Title: PSI 2018/2019 - Statistical Mechanics - Lecture 6

Date: Oct 18, 2018 10:45 AM

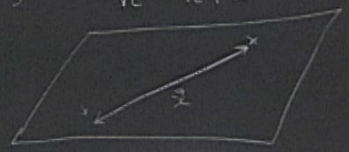
URL: http://pirsa.org/18100028

Abstract:

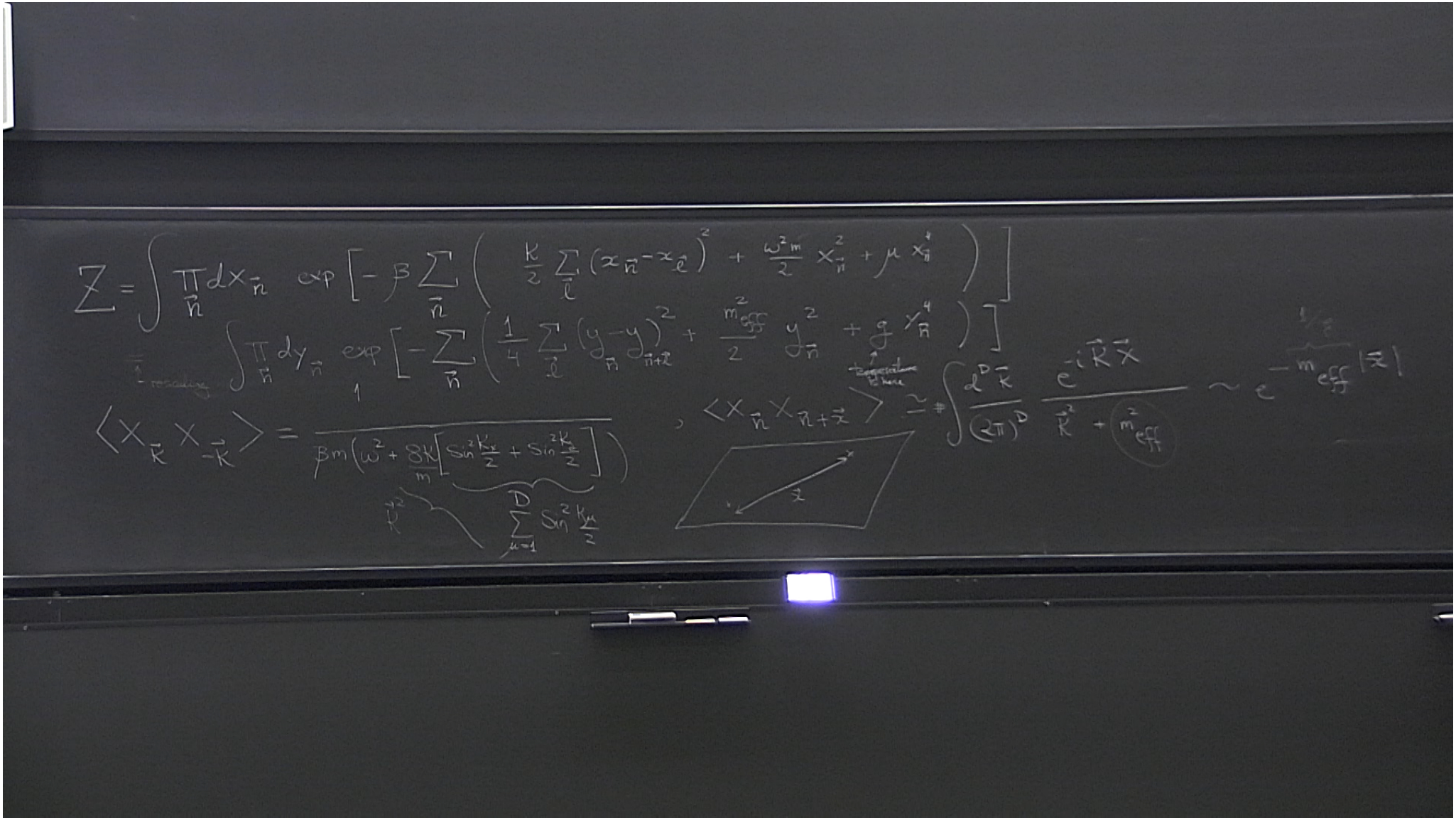
$$Z = \int \prod_{\vec{n}} dx_{\vec{n}} \; \exp\left[-\beta \sum_{\vec{n}} \left(\frac{K}{2} \sum_{\vec{l}} (x_{\vec{n}} - x_{\vec{n}+\vec{l}})^2 + \frac{\omega^2 m^2}{2} x_{\vec{n}}^2 + \mu x_{\vec{n}}^4 \right) \right]$$

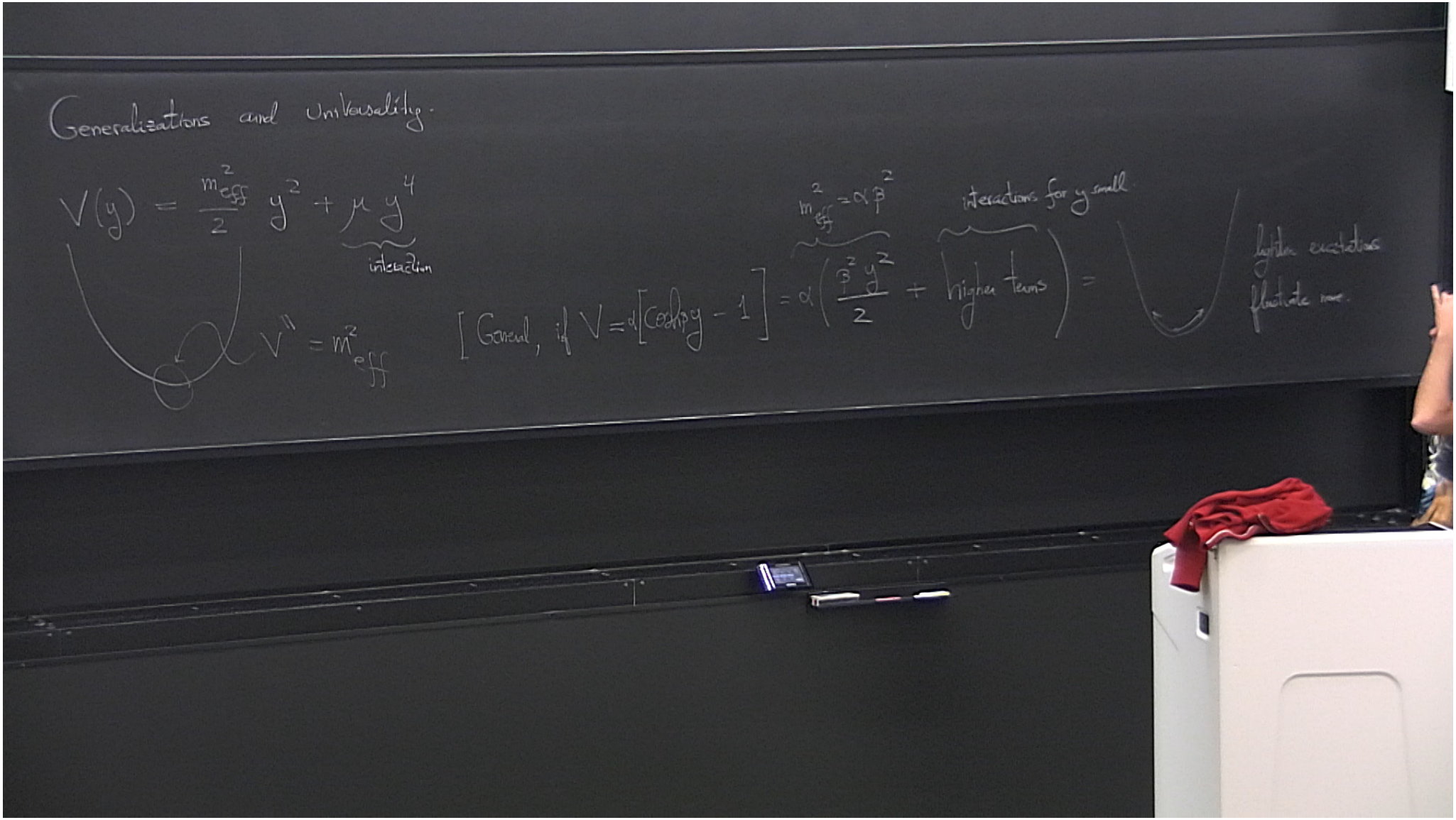
$$\langle x_{\vec{k}} x_{-\vec{k}} \rangle = \frac{1}{\beta m \left(\omega^2 + 8K \underbrace{\left[\sin^2 \frac{k_x}{2} + \sin^2 \frac{k_y}{2} \right]}_{\sum_{\mu=1}^{D} \sin^2 \frac{k_\mu}{2}} \right)}$$

$$\langle x_{\vec{n}} x_{\vec{n}+\vec{z}} \rangle \simeq \int \frac{d^D \vec{k}}{(2\pi)^D} \frac{e^{-i\vec{k}\vec{z}}}{\vec{k}^2 + m_{eff}^2} \sim e^{-m_{eff}|\vec{z}|}$$



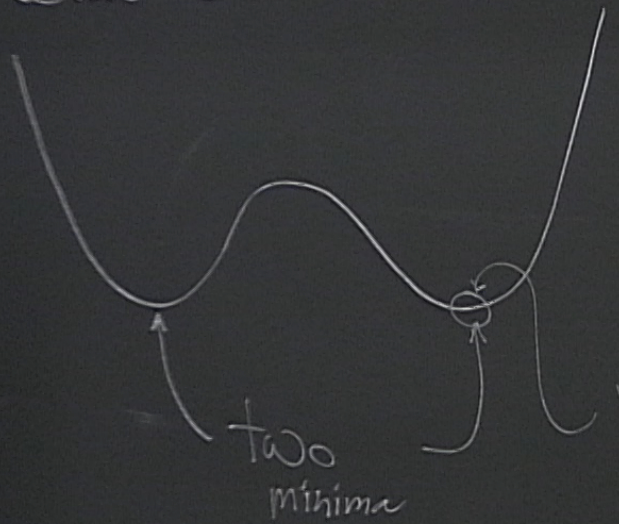



$Z = \int \prod_{\vec{n}} dx_{\vec{n}} \exp\left[-\beta \sum_{\vec{n}} \left(\frac{k}{2} \sum_{\vec{e}} (x_{\vec{n}} - x_{\vec{e}})^2 + \frac{\omega^2 m}{2} x_{\vec{n}}^2 + \mu x_{\vec{n}}^4 \right)\right]$
$\int \prod_{\vec{n}} dy_{\vec{n}} \exp\left[-\sum_{\vec{n}} \left(\frac{1}{4} \sum_{\vec{e}} (y_{\vec{n}} - y_{\vec{n}+\vec{e}})^2 + \frac{m_{eff}^2}{2} y_{\vec{n}}^2 + g\, y_{\vec{n}}^4 \right)\right]$
$\langle x_{\vec{k}} x_{-\vec{k}} \rangle = \frac{1}{\beta m \left(\omega^2 + \frac{8k}{m}\left[\sin^2\frac{k_x}{2} + \sin^2\frac{k_y}{2}\right]\right)}$
$\sum_{\mu=1}^{D} \sin^2\frac{k_\mu}{2}$
$\langle x_{\vec{n}} x_{\vec{n}+\vec{x}} \rangle \simeq \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i\vec{k}\vec{x}}}{\vec{k}^2 + m_{eff}^2} \sim e^{-m_{eff}|\vec{x}|}$

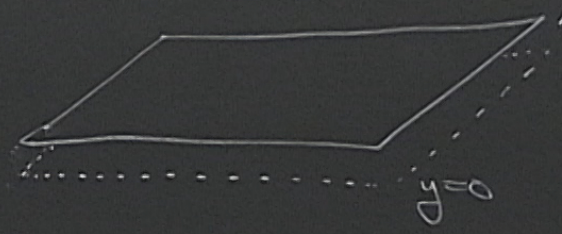

Generalizations and Universality.
$V(y) = \frac{m_{eff}^2}{2} y^2 + \mu y^4$
interaction
$V'' = m_{eff}^2$
[General, if $V = \alpha[\cosh\beta y - 1]$
$m_{eff}^2 = \alpha\beta^2$
interactions for y small
$= \alpha\left(\frac{\beta^2 y^2}{2} + \text{higher terms}\right) =$
lighter excitations
fluctuate more.

$\mu = 1$

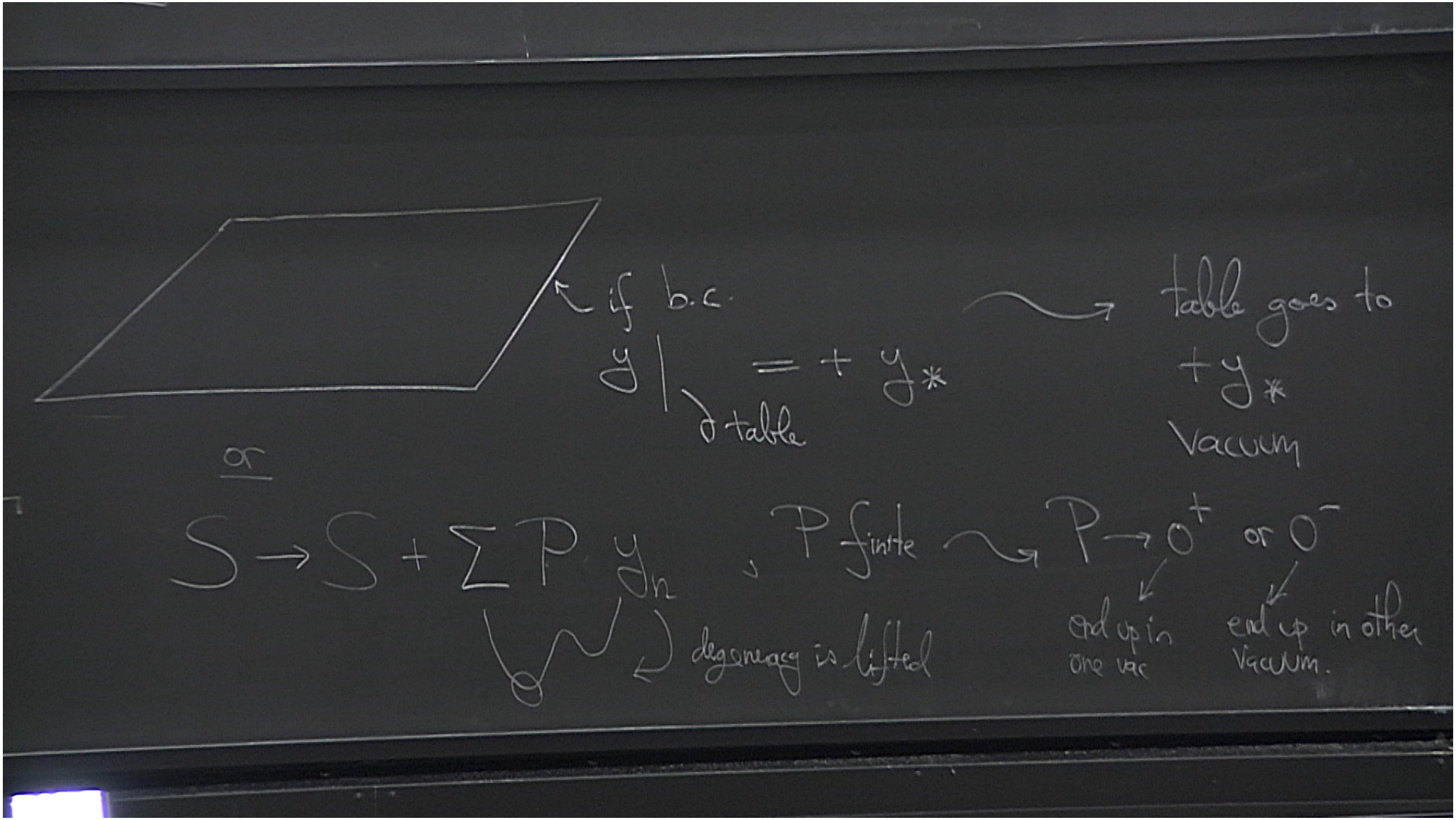
What about $m^2 < 0$ for $\mu > 0$

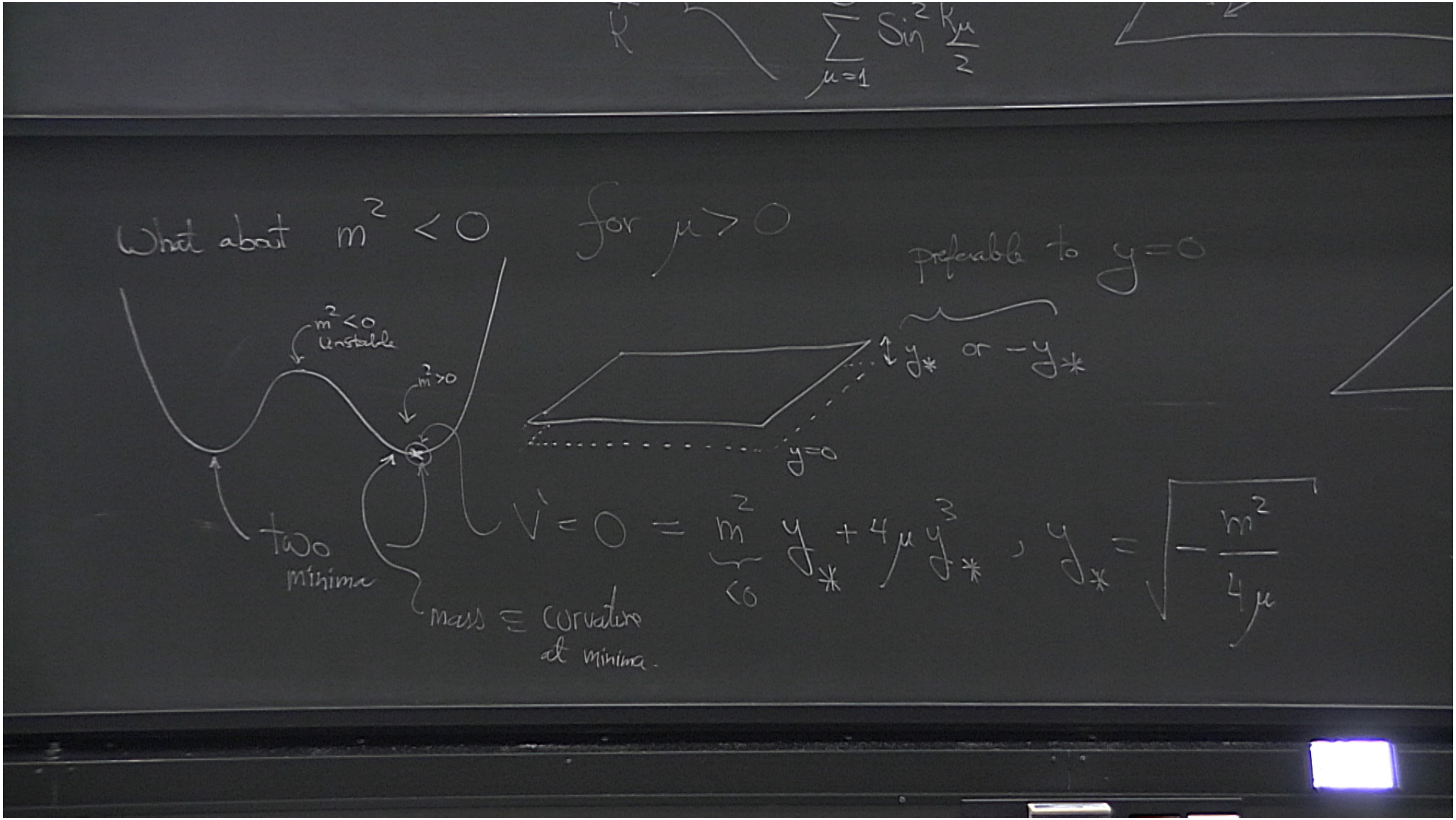


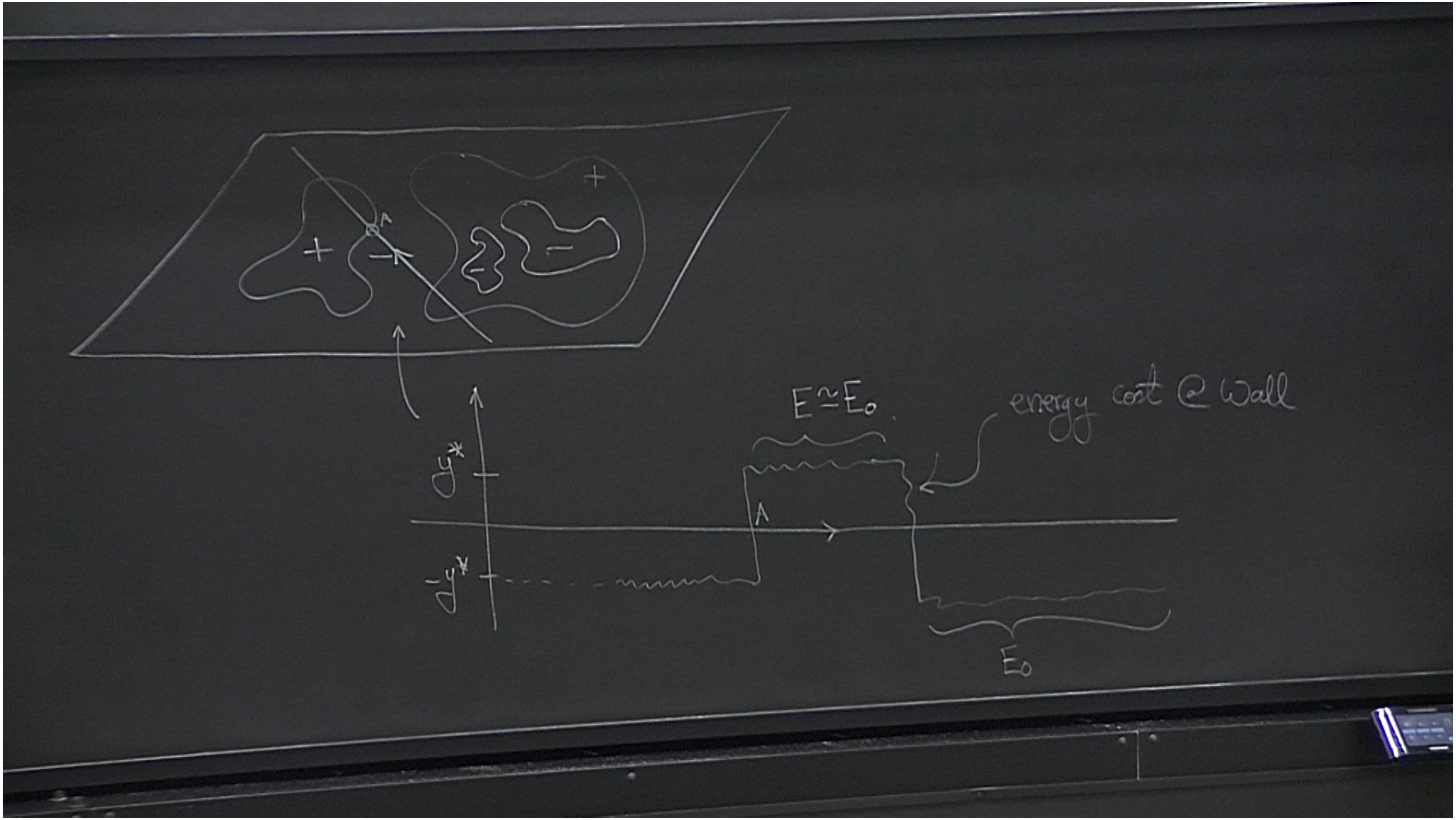
Preferable to $y = 0$

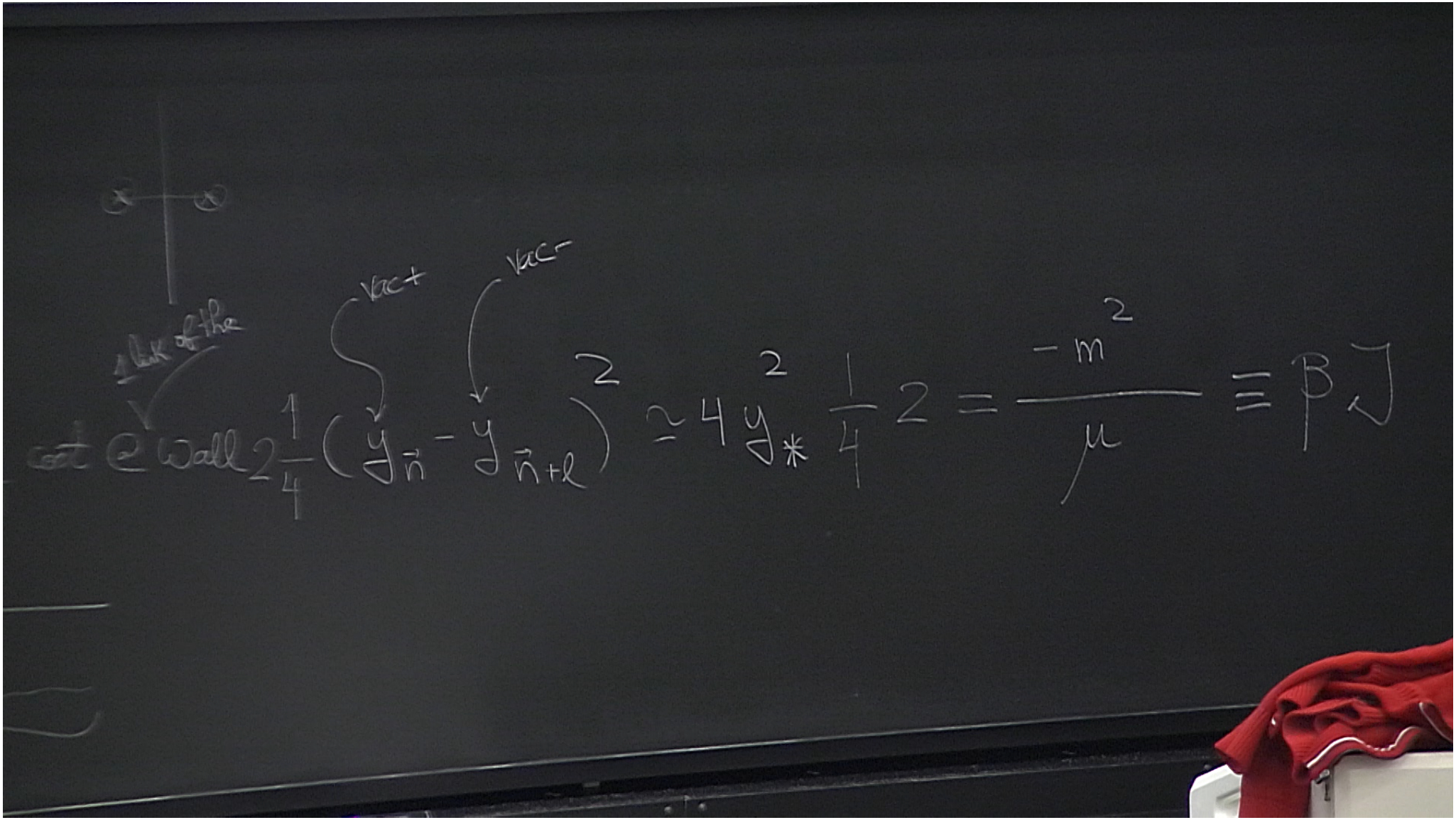


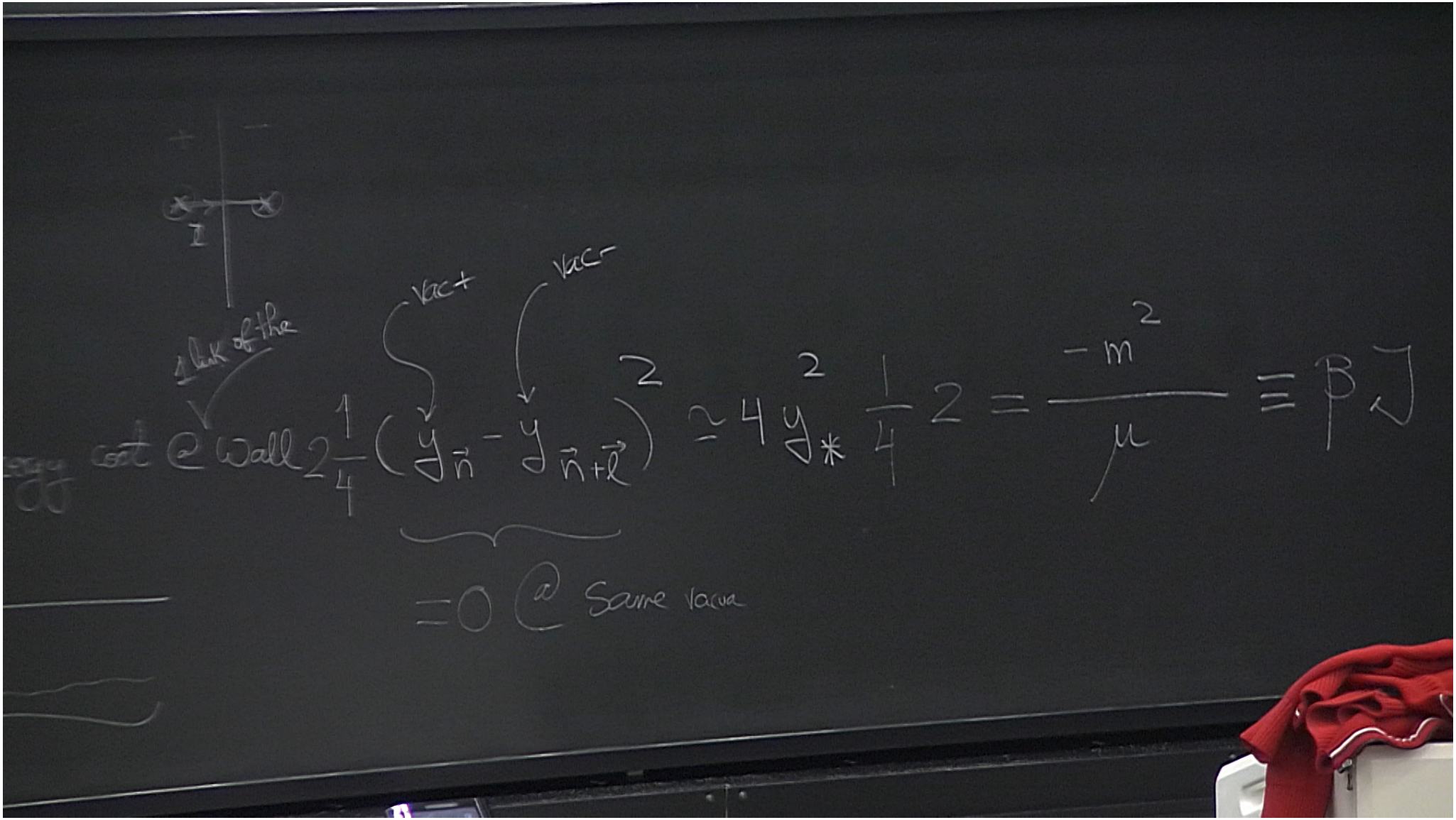
$$V' = 0 = \underbrace{m^2}_{<0} y_* + 4\mu y_*^3, \quad y_* = \sqrt{-\frac{m^2}{4\mu}}$$

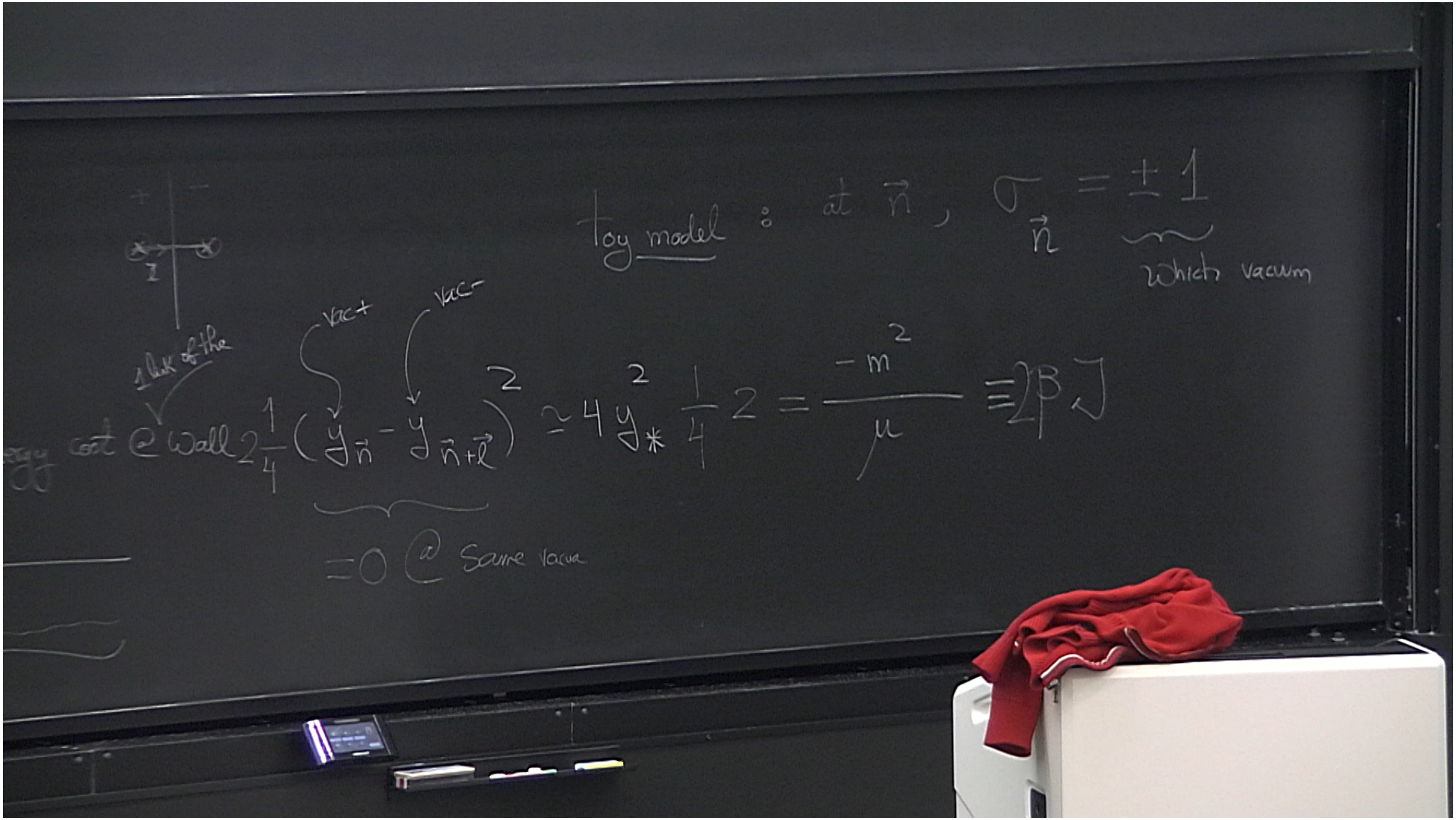

if b.c.
table goes to
+ y*
Vacuum
or
P finite
P → 0+ or 0−
end up in one vac
end up in other vacuum.
degeneracy is lifted

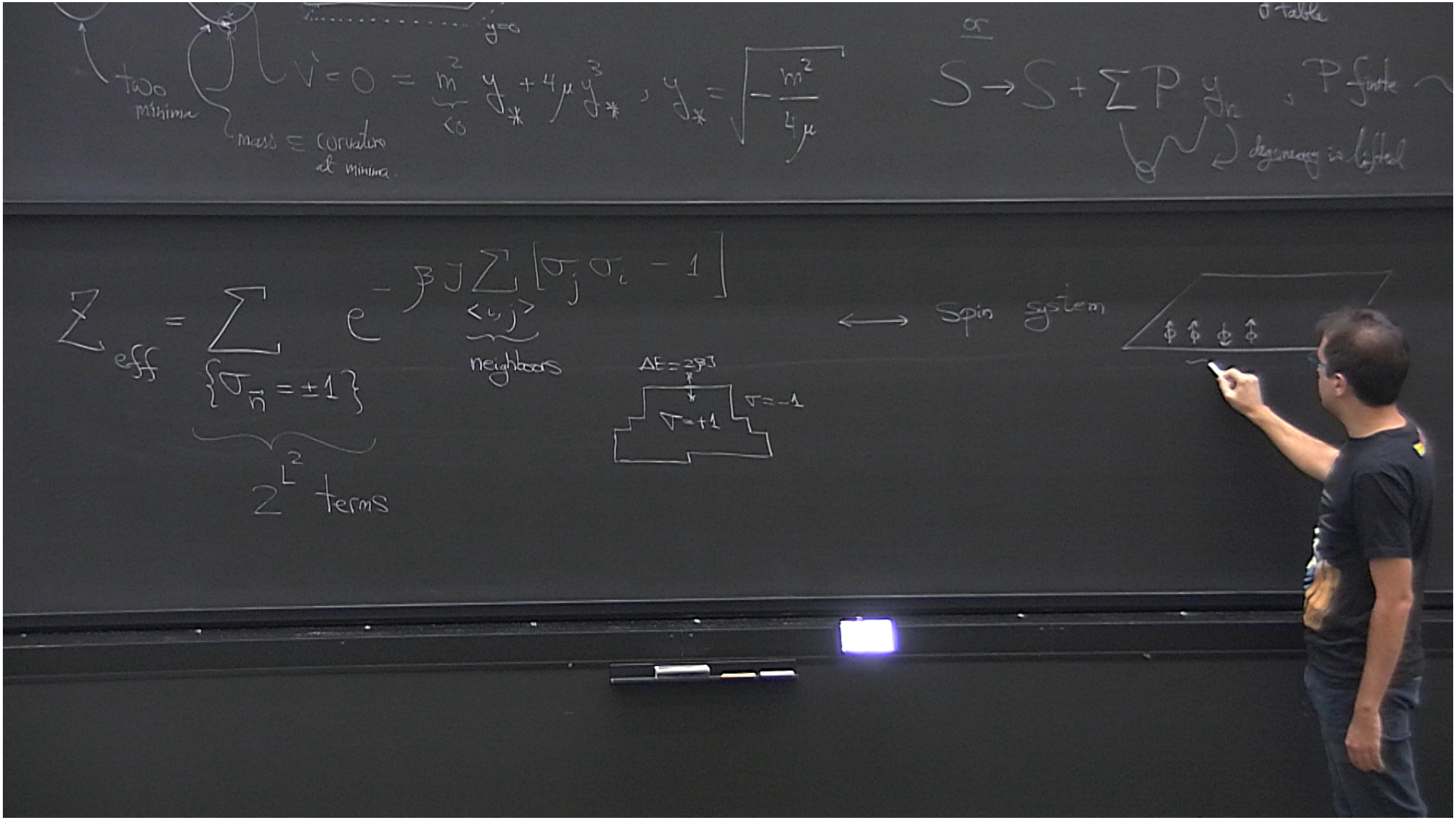

What about $m^2 < 0$ for $\mu > 0$
$m^2 < 0$ unstable
$m^2 > 0$
two minima
mass ∈ curvature at minima.
preferable to $y = 0$
y_* or $-y_*$
$y = 0$
$V' = 0 = m^2 y_* + 4\mu y_*^3$, $y_* = \sqrt{-\frac{m^2}{4\mu}}$

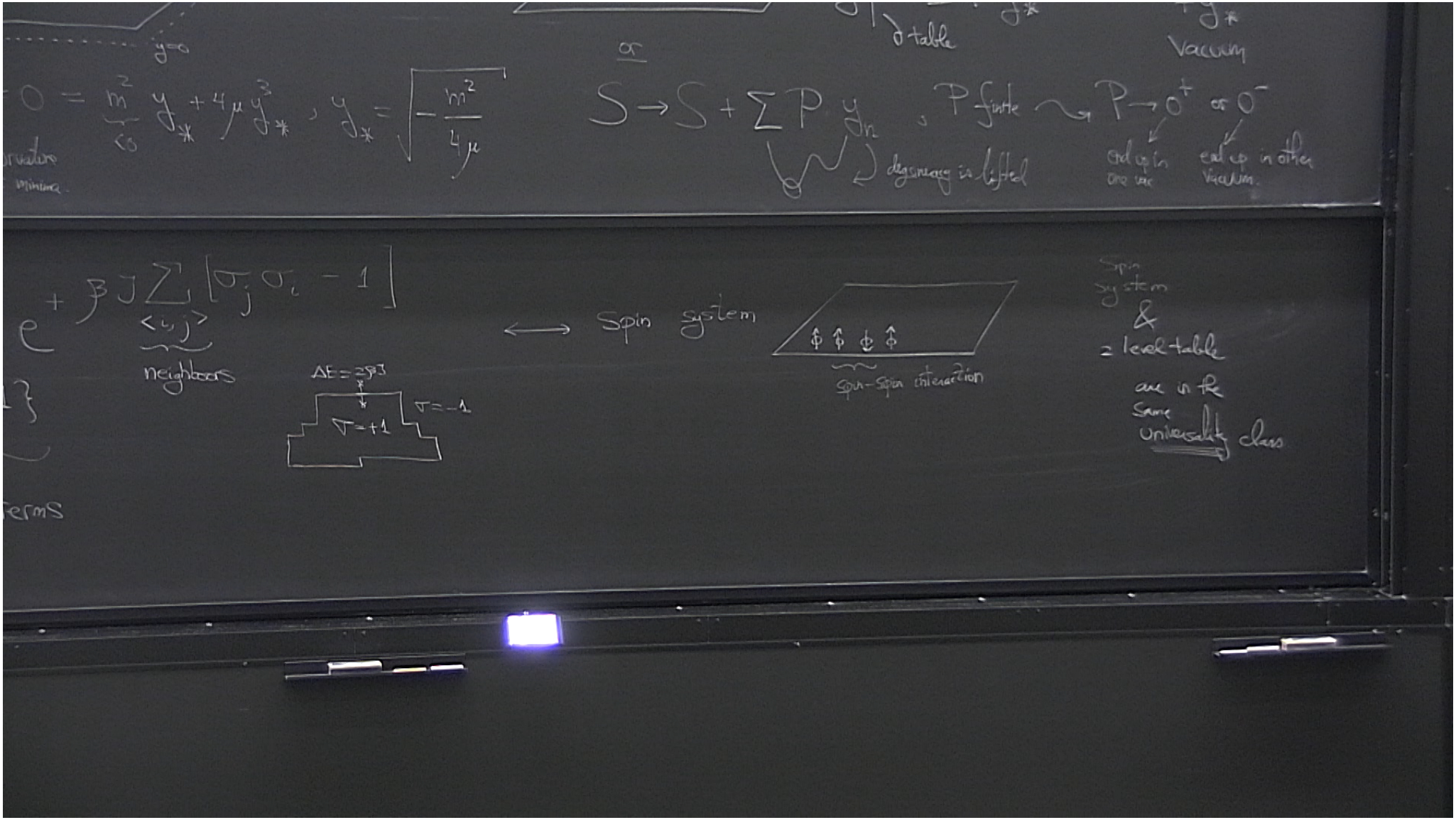

+
+
+
−
−
$E \simeq E_0$
energy cost @ wall
y^*
$-y^*$
E_0

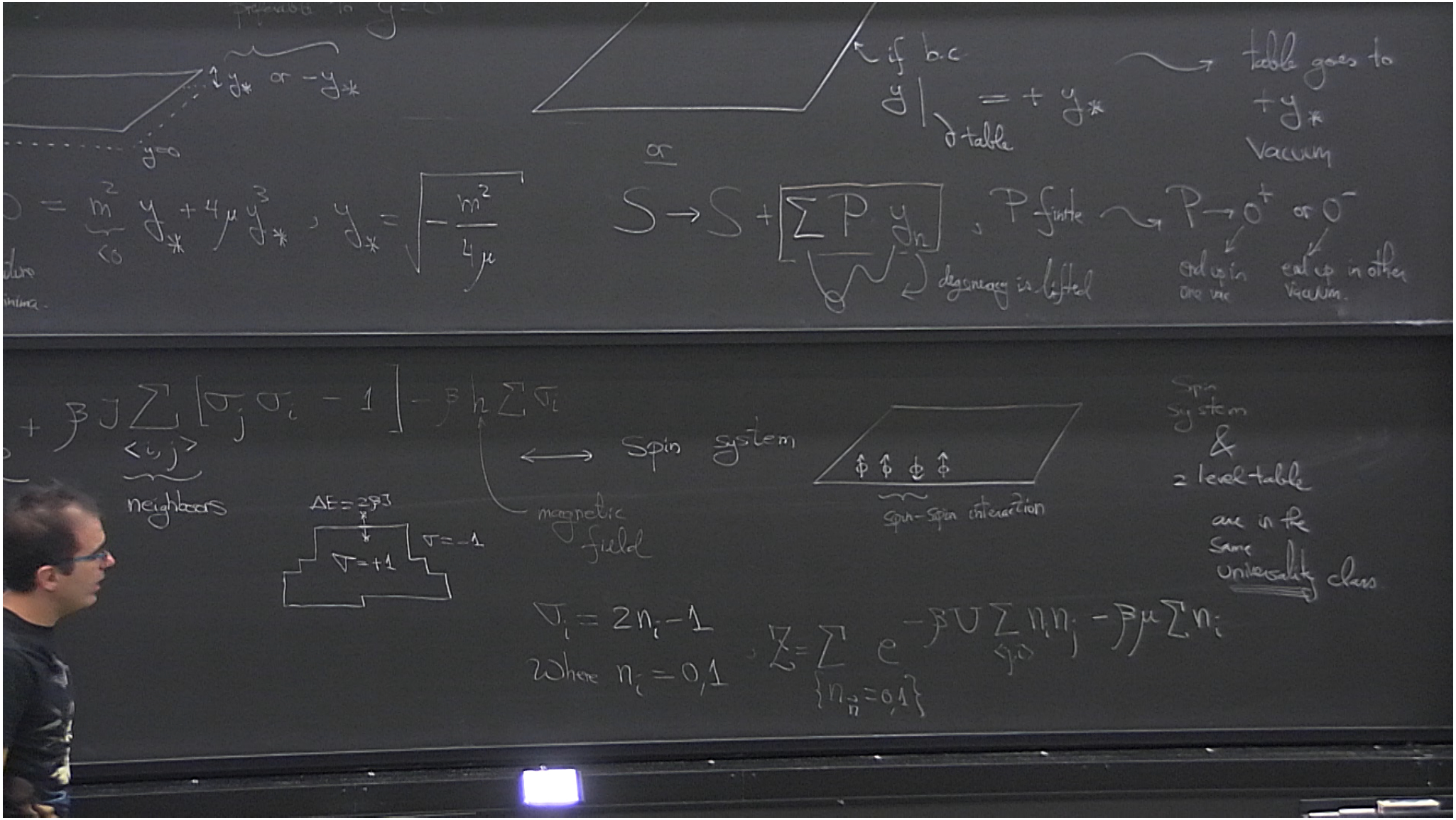

1 link of the
vac+
vac−
at @ wall $2\frac{1}{4}(\vec{y}_{\vec{n}} - \vec{y}_{\vec{n}+\ell})^2 \simeq 4 y_*^2 \frac{1}{4} 2 = \frac{-m^2}{\mu} \equiv \beta J$

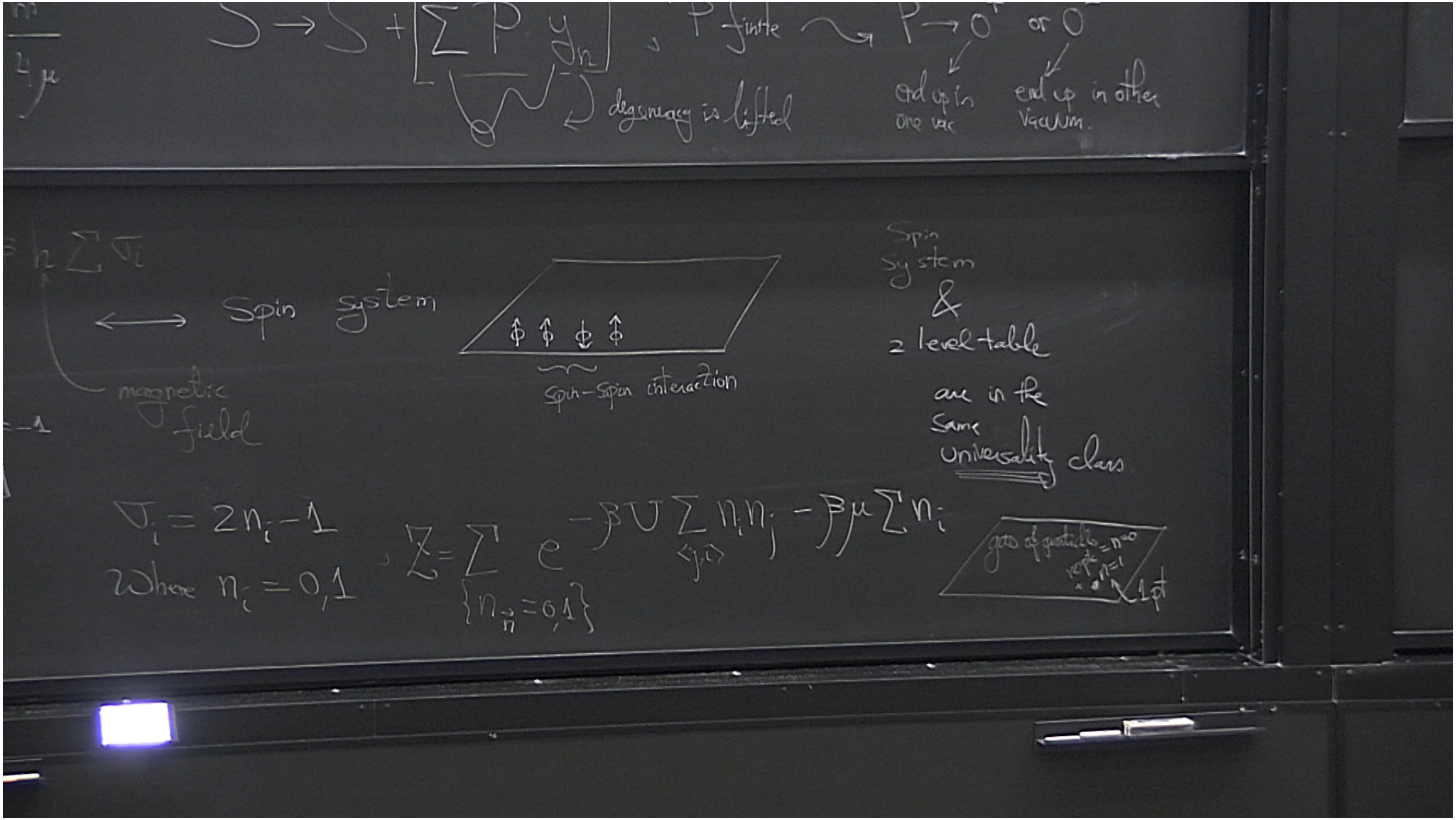

+
−
I
1 link of the
Vac+
Vac−
$\frac{1}{4}\left(\phi_{\vec{n}} - \phi_{\vec{n}+\vec{\ell}}\right)^2 \simeq 4\phi_*^2 \frac{1}{4} 2 = \frac{-m^2}{\mu} \equiv \beta J$
$= 0$ @ Same Vacua

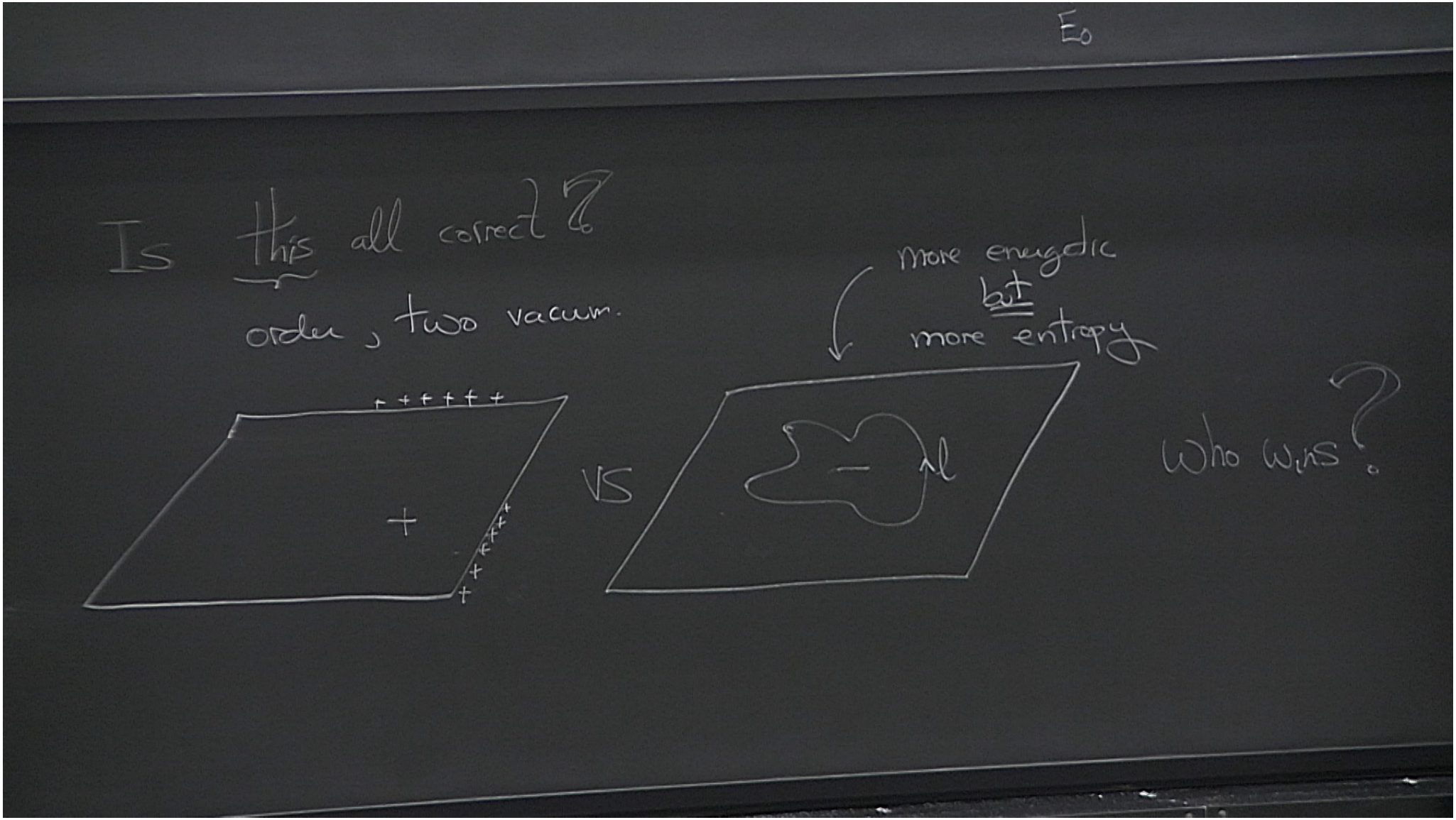

toy model : at $\vec{n}$, $\sigma_{\vec{n}} = \pm 1$
which vacuum
vac+
vac−
1 link of the
$\frac{1}{4}(\phi_{\vec{n}} - \phi_{\vec{n}+\vec{e}})^2 \simeq 4\phi_*^2 \frac{1}{4} 2 = \frac{-m^2}{\mu} \equiv 2\beta J$
$= 0$ @ same vacua

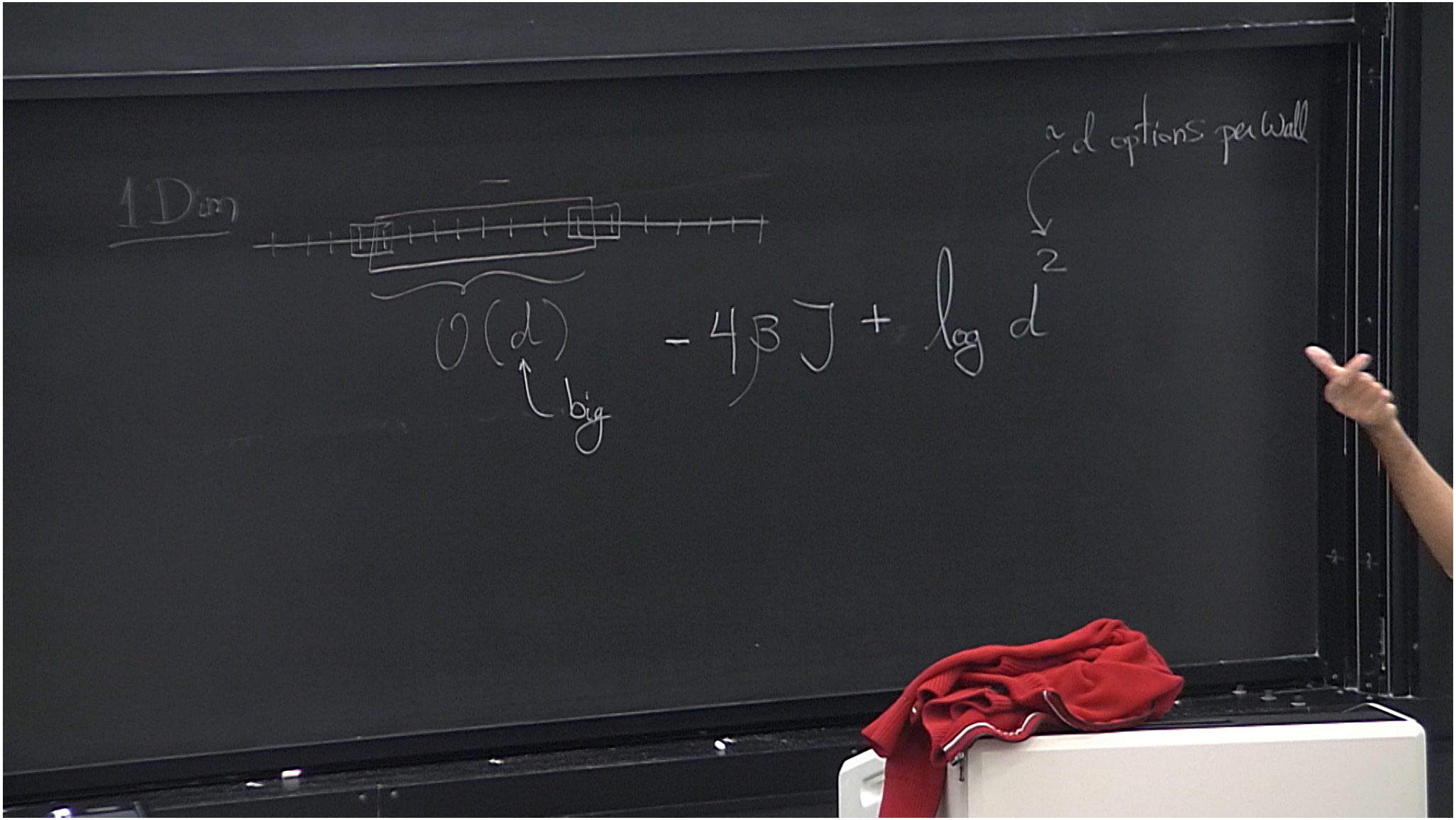

two minima
mass ∈ curvature at minima
$V'=0 = \underbrace{m^2}_{<0} y_* + 4\mu y_*^3 \;,\; y_* = \sqrt{-\frac{m^2}{4\mu}}$
or
$S \to S + \sum P \; y_{P_n}$
P fields
degeneracy is lifted
$Z_{eff} = \sum_{\{\sigma_n = \pm 1\}} e^{-\beta J \sum_{\langle i,j \rangle} [\sigma_j \sigma_i - 1]}$
neighbors
2^{L^2} terms
$\Delta E = 2\beta J$
$\sigma = +1$
$\sigma = -1$
Spin system

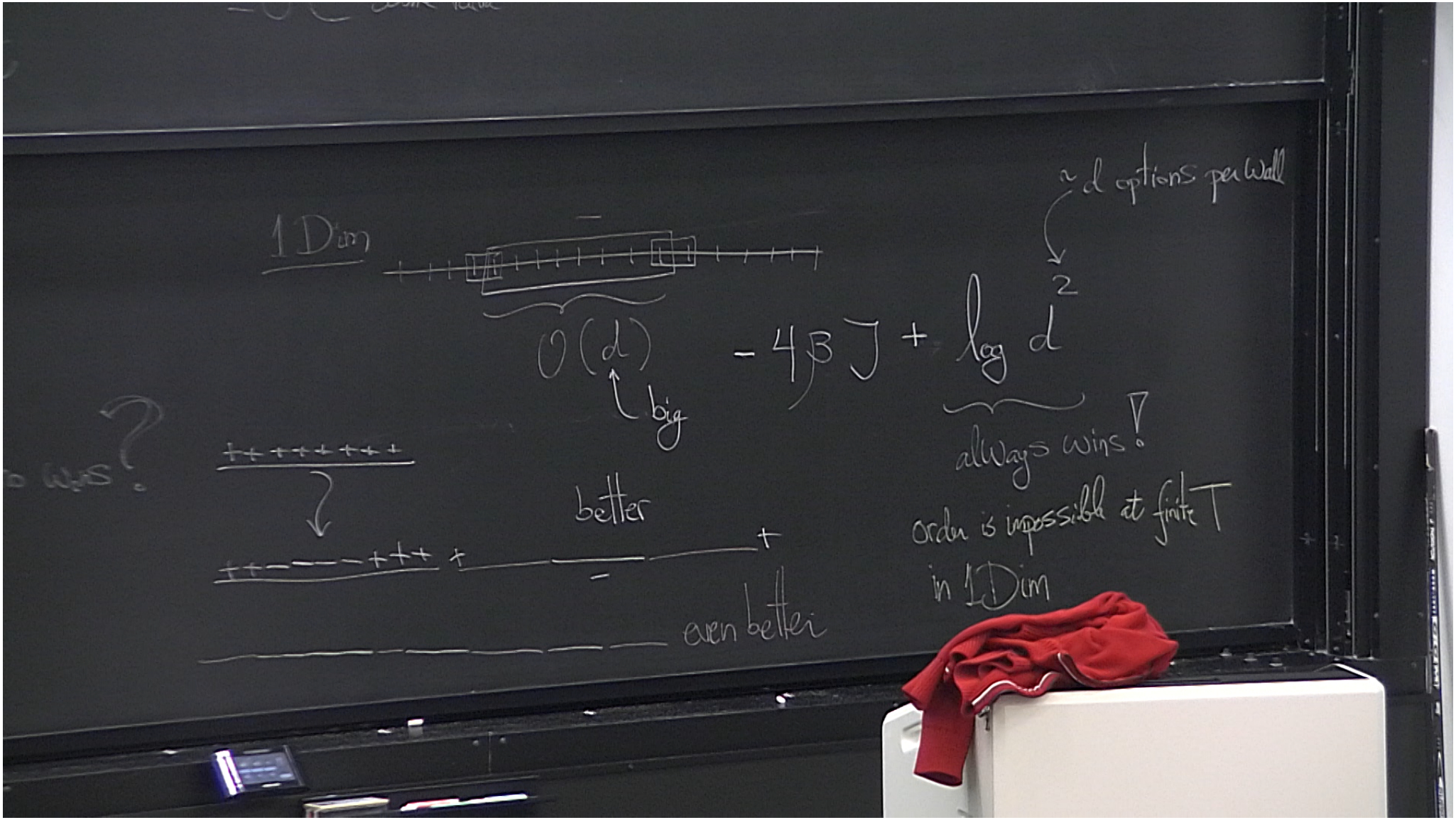

y=0
$0 = \underbrace{m^2}_{c_0} y_* + 4\mu y_*^3$, $y_* = \sqrt{-\frac{m^2}{4\mu}}$
or
$S \to S + \sum P\, y_n$, P finite
degeneracy is lifted
P → 0+ or 0−
end up in one vac
end up in other vacuum
Vacuum
$e^{+\beta J \sum_{\langle i,j \rangle} [\sigma_j \sigma_i - 1]}$
neighbors
ΔE = 2βJ
σ = +1
σ = −1
Spin system
Spin-Spin interaction
Spin system & = level table are in the same universality class
terms

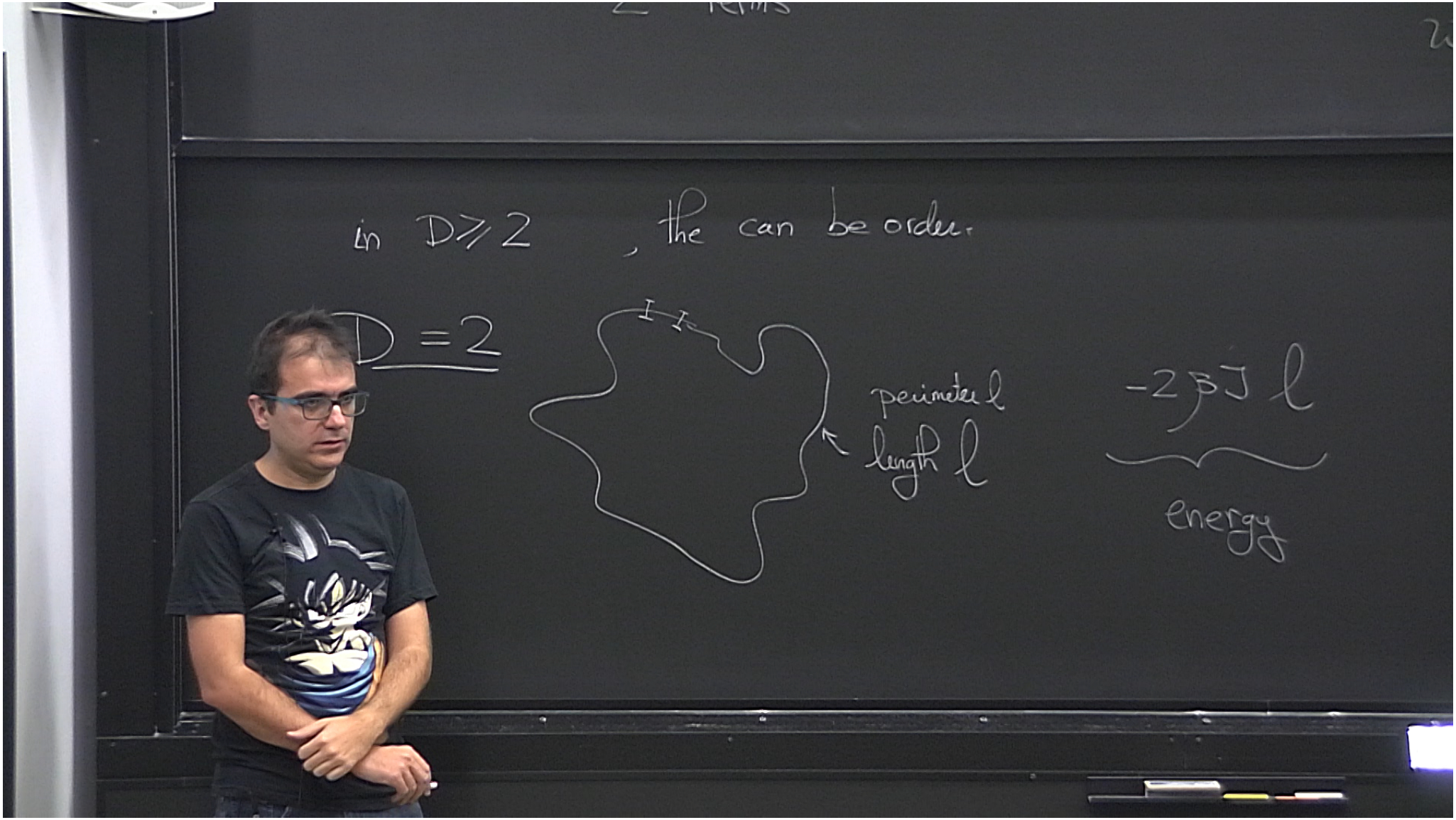

y_* or $-y_*$
$y=0$
if b.c
$\partial_y|_{\partial\,\text{table}} = +y_*$
table goes to $+y_*$ Vacuum
or
$S \to S + \sum P\, y_n$, P finite
degeneracy is lifted
$P \to 0^+$ or 0^-
end up in one vac
end up in other vacuum.
$+\beta J \sum_{\langle i,j\rangle} [\sigma_j \sigma_i - 1] - \beta h \sum \sigma_i$
neighbors
magnetic field
$\longleftrightarrow$ Spin system
Spin-Spin interaction
Spin system & = level table are in the same universality class
$\Delta E = 2\beta J$
$\sigma = -1$
$\sigma = +1$
$\sigma_i = 2n_i - 1$
where $n_i = 0,1$
$Z = \sum_{\{n_{\vec{n}}=0,1\}} e^{-\beta U \sum_{\langle i,j\rangle} n_i n_j - \beta\mu \sum n_i}$

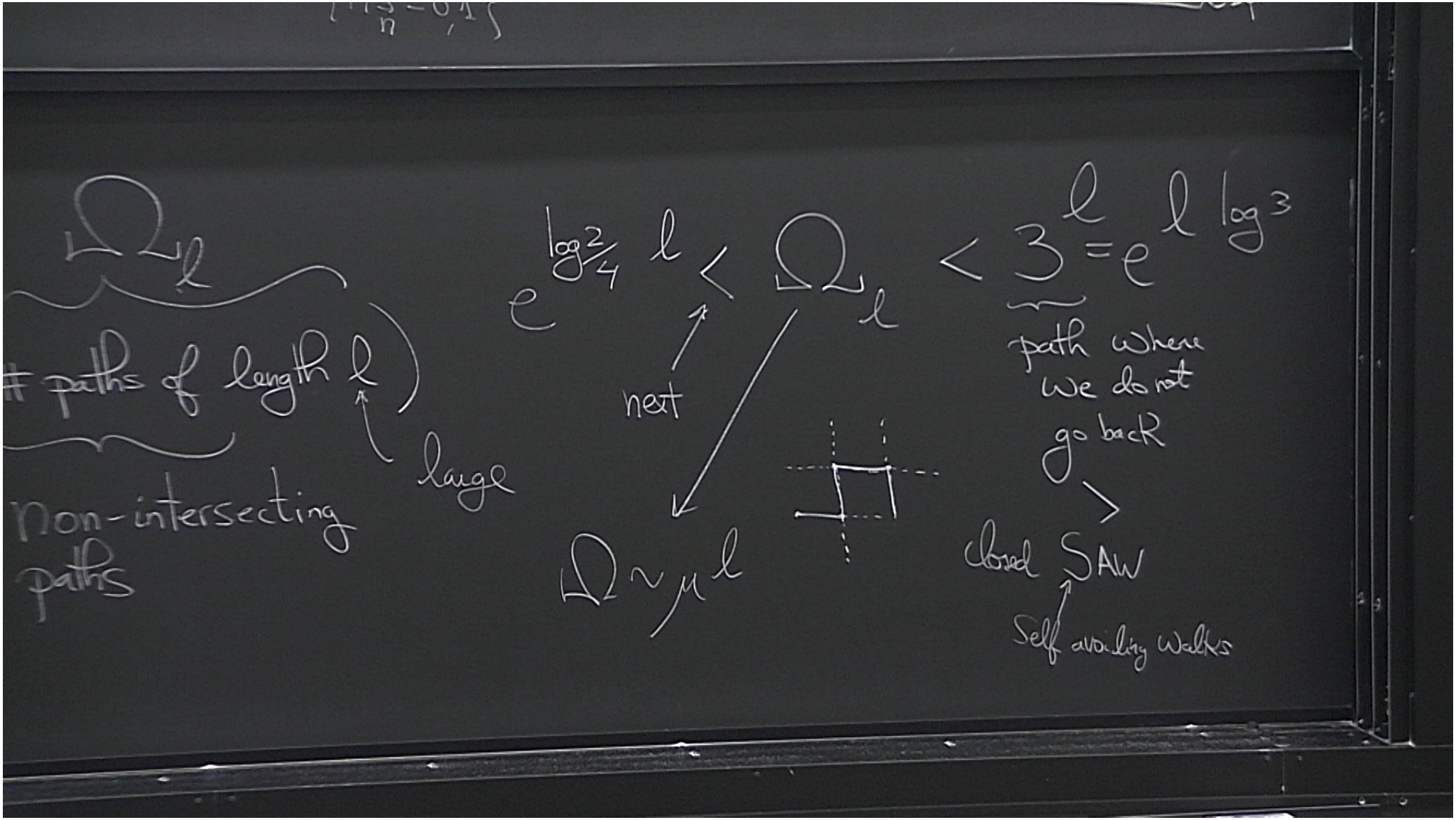

degeneracy is lifted
end up in one vac
end up in other vacuum.
Spin system
magnetic field
Spin system
&
2 level table
are in the same universality class
Spin-Spin interaction
gas of particles
1pt


E_0
Is this all correct?
order, two vacuum.
more energetic
but
more entropy
vs
Who wins?

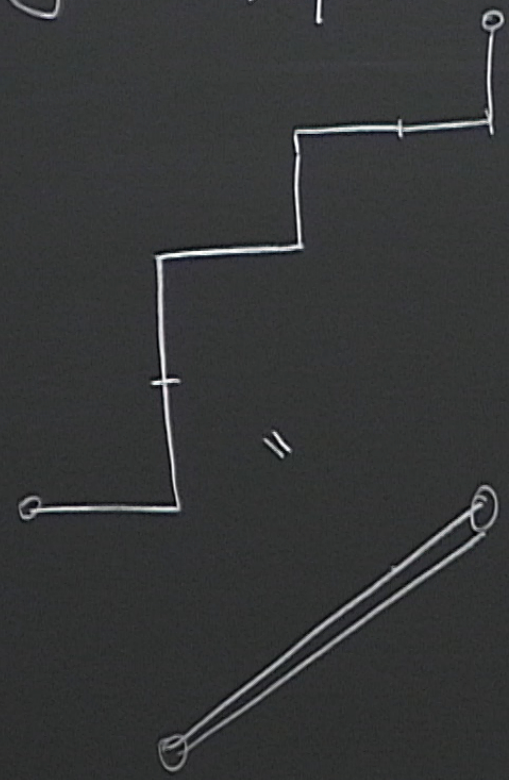

1 Dim
O(d)
big
$-4\beta J + \log d^2$
$\sim d$ options per wall


1Dim
O(d)
big
$-4\beta J + \log d^2$
≈ d options per wall
always wins!
order is impossible at finite T
in 1Dim
better
even better

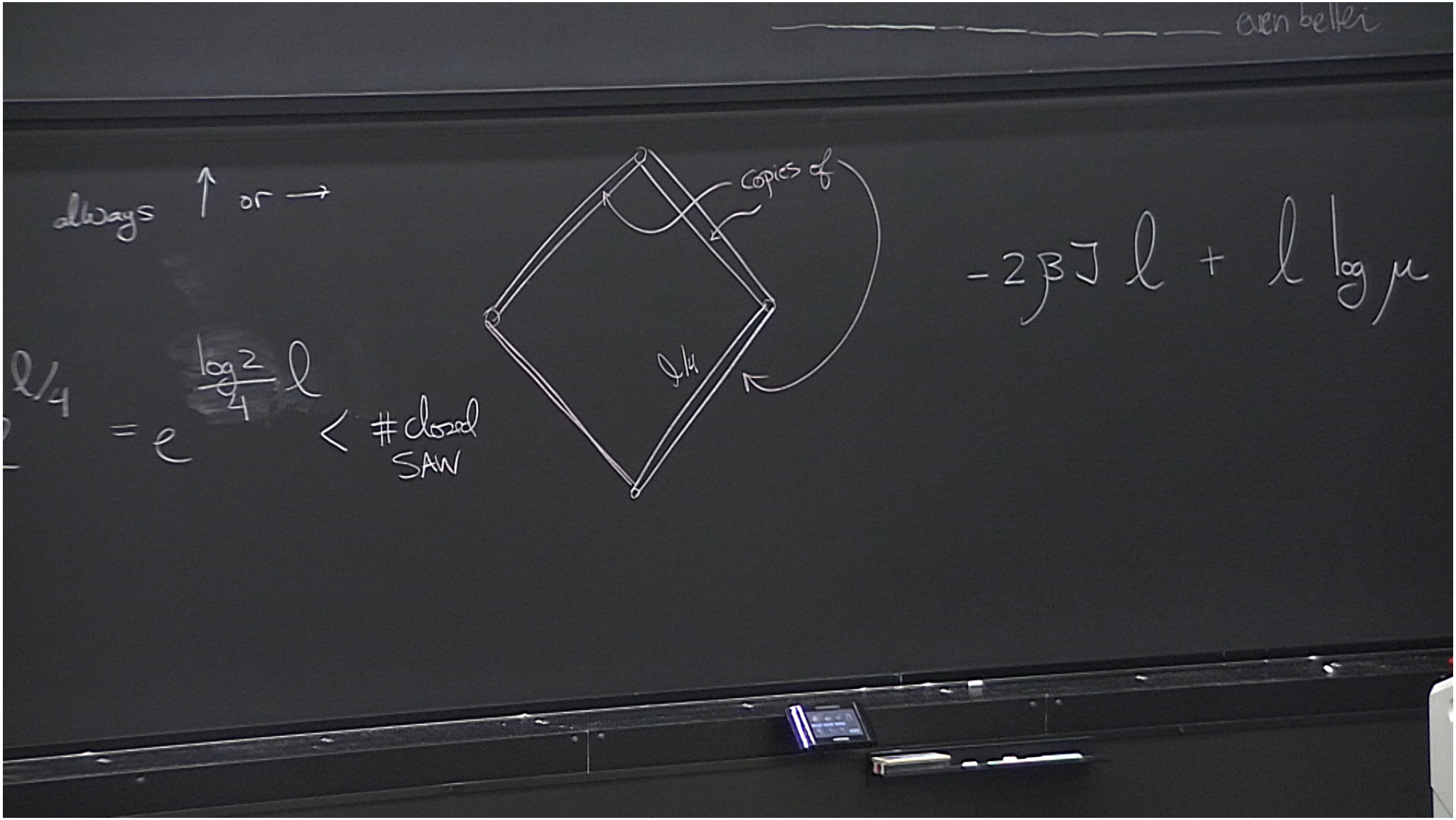

in D≥2
, the can be order.
D=2
perimeter l
length l
-2βJ l
energy

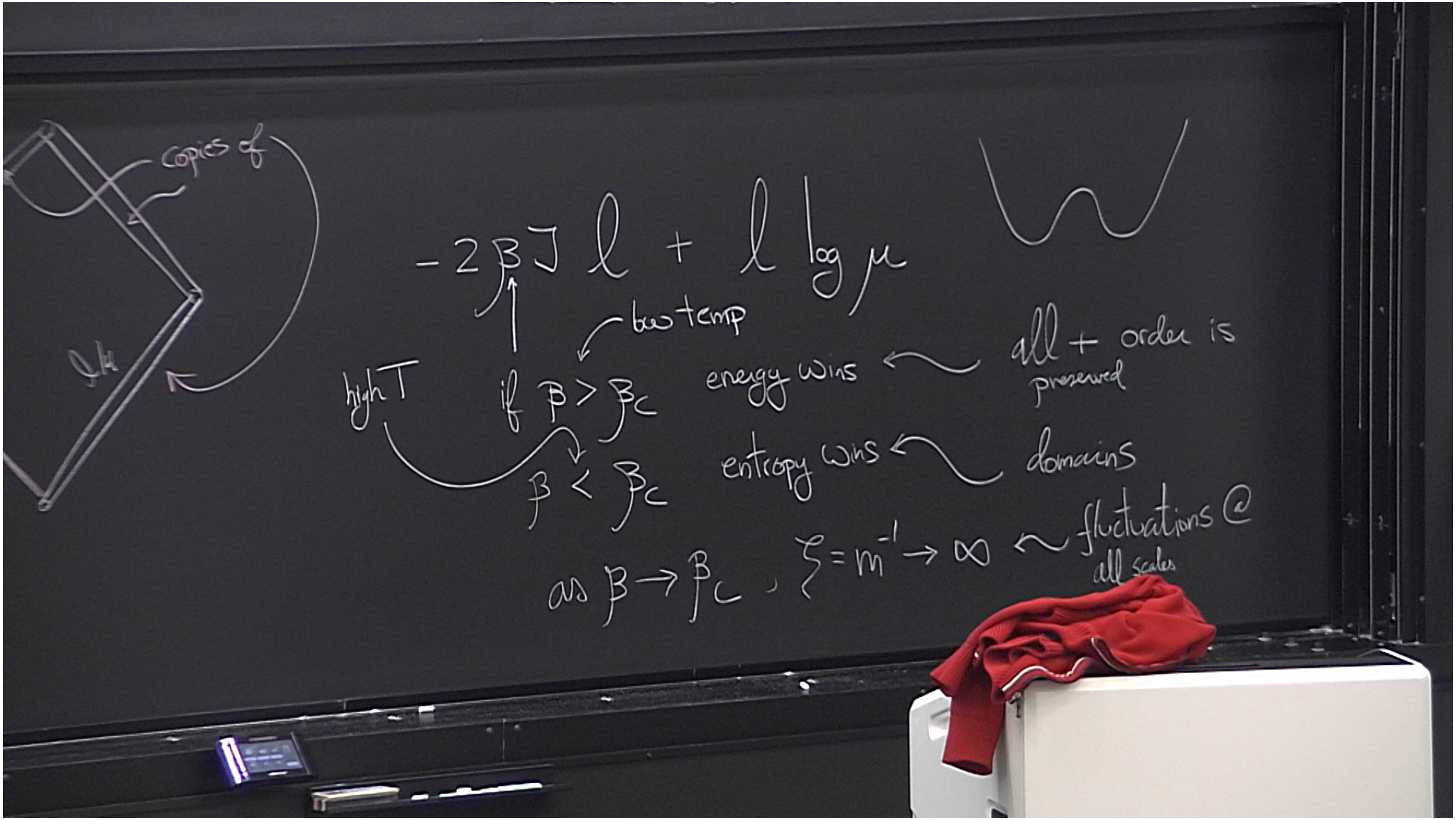

Ω_ℓ
paths of length ℓ
large
non-intersecting paths
$e^{\log\frac{2}{4}\,\ell} < \Omega_\ell < 3^\ell = e^{\ell \log 3}$
next
path where we do not go back
$\Omega \sim \mu^\ell$
closed SAW
Self avoiding Walks

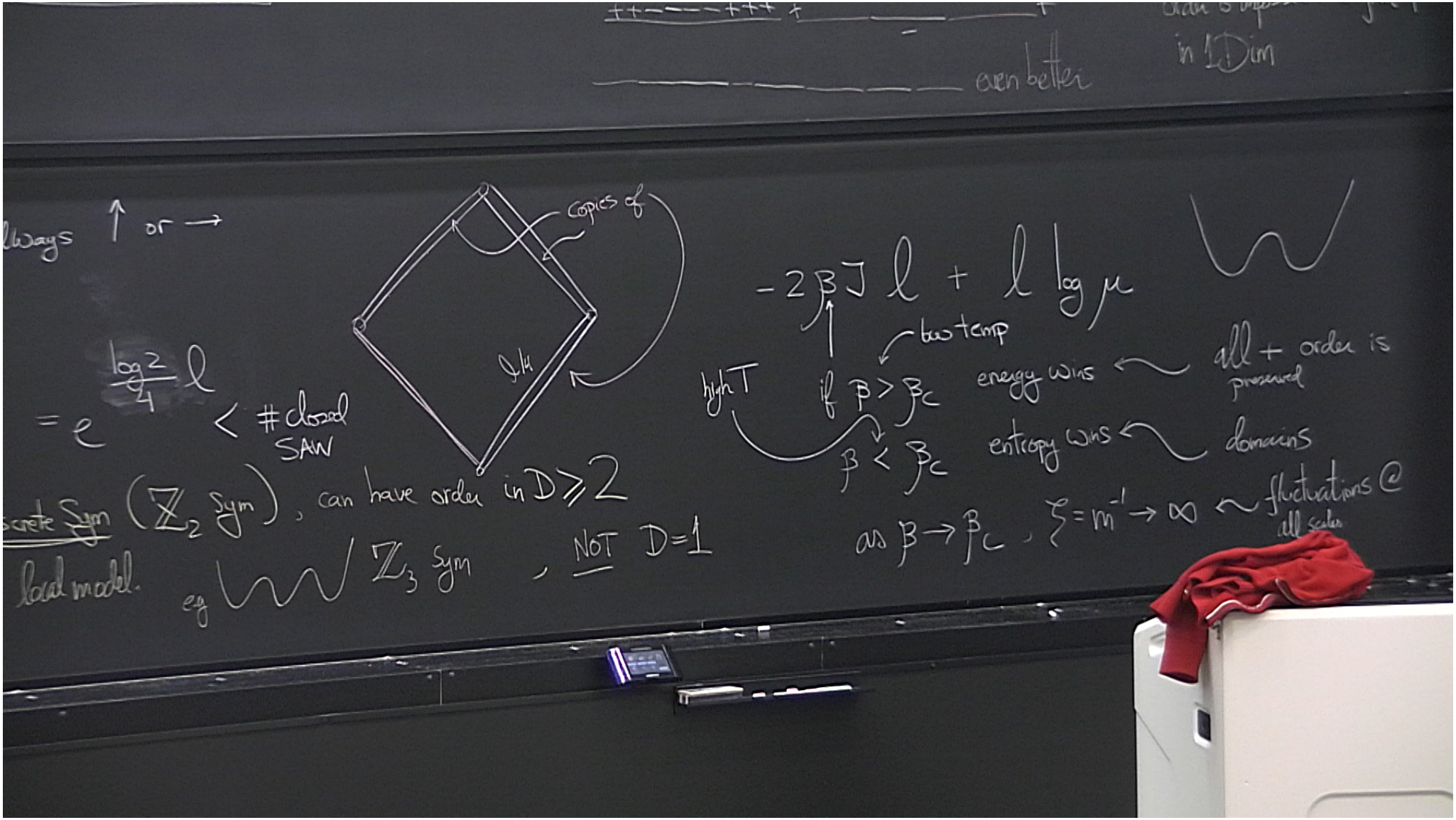
first $\ell/4$ steps always ↑ or →

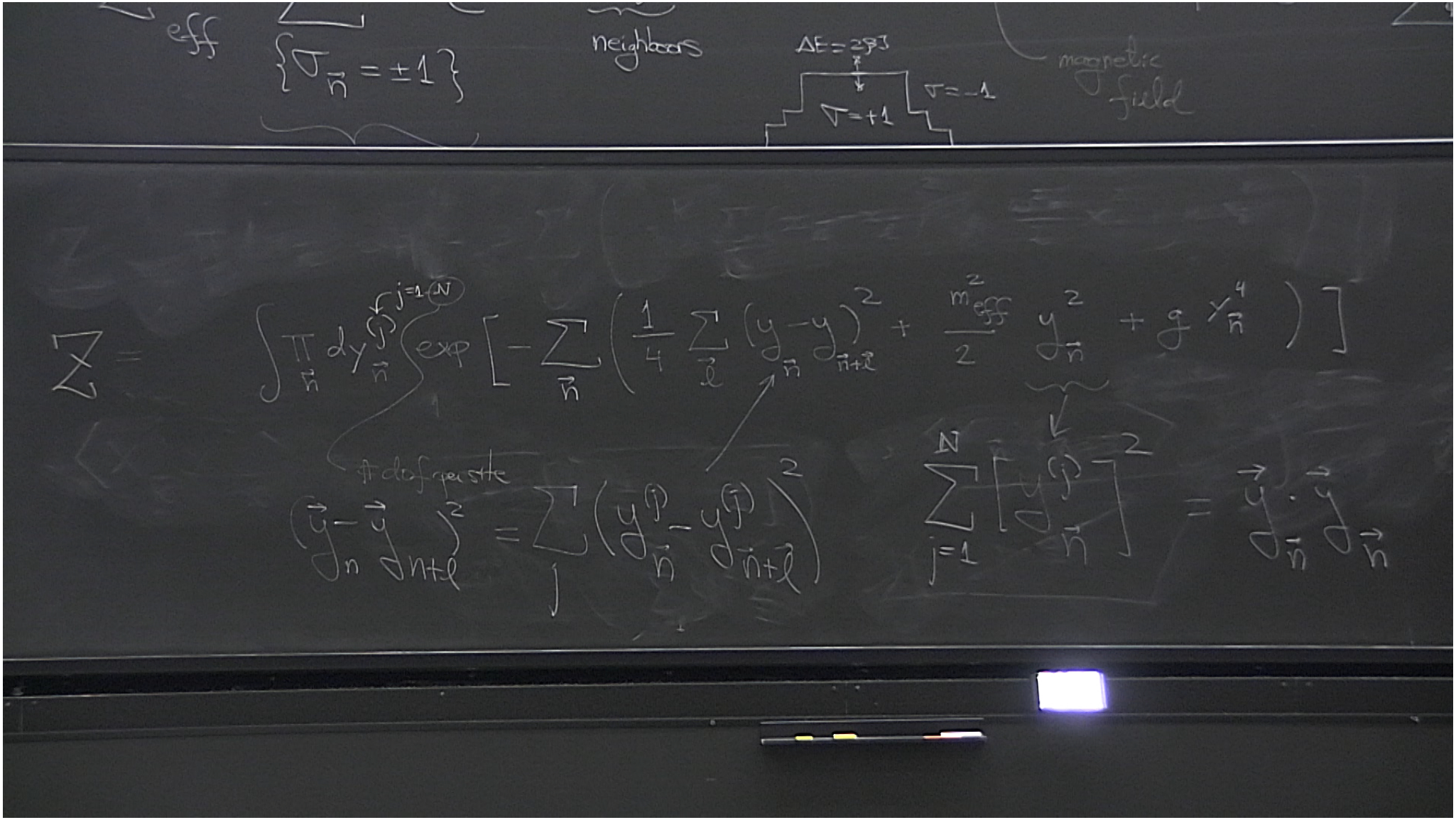


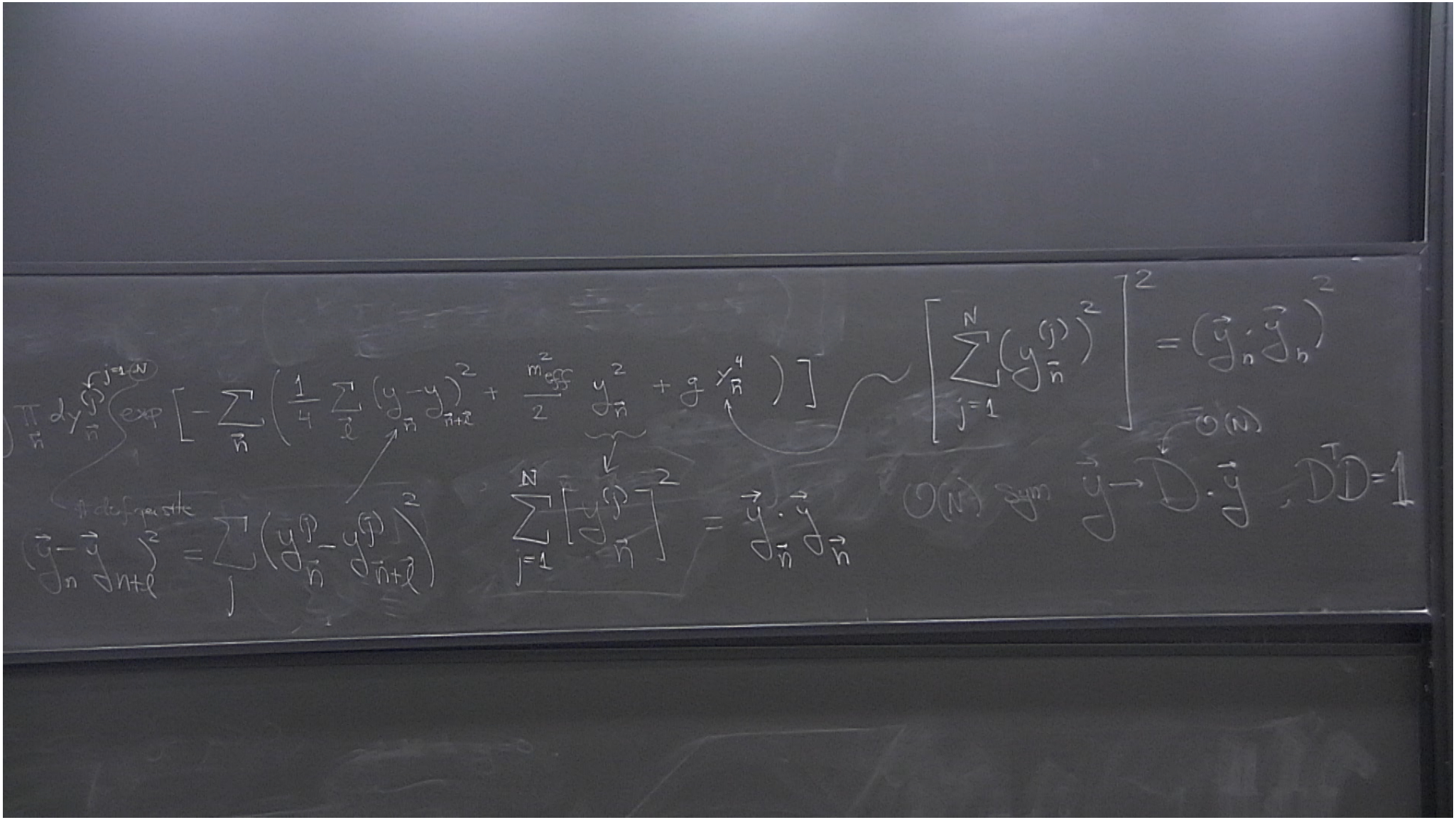
$2^{\ell/4} = e^{\frac{\log 2}{4}\ell}$

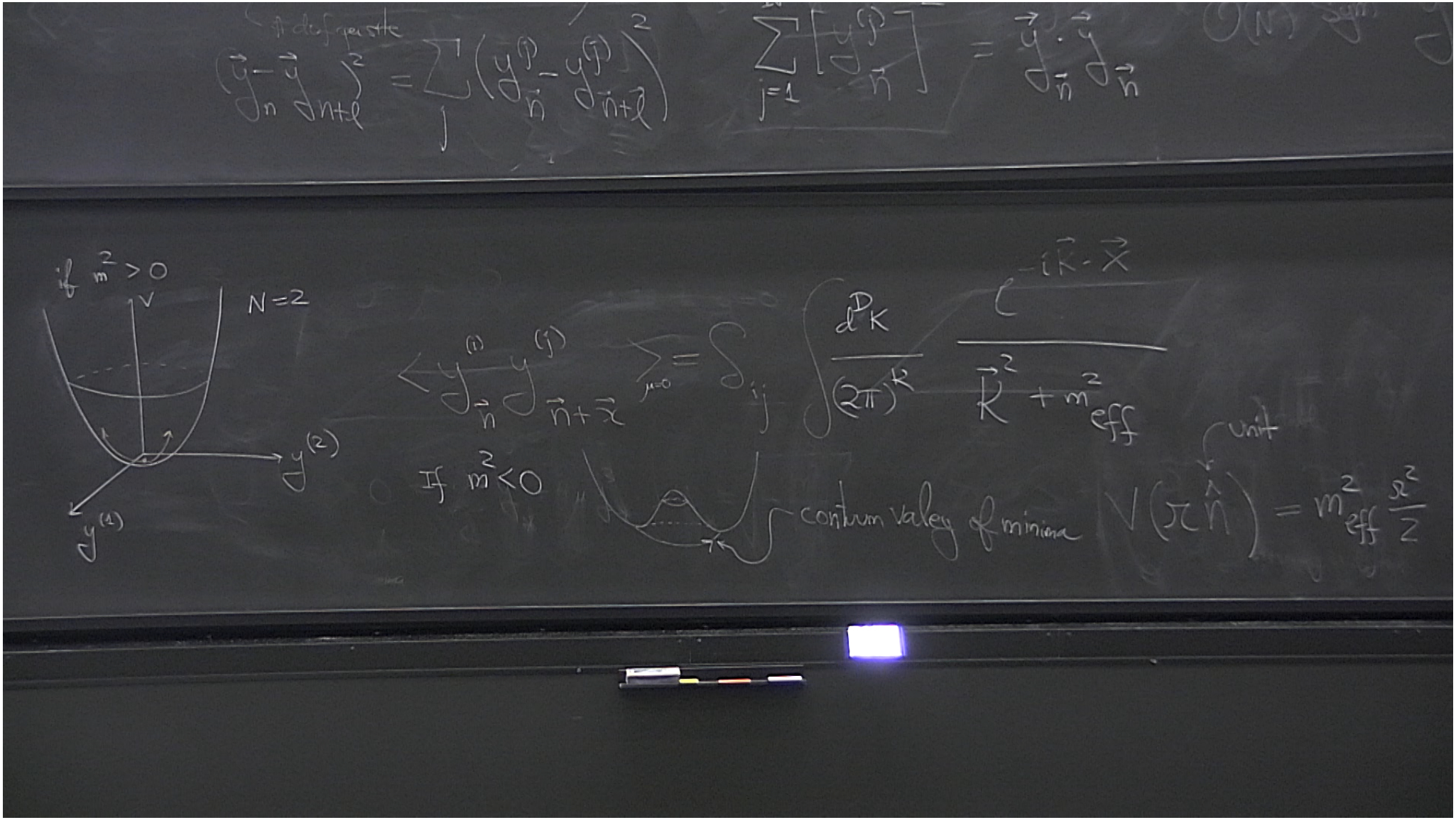

even better
always ↑ or →
copies of
ℓ/4
$-2\beta J\ell + \ell \log \mu$
$\ell/4$
$= e^{\frac{\log 2}{4}\ell}$
< # closed SAW


copies of
$-2\beta J\, l + l \log \mu$
low T
low temp
if $\beta > \beta_c$
energy wins
all + order is preserved
$\beta < \beta_c$
entropy wins
domains
as $\beta \to \beta_c$, $\xi = m^{-1} \to \infty$
fluctuations @ all scales


even better
in 1 Dim
ways ↑ or →
copies of
$-2\beta J\,\ell + \ell \log \mu$
low temp
high T
if $\beta > \beta_c$
$\beta < \beta_c$
energy wins
entropy wins
all + order is preserved
domains
$= e^{\frac{\log 2}{4}\ell} <$ # closed SAW
$\mathbb{Z}_2$ Sym, can have order in $D \geq 2$
$\mathbb{Z}_3$ Sym, NOT $D=1$
as $\beta \to \beta_c$, $\xi = m^{-1} \to \infty$
fluctuations @ all scales


eff
neighbors
ΔE = 2βJ
σ = −1
σ = +1
magnetic field
dof per site


O(N) sym
$\vec{y} \to D \cdot \vec{y}$, $D^T D = 1$
O(N)


if $m^2 > 0$
V
N=2
$y^{(2)}$
$y^{(1)}$
If $m^2 < 0$
continuum valley of minima
unit

$$= \sum_{\vec{n}} \left(y^{(p)}_{\vec{n}} - y^{(p)}_{\vec{n}+\vec{\ell}} \right)^2 \qquad \sum_{j=1}^{} \left[y^{(p)}_{\vec{n}} \right] = \vec{y}_{\vec{n}} \cdot \vec{y}_{\vec{n}}$$

$O(N)$ sym

$$\langle y^{(i)}_{\vec{n}} y^{(j)}_{\vec{n}+\vec{x}} \rangle_{\mu=0} = \delta_{ij} \int \frac{d^D k}{(2\pi)^D} \frac{e^{-i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2_{eff}}$$

$+,-$ vacua

continuous family $\equiv \hat{n}$

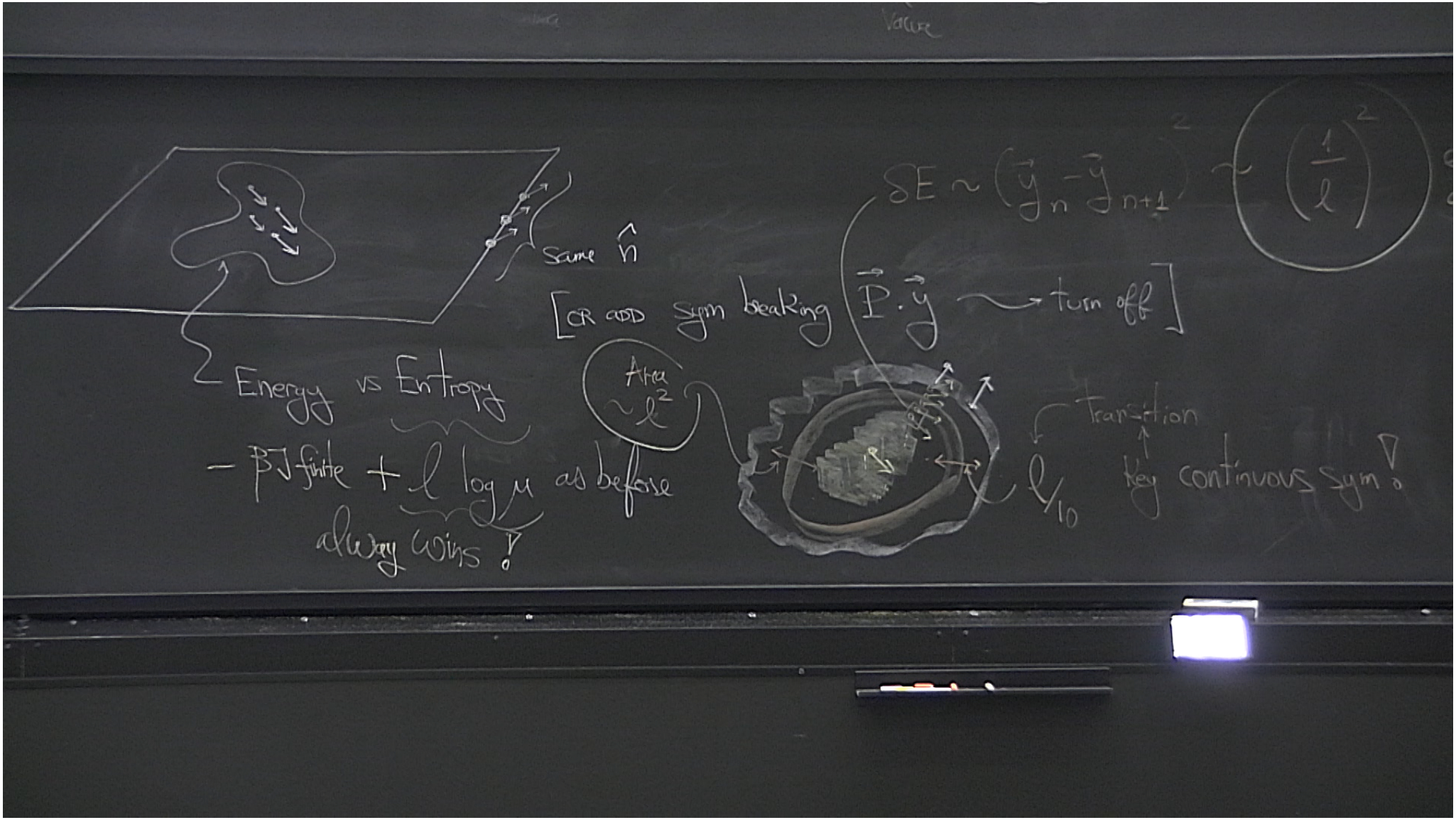
If $m^2 < 0$

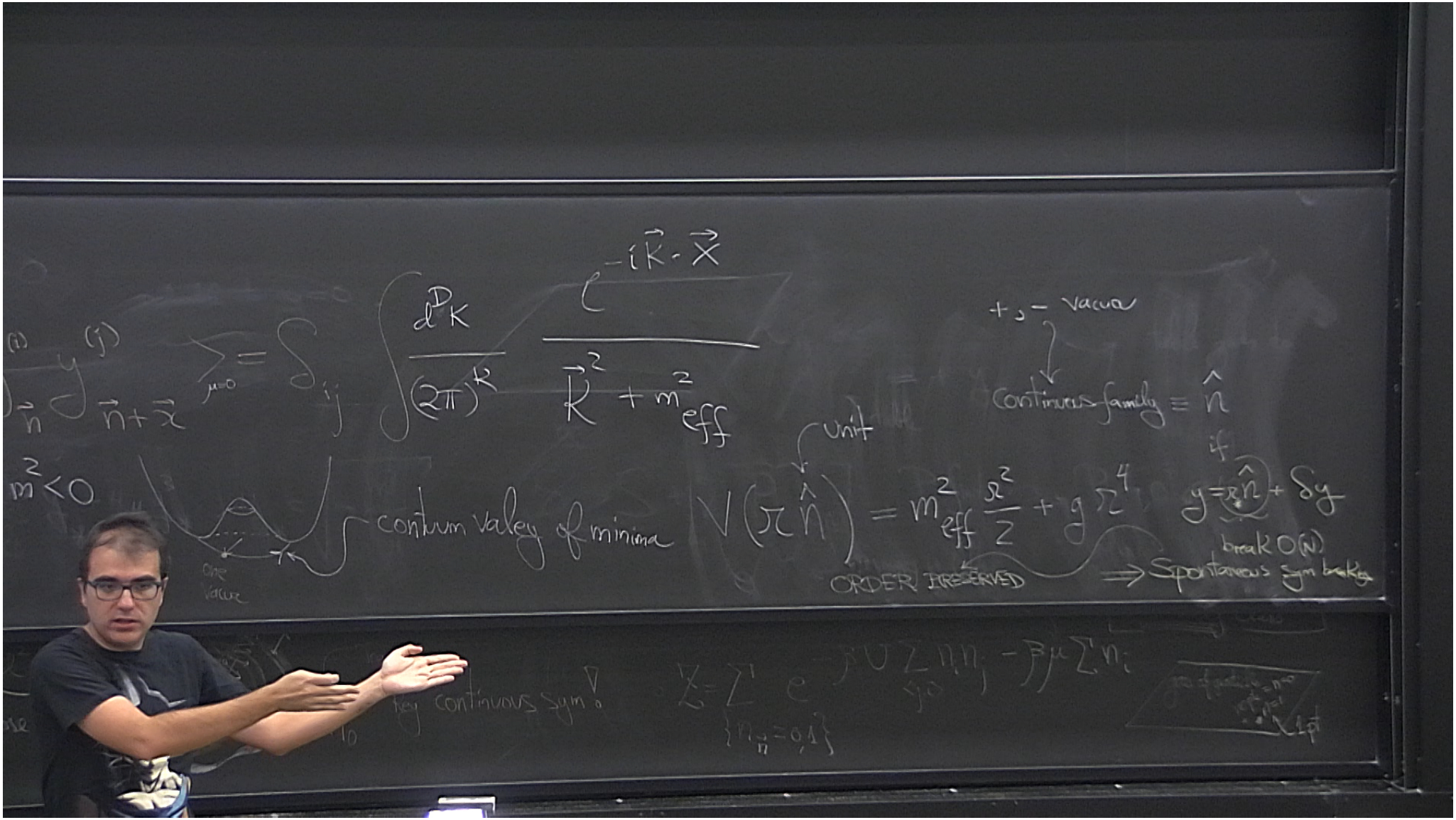
continuum valley of minima

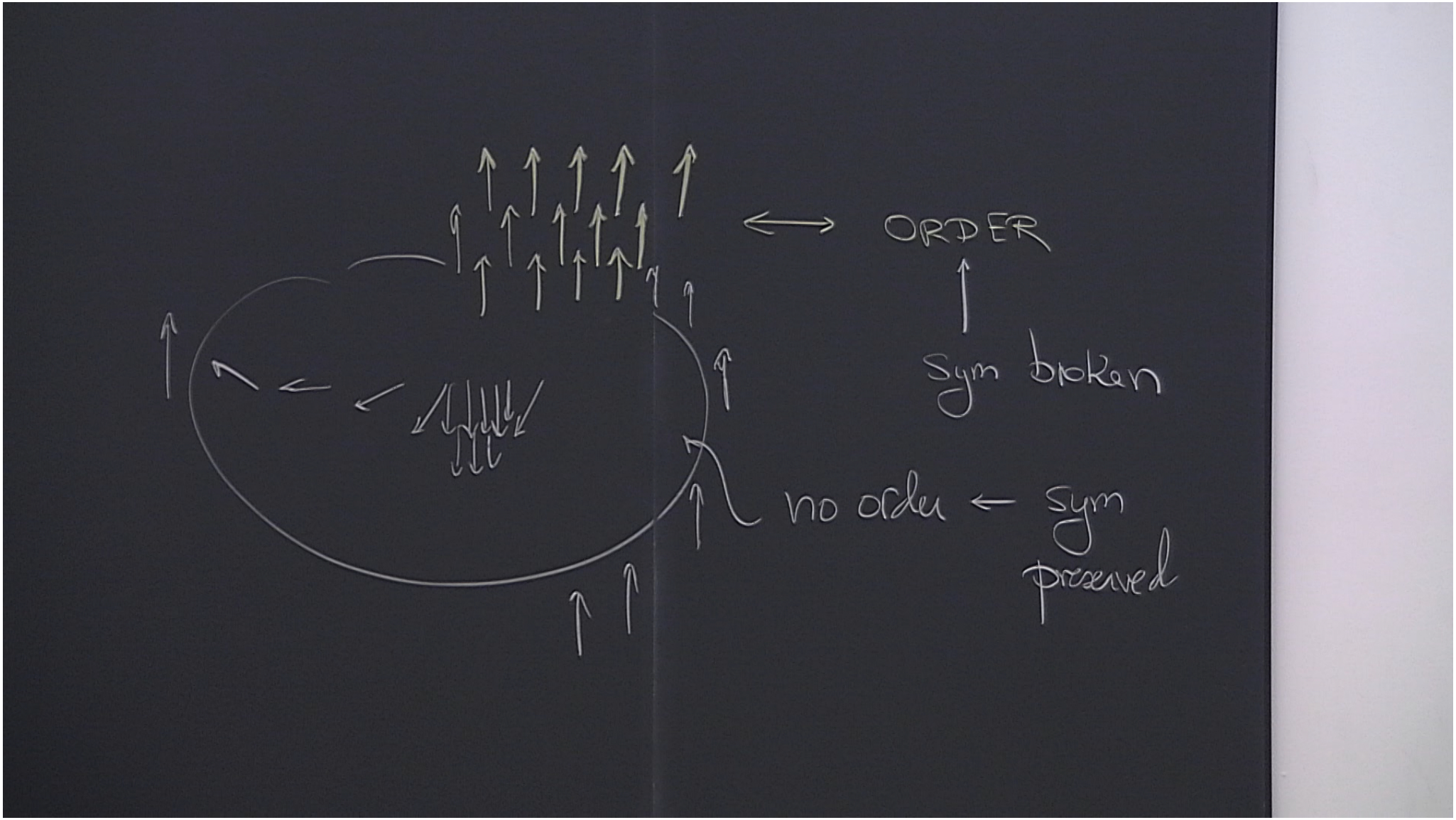
one value

unit

$$V(r\hat{n}) = m^2_{eff} \frac{r^2}{2} + g r^4$$

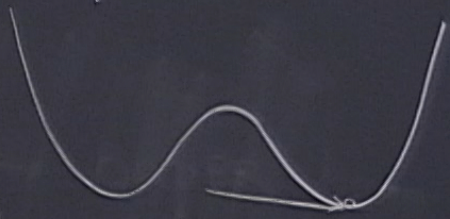

Same $\hat{n}$
[CR add Sym breaking $\vec{P}\cdot\vec{y}$ → turn off]
$\delta E \sim (\vec{y}_n - \vec{y}_{n+1})^2 \sim \left(\frac{1}{\ell}\right)^2$
Energy vs Entropy
$-\beta J$ finite $+ \ell \log \mu$ as before
always wins!
Area $\sim \ell^2$
ℓ_{10}
Transition
key continuous Sym!


$\int \frac{d^D K}{(2\pi)^K} \frac{e^{-i\vec{K}\cdot\vec{X}}}{\vec{K}^2 + m^2_{eff}}$
$+_3$ – Vacua
Continuous family $\equiv \hat{n}$
if
$y = r\hat{n} + \delta y$
$m^2 < 0$
continuum valley of minima
unit
$V(r\hat{n}) = m^2_{eff}\frac{r^2}{2} + g r^4$
ORDER PRESERVED
$\Rightarrow$ Spontaneous Sym broken
break O(N)
one value
key continuous sym!
$Z = \sum_{\{n_i = 0,1\}} e^{\beta U \sum n_i n_j - \beta\mu \sum n_i}$


ORDER
Sym broken
no order ← sym preserved

$$V(\vec{y}) = \frac{m^2}{2}\vec{y}^2 + g(\vec{y}^2)^2 \quad , \quad S = \frac{1}{4}\sum_{n,\ell}(z_n - z_{n+\ell})^2$$

quadratic in $\vec{z}$



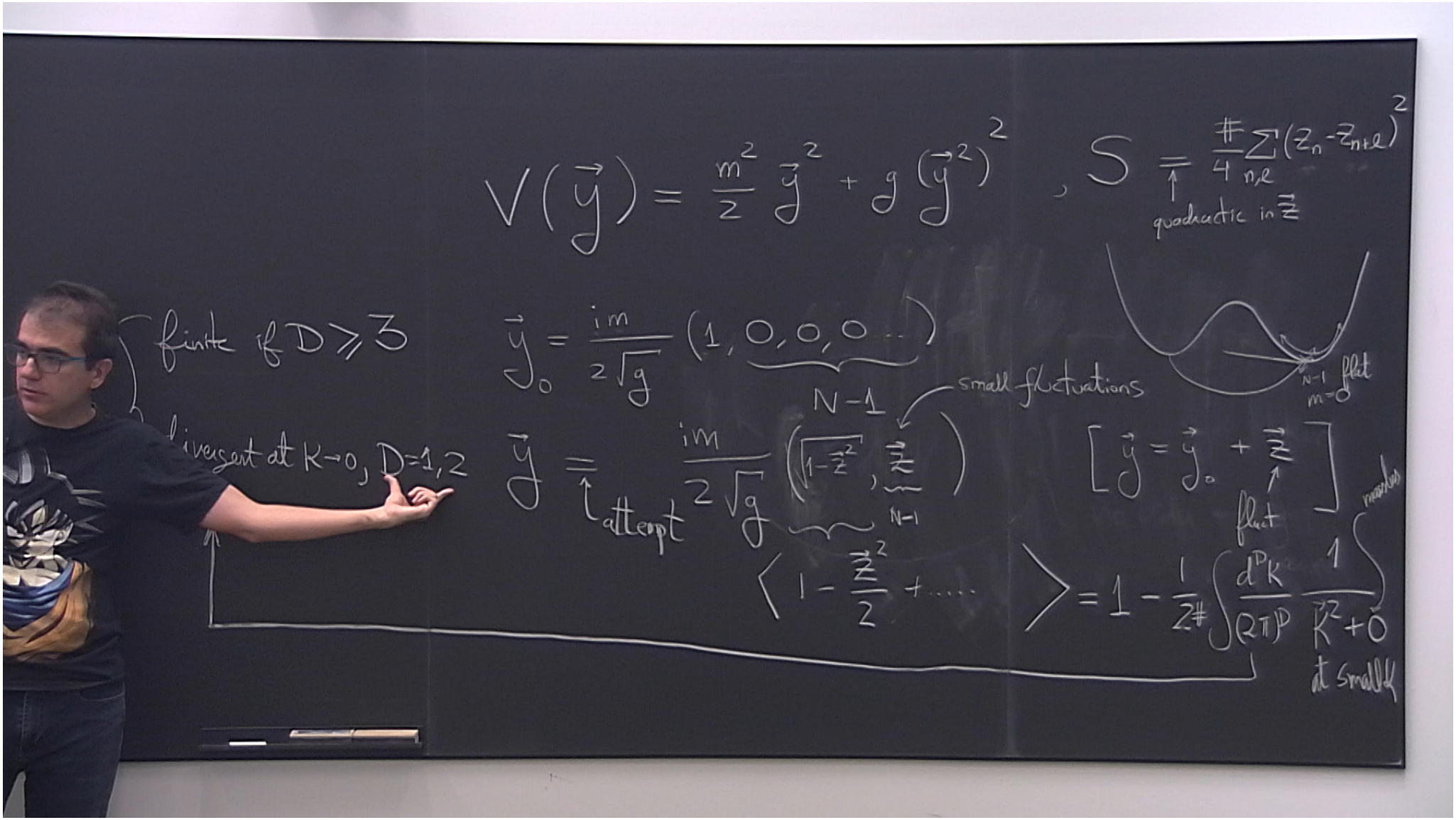
$$\vec{y}_0 = \frac{m}{2\sqrt{g}}(1, \underbrace{0,0,0\ldots}_{N-1})$$

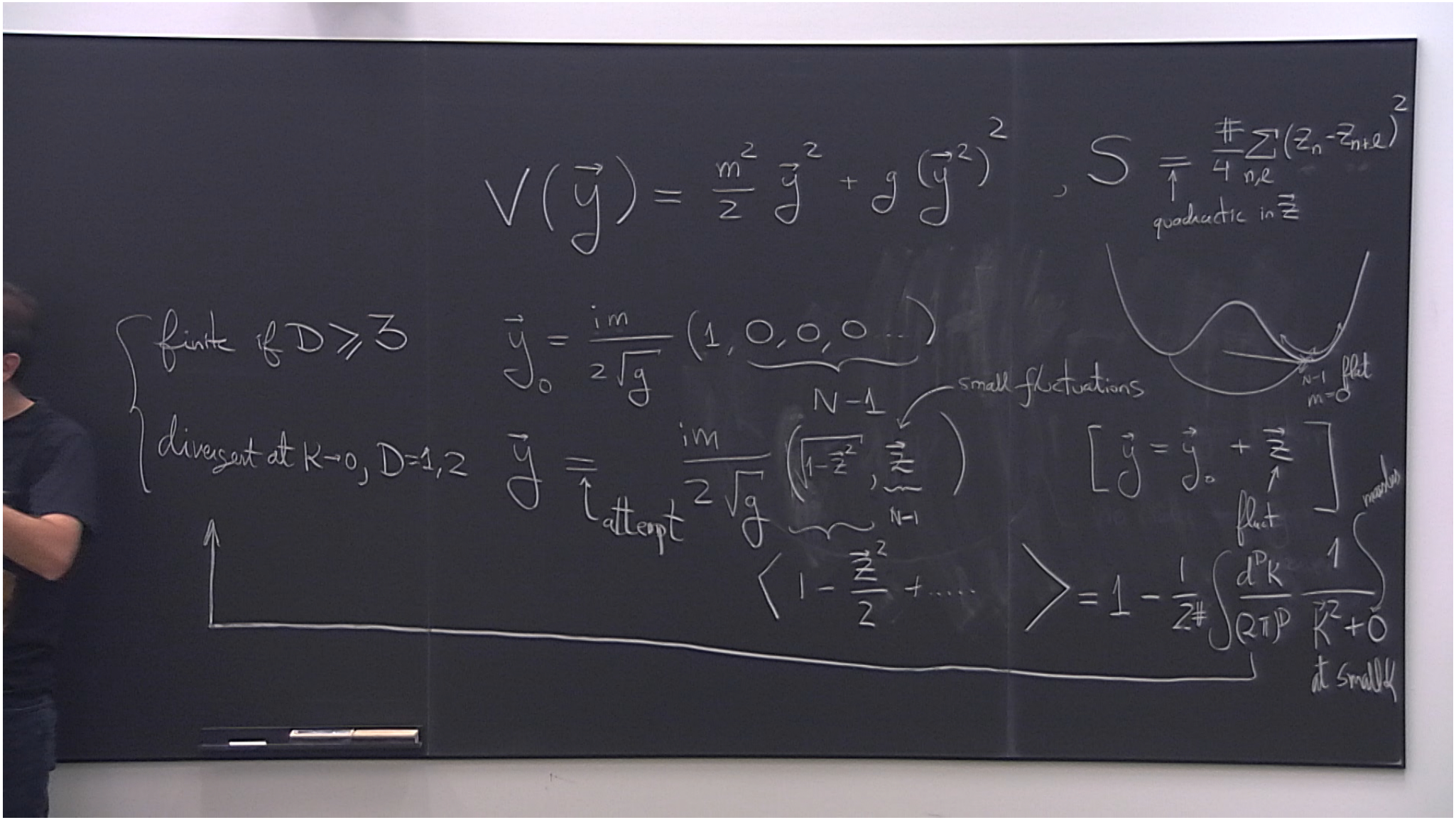
small fluctuations

$$\vec{y} = \vec{y}_{attempt} = \frac{m}{2\sqrt{g}}\left(\sqrt{1-\vec{z}^2}, \underbrace{\vec{z}}_{N-1}\right)$$

$$\sqrt{1-\vec{z}^2} \sim 1 - \frac{\vec{z}^2}{2} + \ldots$$

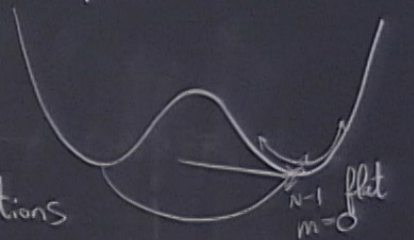
$$\left[\vec{y} = \vec{y}_0 + \vec{z}_{fluct}\right]$$


$V(\vec{y}) = \frac{m^2}{2}\vec{y}^2 + g(\vec{y}^2)^2$
$S = \frac{\#}{4}\sum_{n,\ell}(z_n - z_{n+\ell})^2$
quadratic in $\vec{z}$
finite if $D \geq 3$
diverges at $K \to 0$, $D = 1,2$
$\vec{y}_0 = \frac{im}{2\sqrt{g}}(1, \underbrace{0,0,0\ldots}_{N-1})$
$\vec{y} = \vec{y}_{attempt} = \frac{im}{2\sqrt{g}}\left(\sqrt{1-\vec{z}^2}, \underbrace{\vec{z}}_{N-1}\right)$
small fluctuations
$N-1$ flat $m=0$
$[\vec{y} = \vec{y}_0 + \vec{z}]$
fluct
$\left\langle 1 - \frac{\vec{z}^2}{2} + \ldots \right\rangle = 1 - \frac{1}{2\#}\int \frac{d^D k}{(2\pi)^D} \frac{1}{\vec{k}^2 + 0}$
massless
at small k


$V(\vec{y}) = \frac{m^2}{2}\vec{y}^2 + g(\vec{y}^2)^2$
$S = \frac{\#}{4}\sum_{n,\ell}(z_n - z_{n+\ell})^2$
quadractic in $\vec{z}$
finite if $D \geq 3$
divergent at $K \to 0$, $D = 1,2$
$\vec{y}_0 = \frac{im}{2\sqrt{g}}(1, \underbrace{0,0,0\ldots}_{N-1})$
$\vec{y} = \vec{y}_{attempt} = \frac{im}{2\sqrt{g}}\left(\sqrt{1-\vec{z}^2}, \underbrace{\vec{z}}_{N-1}\right)$
small fluctuations
$N-1$ flat $m=0$
$[\vec{y} = \vec{y}_0 + \vec{z}]$
fluct
$\left\langle 1 - \frac{\vec{z}^2}{2} + \ldots \right\rangle = 1 - \frac{1}{2\#}\int \frac{d^D K}{(2\pi)^D} \frac{1}{\vec{K}^2 + 0}$
massless
at small K

$$V(\vec{y}) = \frac{m^2}{2}\vec{y}^2 + g(\vec{y}^2)^2 \quad , \quad S = \frac{\#}{4}\sum_{n,\ell}(z_n - z_{n+\ell})^2$$

quadratic in $\vec{z}$



- finite if $D \geq 3$
- divergent at $K \to 0$, $D = 1, 2$ ← no spontaneous sym breaking in 2D

$$\vec{y}_0 = \frac{im}{2\sqrt{g}}(1, \underbrace{0,0,0\ldots}_{N-1})$$

small fluctuations

$$\vec{y} = \vec{y}_{attempt} \frac{im}{2\sqrt{g}}\left(\sqrt{1-\vec{z}^2}, \underbrace{\vec{z}}_{N-1}\right) \qquad [\vec{y} = \vec{y}_0 + \vec{z}]$$

$$\left\langle 1 - \frac{\vec{z}^2}{2} + \ldots \right\rangle = 1 - \frac{1}{Z^\#}\int \frac{d^D K}{(2\pi)^D} \underbrace{\frac{1}{K^2 + 0}}_{\text{massless}}$$

$\vec{z}$ fluct

at small K