Title: Towards a categorification of a projection from the affine to the finite Hecke algebra in type A

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Abstract: <p>Work of Bezrukavnikov on local geometric Langlands correspondence and works of Gorsky, NeguÅ£, Rasmussen and Oblomkov, Rozansky on knot homology and matrix factorizations suggest that there should be a categorical version of a certain natural homomorphism from the affine Hecke algebra to the finite Hecke algebra in type A, sending basis lattice elements on the affine side to Jucys-Murphy elements on the finite side. I will try to explain some of the structures involved and will talk about recent progress towards a construction of such a categorification in the setting of Hecke categories.</p>

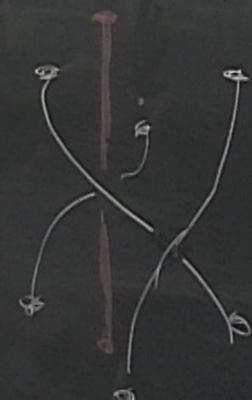
I. Algebraic picture

$$B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}), \varsigma)$$

$$B_n^{\mathrm{aff}} = \pi_1(\mathrm{Conf}_n(\mathbb{C}^\times), \varsigma)$$

$$\mathbb{C}^\times \hookrightarrow \mathbb{C}$$

$$\Phi : B_n^{\mathrm{aff}} \longrightarrow B_n$$

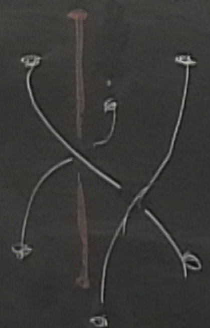


$\pi_1(\mathrm{Conf}_n(\mathbb{C}), \varsigma)$

$= \pi_1(\mathrm{Conf}_n(\mathbb{C}^\times), \varsigma)$

$\hookrightarrow \mathbb{C}$

ff $\longrightarrow B_n$



$W = S_n, \quad S = \{s_1, \dots, s_{n-1}\}$

$\widetilde{W}_n^{aff} = W_n^{aff} \ltimes \mathbb{Z}$

$\rightsquigarrow$

$\mathcal{H}_n, \widetilde{\mathcal{H}}_n^{aff}$ – alg. over $\mathbb{Z}[v, v^{-1}]$

$\mathcal{H}_n$: gen. by t_s, $s \in S$, $t_i := t_{s_i}$

$t_i t_j = t_j t_i, \quad |i-j| > 1$

$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$

$t_i^2 = 1 + (v^{-1} - v) t_i$

$\ldots, s_{n-1} \}$

$\mathbb{Z}[v, v^{-1}]$

$t_i := t_{s_i}$

$\tilde{\mathcal{H}}_n^{\mathrm{aff}}$ is gen. by t_s, $s \in S$,

θ_i, $i = 1, \ldots, n$.

$\langle t_i \rangle$ - gen. finite Hecke alg. $\mathcal{H}_n$,

$\theta_i \theta_j = \theta_j \theta_i$

$t_i \theta_i t_i = \theta_{i+1}$

$t_i \theta_j = \theta_j t_i$, $j \neq i, i+1$

$\varphi:$

$\varphi($

$L_1 = 1$

$$\varphi(\Theta_2) = \varphi(t_1 \theta_1 t_1) = t_1^2 = L_2$$

$$\varphi(\Theta_3) = t_2 t_1^2 t_2 = L_3$$

$$\vdots$$

$$\varphi(\Theta_k) = t_{k-1} t_{k-2} \dots t_1^2 t_2 \dots t_{k-1} = L_k$$

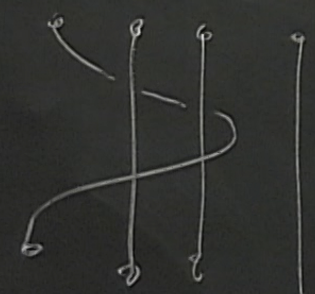
L_i — (mult.) Jucys-Murphy elt's of $\mathcal{H}_n$.

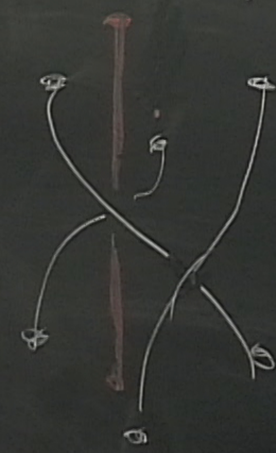
$$L_i L_j = L_j L_i$$

$B_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}), \xi)$

$\mathbb{C}^\times \hookrightarrow \mathbb{C}$

$\Phi : B_n^{\mathrm{aff}} \longrightarrow B_n$





$\mathcal{H}_n$

"$\mathcal{H}_n$: ge

$t_i t_j =$

$t_i t_{i+1} t$

$t_i^2 = 1$

I. Algebraic picture

Centers of $\tilde{\mathcal{H}}_n^{aff}$, $\mathcal{H}_n$

Theorem (Bernstein).

$Z(\tilde{\mathcal{H}}_n^{aff})$ is a free $\mathbb{Z}[v,v^{-1}]$-module

w. basis z_λ, $\lambda \in \mathbb{Z}^n$, $\lambda_1 \geq \lambda_2 \geq \dots$

$$z_\lambda = \sum_{w \in W} \theta_{w(\lambda)}.$$

Theorem (Dipper, James) / $\mathbb{Q}$

$Z(\mathcal{H}_n) = \{$ symm. polyn. in L_i, L_i^{-1} $\}$

$W = S_n$

$\tilde{W}_n^{aff} =$

$\mathcal{H}_n$, $\tilde{\mathcal{H}}_n^{aff}$

$\mathcal{H}_n$: gen. by

$t_i t_j = t_j t_i$,

$t_i t_{i+1} t_i = t_{i+1}$

t^2

picture

$\mathcal{H}_n$

$\mathbb{Z}[v,v^{-1}]$-module

$\lambda_1 \geq \lambda_2 \geq \ldots$

s) / $\mathbb{Q}$

polyn. in L_i, L_i^{-1} }

$$\begin{array}{ccc} Z(\widetilde{\mathcal{H}}_n^{\mathrm{aff}}) & \hookrightarrow & \widetilde{\mathcal{H}}_n^{\mathrm{aff}} \\ \downarrow & & \downarrow \varphi \\ Z(\mathcal{H}_n) & \hookrightarrow & \mathcal{H}_n \end{array}$$

$$\mathrm{Perv}_{G(\mathcal{O})}\left(G(\mathcal{K})/G(\mathcal{O})\right) \longrightarrow D^b_I\left(G(\mathcal{K})/I\right)$$

$$D(CS) \longrightarrow D^b_B(G/B)$$

character sheaves

$\widetilde{\mathcal{H}}_n^{\mathrm{aff}}$ is

θ_i, $i=$

$\langle t_i \rangle$ -

$\theta_i \theta_j = \theta$

$t_i \theta_i t_i$

$t_i \theta_j = \theta$

$K)/I)$

$G = GL_n$

$B \subset G$ – Borel subgroup.

$K = \mathbb{C}[t^{-1}, t]]$.

$\mathcal{O} \subset K = \mathbb{C}[[t]]$.

I – Iwahori subgroup.

$G(\mathcal{O}) \xrightarrow{\text{ev at } t=0} G$

$I = ev^{-1}(B)$,

$x \in \mathbb{Z}^n$

$\theta_x = \theta_1^{x_1} \theta_2^{x_2} \cdots$

$\mathcal{H}_n \hookrightarrow \widetilde{\mathcal{H}}_n^{aff}$

$\varphi : \widetilde{\mathcal{H}}_n^{aff} \longrightarrow \mathcal{H}_n$

$\varphi(\theta_1) = 1, \ \varphi(t_i) =$

e picture

$\tilde{\mathcal{H}}_n^{aff}, \mathcal{H}_n$

ein).

a free $\mathbb{Z}[v,v^{-1}]$-module

$\in \mathbb{Z}^n, \lambda_1 \geq \lambda_2 \geq \dots$

).

James) $/ \mathbb{Q}$

ymm. polyn. in L_i, L_i^{-1} }

$$\begin{array}{ccc} Z(\tilde{\mathcal{H}}_n^{aff}) & \hookrightarrow & \tilde{\mathcal{H}}_n^{aff} \\ \downarrow\!\downarrow & & \downarrow\!\downarrow \varphi \\ Z(\mathcal{H}_n) & \hookrightarrow & \mathcal{H}_n \end{array}$$

Geometric Satake

$$\mathrm{Rep}(GL_n) \simeq \mathrm{Perv}_{G(\mathcal{O})}\left(G(K)/G(\mathcal{O})\right) \longrightarrow D^b_I\left(G(K)/I\right)$$

$$D(CS) \longrightarrow D^b_B(G/B)$$

character sheaves

G

B

$K =$

$\mathcal{O} \subset$

I

$G($

$I =$

$$Z(A_n) \hookrightarrow A_n$$

$\lambda_1 \geq \lambda_2 \geq \ldots$

Geometric Satake

$$\mathrm{Rep}(GL_n) \simeq \mathrm{Perv}_{G(\mathcal{O})}\left(G(K)/G(\mathcal{O})\right) \xrightarrow{\text{Gaitsg. nearby cycles}} D^b_I\left(G(K)/I\right)$$

(1) Q

in L_i }

L_i

$$D(CS) \xrightarrow{\text{HC-functor (Ginzburg Lusztig)}} D^b_B(G/B)$$

$\supset$ character sheaves

$$D_{G\text{-ad}}(G)$$

$(k)/_I)$

$/B)$

Want:

$H_n \supset \{ t_w, w \in W \}$

$\quad \rotatebox{90}{$\in$}$

$t_{s_{i_1}} \cdots t_{s_{i_k}}, \; s_{i_1} \cdots s_{i_k} = w$ - reduced expression

Categorif. of $t_w \rightsquigarrow \Delta_w \in D^b_B(G/B)$

$(!$-ext. from Bruhat cells (shifted))

$x \in \mathbb{Z}$

$\theta_x = \theta,$

$\mathcal{H}_n \hookrightarrow \tilde{\mathcal{H}}$

$\varphi : \tilde{H}_n^{aff} \to$

$\varphi(\theta_1) = 1,$

$t_{s_{i_1}} \dots t_{s_{i_k}}$, $s_{i_1} \dots s_{i_k} = w$ - reduced expression

Categorif. of $t_w \rightsquigarrow \Delta_w \in D^b_B(G/B)$

(!-ext. from Bruhat cells (shifted))

E.g. L_3 should be cat.

$\Delta_{s_2 s_1} * \Delta_{s_1 s_2}$,

Want some central sheaves in $P^b_B(G/B)$

sat. $S_V = \theta_1 + \dots + \theta_n$.

corr. sheaf S_V to have a filtr. by sh. corr. to L_i

$\mathcal{H}_n \hookrightarrow \mathcal{H}_n^{aff}$

$\varphi : \tilde{\mathcal{H}}_n^{aff} \to \mathcal{H}_n$

$\varphi(\theta_1) = 1$, $\varphi(t_i) = t_i$

gebraic picture

rs of $\tilde{\mathcal{H}}_n^{aff}$, $\mathcal{H}_n$

(Bernstein).

ff) is a free $\mathbb{Z}[v,v^{-1}]$-module

z_λ, $\lambda \in \mathbb{Z}^n$, $\lambda_1 \geq \lambda_2 \geq \dots$

$\sum_{\in W} \theta_{w(\lambda)}$.

(Dipper, James) / $\mathbb{Q}$

$) = \{$ symm. polyn. in $L_i^{\pm 1}$ $\}$

$$\begin{tikzcd} Z(\tilde{\mathcal{H}}_n^{aff}) \arrow[r, hook] \arrow[d, two heads] & \tilde{\mathcal{H}}_n^{aff} \arrow[d, two heads, "\varphi"] \\ Z(\mathcal{H}_n) \arrow[r, hook] & H_n \end{tikzcd}$$

Geometric Satake

Gaitsg. nearby cycles

$$\mathrm{Rep}(GL_n) \simeq \mathrm{Perv}_{G(\mathcal{O})}(G(K)/G(\mathcal{O})) \xrightarrow{} D^b_I(G(K)/I)$$

$$D(CS) \xrightarrow{\text{HC-functor (Ginzburg)}} D^b_B(G/B)$$

$\supset$ character sheaves

Want:

$\mathrm{Rep}\, GL_n \to D^b_G(G)$

<u>Lemma</u>: $\mathcal{C}$ – symm. monoidal cat,
s.t. $\mathcal{C}$ is karoubian
(images of all proj. exist in $\mathcal{C}$).
$\rightsquigarrow$
Schur functors S^λ in $\mathcal{C}$ (e.g. $\Lambda^\cdot, S^\cdot, \ldots$)

Assume $X \in \mathcal{C}$
1) $\Lambda^{n+1} X = 0$ $\Rightarrow$ $\exists F: \mathrm{Rep}\, GL(V_n) \to \mathcal{C}$
2) $\Lambda^n X$ is invertible $F(V) = X$.

$= W\}$

$, s_{i_2} \ldots s_{i_k} = w$ - reduced expression,

$\Delta_w \in D^b_B(G/B)$

!-ext. from Bruhat cells (shifted)

cat.

2. sheaves in $P^b_B(G/B)$

$\varphi(\theta_2) = \varphi(t_1 \theta_1 t$

$\varphi(\theta_3) = t_2 t_1^2 t$

$\vdots$

$\varphi(\theta_k) = t_{k-1} t_{k-}$

L_i – (mult.) J

$L_i L_j = L_j L_i$

al cat.

t in $\mathcal{C}$).

g. $\Lambda^{\bullet}, S^{\bullet}, \ldots$)

$GL(V_1) \to \mathcal{C}$

$\mathcal{D}^b_G(G)$

$$\begin{array}{ccc} & G\times G & \xrightarrow{m} G \\ \swarrow & & \searrow \\ G & & G \end{array}$$

$*$ – convolution

$\mathcal{A} * \mathcal{B} = m_!(\mathcal{A} \boxtimes \mathcal{B})$

$\mathcal{D}^b_G(G)$ – braided monoidal cat.

When the centralizer of any $g \in G$ is connected,

$\mathrm{Perv}_G(G)$ – symm. monoidal category.

$\cdot * \cdot = {}^pH^0(\cdot * \cdot)$

cted.

<u>Theorem</u> (joint w. Bezrukavnikov

$\tilde{\mathcal{N}}_P = T^*\mathbb{P}^{n-1}$ — parabolic Springer resol.

U_P

$P = \begin{pmatrix} * \end{pmatrix}$

unipotent radical of P

$\tilde{\mathcal{N}}_P = \{ (g, x \in G/P) : g \in x U_P x^{-1}$

$\tilde{\mathcal{N}}_P = T^* \mathcal{P}$ — parabolic Springer resolution

$$P = \left(\overbrace{\boxed{\quad * \quad}}^{U_P} \right)$$

nected.

unipotent radical of P

$$\tilde{\mathcal{N}}_P = \{ (g, x \in G/P) : g \in x U_P x^{-1} \}$$

$$Spr_P = \pi_* \underline{\mathbb{C}}_{\tilde{\mathcal{N}}_P}[2 \dim U_P].$$

$Spr_P \in Perv_G(G)$,

$\Lambda^{n+1} Spr_P = 0$, $\Lambda^n Spr_P * \hat{S}$ – invertible in CS.

Toy example

$G = GL_1 = \mathbb{C}^\times$

Consider loc. constant sh. on $\mathbb{C}^\times$

w. unipotent monodromy.

$\rightsquigarrow$

$(M, N \in End(M), N\text{-nilp.})$

$A * B \rightsquigarrow M_A \otimes_{\mathbb{C}[[t]]} M_B$

$Z(\tilde{\mathcal{H}}_n^{aff}$

$\Downarrow$

$Z(\mathcal{H}_n$

Geometric Satake

$Rep(GL_n) \simeq Perv_{G(\mathcal{O})}(G$

$D(CS)$

w. unipotent monodromy.

$\rightsquigarrow$

$(M, N \in \mathrm{End}(M), N\text{-nilp.})$

$A * B \rightsquigarrow M_A \otimes_{\mathbb{C}[[t]]} M_B$

$\hat{\delta} = \left(\begin{array}{l} \text{local syst. corr. to} \\ \varprojlim_n k[t]/t^n \end{array} \right)$

$A \in D(\mathbb{C}^\times),\ A * \hat{\delta} - D_{\mathrm{unip.}}(\mathbb{C}^\times)$

$\Downarrow$

$Z(\mathcal{H}_n$

Geometric Satake

$\mathrm{Rep}(GL_n) \simeq \mathrm{Perv}_{G(\mathcal{O})}($

$\downarrow$

$D(CS)$

character

$D_{G\text{-ad}}(G)$

Theorem (Bezrukavnikov)

St – Steinberg variety.

$\mathfrak{g}$ = Lie alg of $\mathfrak{g}$

$\tilde{\mathfrak{g}}$ – Groth.-Springer resol.

$St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}^{\vee}$

$$\underbrace{D^b Coh^G_{\mathcal{N}}(St)}_{\text{set-th. supp. on } \mathcal{N}} \overset{\text{mon. triang cat}}{\simeq} D^b_I(G/I)$$

Want:

$\operatorname{Rep} GL_n \to D^b_G(G)$

Lemma: $\mathcal{C}$ – symm. monoidal s.t. $\mathcal{C}$ is karoubian (images of all proj. exist)

$\Rightarrow$ Schur functors S^λ in $\mathcal{C}$ (e.g.

Assume $X \in \mathcal{C}$

1) $\Lambda^{n+1} X = 0$ $\Rightarrow \exists F : \operatorname{Rep} GL($

2) $\Lambda^n X$ is invertible $F(V) = X$.

$\longrightarrow \mathcal{H}_n^{aff}$

$\downarrow \varphi$

$\longrightarrow H_n$

g. nearby cycles

$G(\mathcal{O})) \quad D^b_I(G(K)/I)$

$D^b_B(G/B)$

functor (Ginzburg Lusztig)

aves

Theorem (Bezrukavnikov)

St - Steinberg variety.

$\mathfrak{g}$ = Lie alg. of $\mathfrak{g}$

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$St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$

$\underbrace{D^b Coh^G_{\mathcal{N}}(St)}_{\text{set. th. supp. on } \mathcal{N}} \underset{\text{cat.}}{\overset{\text{mon. triang}}{\simeq}} D^b_I(G(K)/I)$

Rep ...

Lemma

s.t.

(ima...

Schur func...

Assume X ...

1) $\Lambda^{n+1} X = \dots$

2) $\Lambda^n X$ is i...

y.

sol.

$D^b_I(G(t)/I)$

Want:

Theorem (T.)

There is a monoidal functor

$$D^{b,N}_{perf}(St) \longrightarrow D^b_B(G/B)$$

Compatible Rep (GL_n) action on both sides,

$D^b_G(G)$

$*$ – convolution

$A * B = m_!(A \boxtimes$

$D^b_G(G)$ – braided

When the centraliz

$Perv_G(G)$ – symm. mo

$\cdot * \cdot = {}^pH^0(\cdot * \cdot)$