Title: Toward AGT for general algebraic surfaces

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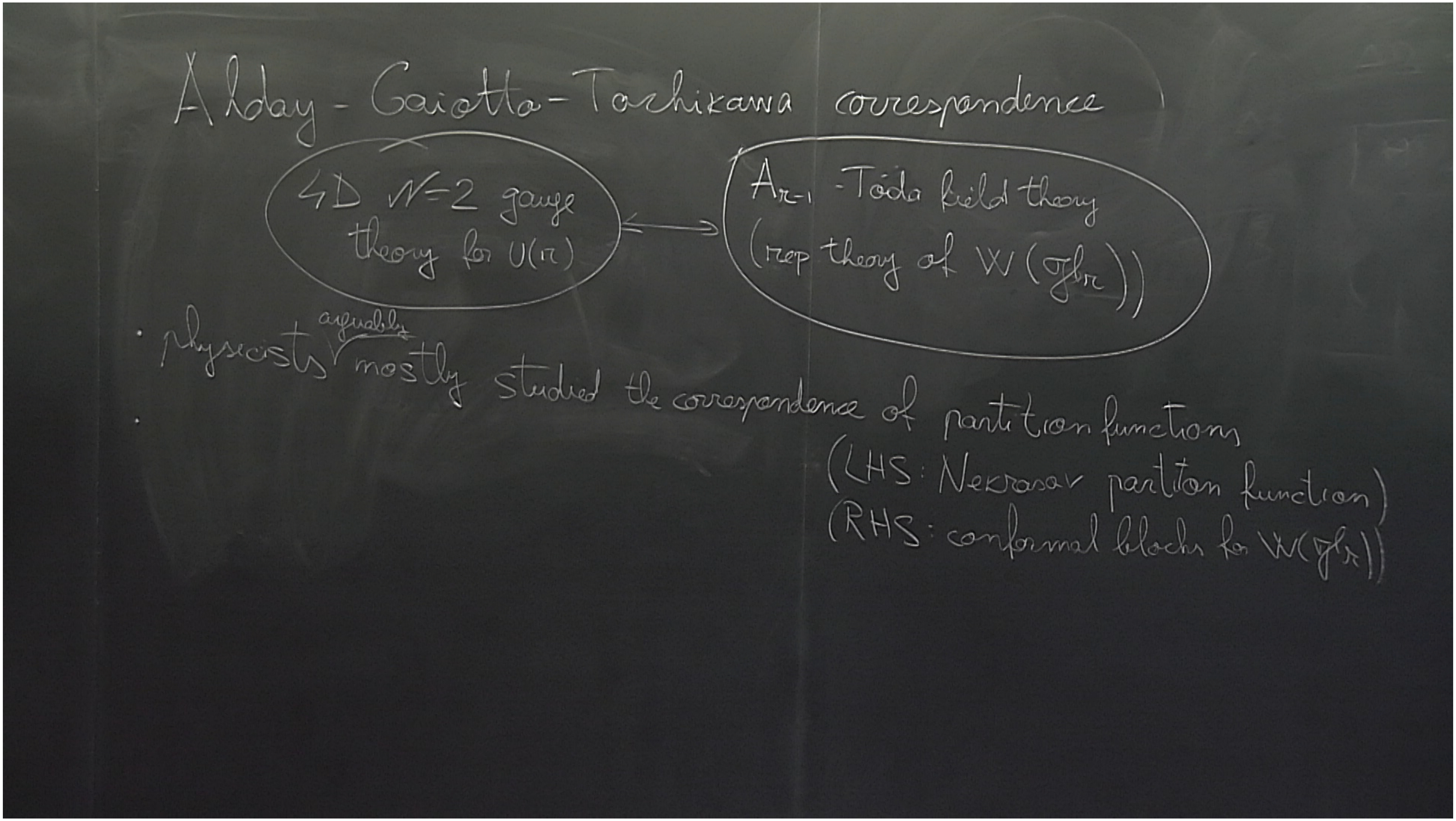
Abstract: <p>The Alday-Gaiotto-Tachikawa correspondence connects gauge theory on a fourfold with conformal field theory. We are interested in a certain algebro-geometric incarnation of this framework, where the fourfold is an algebraic surface and instantons/differential geometry are replaced with sheaves/algebraic geometry. In this talk, we will present a certain approach to AGT that yields partial results for quite general surfaces, and ask questions about what still needs to be done to state and prove the full correspondence in the language of algebraic geometry.</p>



Alday - Gaiotto - Tachikawa correspondence

$$\boxed{\begin{array}{c}\text{4D } \mathcal{N}=2 \text{ gauge}\\ \text{theory for } U(r)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}A_{r-1}\text{-Toda field theory}\\ (\text{rep theory of } W(\mathfrak{gl}_r))\end{array}}$$

- physicists arguably mostly studied the correspondence of partition functions
  (LHS: Nekrasov partition function)
  (RHS: conformal blocks for $W(\mathfrak{gl}_r)$)
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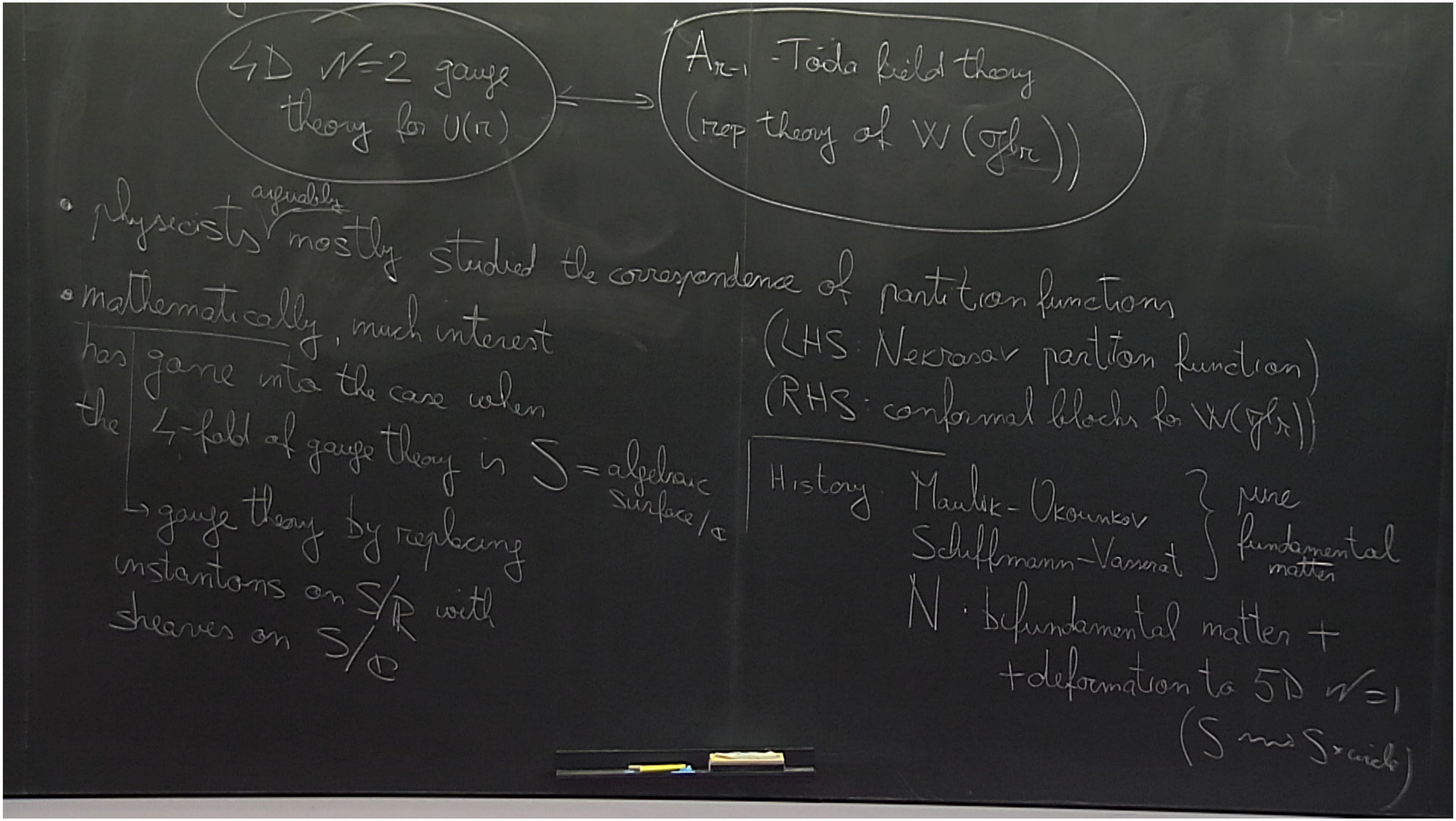
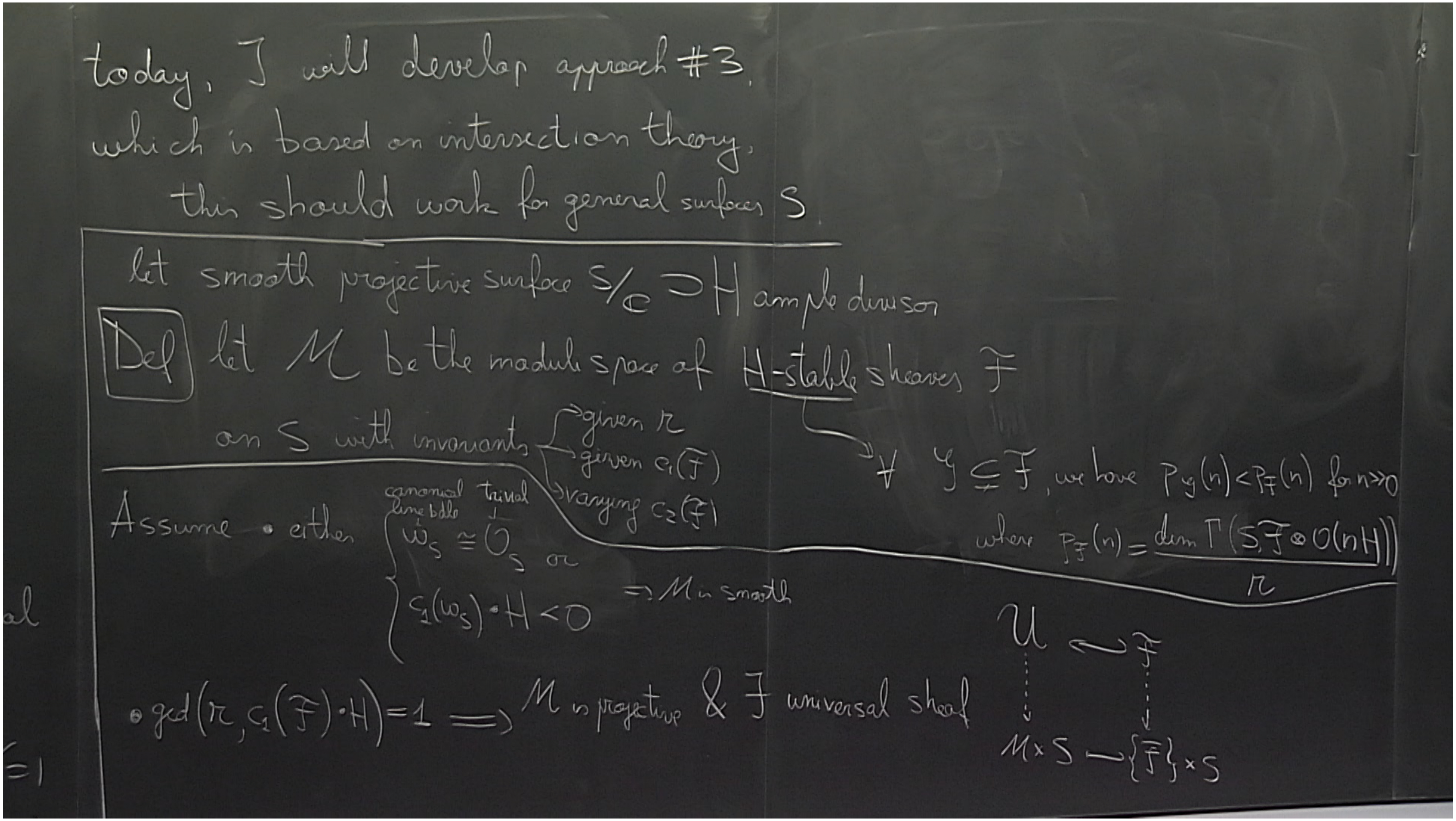


$$\boxed{\text{4D } \mathcal{N}=2 \text{ gauge theory for } U(r)} \longleftrightarrow \boxed{\begin{array}{c} A_{r-1}\text{-Toda field theory} \\ (\text{rep theory of } W(\mathfrak{gl}_r)) \end{array}}$$

- physicists ^arguably^ mostly studied the correspondence of partition functions
  (LHS: Nekrasov partition function)
  (RHS: conformal blocks for $W(\mathfrak{gl}_r)$)
- mathematically, much interest has gone into the case when the 4-fold of gauge theory is S = algebraic surface $/\mathbb{C}$
  → gauge theory by replacing instantons on $S/\mathbb{R}$ with sheaves on $S/\mathbb{C}$

History: Maulik-Okounkov } pure
Schiffmann-Vasserot } fundamental matter
N: bifundamental matter +
+ deformation to 5D $\mathcal{N}=1$
($S \rightsquigarrow S \times$ circle)



today, I will develop approach #3,
which is based on intersection theory,
this should work for general surfaces S

let smooth projective surface $S/_{\mathbb{C}} \supset H$ ample divisor

$\boxed{\text{Def}}$ let $\mathcal{M}$ be the moduli space of <u>H-stable</u> sheaves $\mathcal{F}$
on S with invariants
- → given r
- → given $c_1(\mathcal{F})$
- → varying $c_2(\mathcal{F})$

$\Rightarrow \forall\ \mathcal{G} \subsetneq \mathcal{F}$, we have $P_{\mathcal{G}}(n) < P_{\mathcal{F}}(n)$ for $n \gg 0$

where $P_{\mathcal{F}}(n) = \dfrac{\dim \Gamma(S, \mathcal{F} \otimes \mathcal{O}(nH))}{r}$

Assume • either
$$\begin{cases} \overset{\text{canonical line bdle}}{\omega_S} \cong \overset{\text{trivial}}{\mathcal{O}_S} & \text{or} \\ c_1(\omega_S) \cdot H < 0 \end{cases} \Rightarrow \mathcal{M} \text{ is smooth}$$

• $\gcd(r, c_1(\mathcal{F}) \cdot H) = 1 \Longrightarrow \mathcal{M}$ is projective & $\mathbb{F}$ universal sheaf

$$\begin{tikzcd} \mathbb{U} \arrow[d, dashed] & \mathbb{F} \arrow[l, hook'] \arrow[d, dashed] \\ \mathcal{M} \times S & \{\mathcal{F}\} \times S \arrow[l, hook'] \end{tikzcd}$$

let $\mathcal{M} = \bigsqcup_{c_2 \in \mathbb{Z}} \mathcal{M}_{c_2}$

then $K_{\mathcal{M}} = \bigoplus_{c_2 \in \mathbb{Z}} \underline{K_{alg}(\mathcal{M}_{c_2})} \otimes_{\mathbb{Z}} \mathbb{Q}$

$\|$

free abelian generated by $[\mathcal{V}]$ where $\mathcal{V}$ is a vector bundle on $\mathcal{M}_{c_2}$, modulo $[\mathcal{V}_2] = [\mathcal{V}_1] + [\mathcal{V}_3]$ if $\exists$ s.e.s.

$$0 \to \mathcal{V}_1 \to \mathcal{V}_2 \to \mathcal{V}_3 \to 0$$

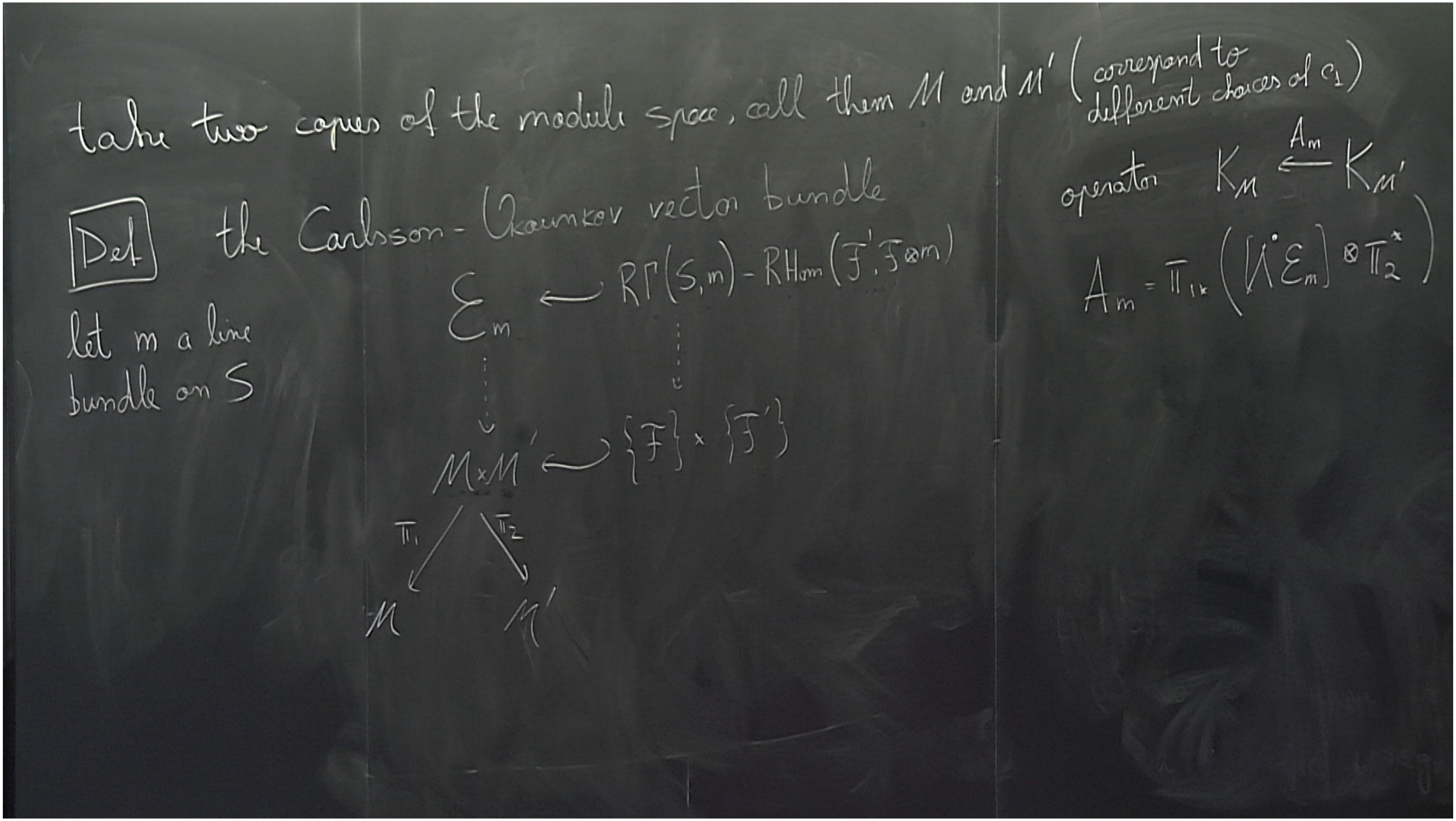
es S

ample divisor

$\underline{H\text{-stable}}$ sheaves $\mathcal{F}$

$\hookrightarrow$ $\forall$ $\mathcal{G} \subsetneq \mathcal{F}$, we have $P_{\mathcal{G}}(n) < P_{\mathcal{F}}(n)$ for $n \gg 0$

$\dim \Gamma(S \mathcal{F} \otimes \mathcal{O}(nH))$



take two copies of the moduli space, call them $\mathcal{M}$ and $\mathcal{M}'$ (correspond to different choices of c_1)

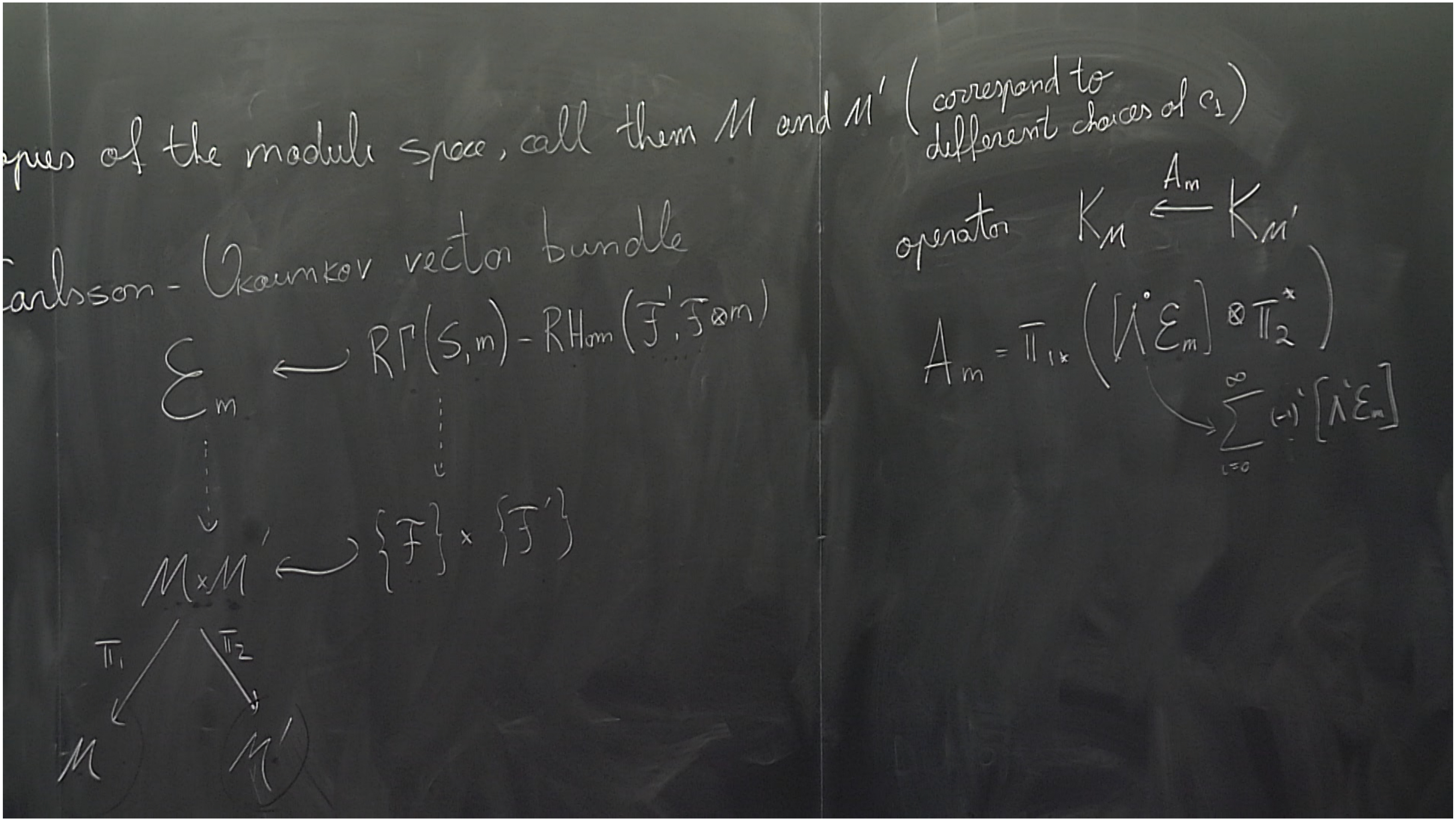
Def the Carlsson-Okounkov vector bundle

let m a line bundle on S

$$\begin{tikzcd}
\mathcal{E}_m \arrow[d, dashed] & R\Gamma(S,m) - R\mathcal{H}om(\mathcal{F}', \mathcal{F}\otimes m) \arrow[l, hook'] \arrow[d, dashed] \\
\mathcal{M}\times\mathcal{M}' \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \{\mathcal{F}\}\times\{\mathcal{F}'\} \arrow[l, hook'] \\
\mathcal{M} & \mathcal{M}'
\end{tikzcd}$$

operator $K_{\mathcal{M}} \xleftarrow{A_m} K_{\mathcal{M}'}$

$$A_m = \pi_{1*}\left([\Lambda^\bullet \mathcal{E}_m] \otimes \pi_2^* \right)$$



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arlsson - Okounkov vector bundle

$$\begin{tikzcd}
 & \mathcal{E}_m \arrow[d, dashed] & & R\Gamma(S,m) - R\mathrm{Hom}(\mathcal{F}', \mathcal{F}\otimes m) \arrow[lll, hook'] \arrow[d, dashed] \\
 & \mathcal{M}\times\mathcal{M}' \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \{\mathcal{F}\}\times\{\mathcal{F}'\} \arrow[ll, hook'] \\
\mathcal{M} & & \mathcal{M}' &
\end{tikzcd}$$

operator $K_{\mathcal{M}} \xleftarrow{A_m} K_{\mathcal{M}'}$

$$A_m = \pi_{1*}\left(\underbrace{[\Lambda^\bullet \mathcal{E}_m]}_{\sum_{i=0}^{\infty}(-1)^i[\Lambda^i \mathcal{E}_m]} \otimes \pi_2^*\right)$$

li space, call

ov vector bundle

$$\begin{tikzcd} R\Gamma(S, m) - R\mathrm{Hom}(\mathcal{F}', \mathcal{F}\otimes m) \arrow[d, dashed] \\ \{\mathcal{F}\} \times \{\mathcal{F}'\} \end{tikzcd}$$

differen

operator $K_{\mathcal{M}} \xleftarrow{A_m} K_{\mathcal{M}'}$

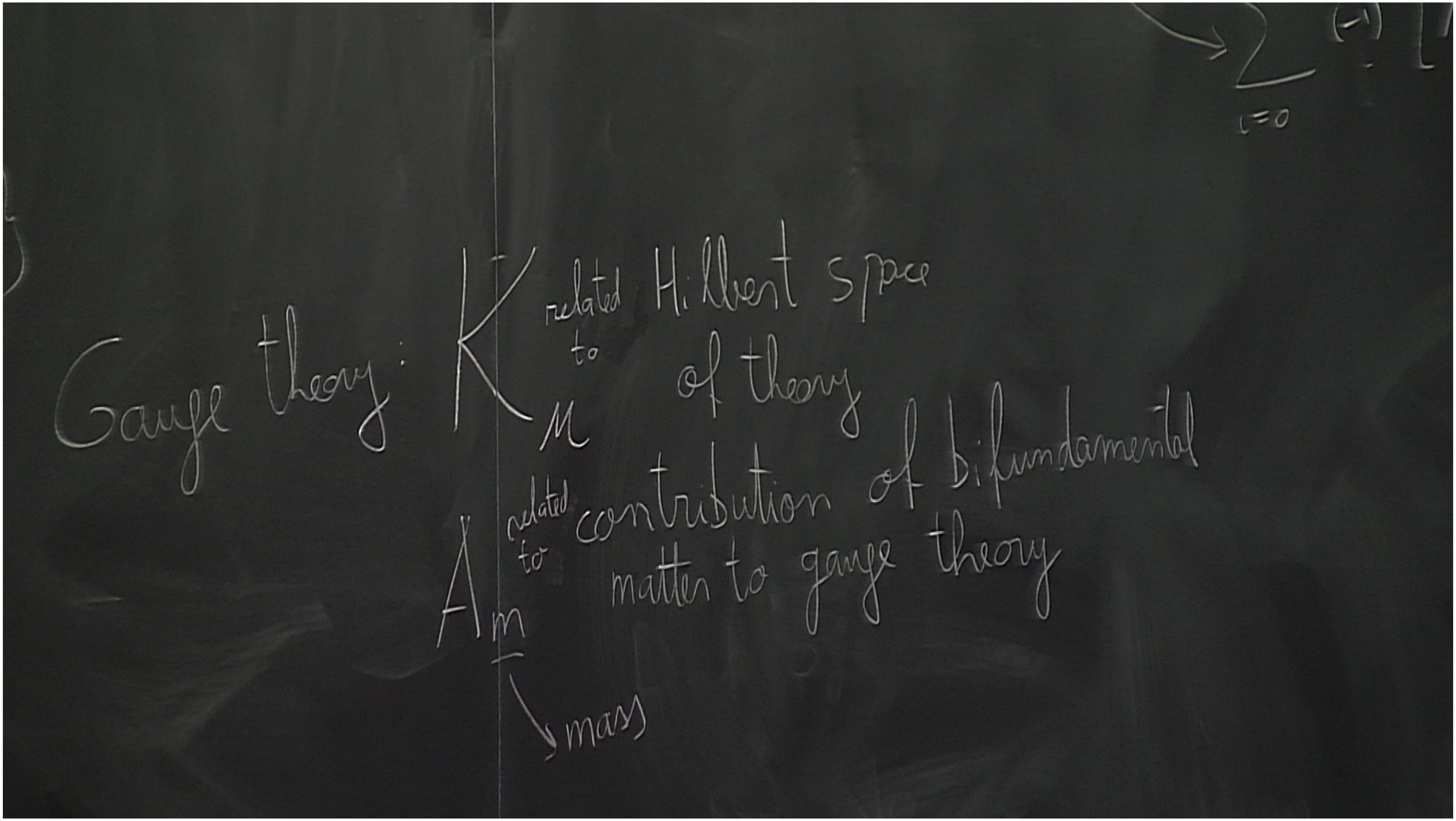
$$A_m = \pi_{1*}\left([\Lambda^{\bullet}\mathcal{E}_m] \otimes \pi_2^{*} \right)$$

$$\sum_{i=0}^{\infty} (-1)^i [\Lambda^i \mathcal{E}_m]$$

Gauge theory: $K_{\mathcal{M}}$ = Hilbert space of theory

$A_{\underline{m}}$ = contribution of bifundamental matter to gauge theory

$\searrow$ mass

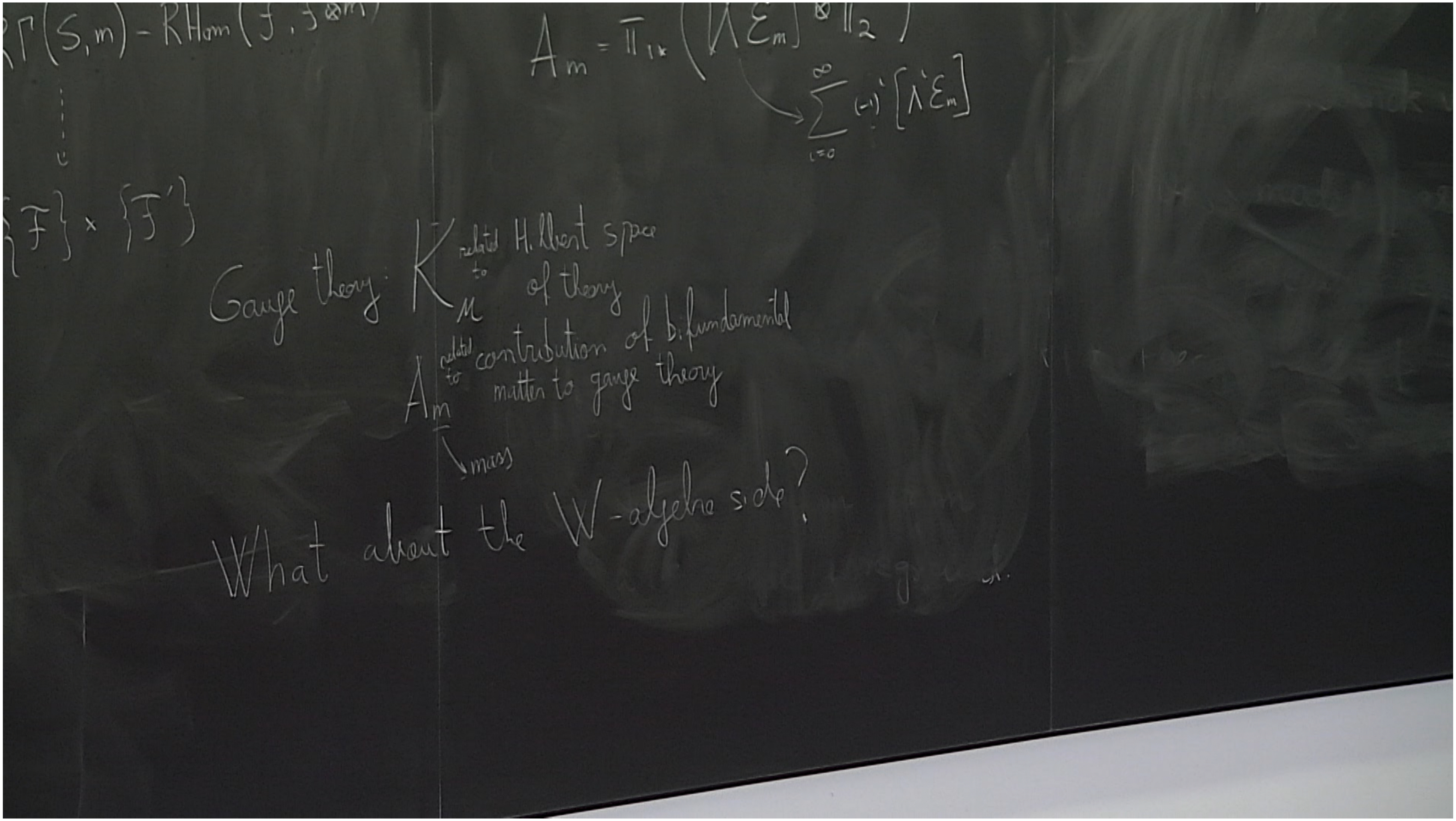


$\rightarrow \sum_{i=0} (-1)^{i} [$

Gauge theory: K_M related to Hilbert space of theory

$\check{A}_{\underline{m}}$ related to contribution of bifundamental matter to gauge theory

$\underline{m} \to$ mass



$\Gamma(S, m) - RHom(\mathcal{F}, \mathcal{F} \otimes m)$

$\{\mathcal{F}\} \times \{\mathcal{F}'\}$

$A_m = \pi_{1*}(\Lambda \mathcal{E}_m \otimes \mathbb{1}_2)$

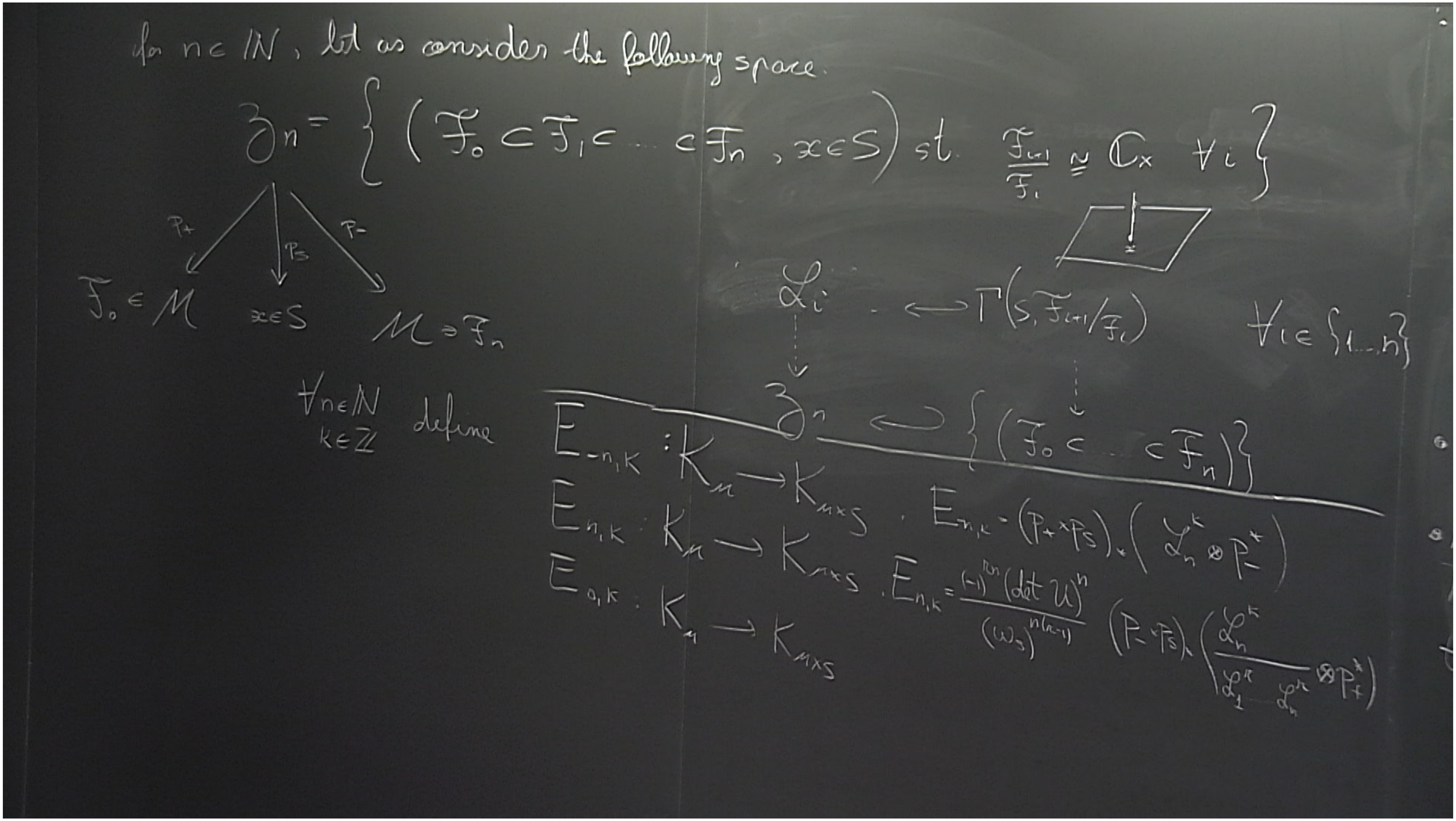
$\sum_{i=0}^{\infty} (-1)^i [\Lambda^i \mathcal{E}_m]$

Gauge theory: K_M related to Hilbert space of theory

$A_{\underline{m}}$ related to contribution of bifundamental matter to gauge theory

$\underline{m}$ → mass

What about the W-algebra side?



for $n \in \mathbb{N}$, let us consider the following space.

$$\mathfrak{Z}_n = \left\{ \left(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \, , x \in S \right) \text{ st. } \frac{\mathcal{F}_{i+1}}{\mathcal{F}_i} \cong \mathbb{C}_x \ \forall i \right\}$$

$$\begin{tikzcd} & \mathfrak{Z}_n \arrow[dl, "p_+"'] \arrow[d, "p_S"] \arrow[dr, "p_-"] & \\ \mathcal{F}_0 \in \mathcal{M} & x \in S & \mathcal{M} \ni \mathcal{F}_n \end{tikzcd}$$

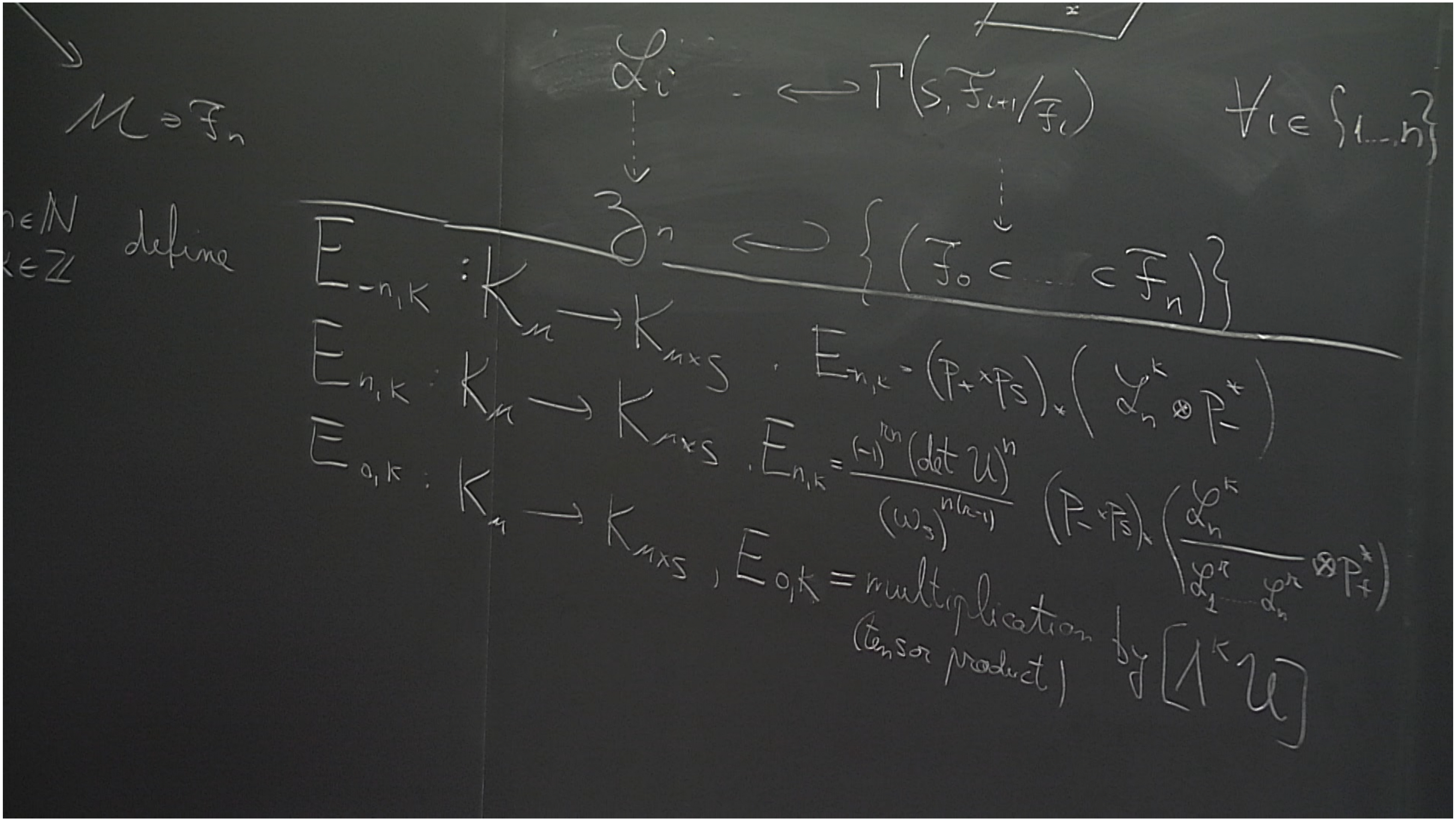
$$\begin{tikzcd} \mathcal{L}_i \arrow[r, phantom, ".\longleftarrow"] \arrow[d, dashed] & \Gamma(S, \mathcal{F}_{i+1}/\mathcal{F}_i) \arrow[d, dashed] \\ \mathfrak{Z}^n \arrow[r, hook'] & \left\{ \left(\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n \right) \right\} \end{tikzcd} \quad \forall i \in \{1, \dots, n\}$$

$\forall n \in \mathbb{N}$, $k \in \mathbb{Z}$ define

$E_{-n,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$. $E_{n,k} = (p_+ \times p_S)_* \left(\mathcal{L}_n^k \otimes p_-^* \right)$

$E_{n,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$. $E_{n,k} = \dfrac{(-1)^{kn} (\det \mathcal{U})^n}{(\omega_S)^{n(k-1)}} (p_- \times p_S)_* \left(\dfrac{\mathcal{L}_n^k}{\mathcal{L}_1^k \cdots \mathcal{L}_n^k} \otimes p_+^* \right)$

$E_{0,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$



$\mathcal{M} \ni \mathcal{F}_n$

$$\begin{tikzcd} \mathcal{L}_i \arrow[r, leftrightarrow] \arrow[d, dashed] & \Gamma(S, \mathcal{F}_{i+1}/\mathcal{F}_i) \arrow[d, dashed] \\ \mathfrak{Z}^n \arrow[r, leftrightarrow] & \{(\mathcal{F}_0 \subset \dots \subset \mathcal{F}_n)\} \end{tikzcd}$$

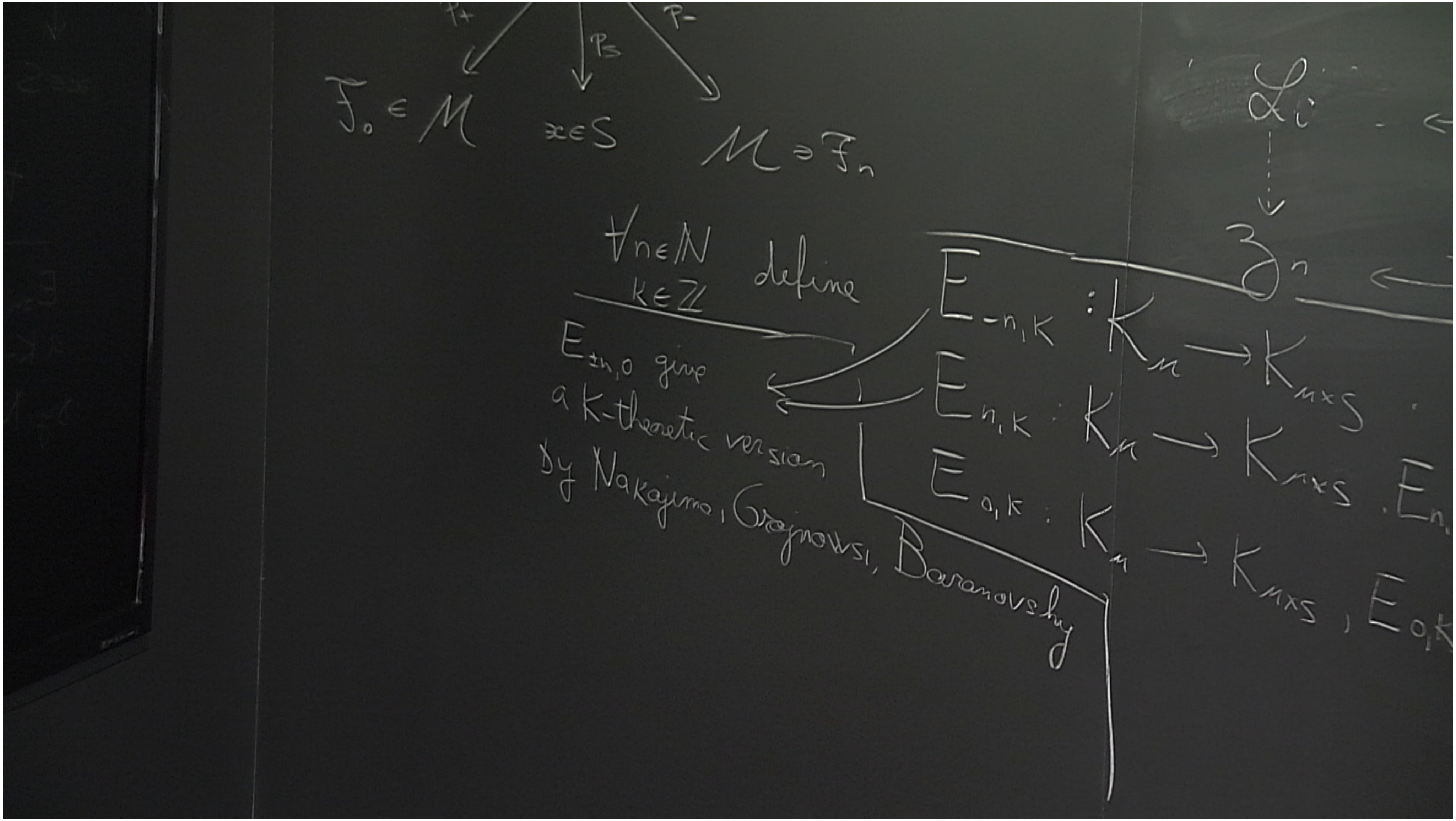
$\forall i \in \{1, \dots, n\}$

$n \in \mathbb{N}$, $k \in \mathbb{Z}$ define

$E_{-n,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$, $E_{-n,k} = (p_+ \times p_S)_*\left(\mathcal{L}_n^k \otimes p_-^*\right)$

$E_{n,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$, $E_{n,k} = \dfrac{(-1)^{kn} (\det \mathcal{U})^n}{(\omega_S)^{n(k-1)}} (p_- \times p_S)_*\left(\dfrac{\mathcal{L}_n^k}{\mathcal{L}_1^k \cdots \mathcal{L}_n^k} \otimes p_+^*\right)$

$E_{0,k} : K_{\mathcal{M}} \longrightarrow K_{\mathcal{M} \times S}$, $E_{0,k} =$ multiplication by $[\wedge^k \mathcal{U}]$
(tensor product)



p_+ p_S p_-

$\mathcal{F}_0 \in \mathcal{M}$ $x \in S$ $\mathcal{M} \ni \mathcal{F}_n$

$\mathcal{L}_i$

$\mathfrak{z}^n$ $\Leftarrow$

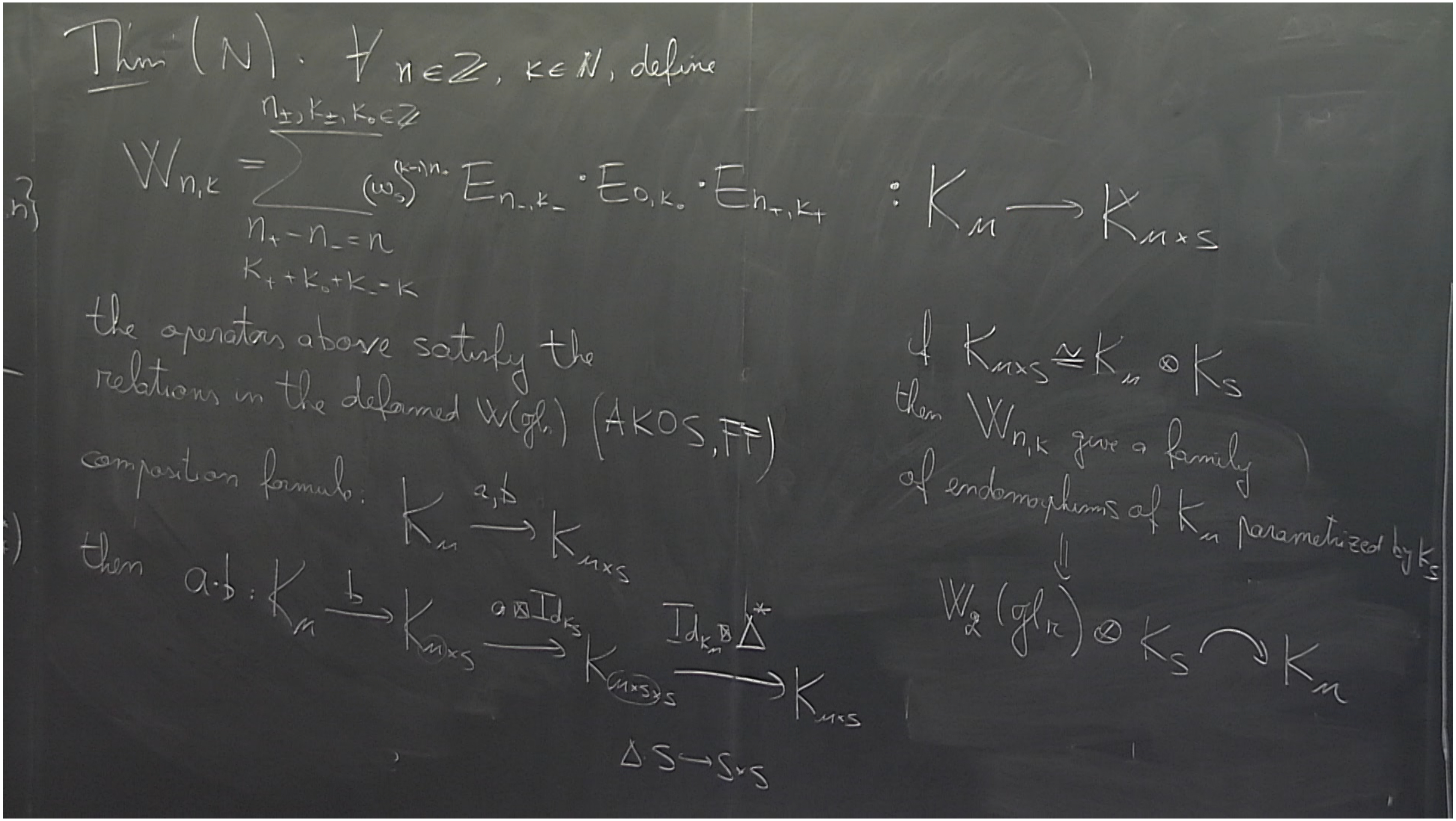
$\forall n \in \mathbb{N}$, $k \in \mathbb{Z}$ define

$E_{-n,k} : K_{\mathcal{M}} \to K_{\mathcal{M}\times S}$

$E_{n,k} : K_{\mathcal{M}} \to K_{\mathcal{M}\times S}$, $E_{n,}$

$E_{0,k} : K_{\mathcal{M}} \to K_{\mathcal{M}\times S}$, $E_{0,k}$

$E_{\pm n,0}$ give a K-theoretic version by Nakajima, Grojnowsi, Baranovsky



Thm (N). $\forall\, n\in\mathbb{Z},\ k\in\mathbb{N}$, define

$$W_{n,k} = \sum_{\substack{n_\pm, k_\pm, k_0\in\mathbb{Z} \\ n_+-n_-=n \\ k_++k_0+k_-=k}} (\omega_s)^{(k-1)n_+} E_{n_-,k_-}\cdot E_{0,k_0}\cdot E_{n_+,k_+} : K_M \longrightarrow K_{M\times S}$$

the operators above satisfy the relations in the deformed $W(gl_r)$ (AKOS, FT)

composition formula: $K_M \xrightarrow{a,b} K_{M\times S}$

then $a\cdot b : K_M \xrightarrow{b} K_{M\times S} \xrightarrow{a\boxtimes Id_{K_S}} K_{M\times S\times S} \xrightarrow{Id_{K_M}\boxtimes\Delta^*} K_{M\times S}$

$\Delta\, S\to S\times S$

if $K_{M\times S}\cong K_M\otimes K_S$

then $W_{n,k}$ give a family of endomorphisms of K_M parametrized by K_S

$\Downarrow$

$W_2(gl_r)\otimes K_S \curvearrowright K_M$

$$W_k(x) = \sum_{n\in\mathbb{Z}} \frac{W_{n,k}}{x^n}, \ \forall k \in \{1,\dots,r\}$$

$\boxed{\text{Thm}}$ (N) operator A_m satisfies the following relation.

$$\frac{m^r}{(\omega_S)^r}\cdot\frac{\overset{\in K_{M\times S}}{\det U}}{\underset{\in K_{M'\times S}}{\det U'}}$$

$$\circledast \quad A_m \cdot W_k(x)\cdot(1-x) = m^k\, W_k(x\gamma)\cdot A_m\left(1-\frac{x}{(\omega_S)^k}\right) \cdot K_{M'} \to K_{M\times S}$$

$\Downarrow$

A_m has a "vertex operator" property

when $S=\mathbb{A}^2$, (*) determines A_m completely

zed by K_S

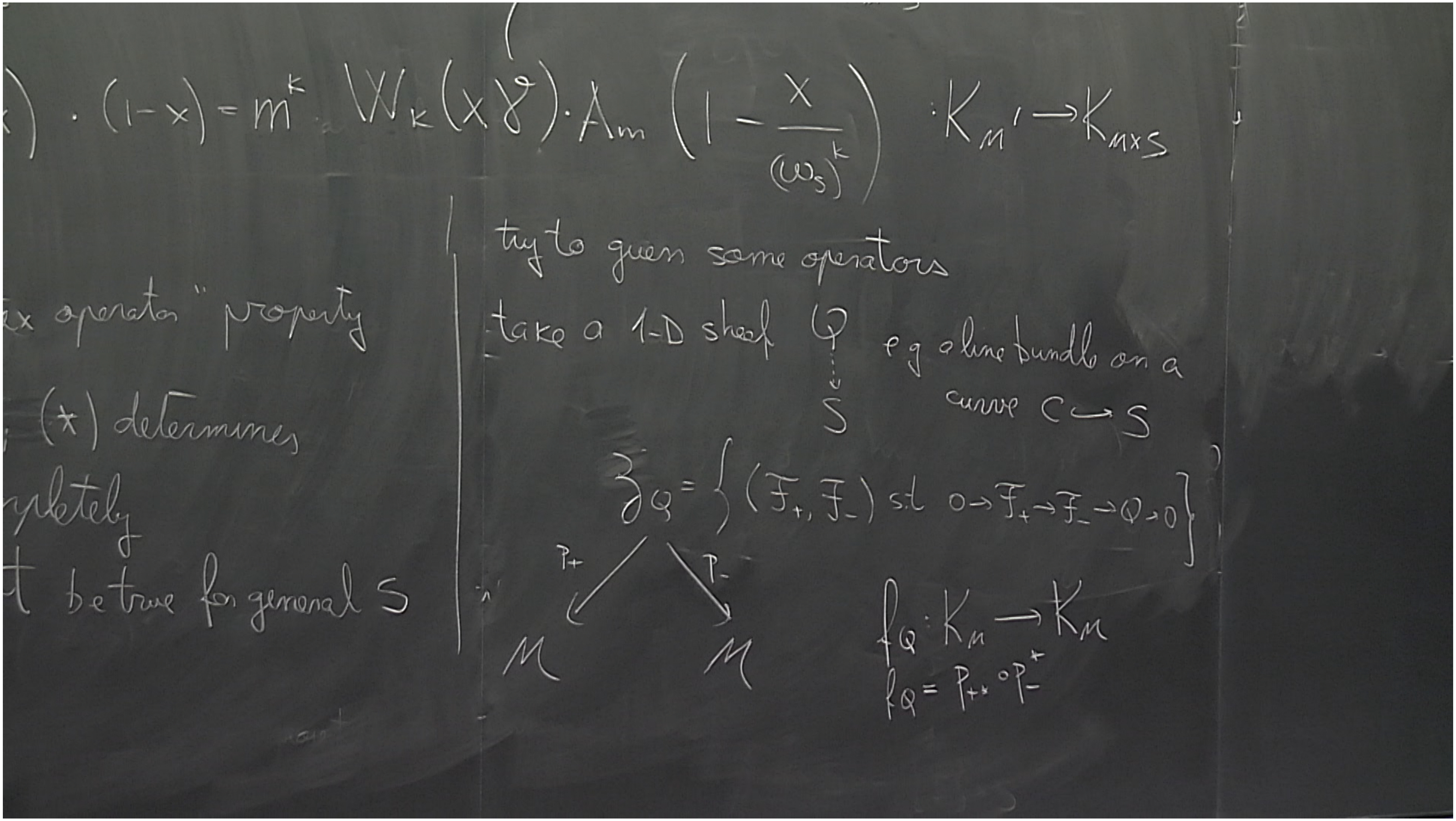
this will not be true for general S

try to guess some operators

take a 1-D sheaf Q e.g. a line bundle on a curve $C \hookrightarrow S$

$$Q \dashrightarrow S$$

$$\mathfrak{Z}_Q = \{(\mathcal{F}_+,\mathcal{F}_-) \text{ s.t. } 0\to\mathcal{F}_+\to\mathcal{F}_-\to Q\to 0\}$$



$$) \cdot (1-x) = m^k \cdot W_k(x\gamma) \cdot A_m\left(1 - \frac{x}{(\omega_S)^k}\right) : K_{M'} \longrightarrow K_{M\times S}$$

ax operator" property

$(\star)$ determines

pletely

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take a 1-D sheaf Q e.g. a line bundle on a curve $C \hookrightarrow S$

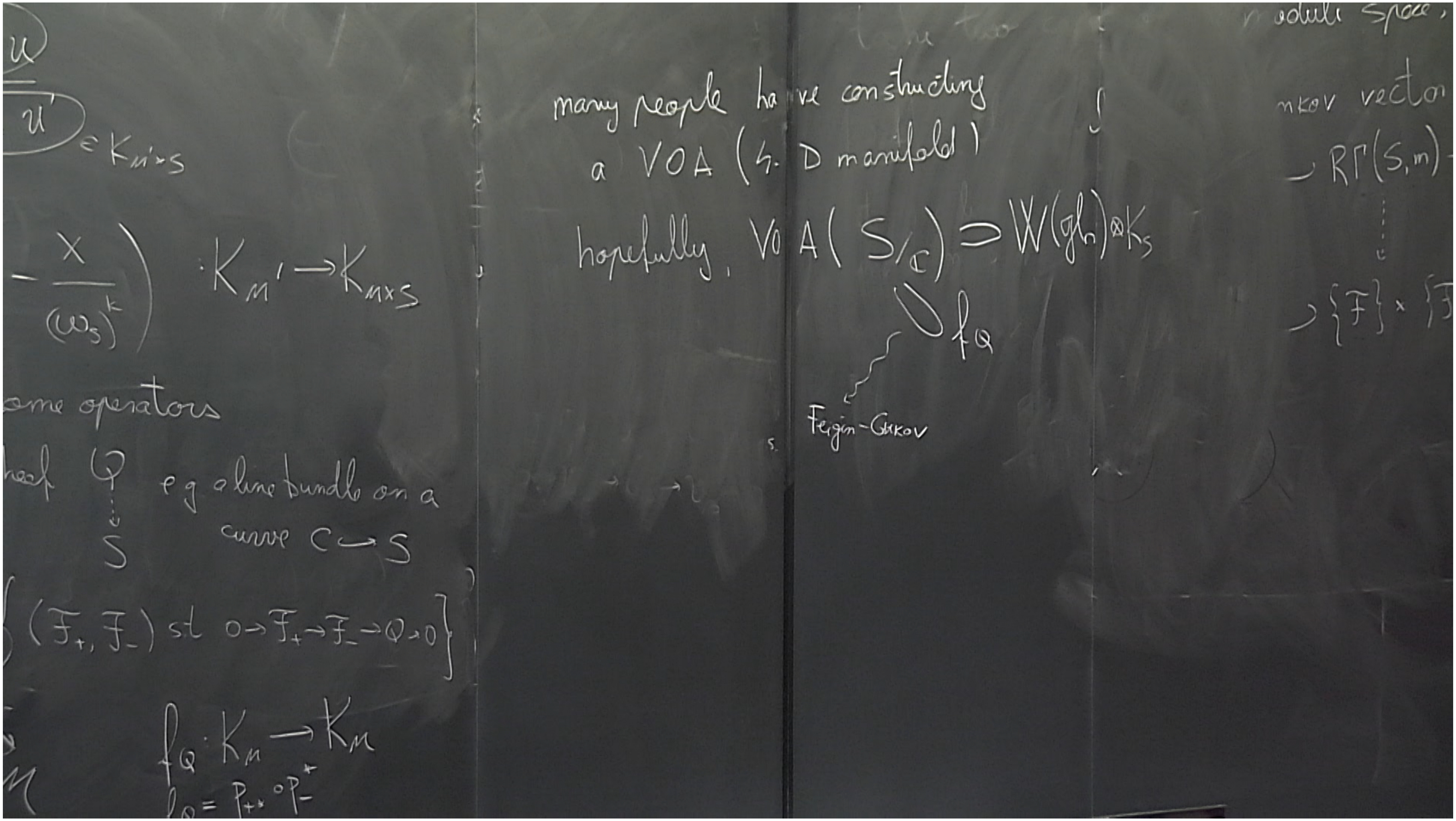
$$\begin{tikzcd} Q \arrow[d, dotted] \\ S \end{tikzcd}$$

$$\mathfrak{Z}_Q = \left\{ (\mathcal{F}_+, \mathcal{F}_-) \text{ s.t. } 0 \to \mathcal{F}_+ \to \mathcal{F}_- \to Q \to 0 \right\}$$

$$\begin{tikzcd} & \mathfrak{Z}_Q \arrow[dl, "p_+"'] \arrow[dr, "p_-"] & \\ M & & M \end{tikzcd}$$

$$f_Q : K_M \longrightarrow K_M$$

$$f_Q = p_{+*} \circ p_-^*$$



$\frac{u}{u'} \in K_{M' \times S}$

$-\frac{x}{(\omega_S)^k}\Big) : K_{M'} \longrightarrow K_{M \times S}$

ome operators

heaf Q e g a line bundle on a

$\vdots$ curve $C \hookrightarrow S$

S

$\{(\mathcal{F}_+, \mathcal{F}_-)$ s.t. $0 \to \mathcal{F}_+ \to \mathcal{F}_- \to Q \to 0\}$

M

$f_Q : K_M \longrightarrow K_M$

$f_Q = p_{+*} \circ p_-^*$

many people ha ve constructing

a VOA (4. D manifold)

hopefully, VOA$(S/\mathbb{C}) \supset W(gl_n) \otimes K_S$

$\cup\ f_Q$

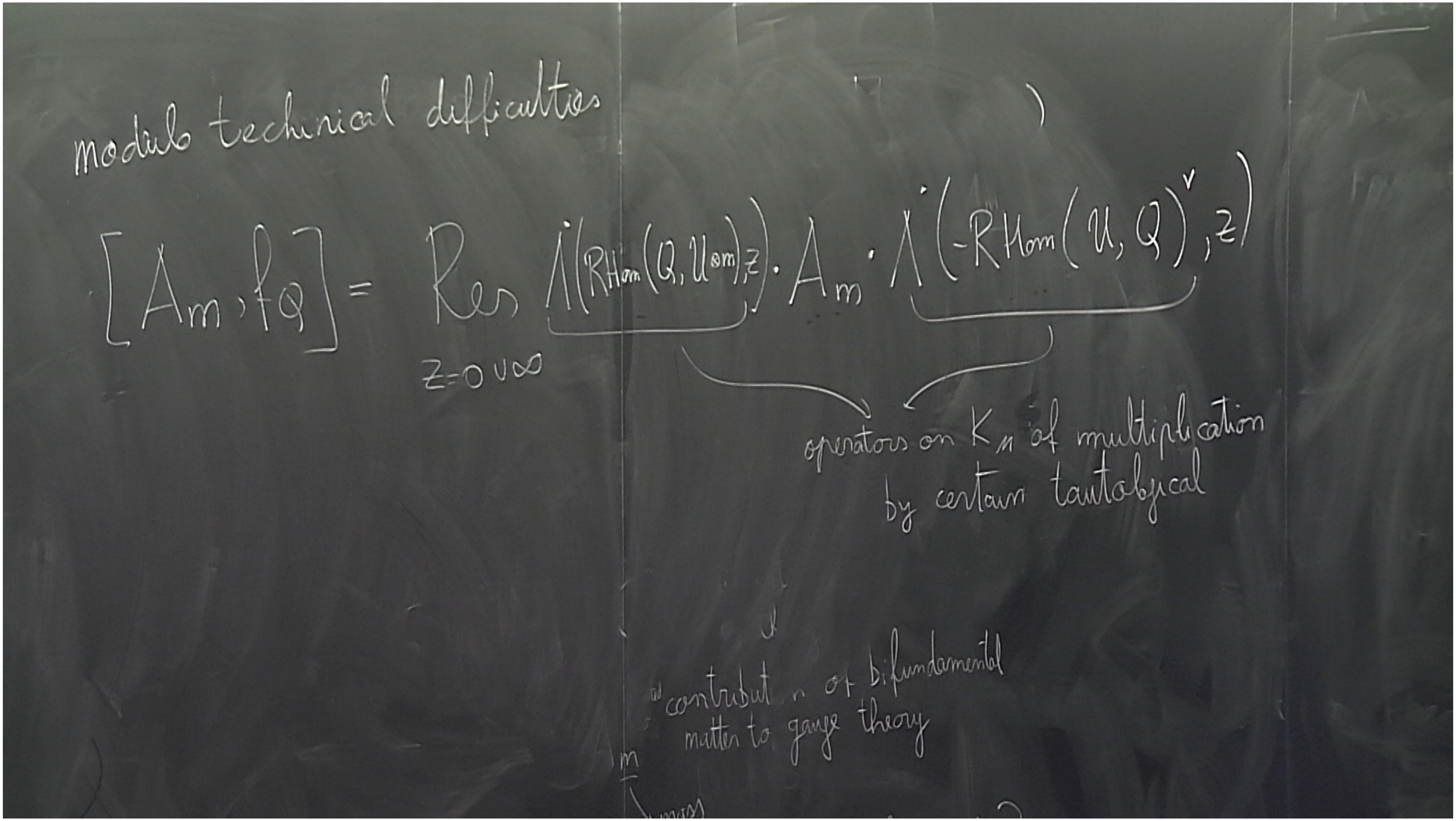
Feigin-Gukov

oduli space,

nkov vector

$RΓ(S, m)$

$\{\mathcal{F}\} \times \{\mathcal{F}$



modulo techinical difficulties

$$[A_m, f_Q] = \operatorname*{Res}_{z=0\cup\infty} \underbrace{\Lambda^\bullet\left(R\mathrm{Hom}(Q, \mathcal{U}\otimes m), z\right)} \cdot A_m \cdot \underbrace{\Lambda^\bullet\left(-R\mathrm{Hom}(\mathcal{U}, Q)^\vee, z\right)}$$

operators on K_n of multiplication by certain tautological

contribution of bifundamental matter to gauge theory

$\underline{m}$

mass