Title: Entanglement Content of Particle Excitations

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Abstract: <p>In this talk I will review the results of recent work in collaboration with Cecilia De Fazio, Benjamin Doyon and IstvÃ¡n M. SzÃ©csÃ©nyi. We studied the entanglement of excited states consisting of a finite number of particle excitations. More precisely, we studied the difference between the entanglement entropy of such states and that of the ground state in a simple bi-partition of a quantum system, where both the size of the system and of the bi-partition are infinite, but their ratio is finite. We originally studied this problem in massive 1+1 dimensional QFTs where analytic computations were possible. We have found the results to apply more widely, including to higher dimensional free theories. In all cases we find that the increment of entanglement is a simple function of the ratio between region's and system's size only. Such function, turns out to be exactly the entanglement of a qubit state where the coefficients of the state are simply associated with the probabilities of particles being localised in one or the other part of the bi-partition. In this talk I will describe the results in some detail and discuss their domain of applicability. I will also highlight the main QFT techniques that we have used in order to obtain them analytically and present some numerical data. </p>

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Entanglement Content of Particle Excitations

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City, University of London

Perimeter Institute for Theoretical Physics
Waterloo, 18 September 2018

Background:

This talk is based on two papers:

OC-A, Cecilia De Fazio, Benjamin Doyon and István M. Szécsényi, *Entanglement Content of Quasi-Particle Excitations*, 1805.04948

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OC-A, Cecilia De Fazio, Benjamin Doyon and István M. Szécsényi, *Entanglement Content of Quantum Particle Excitations I. Free Field Theory*, 1806.03247

I would like to start by thanking my collaborators:

Benjamin

Cecilia

István

1. Entanglement Entropies

Consider the scaling limit (QFT) of a gapped periodic quantum chain of length L at $T = 0$ (e.g. a massive QFT) subdivided into two regions of length ℓ and $L - \ell$

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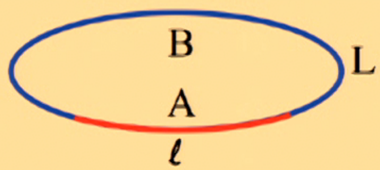

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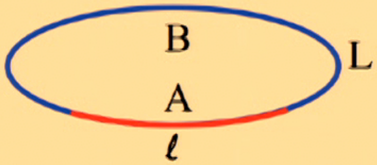

nth Rényi Entropy & Reduced Density Matrix

$$S_n^\Psi(\ell, L) = \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n} \quad \text{with} \quad \rho_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$

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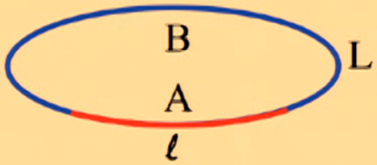

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Here $|\Psi\rangle$ will be an excited state of the QFT. Also [Bennett et al.'96; Eisert & Cramer'05; Peschel & Zhao'05]

Von Neumann & Single Copy Entropies

$$S_1^\Psi(\ell, L) := \lim_{n\to 1} S_n^\Psi(\ell, L), \quad S_\infty^\Psi(\ell, L) := \lim_{n\to\infty} S_n^\Psi(\ell, L)$$

2. Nature of the Excited States

- In our work we have considered zero particle-density states: finite number of single-particle excitations.
- In massive QFT we can parametrize such states by their rapidities and other quantum numbers: $|\theta_1, \theta_2, \ldots, \theta_k\rangle_{\mu_1\mu_2\ldots\mu_k}$

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- For free Bosons, we may have states involving repeated rapidities whereas such states are not allowed for free Fermions.
- Other extensive studies of excited states include low-lying excitations in CFT [Alcaraz, Berganza & Sierra'11'12], highly excited states of the critical XY and XXZ chains [Alba, Fagotti & Calabrese'09], and the gapped XXZ chain [Mölter, Barthel, Schollwöck & Alba'14]

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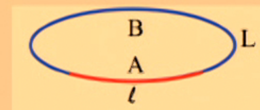
- We consider the scaling limit in which

Scaling Limit

$$\ell, L \to \infty \quad \text{while} \quad r = \frac{\ell}{L} \quad \text{fixed}$$



- In this limit we found analytically in free massive QFT:

$$\lim_{L\to\infty} \Delta S_n^{\Psi}(rL, L) = \lim_{L\to\infty} [S_n^{\Psi}(rL, L) - S_n^{0}(rL, L)]$$

$$:= \Delta S_n^{\Psi}(r), \qquad 0 \leq r \leq 1$$

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$$\Delta S_n^1(r) = \frac{\log(r^n + (1-r)^n)}{1-n}, \quad \Delta S_1^1(r) = -r\log r - (1-r)\log(1-r)$$

$$\Delta S_\infty^1(r) = \begin{cases} -\log(1-r) & \text{for} \quad 0 \leq r < \frac{1}{2} \\ -\log r & \text{for} \quad \frac{1}{2} \leq r \leq 1 \end{cases}$$

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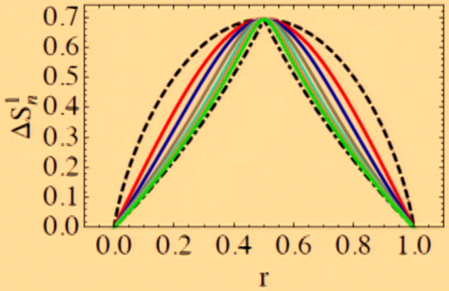
All functions are maximal at $\Delta S_n^1(1/2) = \log 2$ [Pizorn'12; Mölter, Barthel, Schollwöck and Alba'14]



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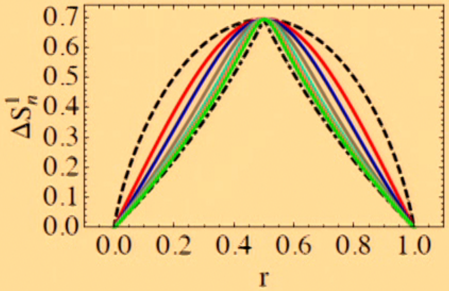

These are exactly the entanglement entropies of the state:

$$|\Psi_{\rm qb}\rangle = \sqrt{r}|1\rangle\otimes|0\rangle + \sqrt{1-r}|0\rangle\otimes|1\rangle$$

5. Many-Particle Excitations of Distinct Momenta

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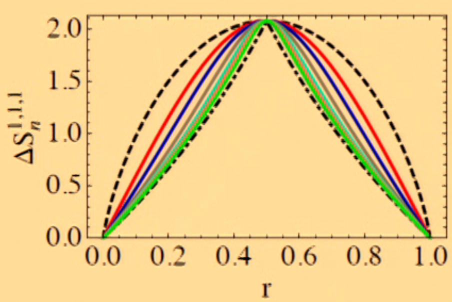

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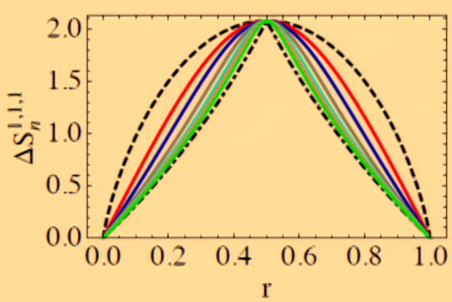

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$$|\Psi_{\rm qb}\rangle = \sqrt{r^3}|1\rangle^{\otimes 3}\otimes|0\rangle^{\otimes 3} + \sqrt{(1-r)^3}|0\rangle^{\otimes 3}\otimes|1\rangle^{\otimes 3} +$$
$$\sqrt{r^2(1-r)}(|1\rangle\otimes|1\rangle\otimes|0\rangle + \text{perm.})\otimes(|1\rangle\otimes|0\rangle\otimes|0\rangle + \text{perm.})$$
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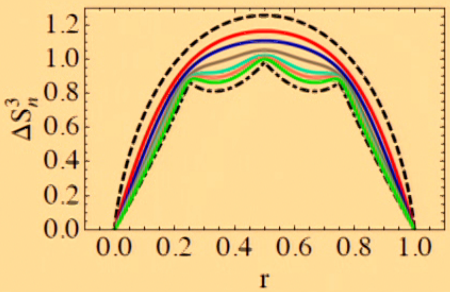

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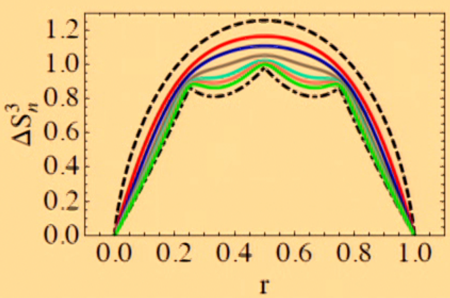

These are the entropies of the state:

$$|\Psi_{\rm qb}\rangle = \sqrt{r^3}|3\rangle \otimes |0\rangle + \sqrt{3r^2(1-r)}|2\rangle \otimes |1\rangle + \sqrt{3r(1-r)^2}|1\rangle \otimes |2\rangle + \sqrt{(1-r)^3}|0\rangle \otimes |3\rangle$$

7. Generic States

Let $\Delta S_n^{k_1,k_2,\cdots}(r)$ be the increment of Rényi entropies for a $\sum_i k_i$-particle state where k_i particles have momentum p_i for each i and $p_i \neq p_j$ for $i \neq j$.

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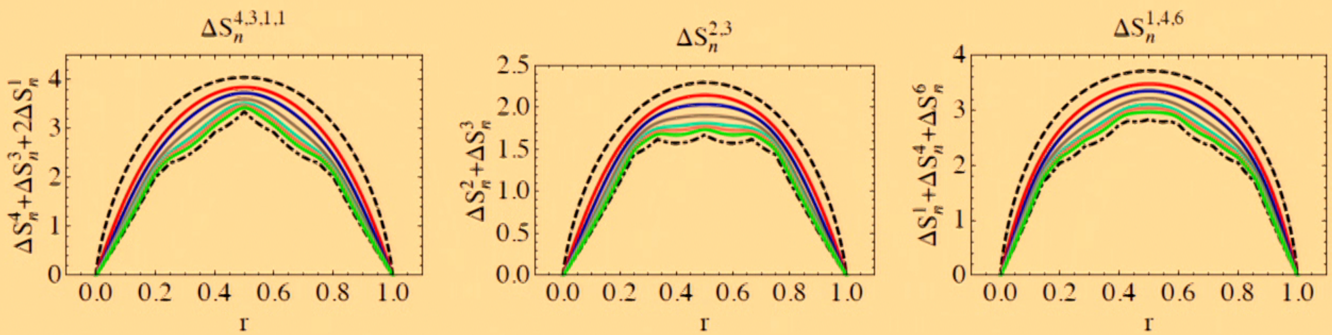

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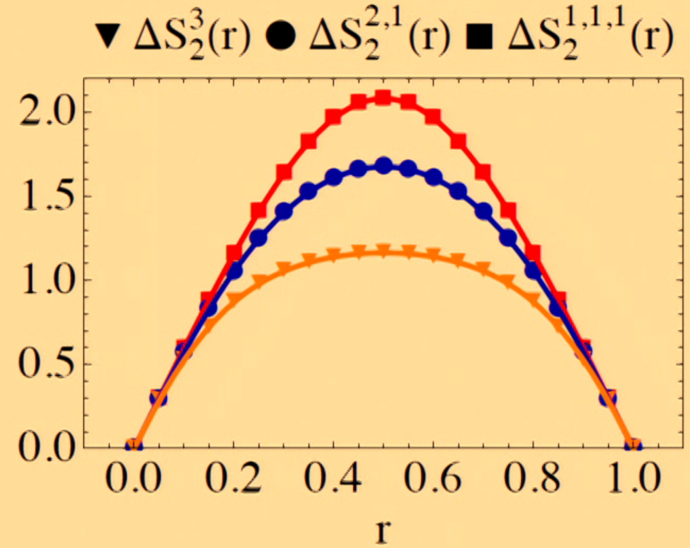
- In the figure $\Delta x = 0.01$, $L = 10$, $m = 1$ and p_1, p_2, p_3 are between 10 and 50, so this is roughly the QFT regime.



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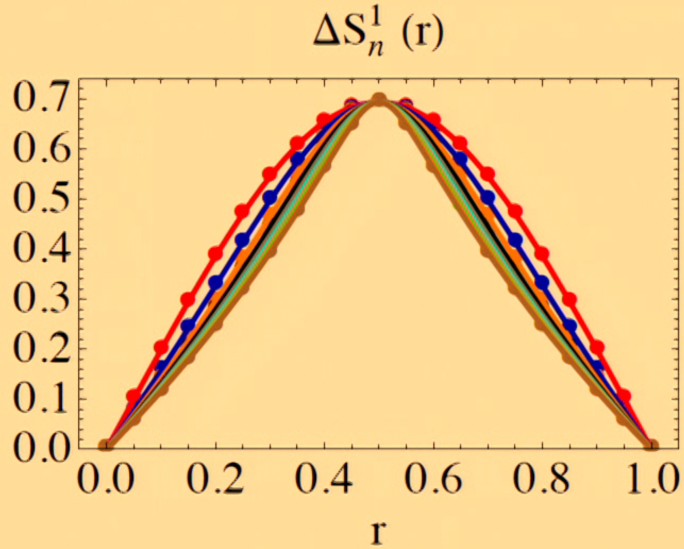
- In the figure the momentum is $p = 314$, $\Delta x = 0.01$ and $L = 5$, $m = 1$. This is a regime where $\frac{2\pi}{p} \sim \Delta x$ but the agreement is still perfect.



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- In general

Localised Quasiparticle Interpretation

$$\min(\xi, \frac{2\pi}{|p_i|}) \ll \min(\ell, L - \ell)$$

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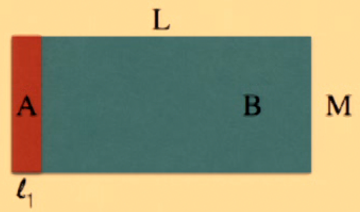

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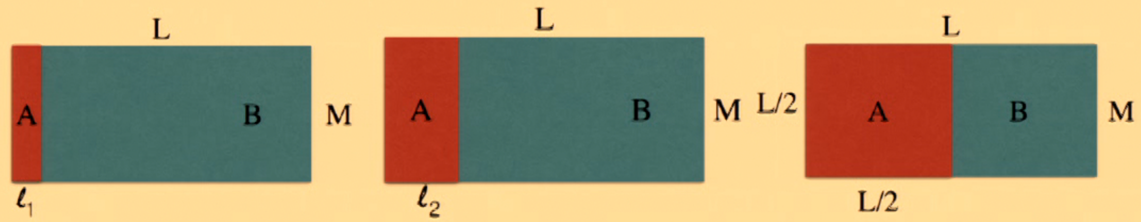

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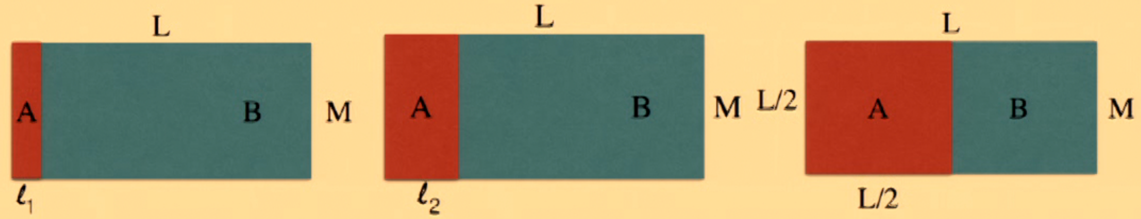

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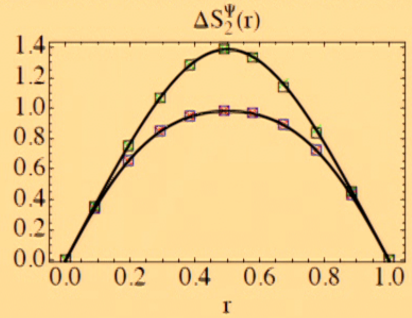

- The figure shows the increment of 2nd Rényi entropy in the toric lattice $[0; L]^2$ with $L = 50$ and lattice spacing $\Delta x = 1$.

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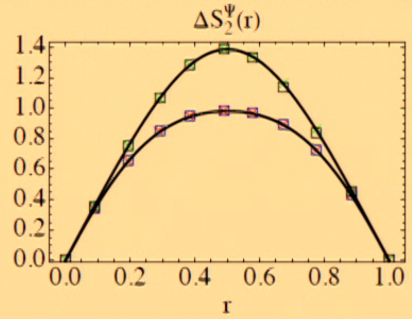

- The figure shows the increment of 2nd Rényi entropy in the toric lattice $[0; L]^2$ with $L = 50$ and lattice spacing $\Delta x = 1$. Squares are for mass $m = 1$ and small momenta, crosses are for mass $m = 0.001$ and large momenta.

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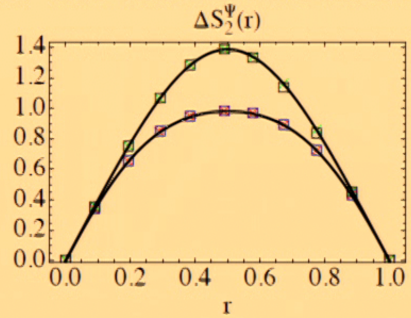
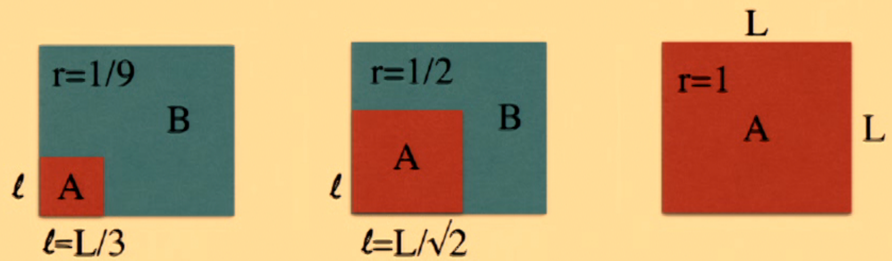


- The figure shows the increment of 2nd Rényi entropy in the toric lattice $[0;L]^2$ with $L = 50$ and lattice spacing $\Delta x = 1$. Squares are for mass $m = 1$ and small momenta, crosses are for mass $m = 0.001$ and large momenta. The region grows as



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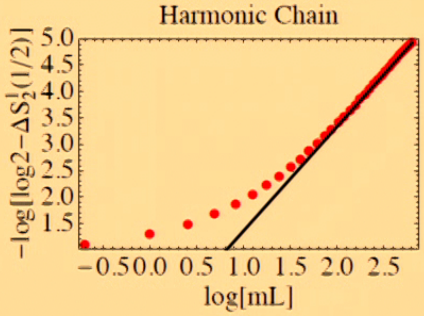
- We have investigated the large-volume corrections to our formulae in the harmonic chain and the harmonic lattice:



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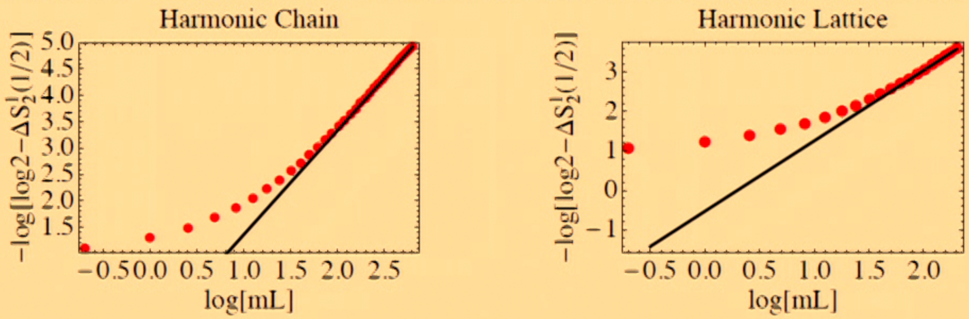

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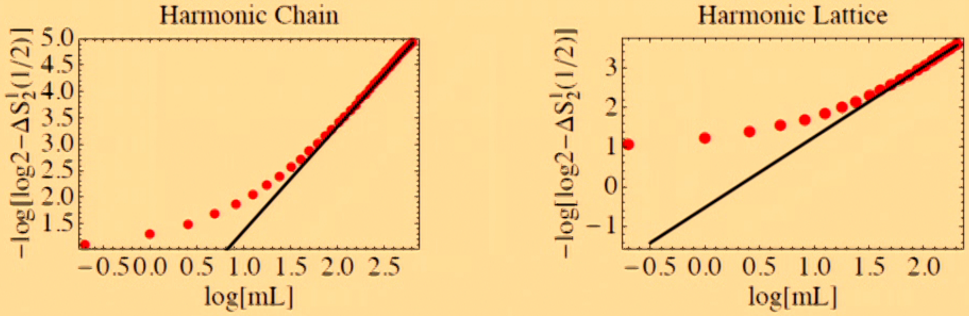

- Corrections to saturation ($\Delta S_2^1(1/2) = \log 2$) in one and two dimensions. In both cases we have power-law corrections of the form β/L^α.
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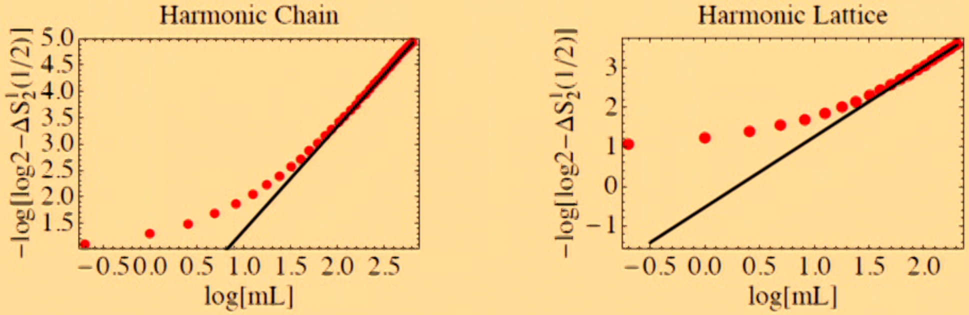

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- For the harmonic chain the fit is $-0.613 + 1.974 \log(mL)$, which suggests (negative) leading corrections of order $1/L^2$.
- For the harmonic lattice the fit is $-0.527 + 1.783 \log(mL)$, which suggests power-law behaviour with a non-integer power (more investigation is needed).

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$$\Delta S_n^{\Psi}(r) = \lim_{L\to\infty} \frac{1}{1-n} \log \left[\frac{{}_L\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(rL)|\Psi\rangle_L}{{}_L\langle 0|\mathcal{T}(0)\tilde{\mathcal{T}}(rL)|0\rangle_L} \right]$$

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- 2) The finite volume form factor approach [Pozsgay & Takacs'08] to finite volume correlators.
- Finite volume is required because in infinite volume the correlator ${}_L\langle \Psi | \mathcal{T}(0)\tilde{\mathcal{T}}(rL)|\Psi\rangle_L$ has singularities which are regularized in finite volume.

13. Branch Point Twist Fields in Finite Volume

- The branch point twist field is defined by

$$\begin{aligned} \mathcal{T}(x)\mathcal{O}_i(y) &= \mathcal{O}_{i+1}(y)\mathcal{T}(x) \quad \text{for} \quad y^1 > x^1 \\ &= \mathcal{O}_i(y)\mathcal{T}(x) \quad \text{for} \quad x^1 > y^1 \end{aligned}$$

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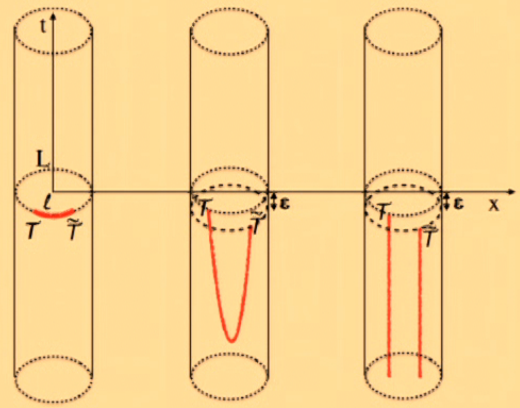
Twist fields sit at the origin of branch cuts propagating towards the right. Problem in finite volume... But there is a solution!

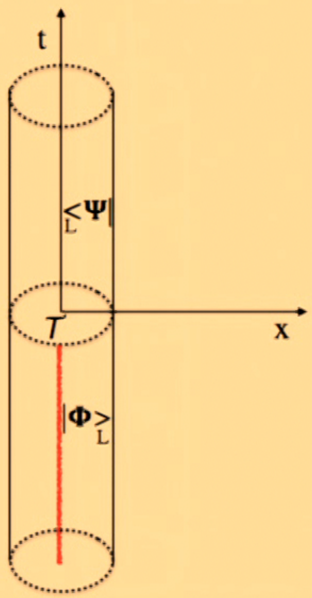


14. Form Factor Expansion in Finite Volume

- As usual, we may compute the two-point function on a excited state by introducing a sum over a complete set of intermediate states. Schematically

$$_L\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\Psi\rangle_L = \sum_{\Phi} e^{-iP_\Phi \ell}|_L\langle\Psi|\mathcal{T}(0)|\Phi\rangle_L|^2$$

- Each matrix element corresponds to the picture



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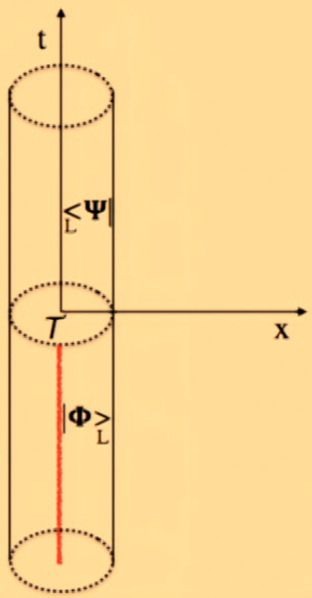

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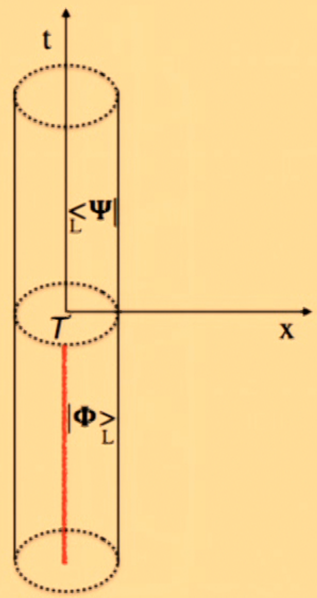

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The quantization condition is affected by the branch cut in a non-trivial way!

15. Diagonalization of Twist Field Action

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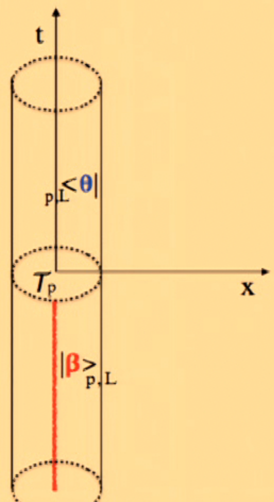

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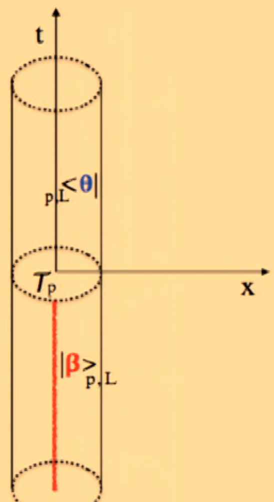

States are characterized by their rapidities θ, β. These are quantized according to the Bethe-Yang equations:

$$m \sinh\theta = 2\pi I \quad m\sinh\beta = 2\pi J \pm \frac{2\pi p}{n}$$

for $I, J \in \mathbb{Z}$

17. Entanglement Entropies

- Putting all these ideas together, as well as a prescription for finite volume for factors, it can be shown that the function ${}_L\langle\Psi|\mathcal{T}(0)\tilde{\mathcal{T}}(\ell)|\Psi\rangle_L$ has a natural $1/L$ expansion.
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$C_n(\{N^\pm\})$ are coefficients that characterize the state and

$$g^n_p(r) = \frac{\sin^2\frac{\pi p}{n}}{\pi^2}\sum_{J\in\mathbb{Z}} \frac{e^{2\pi i r(J-\frac{p}{n})}}{(J-\frac{p}{n})^2} = 1-(1-e^{-\frac{2\pi i p}{n}})r\,.$$

- These are functions that arise naturally when considering the behaviour of form factors near kinematic poles.

18. Summary of Results

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- Some of the formulae had been shown to work in the semi-classical approximation in numerical work [Mölter, Barthel, Schollwöck & Alba'14]. Ours is the first analytical derivation of such results in QFT.
- Our derivation incorporates the quantum effect of indistinguishability of particles and shows how it plays a role in entanglement results.

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where

$$S_{j_1 j_2} = (S^*_{j_1 j_2})^{-1} = \begin{cases} e^{i\varphi} & \text{for} \quad j_1 > j_2 \\ 1 & \text{for} \quad j_1 < j_2 \\ 0 & \text{for} \quad j_1 = j_2 \end{cases}$$

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- These are excited states of spin-preserving quantum chains, integrable or not. How about particle production?
- For one-particle excitations (where the scattering matrix is not involved) we expect the result to be general [Pizorn'12]

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