Title: The Uses of Lattice Topological Defects

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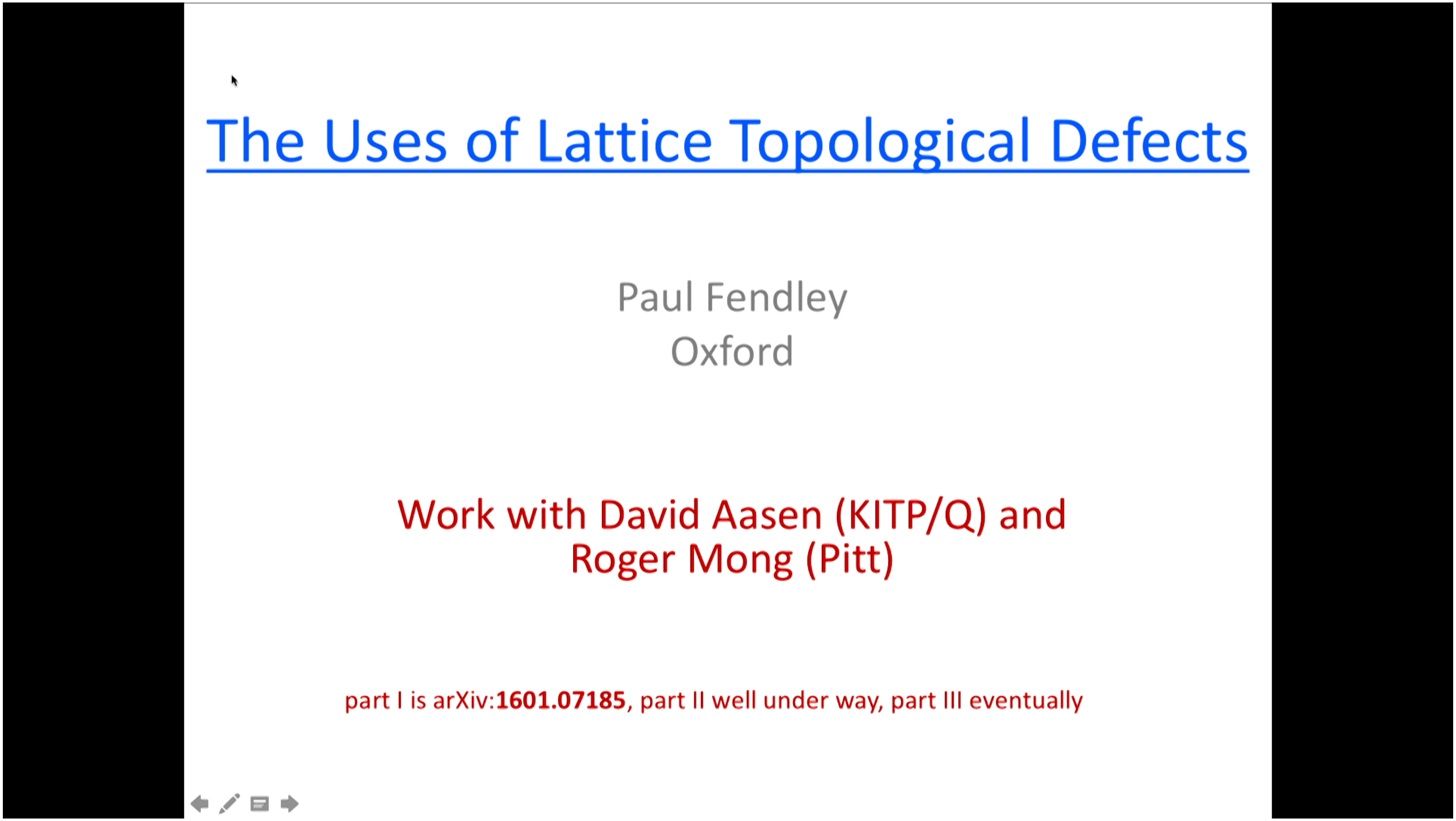
Abstract: <p>I give an overview of work with Aasen and Mong on topologically invariant defects in two-dimensional classical lattice models, quantum spin chains and tensor networks. We show how to find defects that satisfy commutation relations guaranteeing the partition function depends only on their topological properties. These relations and their solutions can be extended to allow defect lines to branch and fuse, again with properties depending only on topology. These lattice topological defects have a variety of useful applications. In the Ising model, the fusion of duality defects allows Kramers-Wannier duality to be enacted on the torus and higher genus surfaces easily, implementing modular invariance directly on the lattice. These results can be extended to a very wide class of models, giving generalised dualities previously unknown in the statistical-mechanical literature. A consequence is an explicit definition of twisted boundary conditions that yield the precise shift in momentum quantization and thus the spin of the associated conformal field. Other universal quantities we compute exactly on the lattice are the ratios of g-factors for conformal boundary conditions. </p>



The Uses of Lattice Topological Defects

Paul Fendley
Oxford

Work with David Aasen (KITP/Q) and
Roger Mong (Pitt)

part I is arXiv:**1601.07185**, part II well under way, part III eventually

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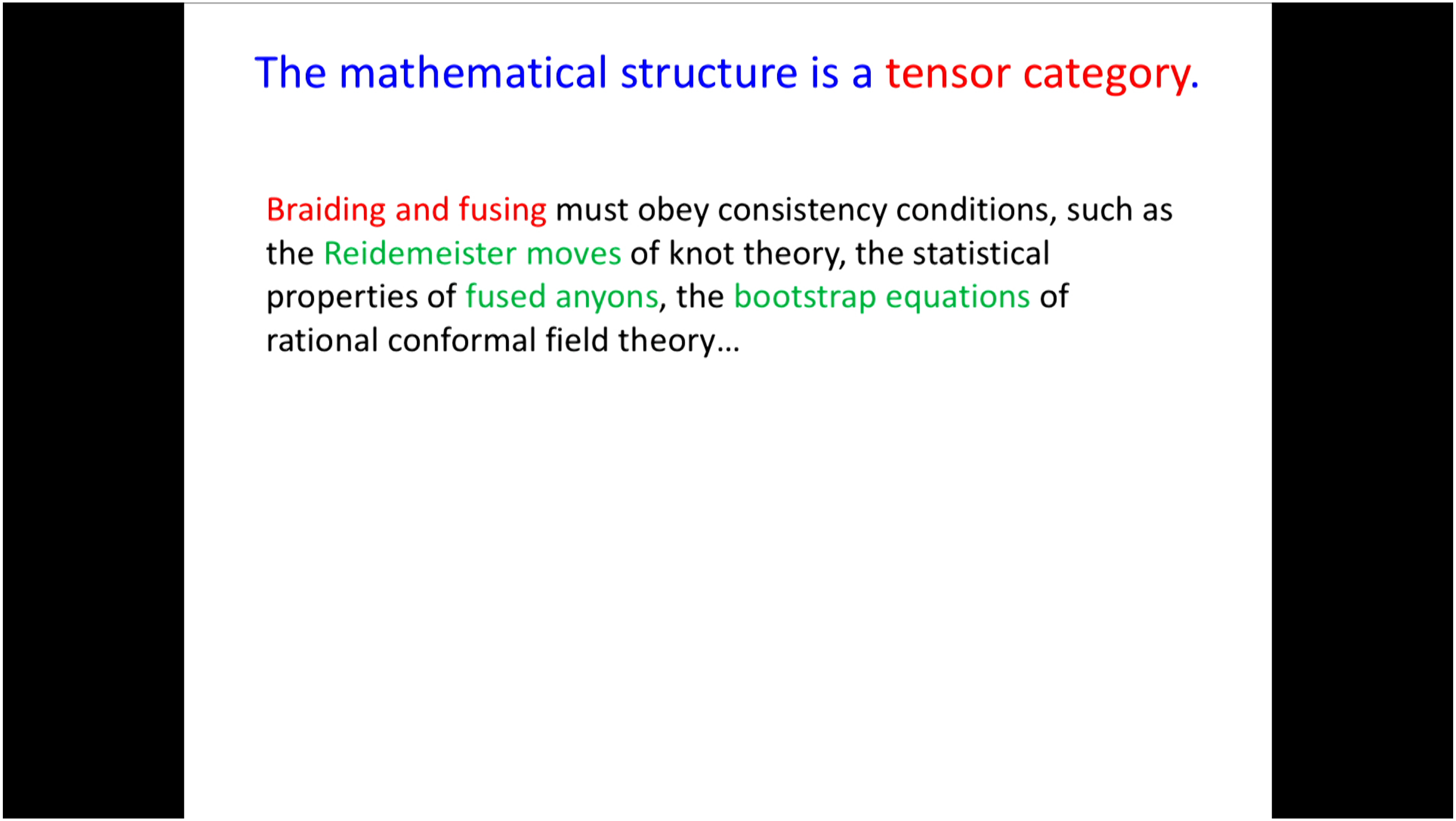
- Last (in this list), least (as judged by number of citations), but first (as in time) are the deep connections to lattice models of statistical mechanics.



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Which type of categories depends precisely on which structure you want, get beasts such as fusion categories (no braiding required), modular tensor category (works on the torus)…

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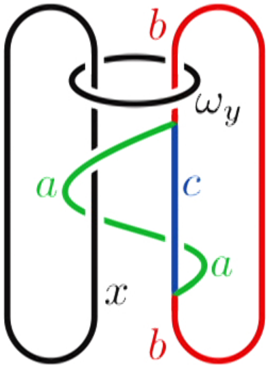
The category allows e.g. matrix elements to defined precisely via pictures:

$$M^*_{xy}[Z^b_{ac}] \equiv \frac{\theta_a}{d_y}$$



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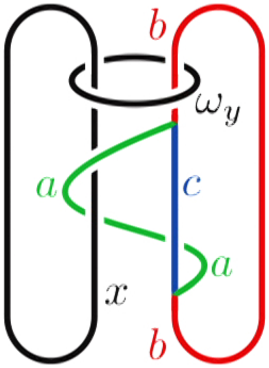
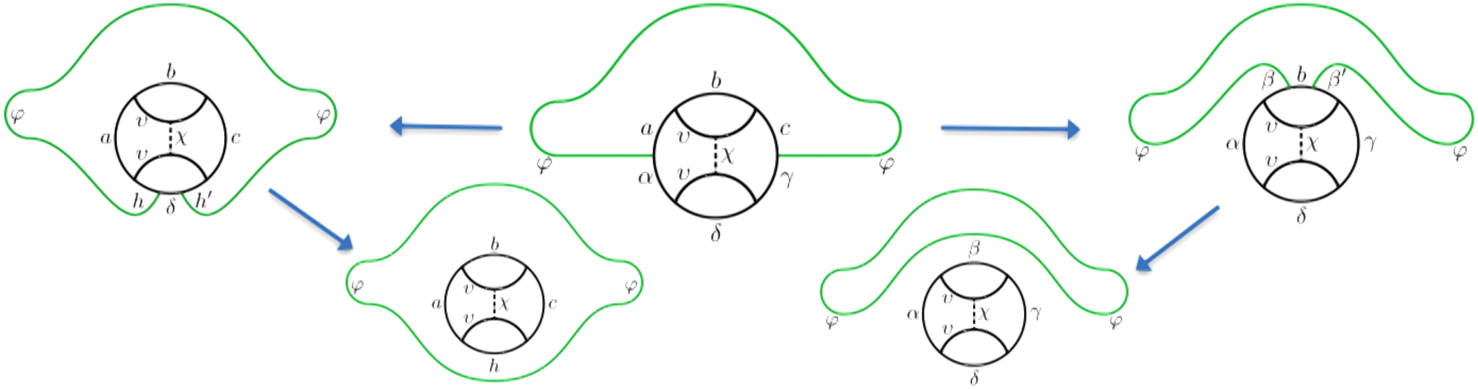
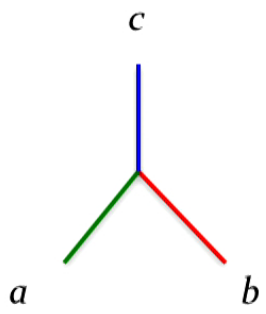


To compute the numbers, the rules allow various manipulations of the graphs to relate the topological invariants, e.g.



Specifying the category means you specify the allowed labels on the edges of the diagram, and which vertices are allowed:



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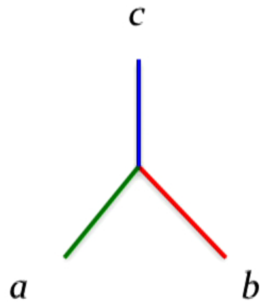

Non-negative integers

A convenient way of doing this is via a fusion algebra: $a \otimes b = \sum_c N_{ab}^c \, c$

When $N_{ab}^c > 0$, the *abc* vertex is allowed.

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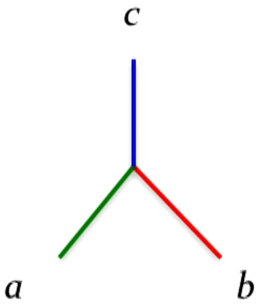

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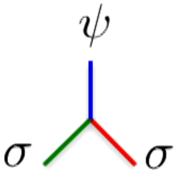
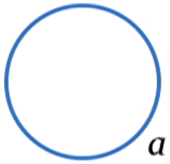
For the Ising fusion category, there are three objects, with labels $1, \ \sigma, \ \psi$

identity

$$\sigma \otimes \sigma = 1 + \psi$$

Only vertex not involving identity is  and its rotations.

Some key rules:

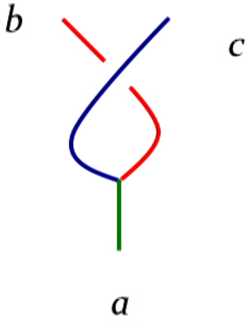


$$= d_a$$

quantum dimension

For Ising:

$$d_1 = 1$$
$$d_\psi = 1$$
$$d_\sigma = \sqrt{2}$$



$$= e^{i\pi(h_a - h_b - h_c)}$$

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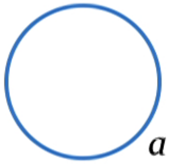

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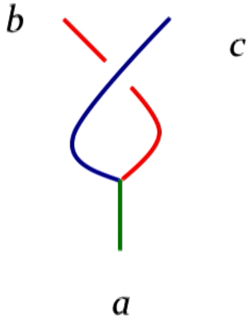

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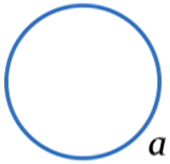

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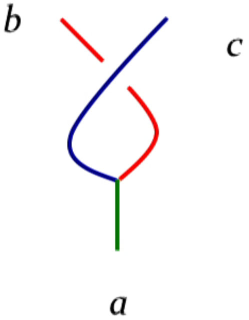

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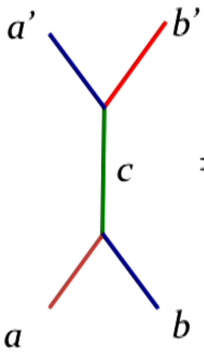
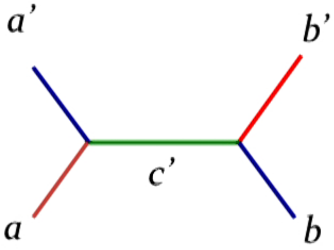


$$= \sum_{c,c'} (F^{a'b'}_{ab})_{cc'}$$



$$F^{\sigma\sigma}_{\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

F-symbols

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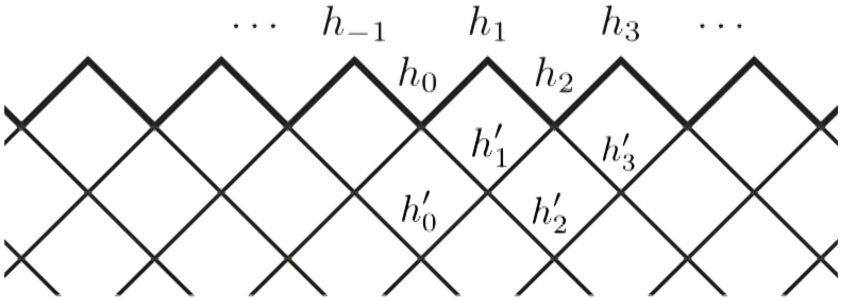
- Many generalisations of Kramers-Wannier duality, given explicitly and exactly
- Exact lattice derivation of g-factors for conformal boundary conditions
- By doing modular transformations on the lattice, get momentum quantization conditions that yield exact dimensions of operators in the continuum limit

Defining the models

Degrees of freedom are Integer-valued ``heights'' living on the sites of a square lattice:



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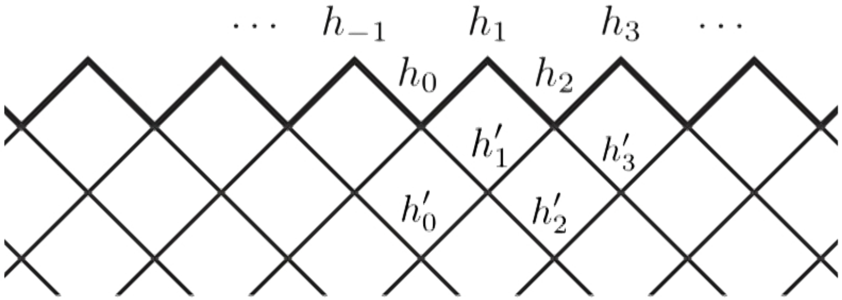

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$$a \Diamond b = e^{K_x \delta_{ab}} \qquad \overset{b}{\underset{a}{\Diamond}} = e^{K_y \delta_{ab}}$$

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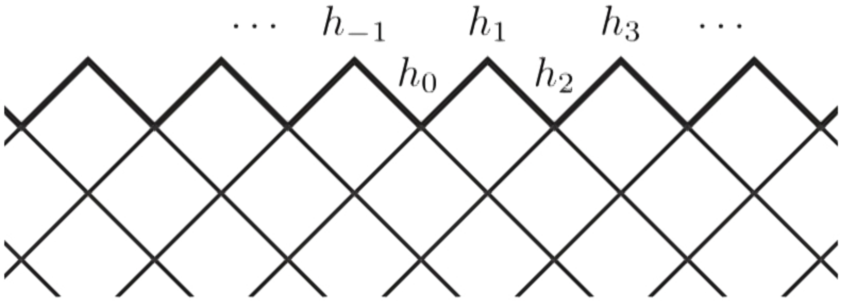

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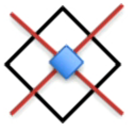

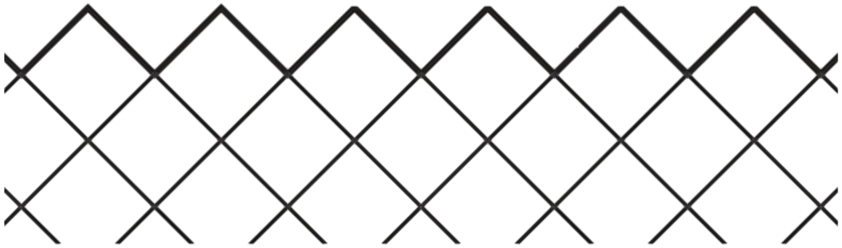

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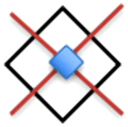

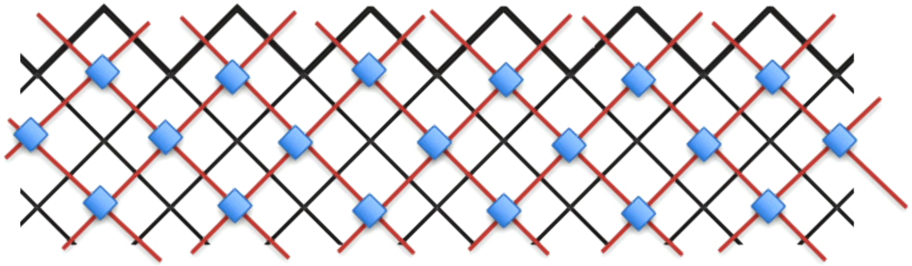

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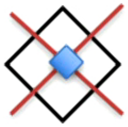

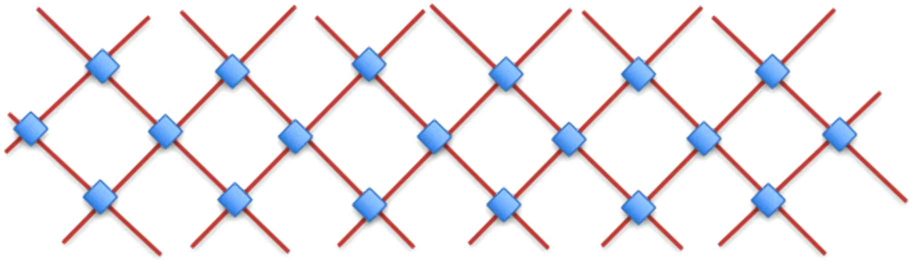

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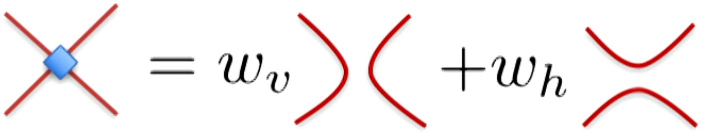
Many important models can be written in this way, including critical lattice models yielding (probably) all rational CFT in the continuum limit

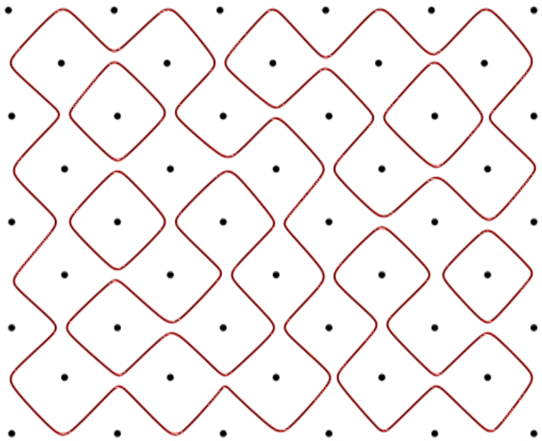
e.g. lattice height models corresponding to the minimal CFTs can rewritten as loop models:

Temperley-Lieb; Fortuin-Kasteleyn; Baxter; Andrews-Baxter-Forrester; Pasquier

In math literature, called shadow world (Turaev)



$$= w_v \,)(\, + w_h \asymp$$



$$Z = \sum_{\text{Loop configs}} d^{\#\text{loops}} w_v^{\#)(} \, w_h^{\#\asymp}$$

$$= \sum \text{(topological weight) x (local weights)}$$

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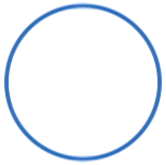

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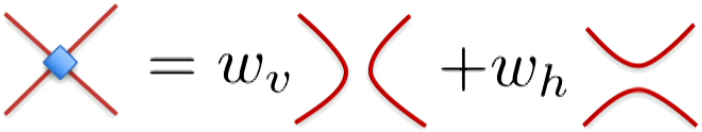

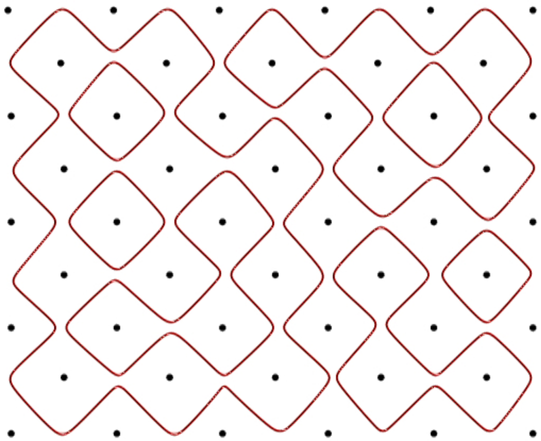

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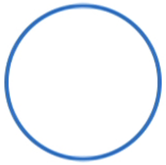

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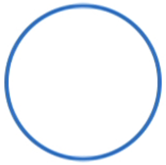

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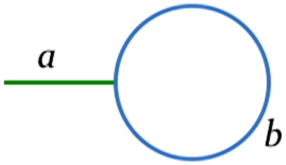

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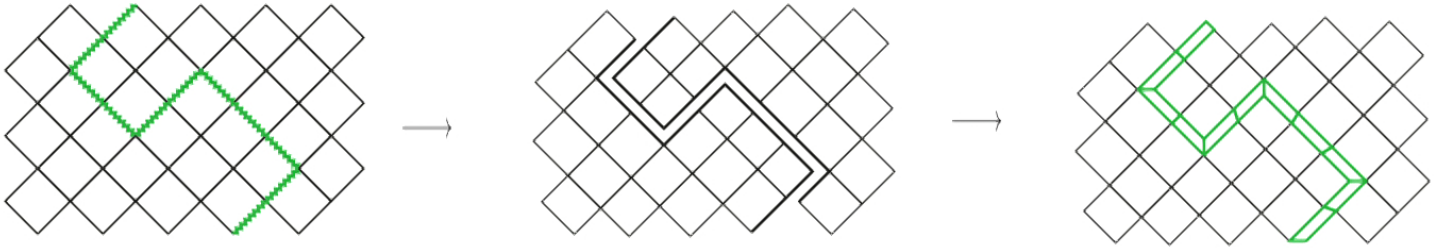
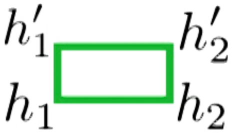
This setup allows topological defects to be defined, and interesting and exact properties to be derived for them.

Inserting a defect



The defects have a weight depending on the adjacent heights:



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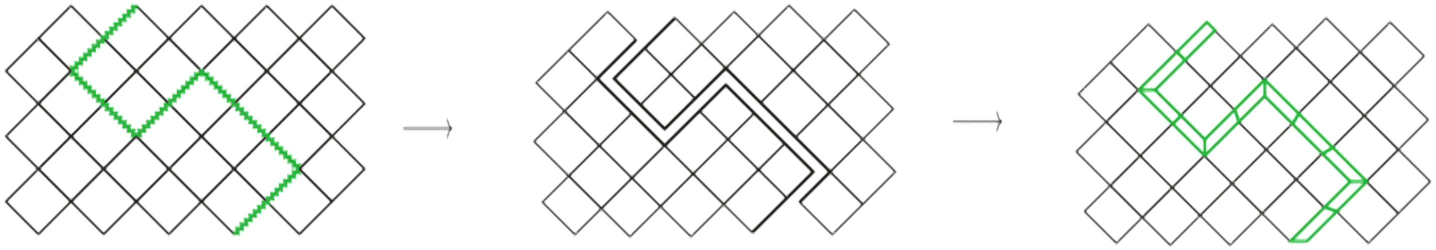

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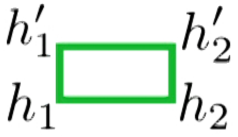
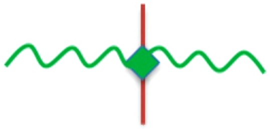


In the tensor network picture:

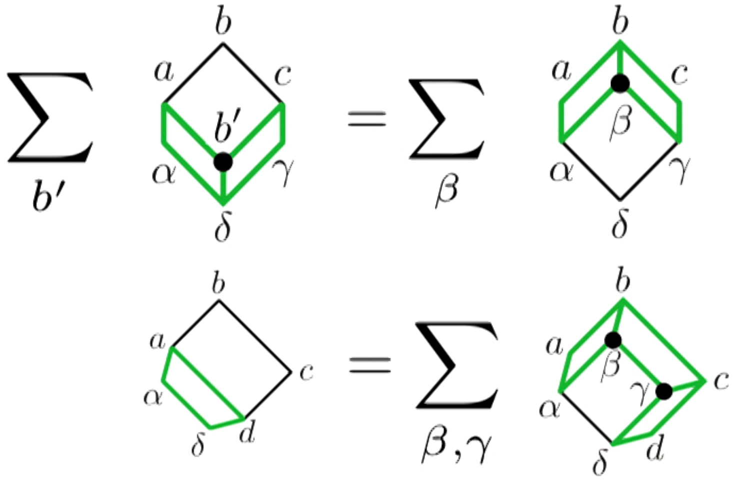


In the presence of the defect, the partition function is modified to

$$\mathcal{Z} = \sum_{\text{heights}} \prod_{\text{faces}} \Diamond \times \prod_{\text{along defect}} \square$$

Topological defects

For $\mathcal{Z}$ to be invariant under deformations of the defect's path:



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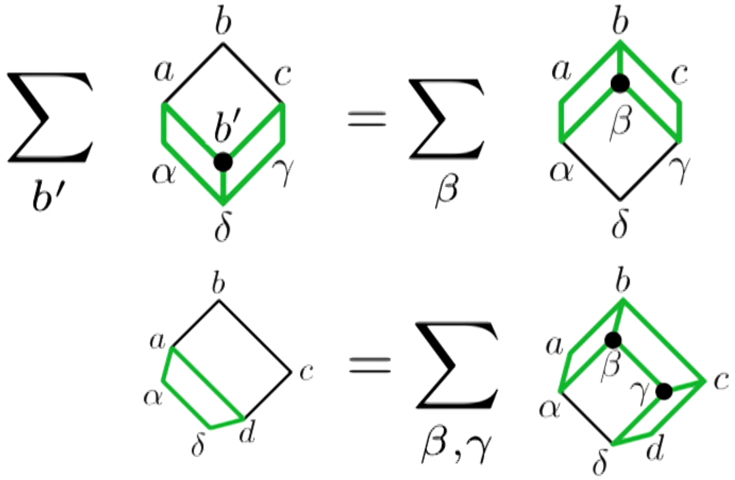
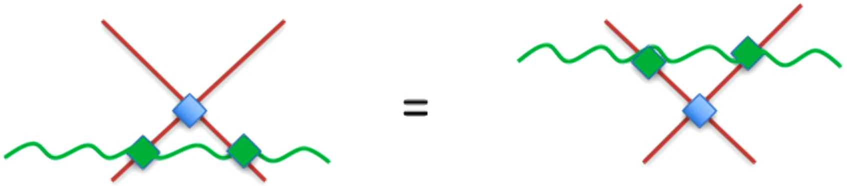


In tensor networks, these have been dubbed ``pulling-through'' conditions:



Verstraete et al

Two types of topological defects in Ising

Recall that for Ising, include degrees of freedom on only half the sites:

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$$= e^{K_y \delta_{ab}}$$

spin-flip defect:

$$= 1 - \delta_{ab}$$

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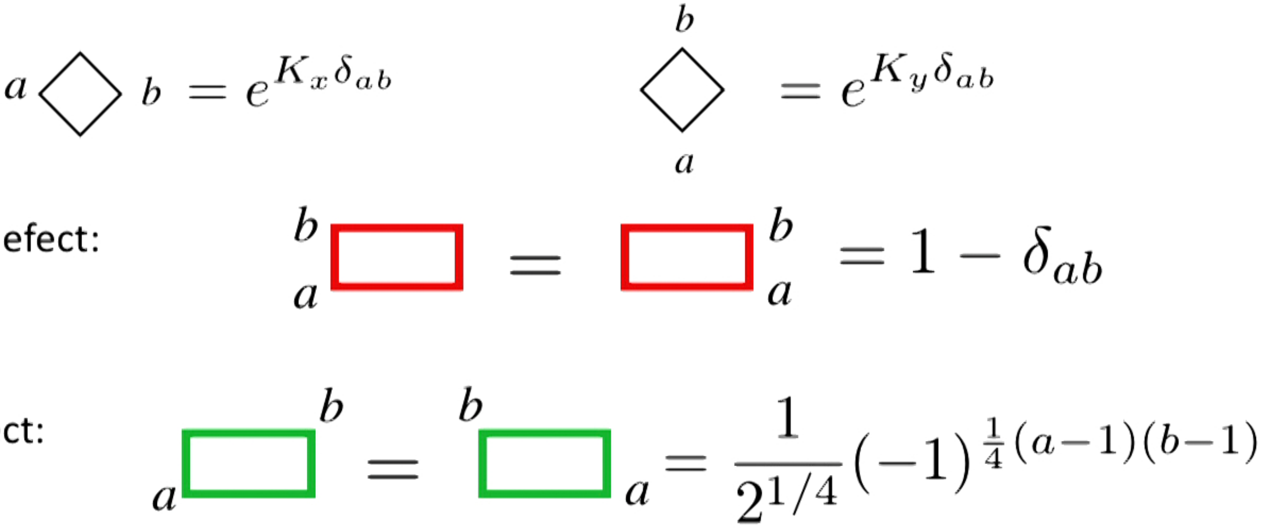

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spin-flip defect:

$${}^{b}_{a}\square = \square^{b}_{a} = 1 - \delta_{ab}$$

duality defect:

$${}_{a}\square^{b} = {}^{b}\square_{a} = \frac{1}{2^{1/4}}(-1)^{\frac{1}{4}(a-1)(b-1)}$$

Two types of topological defects in Ising

Recall that for Ising, include degrees of freedom on only half the sites:

$$a \Diamond b = e^{K_x \delta_{ab}} \qquad \overset{b}{\underset{a}{\Diamond}} = e^{K_y \delta_{ab}}$$

spin-flip defect:

$${}^{b}_{a}\boxed{\quad\quad} = \boxed{\quad\quad}^{b}_{a} = 1 - \delta_{ab}$$

duality defect:

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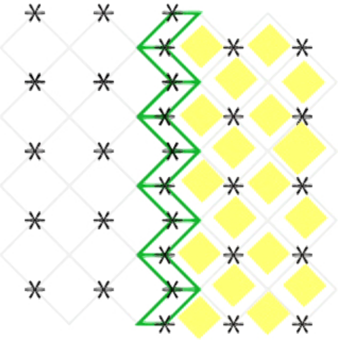
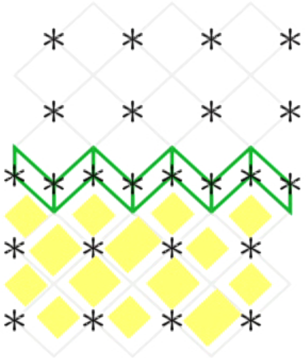
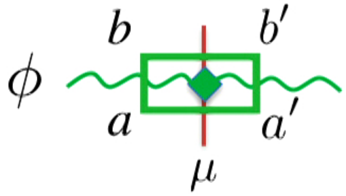


Couplings on one side of defect are dual values of those on the other!



Useful application 1: generalised duality

For each object in the category, there are defect weights satisfying the commutation relations. To find them, use braiding if it exists, or better the Drinfeld centre.



$$= \frac{1}{\sqrt{d_a d_{b'}}} \left(F^{\phi a}_{a'\mu}\right)_{b'b}$$

In Ising, spin-flip defect is labeled by ψ, while the duality defect by σ.

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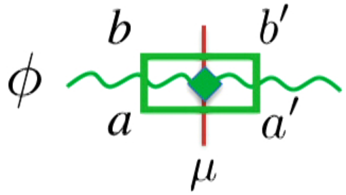

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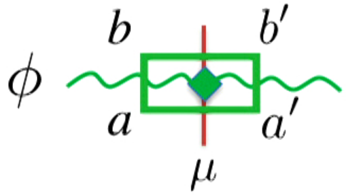

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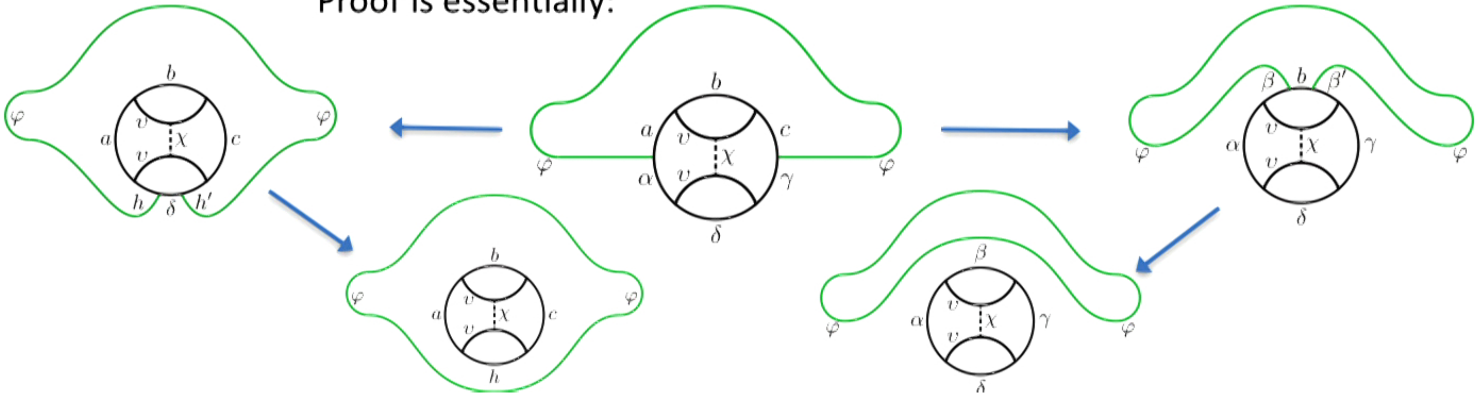
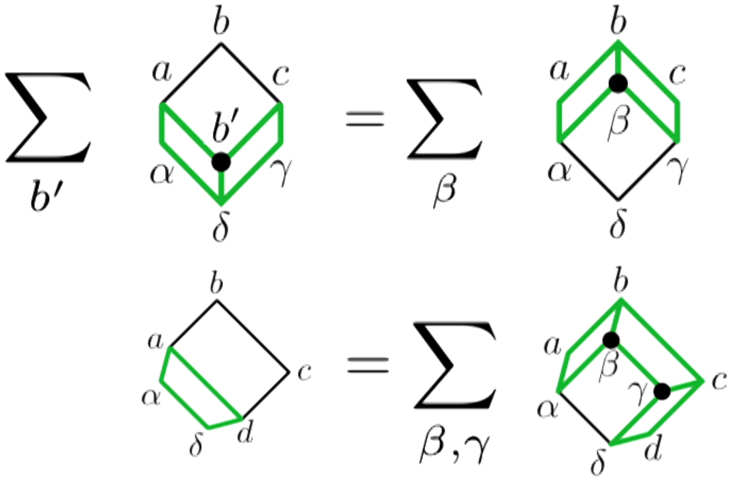
Proof is essentially:



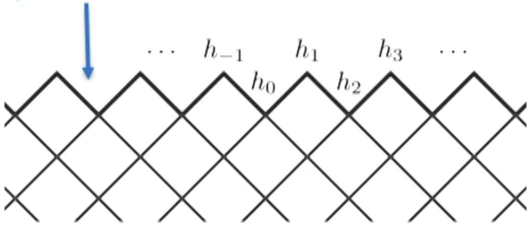
Topological defects

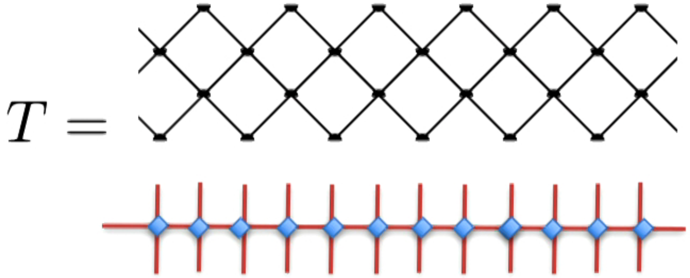
For $\mathcal{Z}$ to be invariant under deformations of the defect's path:



The transfer matrix is (the original?) matrix product operator.

Vector space on which T acts





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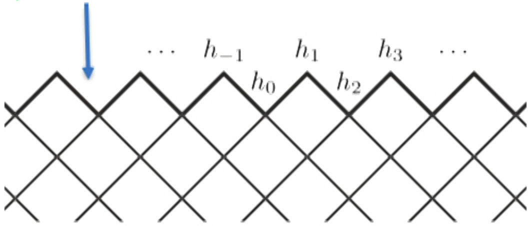
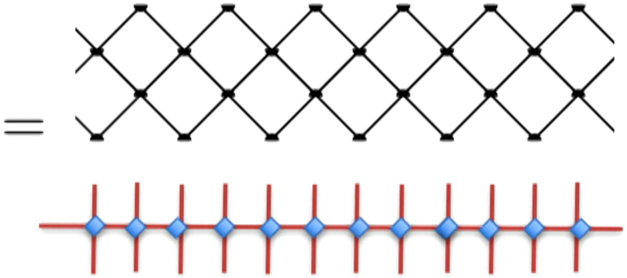
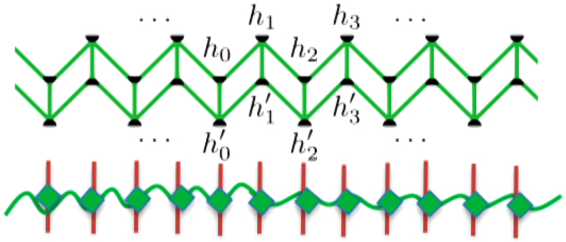


$T =$



Then defect also is an MPO:

$\mathcal{D}_\phi =$



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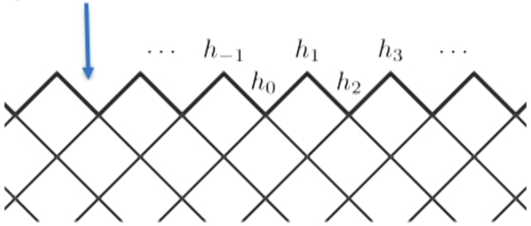
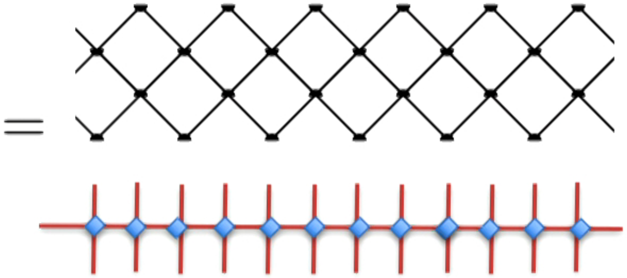

Vector space on which *T* acts

$$T =$$



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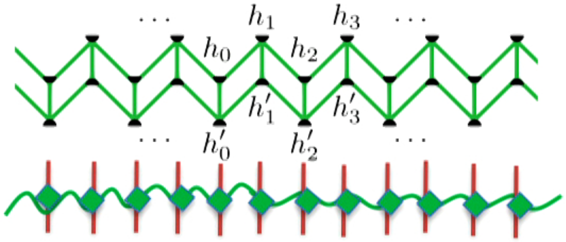
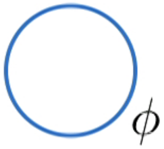


For any topological defect, the defect commutation relations ensure that

$$T\mathcal{D}_\phi = \mathcal{D}_\phi T$$

Deforming turns microscopic into macroscopic

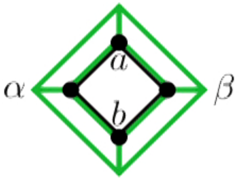
Recall



$$= d_\phi$$

Can nucleate a defect loop around a single face, find

$$\sum_{a,b}$$

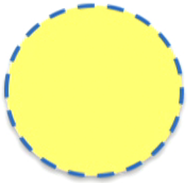


$$= d_\phi$$

Corresponding partition functions on a disc are related as

$$\mathcal{Z} = d_\phi Z$$

$$= d_\phi$$



Duality is not a symmetry

Fusing defects together obeys the same rules as the objects in category.

In Ising category (and CFT):

$$\sigma \otimes \sigma = 1 + \psi$$

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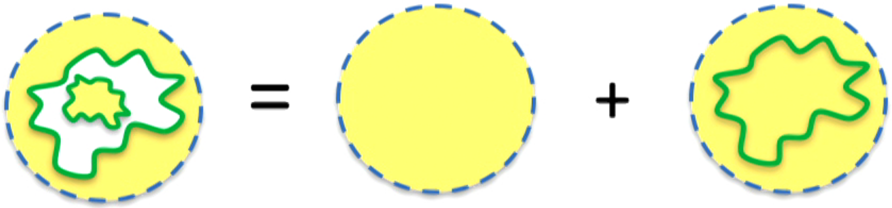

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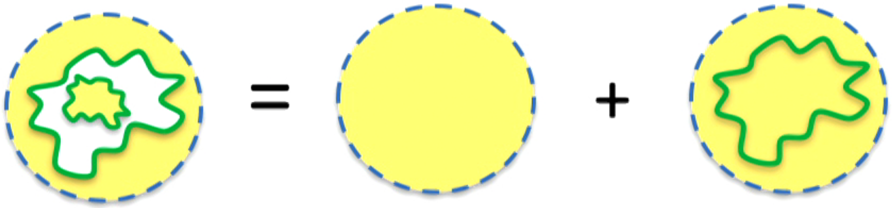
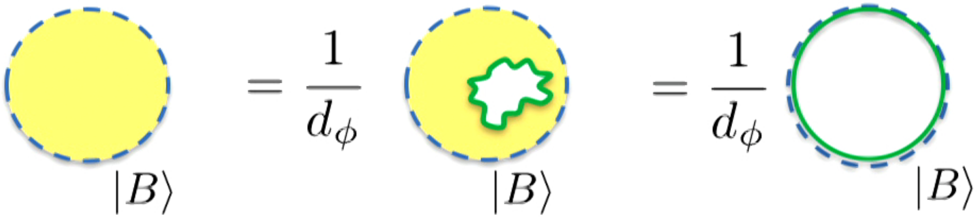


If $d_\phi \neq 1$, then $\mathcal{D}_\phi$ is not unitary.

In Ising, duality is not even invertible! $(\mathcal{D}_\psi)^2 = 1 \quad \Rightarrow \quad (\mathcal{D}_\sigma)^4 = 2(\mathcal{D}_\sigma)^2$

Useful application 2:
g-factors for conformal boundary conditions



Consider vector space of all configurations of spins/heights near edge. Each vector $|B\rangle$ corresponds to a boundary condition, e.g. $|\mathrm{fixed\ up}\rangle = |+++++\ldots\rangle$

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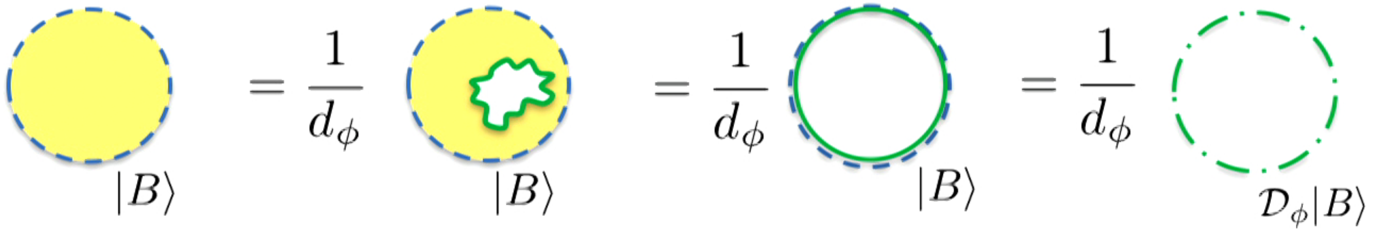


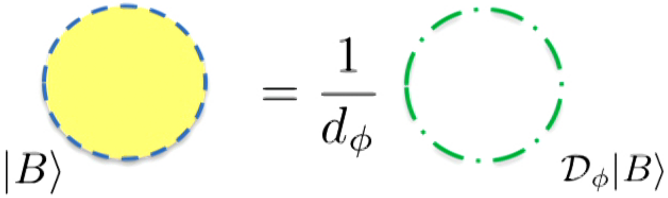
Consider vector space of all configurations of spins/heights near edge. Each vector $|B\rangle$ corresponds to a boundary condition, e.g. $|\text{fixed up}\rangle = |+++++\dots\rangle$

Acting with $\mathcal{D}_\phi$ absorbs the defect into the edge, changing the boundary condition

$$|B\rangle \to \mathcal{D}_\phi|B\rangle$$

In Ising the absorbing the duality defect gives e.g.

$$\mathcal{D}_\sigma|\text{fixed up}\rangle = |\text{free}\rangle \qquad \mathcal{D}_\sigma|\text{free}\rangle = |\text{fixed up}\rangle + |\text{fixed down}\rangle$$



Thus for $Z_{|B\rangle}$ the partition function on the disc with boundary condition $|B\rangle$, we have proved directly on the lattice

$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi$$

In Ising, $$Z_{\text{free}}(K) = \sqrt{2}\, Z_{\text{fixed}}(\widehat{K})$$

where dual coupling is defined by $\sinh(2K)\sinh(2\widehat{K}) = 1$

For conformal boundary conditions, this ratio of partition functions is by definition

$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi = \frac{g_{\mathcal{D}_\phi|B\rangle}}{g_{|B\rangle}}$$

where $-T \ln g_{|B\rangle}$ is the subleading term in the free energy, which depends on boundary condition. This Affleck-Ludwig g-factor is universal, and computable in CFT once the Cardy boundary states have been identified.

Calculation is much more direct here!

Moreover, gives precise lattice expressions for boundary states.

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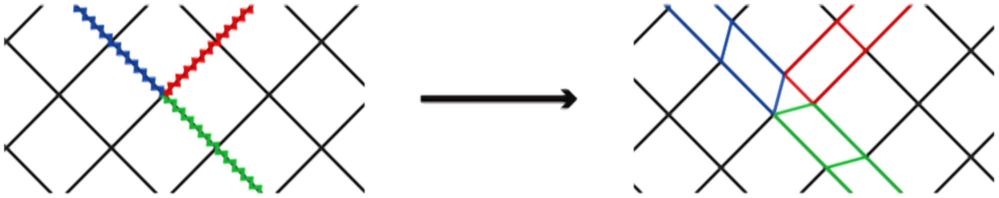
For lattice model in the tricritical Ising model universality class: $d_\tau = (1+\sqrt{5})/2$

For another model in the same universality class, a different defect: $d_\epsilon = \sqrt{2}$

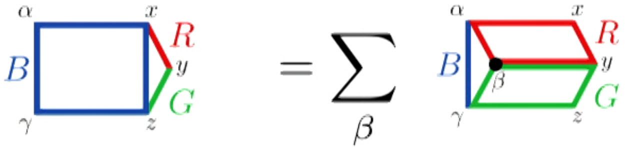
Both agree with ratios found from CFT by Chim, Affleck

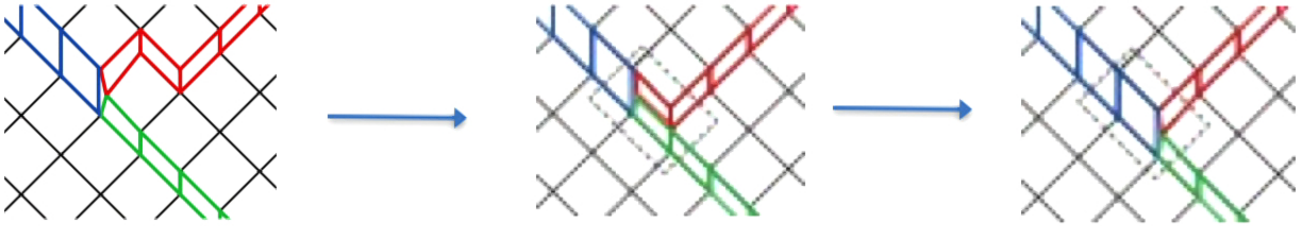
Branching and fusing

Straightforward to define junctions of these topological defects



and show that they obey





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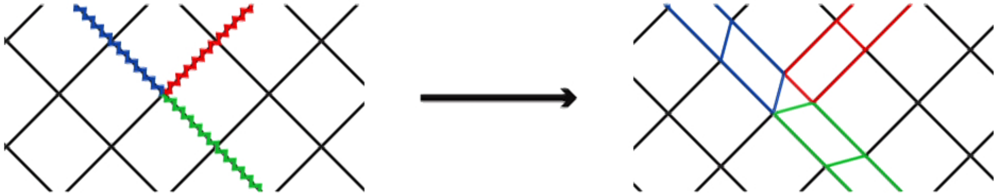

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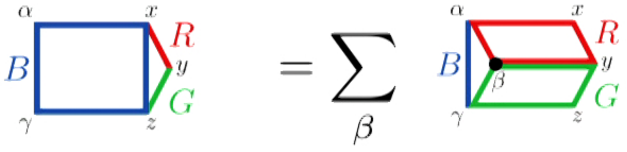
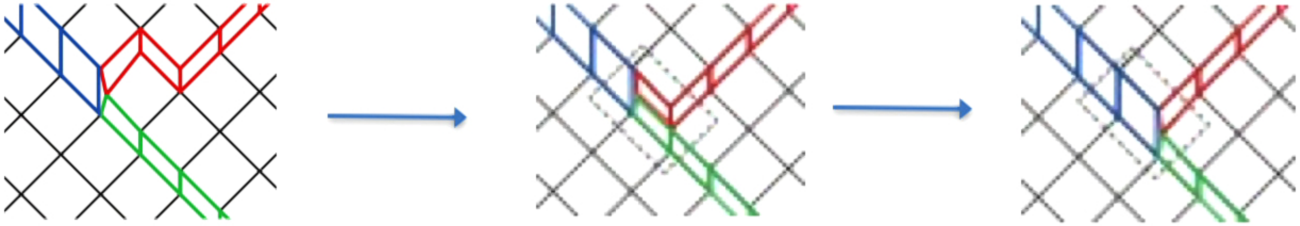
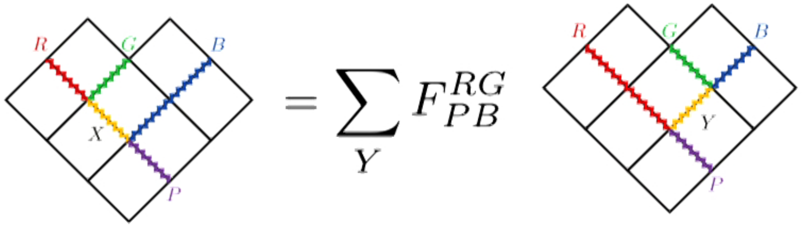


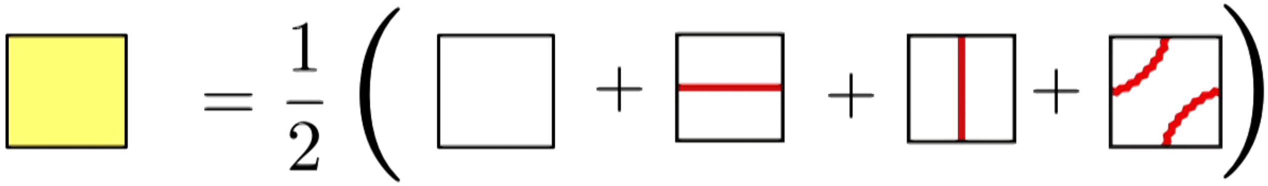


Another microscopic to macroscopic relation: topological defects obey *F*-moves:



Useful application 3:
duality and modular transformations on the torus

Use these F-moves to give an easy graphical proof of the Ising relation:



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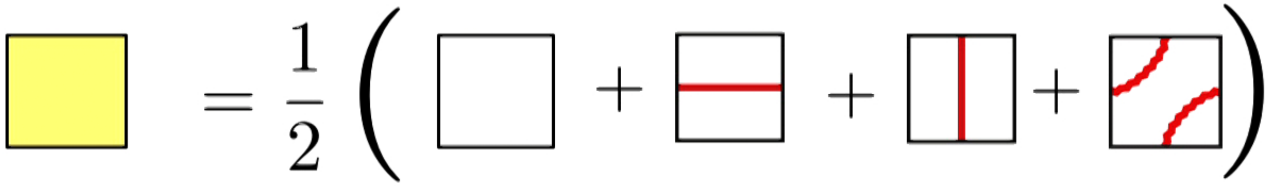
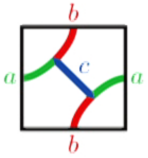


A general basis for the toroidal partition functions is $Z^b_{ac} \equiv$



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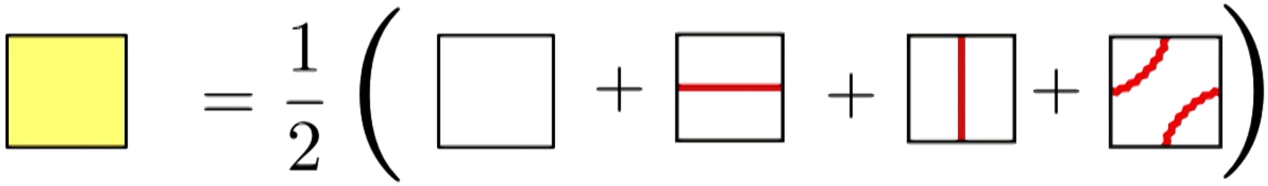

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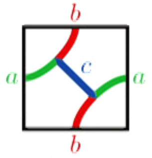


Lattice to CFT

RCFT toroidal partition functions are of the form

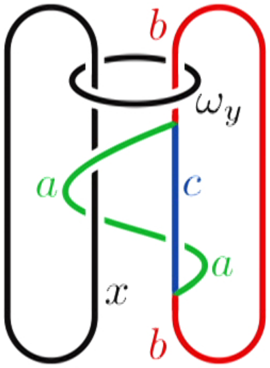
$$Z_M = \sum_{ij} \bar{\chi}_i(\bar{\tau}) M_{ij} \chi_j(\tau)$$

$\chi_j(\tau)$ are characters of Virasoro or some extended algebra.

The category used to build the lattice model is a subcategory of that describing the chiral operators in the RCFT.

By matching lattice and continuum modular transformations, we conjecture

$$M^*_{xy}[Z^b_{ac}] \equiv \frac{e^{2\pi i h_a}}{d_y}$$

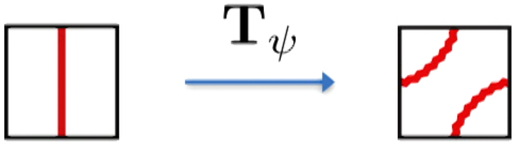


Useful application 4:
scaling dimensions from Dehn twists

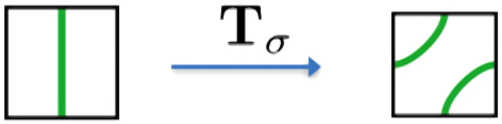
In the presence of twisted boundary conditions ϕ, the eigenvalues t_ϕ of the Dehn twist are the related to the shift in momentum quantization:

$$e^{\frac{2\pi i \mathcal{P}_\phi}{L}} = t_\phi$$

Dehn twist with twisted Ising boundary conditions



Duality-twisted boundary conditions



Find

$$(\mathbf{T}_\sigma)^4 = \sqrt{2}(\mathbf{T}_\sigma)^2 - 1_\sigma \ , \qquad (\mathbf{T}_\sigma)^8 = -1_\sigma \ , \qquad (\mathbf{T}_\sigma)^{16} = 1_\sigma$$

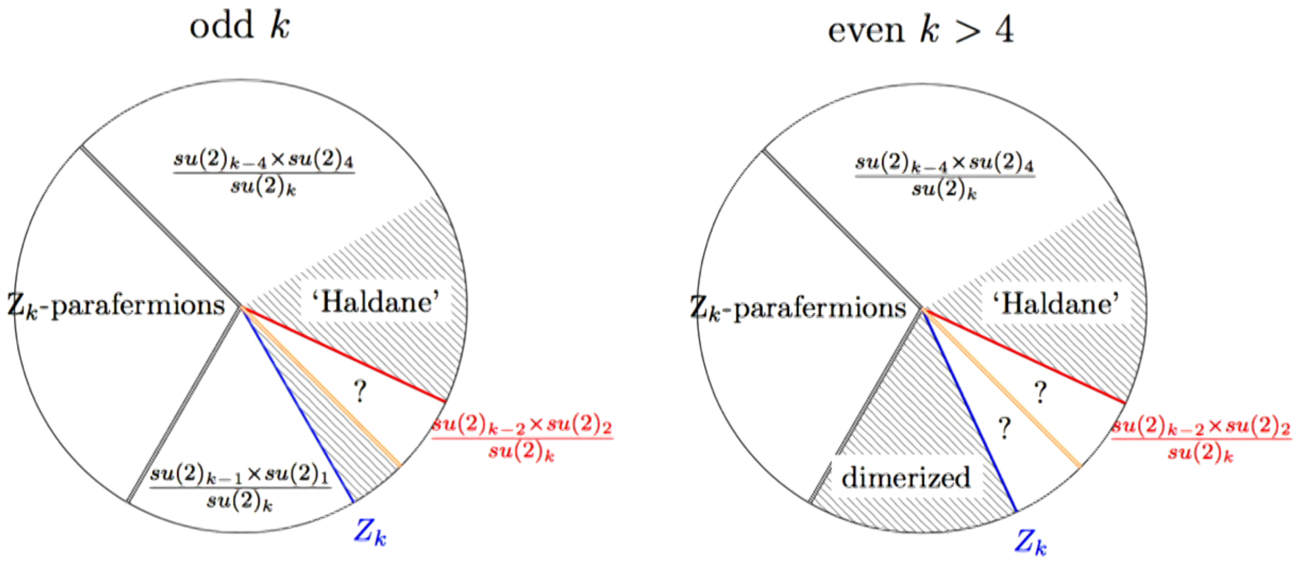
Thus in CFT expect twist field of chiral dimension $\pm\frac{1}{16} \pm \frac{1}{2}$

This is a completely rigorous and exact lattice calculation.
Only the pictures are schematic.

The CFT identification of course uses the standard assumption that lattice model scales to continuum field theory. Since Ising at criticality has been rigorously proven to be Ising CFT, maybe the proof can be extended to cover these defects and twist operators?

I mentioned different categories and hence different lattice models give same CFT.

Same category can give different CFTs, by choosing different Boltzmann weights; some may be integrable, some not. People have studied various phase diagrams, e.g.



Vernier, Jacobsen and Saleur

The same topological defects occur throughout the phase diagram. Thus the same ratios of *g*-factors and same Dehn twist eigenvalues throughout.

Latter predicts that at any critical point or phase, see $SU(2)_k$ critical exponents.

Another useful application : explains peculiar ground-state and kink degeneracies in perturbed CFTs (integrable in continuum but not on lattice).

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Lots of generalisations: orbifold defects, defects between different theories

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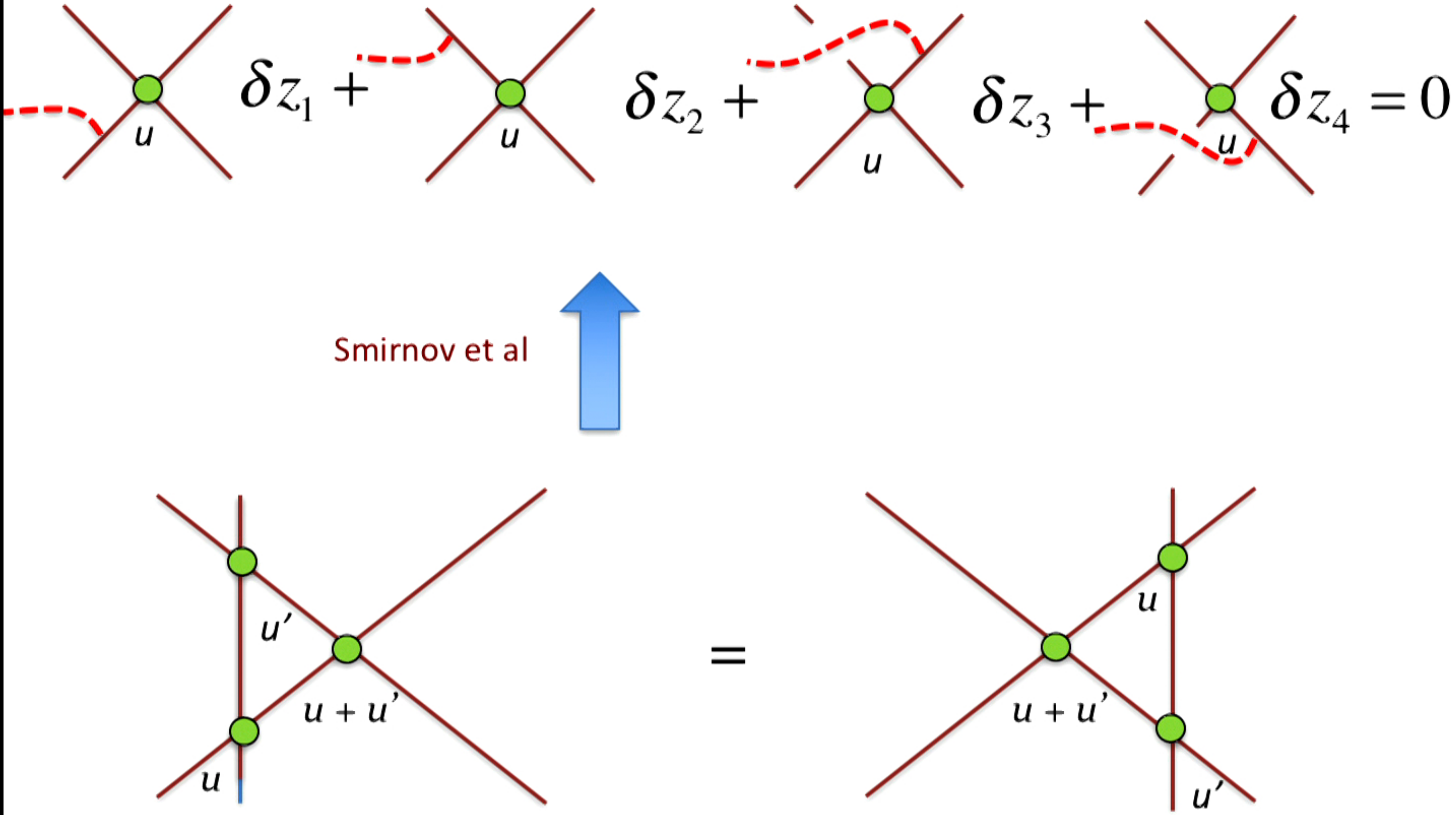
Lots of generalisations: orbifold defects, defects between different theories

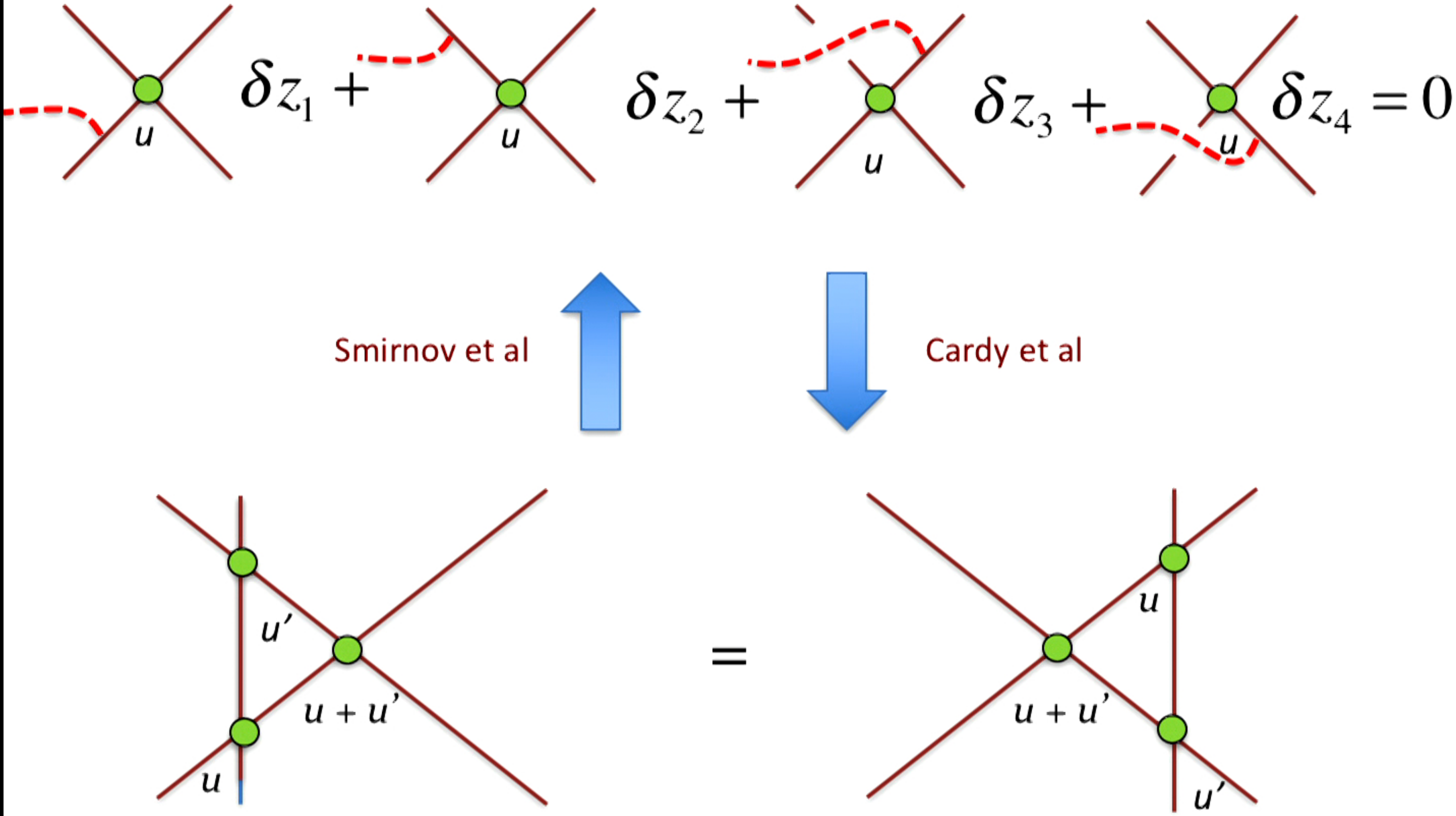
Higher dimensions (a la p-form symmetries) ?

Can terminate defects to make chiral vertex operators, gives nice way of yielding conserved currents (linear way of finding solution to Yang-Baxter).



Smirnov et al

In all the examples studied, the solution of this linear equation gives weights solving the YBE!



Smirnov et al

Cardy et al