Title: Generic ways of quantifying resources

Date: Sep 04, 2018 03:30 PM

URL: http://pirsa.org/18090035

Abstract: <p>Studying the usefulness of resources can be formalized via the framework of a resource theory. However, the complete answer to the question whether a certain resource is more useful than another one is often hard to find in many of the numerous applications of the framework. Approximate answers can be found by identifying so-called monotonesâ€”measures of "resourcefulness". I will present several generic constructions of monotones, of which many monotones known in the literature are concrete examples of. These constructions provide a way to relate monotones in different resource theories, thus enabling for the translation of results between them.</p>

Generic Ways of Quantifying Resources

1. Motivation, the framework, monotones
2. Generalized cost of formation & yield
3. k-monotones. weight & robustness
4. How to tell good and bad monotones apart?

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1a) Why care?

Why quantify?

Why generic?

of Quantifying
urces

the framework, monotones

f formation & yield

weight & robustness

good and

ones apart?

Resource Theory

$(R, R_{free}, *)$

R – all resources

$R_{free} \subseteq R$ – free resources

$* : R \times R \longrightarrow P(R)$

- commutative
- associative
- 0, $0 * r = \{r\}$
- $R_{free} * R_{free} = R_{free}$

...alized cost of formation & yield

3. k-monotones. weight & robustness
4. How to tell good and bad monotones apart?

$$S \ast T = \bigcup_{\substack{s \in S \\ t \in T}} s \ast t$$

1a) Why care?

Why quantify?

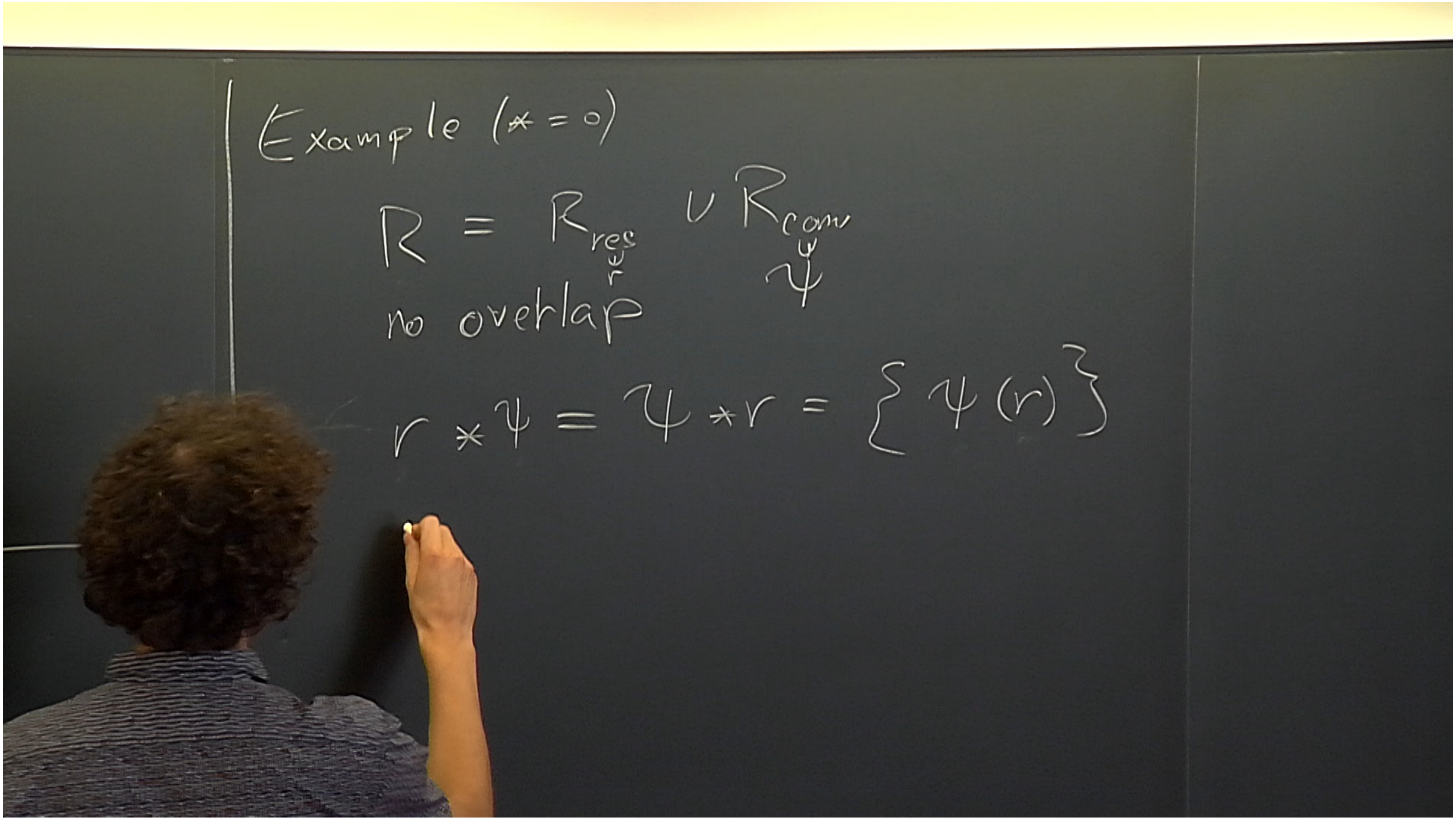
Why generic?

$r * s \subseteq R$

Example ($* = 0$)

$R = R_{res} \cup R_{com}$

no overlap


Example ($\hbar = 0$)
$R = R_{res} \cup R_{com}$
$r \in R_{res}$, $\psi \in R_{com}$
no overlap
$r * \psi = \psi * r = \{\psi(r)\}$

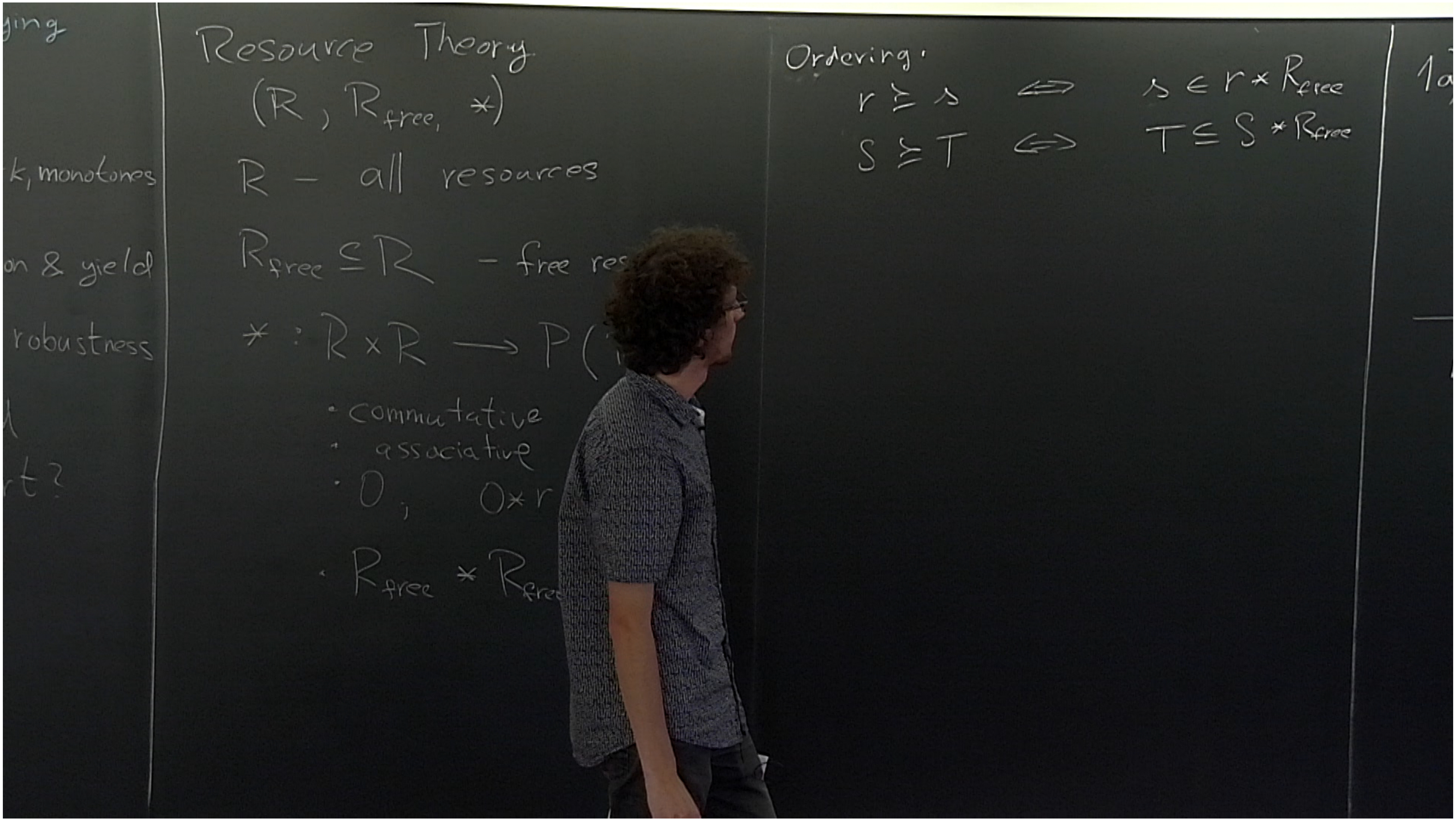
Example ($\hbar = 0$)

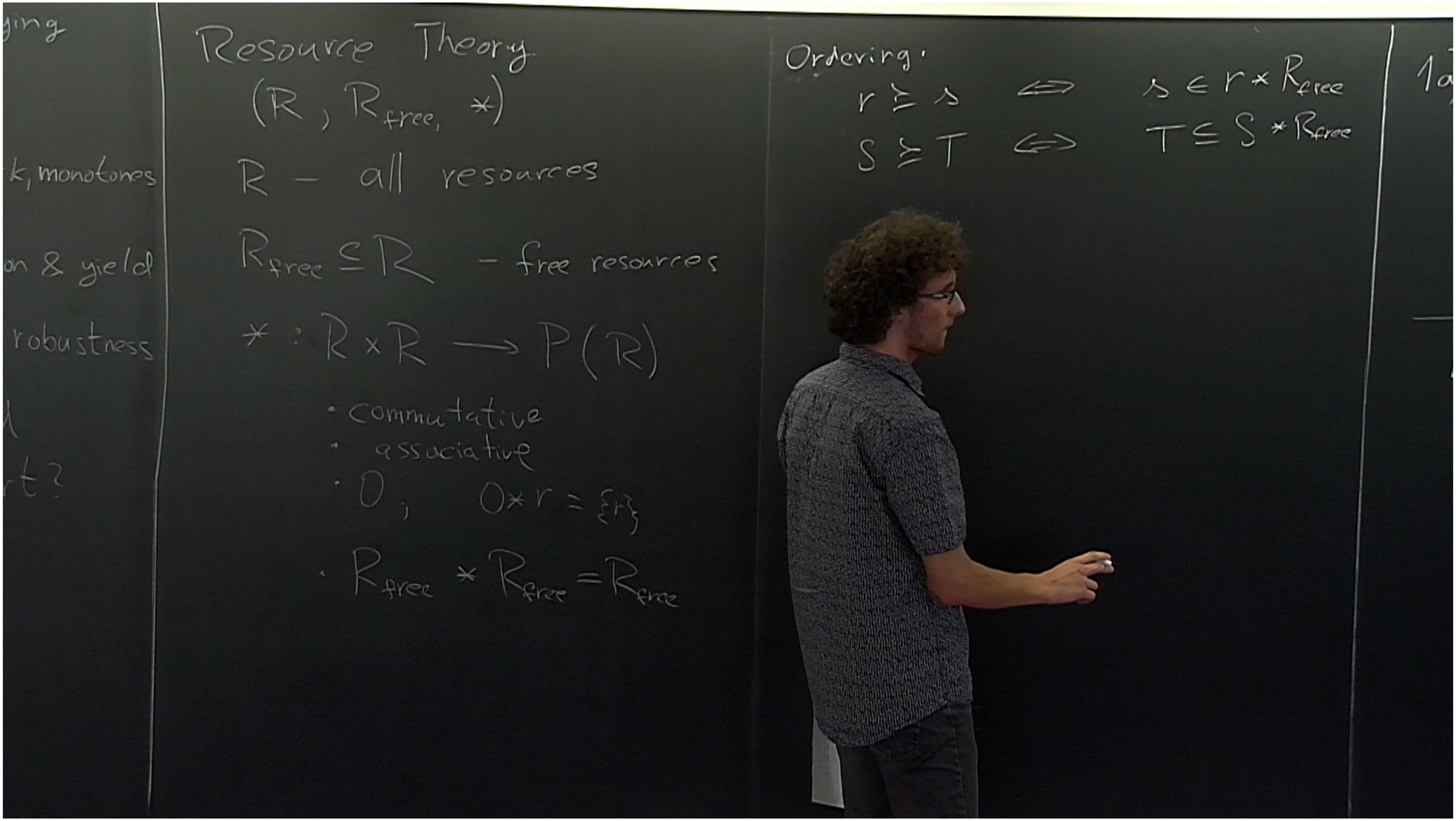
$$R = \underset{\substack{\cup \\ r}}{R_{res}} \cup \underset{\substack{\cup \\ \psi}}{R_{conv}}$$

no overlap

$$r * \psi = \psi * r = \{\psi(r)\}$$

$$r * r' = \emptyset$$


Resource Theory
(R, R_free, *)
R – all resources
R_free ⊆ R – free res
* : R × R ⟶ P(
• commutative
• associative
• 0, 0 * r
• R_free * R_free
Ordering:
r ≥ s ⟺ s ∈ r * R_free
S ≥ T ⟺ T ⊆ S * R_free
k, monotones
on & yield
robustness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Example ($\ast = 0$)

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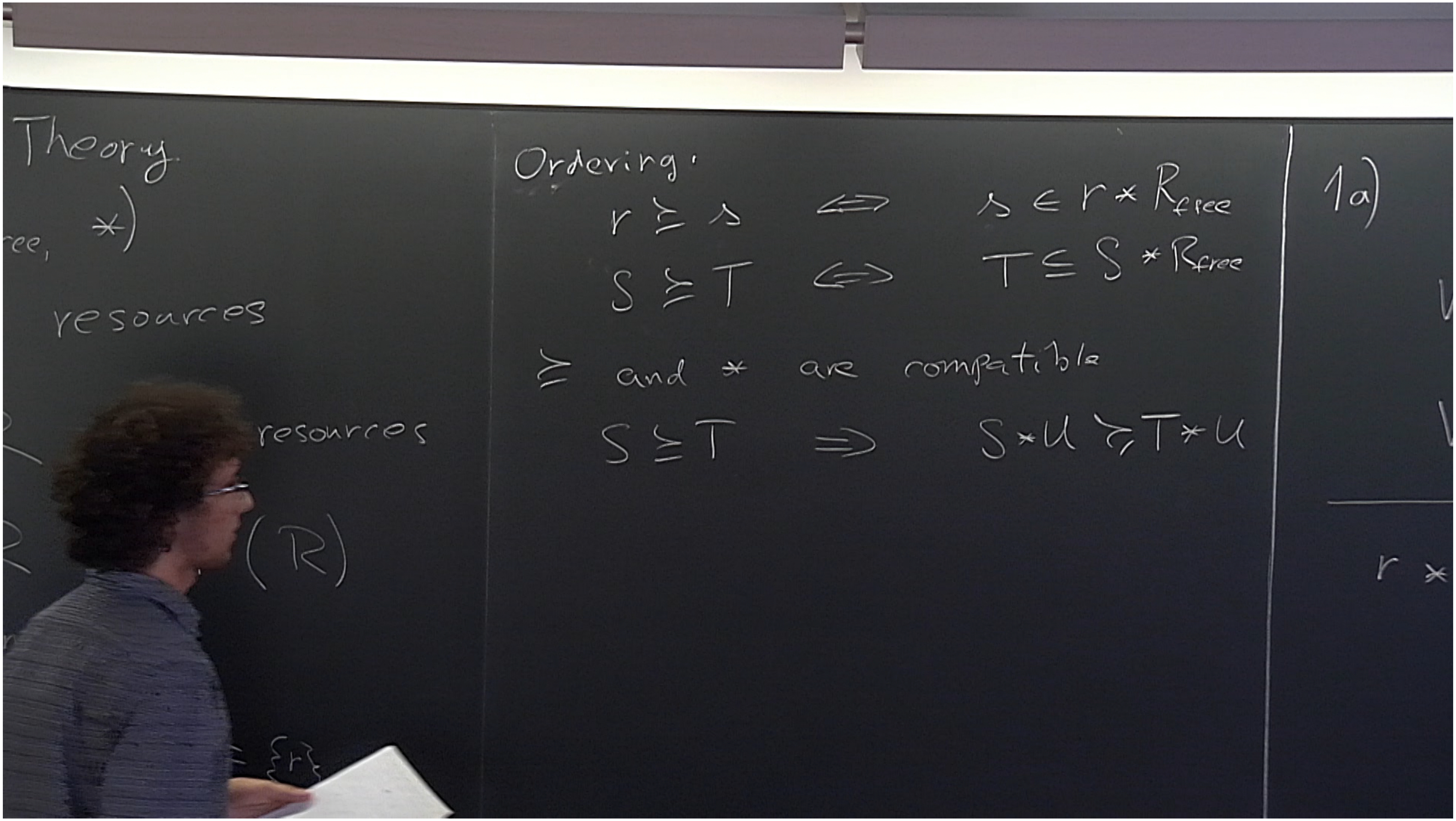
no overlap

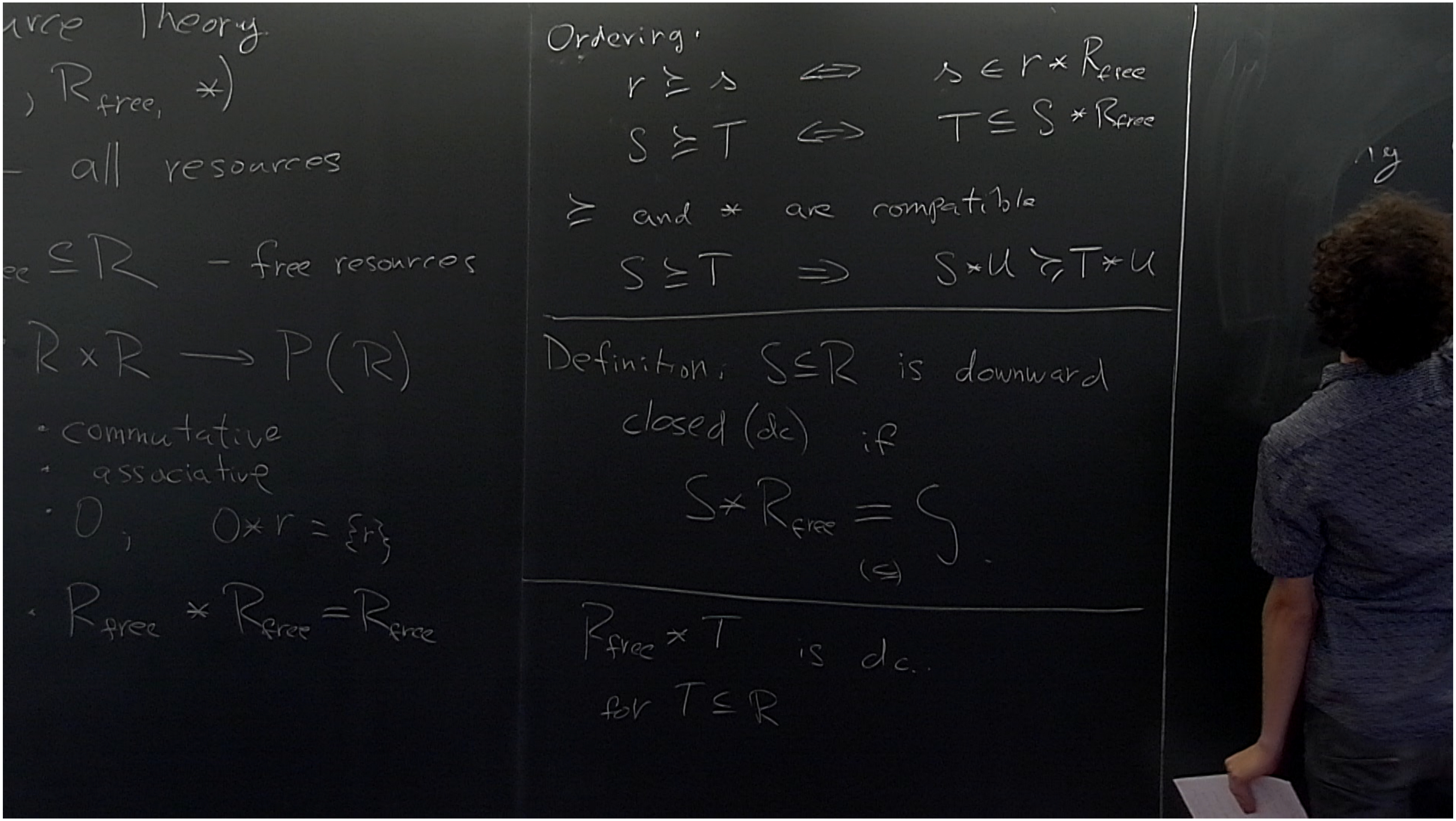
$$r \ast \psi = \psi \ast r = \{\psi(r)\}$$

$$r \ast r' = \emptyset$$

Alternative

$$r \ast \psi = \{\psi(r), \psi \otimes r\}$$


Theory.
resources
resources
Ordering:
$r \geq s \iff s \in r * R_{free}$
$S \succeq T \iff T \subseteq S * R_{free}$
$\succeq$ and $*$ are compatible
$S \succeq T \implies S * U \succeq T * U$
1a)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≥ and * are compatible
S ≥ T ⟹ S*U ≥ T*U
Definition: S⊆R is downward closed (dc) if
S * R_free = S
(⊆)
R_free * T is dc.
for T ⊆ R

R_{free}
R_{free}

$T * U$

rd

Monotones

$f : R \to \mathbb{R}_+$ is a monotone

if $r \succeq s \Rightarrow f(r) \geq f(s)$

$\forall\, r, s \in R.$

$f : P(R) \to \mathbb{R}_+ \ldots$

$S \succeq T \Rightarrow f(S) \geq f(T)$

Example $(* = \circ)$

$R = R_{res} \cup R_{conv}$, with $r \in R_{res}$ and $\psi \in R_{conv}$

no overlap

$r * \psi = \psi * r = \{\psi(r)\}$

$r * r' = \emptyset$

Alternative

$r * \psi = \{\psi(r), \psi \otimes r\}$

…ifying

work, monotones

…ation & yield

& robustness

and

…part?

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$S \succeq T \iff T \subseteq S * R_{free}$

$\succeq$ and $*$ are compatible

$S \succeq T \implies S * U \succeq T * U$

Definition: $S \subseteq R$ is downward closed (dc) if

$$S * R_{free} \underset{(\subseteq)}{=} S.$$

$R_{free} * T$ is dc. for $T \subseteq R$

$\leq r * R_{free}$

$\leq S * R_{free}$

ble

$U \succeq T * U$

ownward

Monotones

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if $r \succeq s \Rightarrow f(r) \geq f(s)$

$\forall\ r, s \in \mathcal{R}.$

$f : \mathcal{P}(\mathcal{R}) \to \mathbb{R}_+$ . . .

$S \succeq T \Rightarrow f(S) \geq f(T)$

tone

$f(s)$

$f(T)$

② $f : R \to \mathbb{R}_+$ is a function

$S \subseteq R$ is d.c.

Definition: The f-cost of formation relative to S is

$${}_S\Sigma_f : R \longrightarrow \mathbb{R}_+ \quad \text{defined by}$$

$${}_S\Sigma_f(r) := \inf\{f(s) : r \in S *_s\}$$

a monotone

$f(r) \geq f(s)$

$f(S) \geq f(T)$

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The f-yield:

$${}_S\Lambda_f(r) := \sup_{s \in R} \{ f(s) : s \in S * r \}$$

is a monotone

$\Rightarrow \quad f(r) \geq f(s)$

$\subseteq R.$

R_+

$\Rightarrow \quad f(S) \geq f(T)$

(2) $f : R \to \mathbb{R}_+$ is a function

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are monotones.

$\in r * R_{free}$

$\subseteq S * R_{free}$

tible

$* U \succeq_? T * U$

downward

).

Monotones

$f : R \to \mathbb{R}_+$ is a monotone if $r \succeq s \Rightarrow f(r) \geq f(s)$ $\forall\, r, s \in R$.

$f : P(R) \to \mathbb{R}_+$

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are monotones

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③ k-monotones

$(* = 0)$

Definition: (2-monotone)

$f^{(2)} : R \times R \longrightarrow R_+$ is a 2-monotone if

$$f^{(2)}(r,s) \geq f^{(2)}(\psi(r), \psi(s)) \quad \forall r,s \in R, \ \psi \in R_{free}$$

Note it is NOT

$$r \geq r' \wedge s \geq s' \Rightarrow f^{(2)}(r,s) \geq f^{(2)}(r',s')$$

e

onotones.

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k-monotone $\longrightarrow$ m-monotones $(m \leq k)$

Theorem: $f^{(2)}$ is a 2-monotone, $S \subseteq R$ is d.c. Then

$$f^{(1)}(r) := \inf\{ f^{(2)}(r,s) : s \in S \}$$

are monotones

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Theorem: $f^{(2)}$ is a 2-monotone, $S \subseteq R$ is d.c. Then

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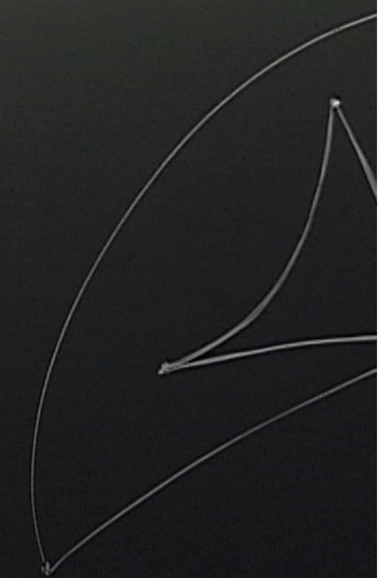
is a 1-monotone.



a monotone

$f(r) \geq f(s)$

$f(S) \geq f(T)$

Weight & Robustness.

R has a convex structure.

R_{conv} are convex-linear.

$$M : R \times R \times R \longrightarrow \mathbb{R}_{+,\infty}$$

$$M(r,s,t) = \begin{cases} \lambda & \text{if } \quad r = \lambda s + (1-\lambda) t \ ; \ \lambda \in [0,1] \\ \infty & \text{otherwise} \end{cases}$$

is a 3-monotone.

3

is a monotone

$\Rightarrow \quad f(r) \geq f(s)$

R.

$\Rightarrow \quad f(S) \geq f(T)$

Weight & Robustness.

$\mathcal{R}$ has a convex structure.

$\mathcal{R}_{conv}$ are convex-linear.

$$M : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathbb{R}_{+,\infty}$$

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is a 3-monotone.

Use theorem to get 1-monotones:

- $M_w(r) = \inf\{ M(r,s,t) \;:\; s \in \mathcal{R},\; t \in \mathcal{R}_{free} \}$
- robustness

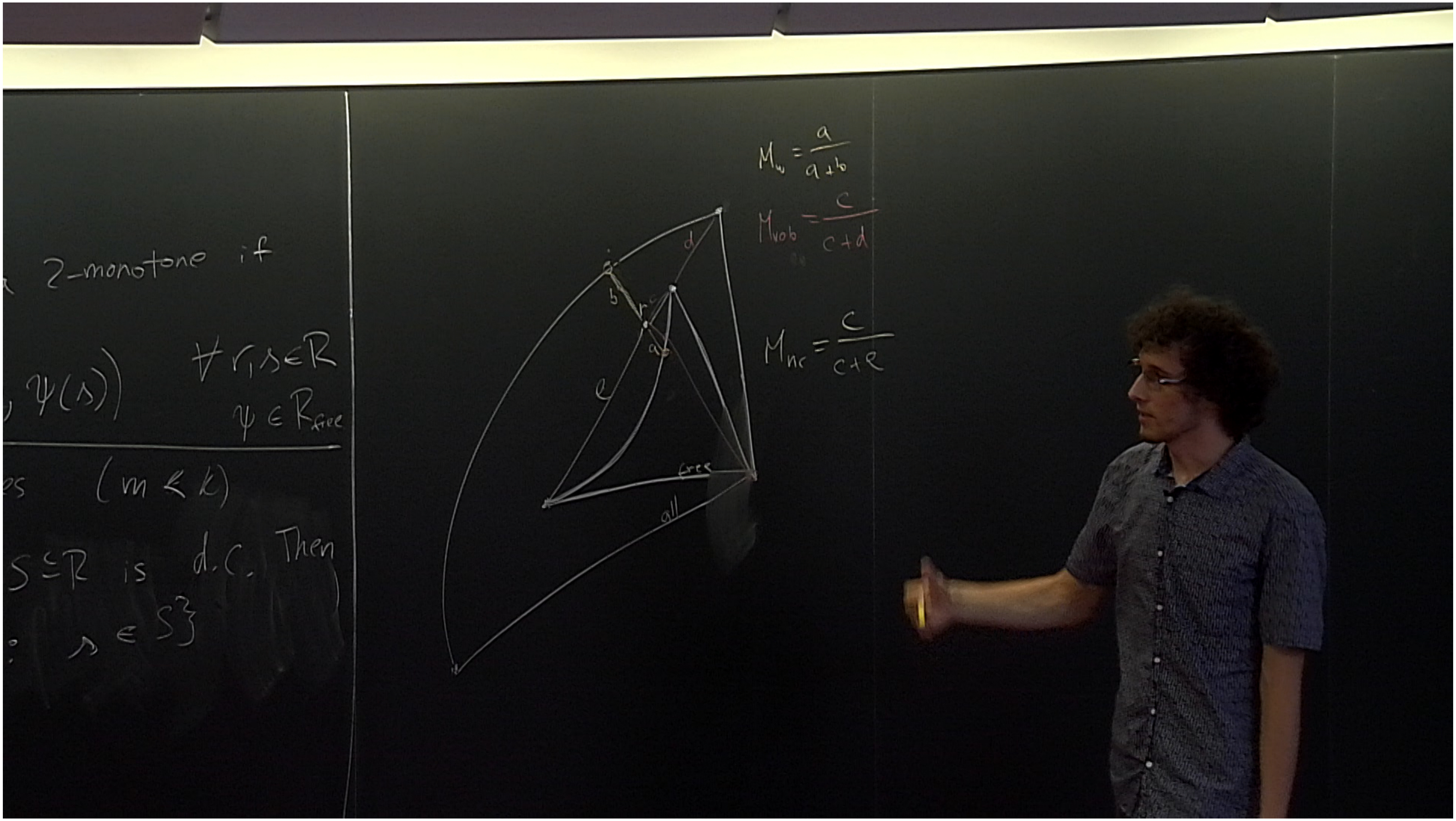
③ k

Defin

f

$\rightarrow$ k

Theor



2-monotone if
$\forall r, s \in R$
$\psi(s))$
$\psi \in R_{free}$
$(m \leq k)$
$S \subseteq R$ is d.c. Then
$s \in S\}$
$M_w = \frac{a}{a+b}$
$M_{rob} = \frac{c}{c+d}$
$M_{nc} = \frac{c}{c+e}$
free
all