

loops

orientation

length of loop

$$\sim \left(\frac{8\pi}{L}\right)$$

$$Z = \text{Gen. Loop Gas}$$

Gas of Loops.

What about ...? Indeed

$$Z_{\text{Ising}} = \text{Loop Gas}$$

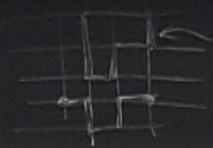
loops

integral sum

$$+1$$

Gas

$$e^{-\beta J} \text{ for } \sigma_i \sigma_j = -1 \quad \sum_{\sigma_i = \pm 1} \sigma_i \sigma_j = 0 = \sigma_j - \sigma_j$$



α small in high T , $\beta \rightarrow 0$

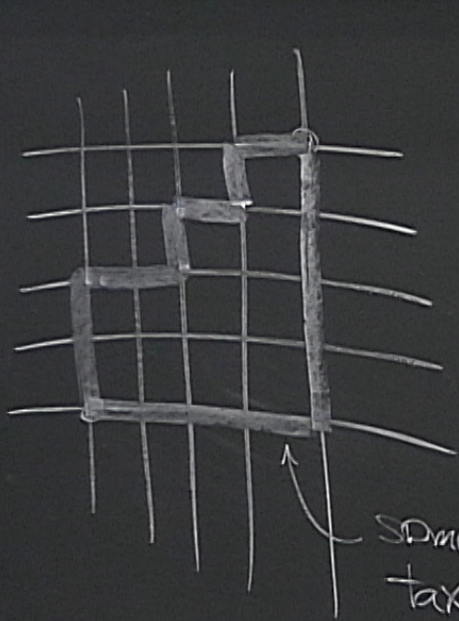
$$\left(\begin{array}{l} A+B=e^{\beta J} \\ A-B=e^{-\beta J} \end{array} \right) = A^{\# \text{ links}} \sum_{\{\sigma = \pm 1\}} \prod_{\langle ij \rangle} (1 + \alpha \sigma_i \sigma_j) = A^{\#} (1 + \dots)$$

$$\dots = \sum_{\text{loops}} \left(\frac{1}{2} \right)^{\text{length}} \left(\underbrace{\sigma_i \sigma_j \sigma_k \dots}_{\substack{\text{interval} \\ \text{sum} \\ +1}} \right)$$

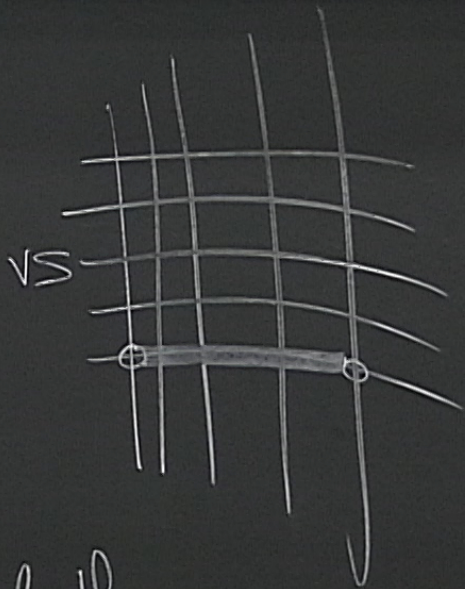
$2D \text{ Ising} = \text{Loop Gas}$

connect
j and j+n

$$\langle S_j \cdot S_{j+n} \rangle = \left(\frac{B J}{2} \right)$$

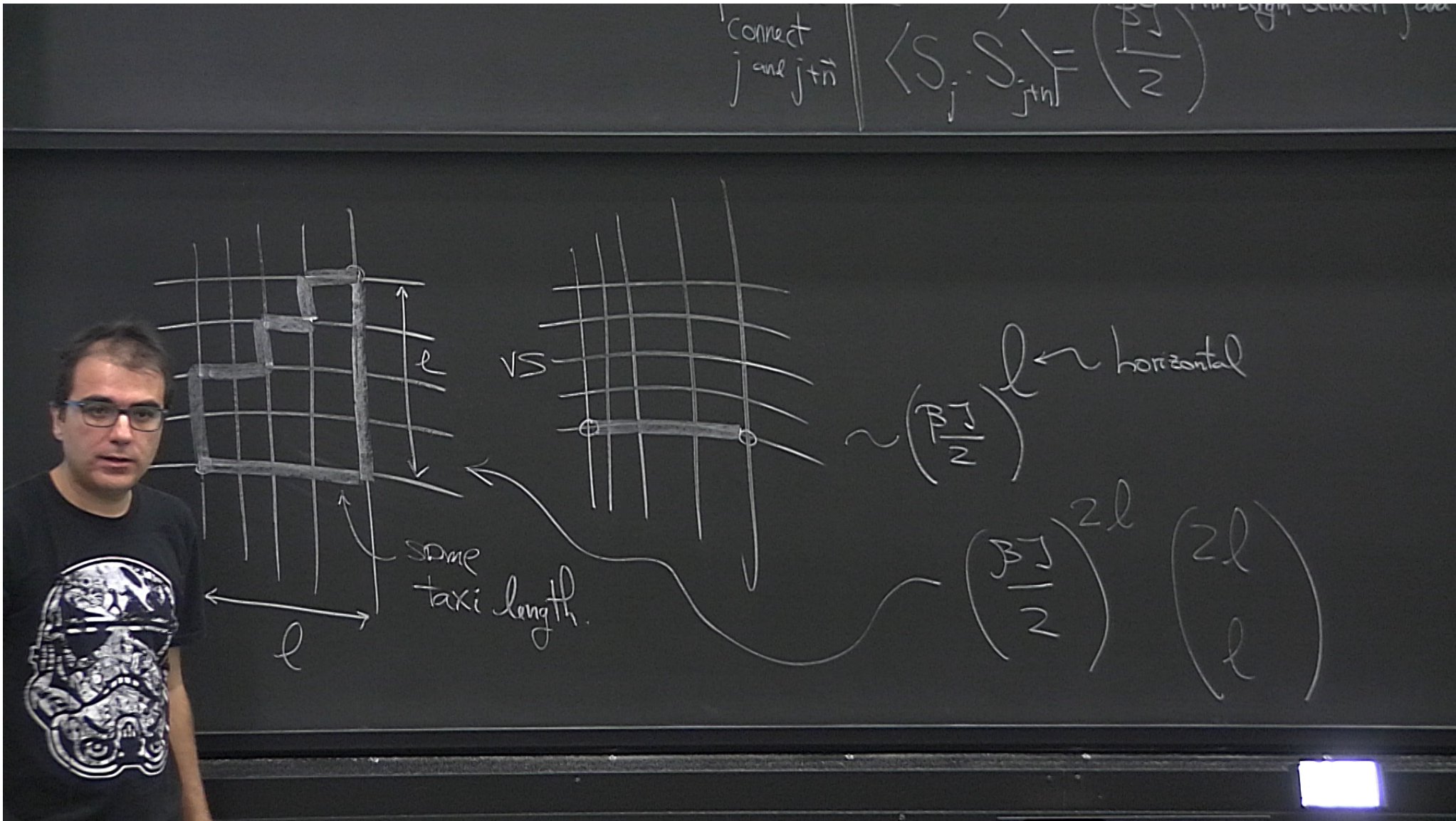


some
taxi length.



VS

$$\sim \left(\frac{B J}{2} \right)^l \leftarrow \text{horizontal}$$



$$\langle S_j \cdot S_{j+n} \rangle = \binom{\beta J}{\frac{1}{2}}^{\text{Min. Length Between } j \text{ and } j+n} N(\text{paths with that length}) \quad \leftarrow \text{exponentially sep.}$$

$$\binom{\beta J}{\frac{1}{2}}^l \sim e^{-l/\xi} \quad \leftarrow \text{horizontal}$$

$$\binom{\beta J}{\frac{1}{2}}^{2l} \binom{2l}{l} \sim e^{-l/\xi}$$

$$\langle S_j \cdot S_{j+n} \rangle = \binom{\beta J}{2}^{\text{Min. Length Between } j \text{ and } j+n} N(\text{paths with that length}) \quad \leftarrow \text{exponentially sig.}$$

$$\binom{\beta J}{2}^l \sim e^{-l/\xi} \quad \begin{array}{l} \text{horizontal} \\ \text{dist} \end{array}$$

$$\binom{\beta J}{2}^{2l} \binom{2l}{l} \sim e^{-4l} \sim e^{-\sqrt{2}l} \quad \begin{array}{l} \text{dist} \\ \text{dist}/\xi \end{array}$$

$$\langle S_j \cdot S_{j+n} \rangle = \binom{\beta J}{2}^{\text{Min. Length Between } j \text{ and } j+n} N(\text{paths with that length}) \quad \leftarrow \text{exponentially sup.}$$

$$\begin{aligned} & \binom{\beta J}{2}^l \sim e^{-l/\xi} \quad \leftarrow \text{horizontal} \\ & \binom{\beta J}{2}^{2l} \binom{2l}{l} \sim e^{-4l} e^{-\sqrt{2}l} \quad \leftarrow \text{exp suppressed} \end{aligned}$$

$\begin{matrix} \nearrow = \text{dist} \\ \searrow = \text{dist} \end{matrix}$

path that connect j and $j+n$

Min Length Between j and $j+n$

N (paths with

$\langle S_j \cdot S_{j+n} \rangle = \left(\frac{\beta J}{2} \right)^n$

$e^{-l \log \frac{2}{\beta J}}$

$\sim \left(\frac{\beta J}{2} \right)^l$ $\leftarrow$ horizontal

$\sim \left(\frac{\beta J}{2} \right)^{2l}$

$\sim \left(\frac{\beta J}{2} \right)^{2l} \binom{2l}{l} \sim e^{-\frac{\text{dist}}{e}}$

$\sim e^{-l/\xi}$

$\sim e^{-\sqrt{2}l}$

$\sim e^{-\frac{\text{dist}}{e}}$

some taxi length

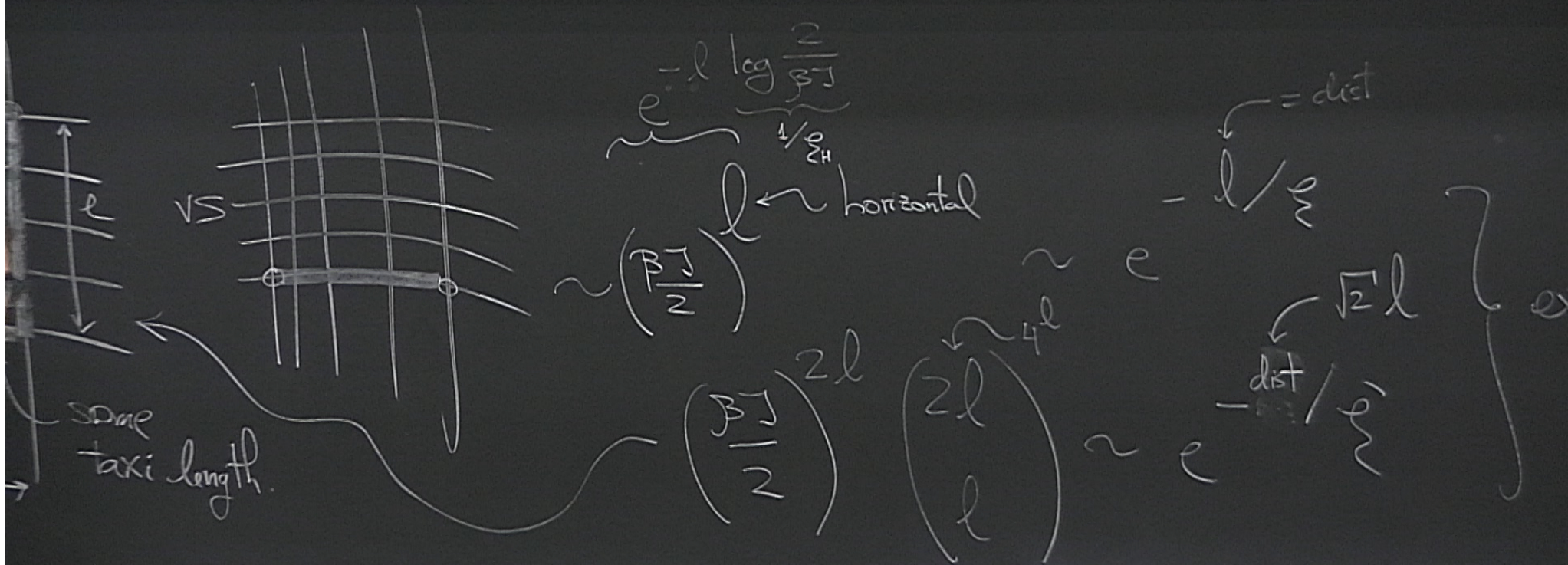
vs

path that
connect
 j and $j+n$

$$\langle S_j \cdot S_{j+n} \rangle = \left(\frac{\beta J}{2} \right)^n$$

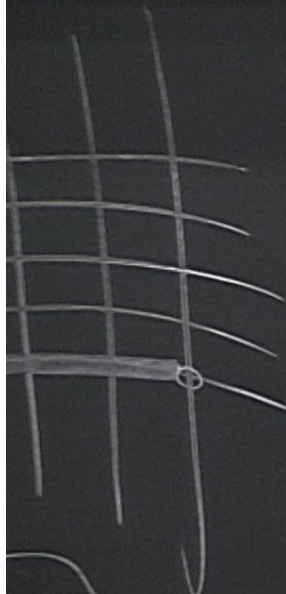
Min Length Between j and $j+n$

N (paths with



path that connect j and $j+n$

$$\langle S_j \cdot S_{j+n} \rangle = \left(\frac{\beta J}{2} \right)^{\text{Min Length Between } j \text{ and } j+n} N(\text{paths with that length})$$



$$\sim \left(\frac{\beta J}{2} \right)^l$$

$$\sim \left(\frac{\beta J}{2} \right)^{2l}$$

$$\sim \left(\frac{2l}{l} \right)^{4l} \sim e^{-\frac{\text{dist}}{\xi_D}}$$

$$e^{-l \log \frac{2}{\beta J}}$$

$$e^{-l/\xi_H}$$

$$e^{-\frac{\sqrt{2}l}{\xi_D}}$$

horizontal
 = dist
 exp suppressed

$n \vec{j}$ and $\vec{j} + n$ $N(\text{paths with that length}) \parallel \leftarrow \text{exponentially suppressed @ large dist.}$

$$\begin{aligned}
 & e^{-l/\xi_H} \\
 & \sim e^{-\frac{\sqrt{2}l}{\xi_D}}
 \end{aligned}
 \left. \begin{array}{l} \text{--- dist} \\ \text{--- } \sqrt{2}l \\ \text{--- dist} \end{array} \right\} \text{exp suppressed @ large } T$$

2D Ising

$$e^{\beta J}, \sigma_i \sigma_j = +1 \quad \alpha e^{-\beta J}, \sigma_i \sigma_j = -1$$



$\sigma_i = \pm 1$

Small in high

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\langle ij \rangle} [e^{\beta J \sigma_i \sigma_j} = A + B \sigma_i \sigma_j] = A [1 + \frac{B}{A} \sigma_i \sigma_j]$$

$$\begin{pmatrix} A+B=e^{\beta J} \\ A-B=e^{-\beta J} \end{pmatrix}$$

#links

$$\sum_{\{\sigma_i = \pm 1\}} \prod_{\langle ij \rangle} (1 + \alpha \sigma_i \sigma_j) = A^{\#} (1 + \dots)$$

2D Ising = Loop Gas

At low T (large β)

$$Z \approx \int \pi d\vec{\theta}_n e^{\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \sim \beta J \sum_{\langle ij \rangle} \left(1 - \frac{(\theta_i - \theta_j)^2}{2} + \dots \right)$$

$$\propto \int \pi d\vec{\theta}_n e^{-\beta J \sum_{\langle ij \rangle} \frac{(\theta_i - \theta_j)^2}{2}}$$

massless particle in 2D $m_{\text{eff}} = 0$

Gaussian Model
Spin Wave
↑↑↑↑↑↑↑↑

massless particle in 2D
 $m_{\text{eff}}^2 = 0$

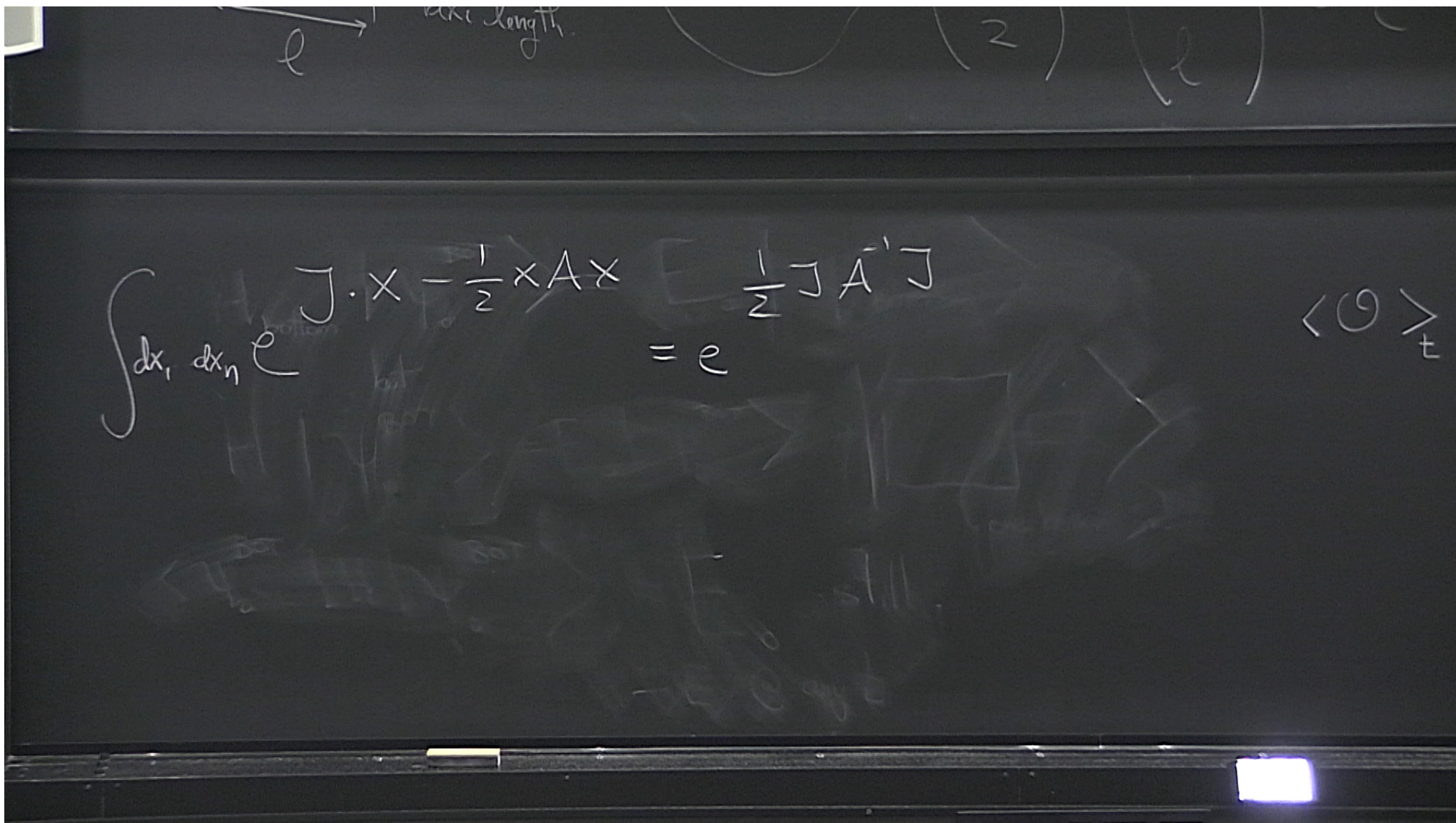
$$\propto \int \prod d\vec{\theta}_n e^{-\frac{\beta J}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2}$$

~~$+\frac{m_j^2}{2} \theta_j^2$~~

At low T (large β)

$$Z \approx \int \prod d\vec{\theta}_{\vec{n}} e^{\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)} \sim \beta J \sum_{\langle ij \rangle} \overset{\text{norm}}{-\frac{(\theta_i - \theta_j)^2}{2}} + \dots$$

$$\langle S_i S_{j+\vec{n}} \rangle = \langle e^{i\theta_j - i\theta_{j+\vec{n}}} \rangle$$

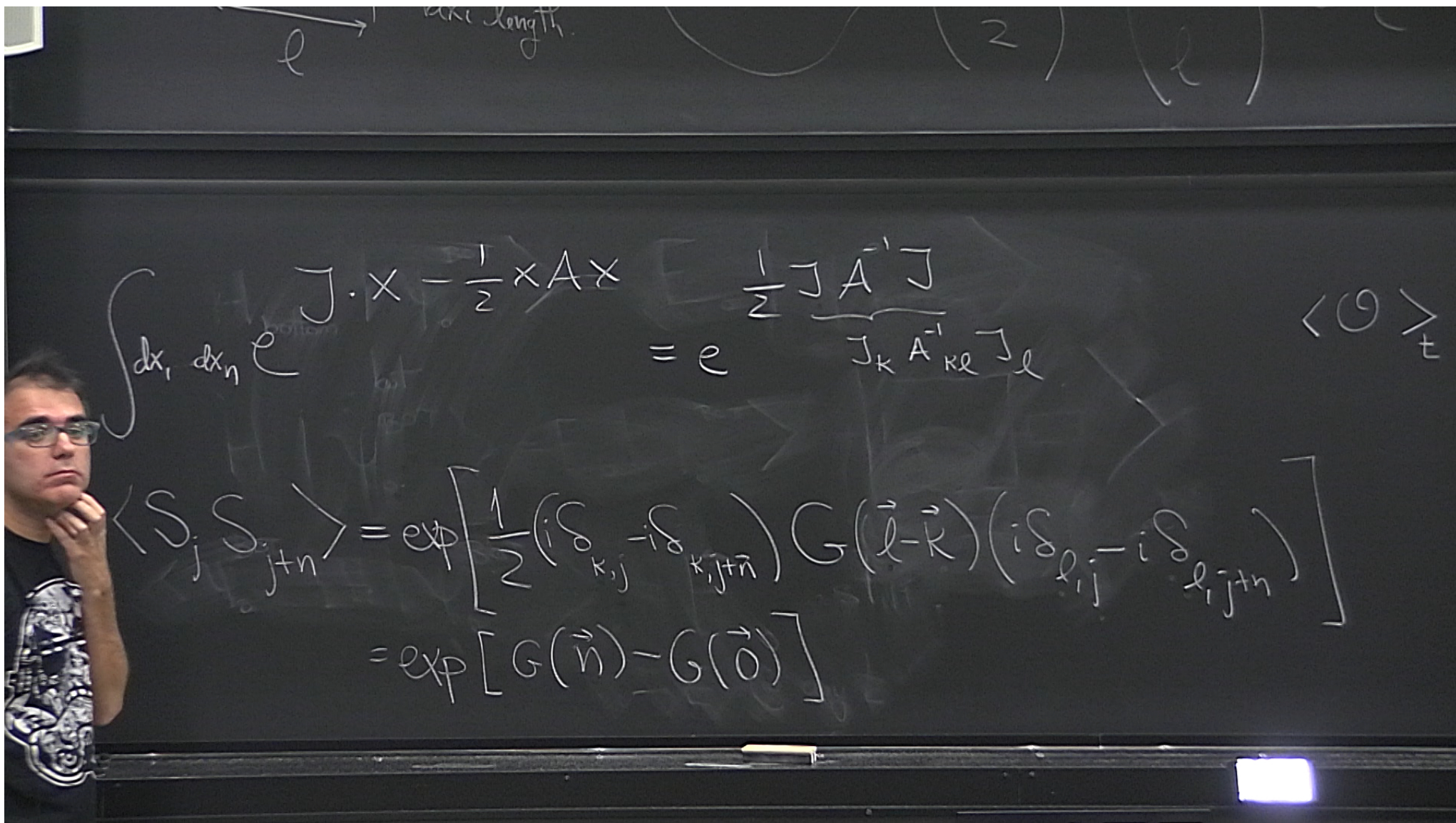


$$\beta J \sum_{\langle ij \rangle} \left(1 - \frac{(\theta_i - \theta_j)^2}{2} + \dots \right) \propto \int \pi d\vec{\theta}_n e^{-\beta J \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2}$$

Gaussian Model
 Spin Wave
 $\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$

massless particle in 2D
 $m_{\text{eff}}^2 = 0$

$\left| \Theta_{j+n} \right\rangle_{\text{GAUSSIAN}} = \left\langle e^{J \cdot \Theta} \right\rangle, J_{\vec{K}} = i\delta_{K,j} - i\delta_{K,j+n}$



ℓ → wave length.

$$dx_1 dx_n e^{J \cdot x - \frac{1}{2} x A x} = e^{\frac{1}{2} J A^{-1} J}$$

$$= e^{\frac{1}{2} J_K A_{KL}^{-1} J_L}$$

$$\langle \mathcal{O} \rangle_t = \sum_n$$

$$\langle S_{j+n} \rangle = \exp \left[\frac{1}{2} (i\delta_{k,j} - i\delta_{k,j+n}) G(\vec{\ell} - \vec{k}) (i\delta_{\ell,j} - i\delta_{\ell,j+n}) \right]$$

NEXT LECTURE

$$= \exp [G(\vec{n}) - G(\vec{0})] = \exp \left(\frac{-\#}{\beta J} \log(\vec{n}) \right) \sim \frac{1}{|\vec{n}|^{\#/\beta J}}$$



re J_l

$(i\delta_{l,j} - i\delta_{l,j+n})$

LECTURE

$\exp\left(\frac{-\#}{\beta J} \log(\vec{n})\right) \sim \frac{1}{|\vec{n}|^{\#/\beta J}}$

$|C_n|^2 + \sum_{m \neq n} \underbrace{\langle n|O|m \rangle}_{\substack{2) \text{ Small}}} C_n^* C_m e^{-i(E_m - E_n)t}$

$\langle O \rangle_t = \sum_n \underbrace{\langle n|O|n \rangle}_{\substack{\approx \langle O \rangle_{\text{micro}}(E_n)}} |C_n|^2$

$\langle O_{\text{extreme}} \rangle_{N_{\text{def}}}$

$1) \& 2) \iff ETH$

$\langle O \rangle_{\text{canonical}} = \langle O \rangle_{\text{microcanonical}} = \sum_n \frac{\langle n|O|n \rangle}{\sum_n \langle n|n \rangle}$

@ fixed $E \simeq E_0$

← POWER LAW DECAY WITH CONT. EXP.

re J l

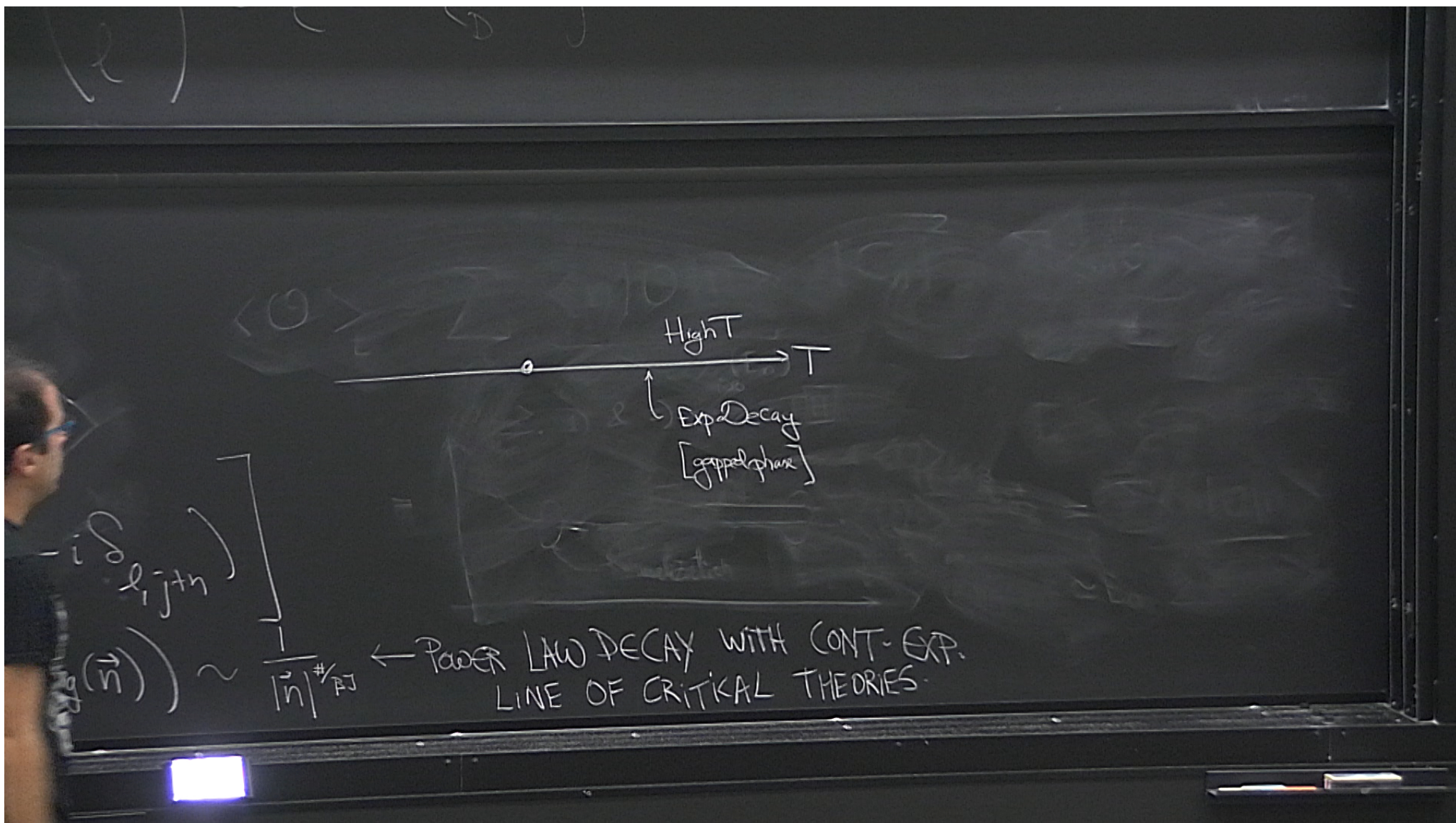
$(i\delta_{l,j} - i\delta_{l,j+n})$
 AT LECTURE
 $\exp\left(\frac{-\#}{\beta J} \log(\vec{n})\right) \sim \frac{1}{|\vec{n}|^{\#/\beta J}}$

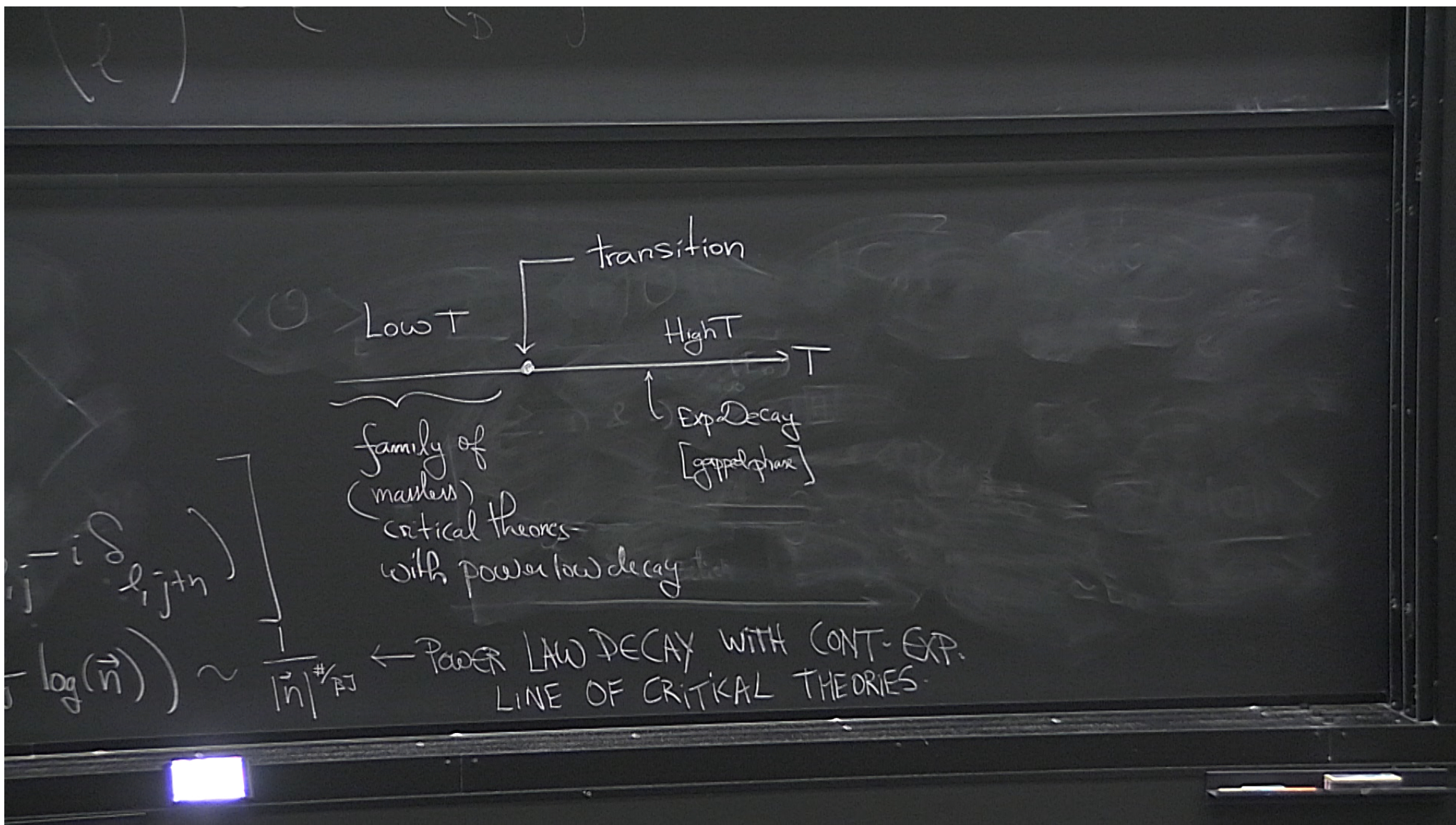
$$\langle \mathcal{O} \rangle_t = \sum_n \underbrace{\langle n | \mathcal{O} | n \rangle}_{\substack{\approx \langle \mathcal{O} \rangle_{\text{micro}}(E_n)}} |C_n|^2 + \sum_{m \neq n} \underbrace{\langle n | \mathcal{O} | m \rangle}_{\substack{2) \text{ small}}} \frac{C_n^* C_m}{e^{-i(E_m - E_n)t}}$$

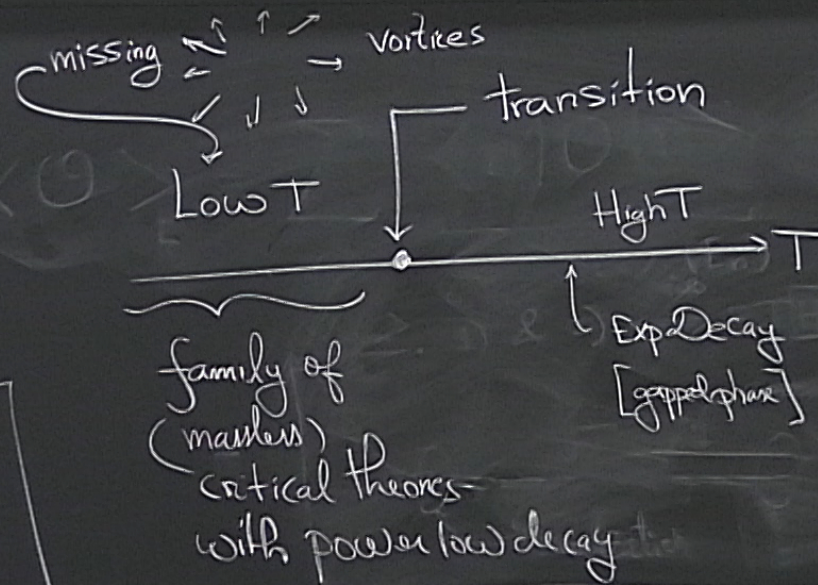
1) & 2) $\leftrightarrow$ ETH

$E_0 - \delta < E_n < E_0 + \delta$
 $\langle \mathcal{O} \rangle_{\text{microcanonical}} = \frac{\sum \langle n | \mathcal{O} | n \rangle}{\sum \langle n | n \rangle}$

← POWER LAW DECAY WITH CONT. EXP.
LINE OF CRITICAL THEORIES.



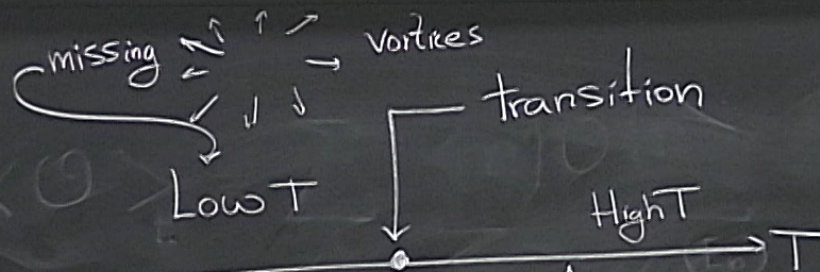




$$-i\delta_{l,j+n}$$

$$\log(\vec{n}) \sim \frac{1}{|\vec{n}|^{\frac{1}{d}}}$$

← POWER LAW DECAY WITH CONT. EXP.
LINE OF CRITICAL THEORIES.



Guess :

As $T \uparrow$

↓
Proliferation of vortices
driving a phase
transition.

family of
(massless)
critical theories
with power law decay

Exp. Decay
[gapped phase]

← POWER LAW DECAY WITH CONT. EXP.
LINE OF CRITICAL THEORIES

$$-i\delta_{\ell,j+n} \log(\vec{n}) \sim \frac{1}{|\vec{n}|^{\frac{d-2}{2}}}$$

ℓ → *link length*

$$J \cdot x - \frac{1}{2} x A x = \frac{1}{2} J A^{-1} J$$

$\downarrow$ Same in $D \geq 2$

$$= e^{J_k A_{kl}^{-1} J_l}$$

$\downarrow$

$$\langle S_i S_{j+n} \rangle = \exp \left[\frac{1}{2} (i\delta_{k,j} - i\delta_{k,j+n}) G(\vec{l} - \vec{k}) (i\delta_{l,j} - i\delta_{l,j+n}) \right]$$

$\downarrow$ NEXT LECTURE

$$= \exp [G(\vec{n}) - G(\vec{0})] = \exp \left(\frac{-\#}{\beta J} \log(\vec{n}) \right) \sim \frac{1}{|\vec{n}|^{\#/\beta J}} \leftarrow \text{Power Law}$$

missing $\uparrow$
Low

family
(master)
critical
with