Title: Emergent Hyperbolic Network Geometry and Dynamics

Date: Sep 04, 2018 02:00 PM

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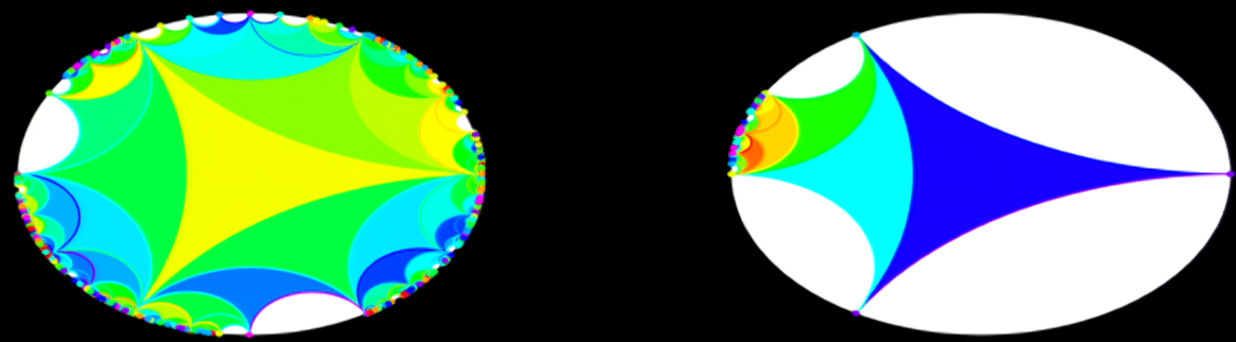
Abstract: <p>Simplicial complexes naturally describe discrete topological spaces. When their links are assigned a length they describe discrete geometries. As such simplicial complexes have been widely used in quantum gravity approaches that involve a discretization of spacetime. Recently they are becoming increasingly popular to describe complex interacting systems such a brain networks or social networks. In this talk we present non-equilibrium statistical mechanics approaches to model large simplicial complexes. We propose the simplicial complex model of Network Geometry with Flavor (NGF), we explore the hyperbolic nature of its emergent geometry and their relation with Tree Tensor Networks. Finally we reveal the rich interplay between Network Geometry with Flavor and dynamics. We investigate the percolation properties of NGF using the renormalization group finding KTP and discontinuous phase transitions depending on the dimensionality simplex. We also comment on the synchronization properties of NGF and the emergence of frustrated synchronization.</p>



Perimeter Institute
4 September 2018

Emergent hyperbolic geometry and dynamics

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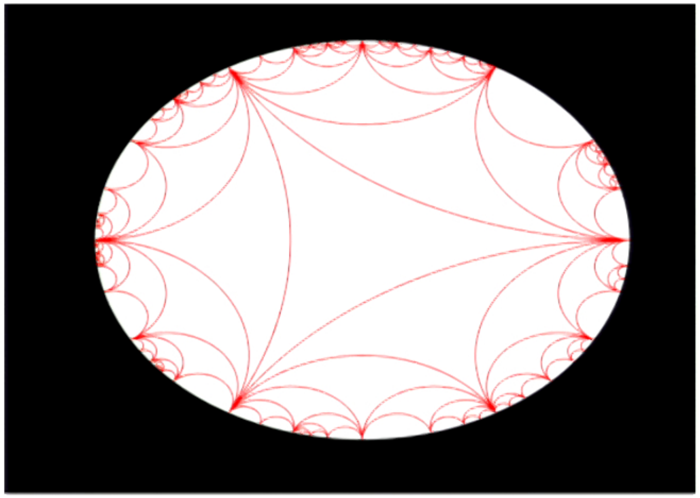
Queen Mary
University of London

Network theory
Tensor Networks
A case for convergent evolution?

Network Geometry with Flavor (NGF)

Emergent network geometry (simplicial complexes)



Rich interplay between network geometry and dynamics

Their deterministic version allow for the renormalization group

Outlook of the talk

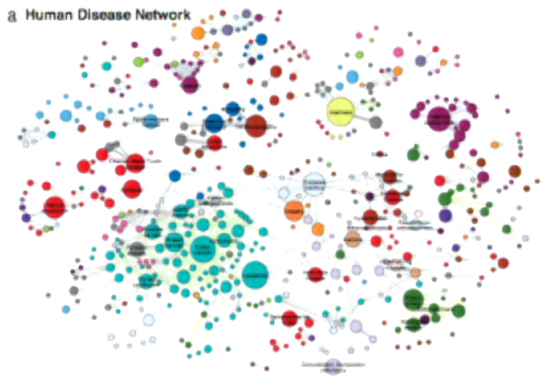
- Motivation and introduction to Network Theory
- Emergent hyperbolic geometry and dynamics

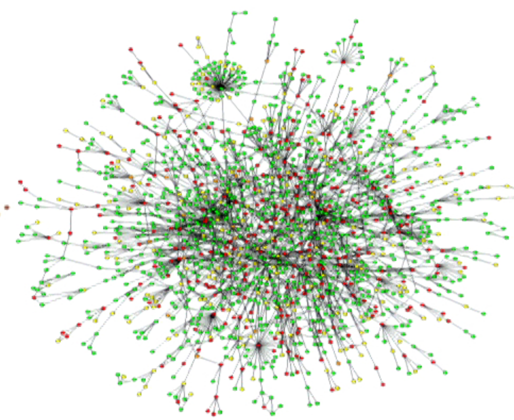
  -Network Geometry with Flavor (NGF)

  -Topological percolation and renormalization group

  -Localization of modes in NGF and frustrated synchronization

Complex networks





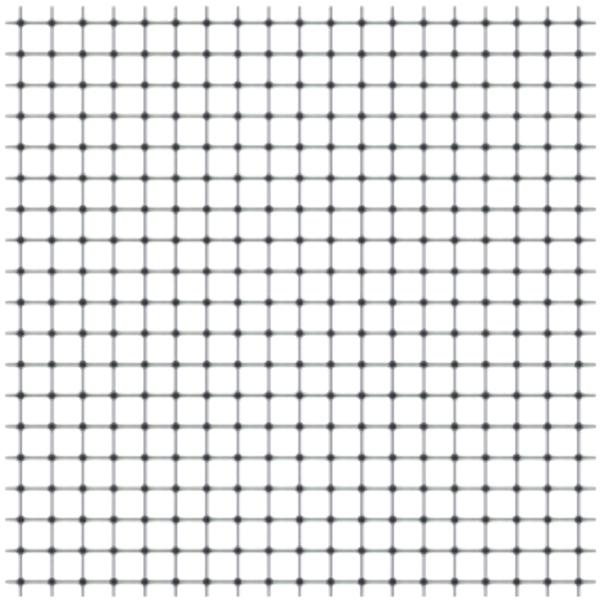

describe

the interactions between the elements of large complex

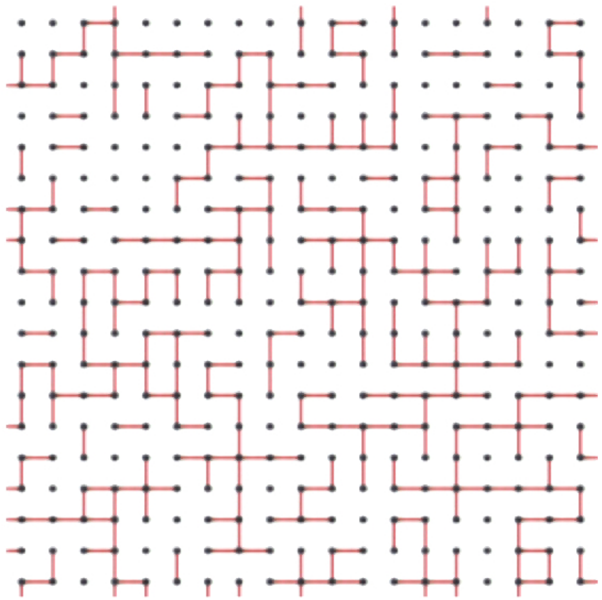
Biological, Social and Technological systems.

Randomness and order
Percolation

p=1

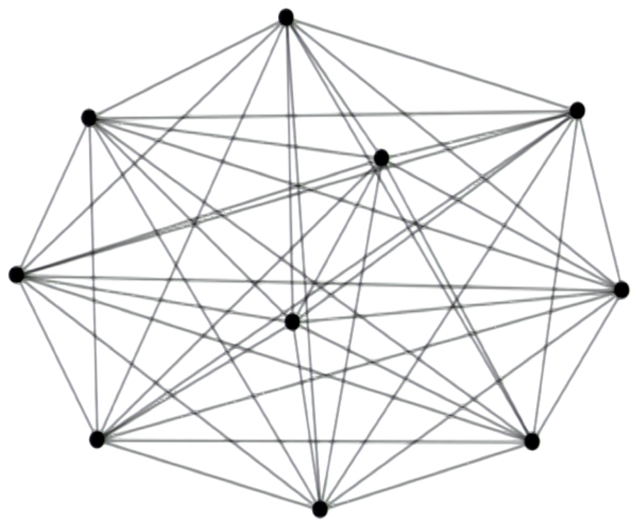


p=0.4



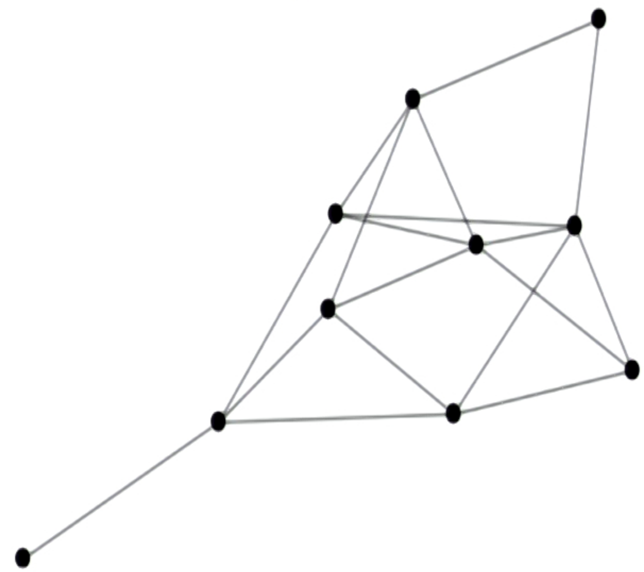
Randomness and order
Random graph

p=1



Complete graph

p=0.4

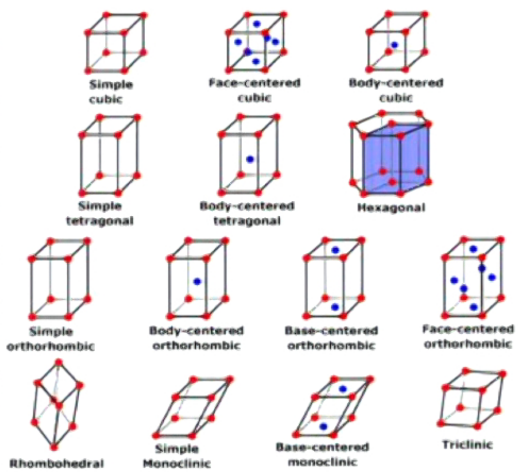


Random graph

Randomness and order
Complex networks

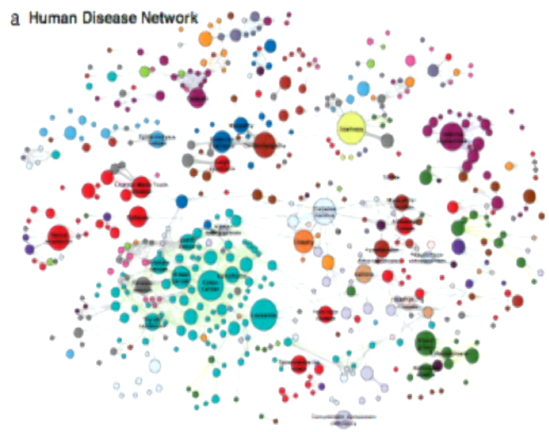
LATTICES | **COMPLEX NETWORKS** | **RANDOM GRAPHS**

LATTICES



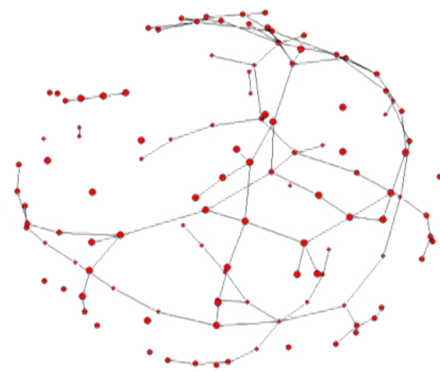
Regular networks
Symmetric

COMPLEX NETWORKS



Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

RANDOM GRAPHS



Totally random
Binomial degree distribution

Universalities

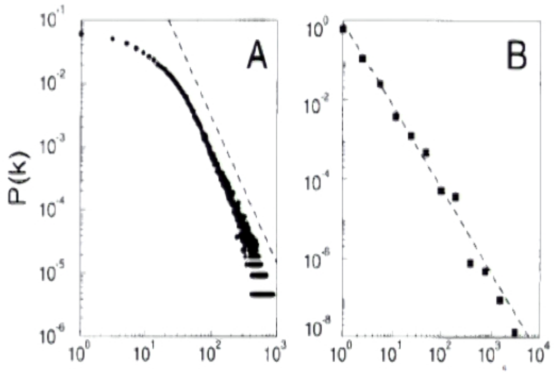
- Small-world: $D \propto \log N$

[Watts & Strogatz 1998]

- Scale-free: $P(k) \propto k^{-\gamma}$

[Barabasi & Albert 1999]

$\gamma \in (2,3]$



- Modularity: Local communities of nodes

[Fortunato 2010]

Models

- Maximum entropy ensembles:

  maximum random graphs satisfying a set of constraints

  -Configuration model, Exponential Random Graphs

- Deterministic models:

  Hierarchical models

  -Apollonian network, Pseudo-fractal network

- Non-equilibrium growing network models:

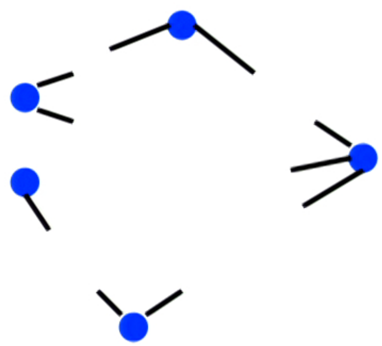
  Explanatory of emergent properties of complex networks

  -BA model, BB model

Networks with given degree sequence

Microcanonical ensemble

$$P(G)=\frac{1}{\aleph}\prod_i \delta\left(k_i-\sum_j a_{ij}\right)$$



Ensemble of network with exact degree sequence

Configuration model

Canonical ensemble

$$P(G)=\frac{1}{Z}e^{-\sum_i \lambda_i \sum_{j=1..N} a_{ij}} = \prod_{i<j} p_{ij}^{\,a_{ij}}(1-p_{ij})^{1-a_{ij}}$$

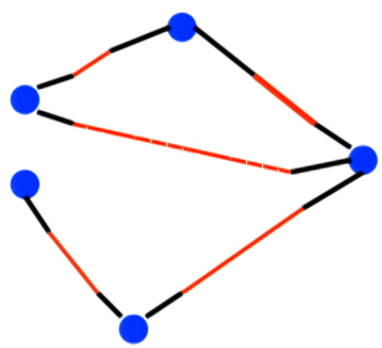
Ensemble of networks given expected degree sequence

Exponential Random Graph

Networks with given degree sequence

Microcanonical ensemble

$$P(G) = \frac{1}{\aleph}\prod_i \delta\left(k_i - \sum_j a_{ij}\right)$$

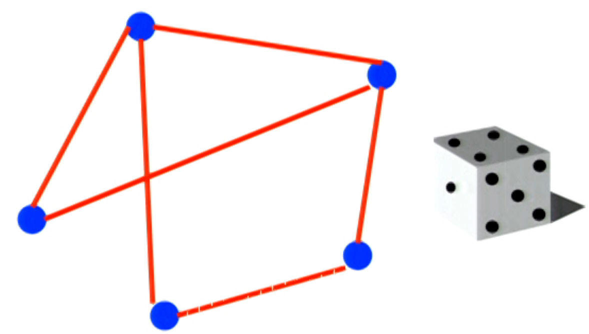


Ensemble of network with exact degree sequence

Configuration model

Canonical ensemble

$$P(G) = \frac{1}{Z} e^{-\sum_i \lambda_i \sum_{j=1..N} a_{ij}} = \prod_{i<j} p_{ij}^{a_{ij}} (1 - p_{ij})^{1-a_{ij}}$$



Ensemble of networks given expected degree sequence

Exponential Random Graph

Critical phenomena on scale-free networks

Scale free networks: $\langle k \rangle$ *finite* $\langle k^2 \rangle \to \infty$

- **Percolation:**

Percolation threshold

$$p_c = \frac{\langle k \rangle}{\langle k^2 \rangle} \to 0$$

Scale free networks are always percolating

- **Ising model:**

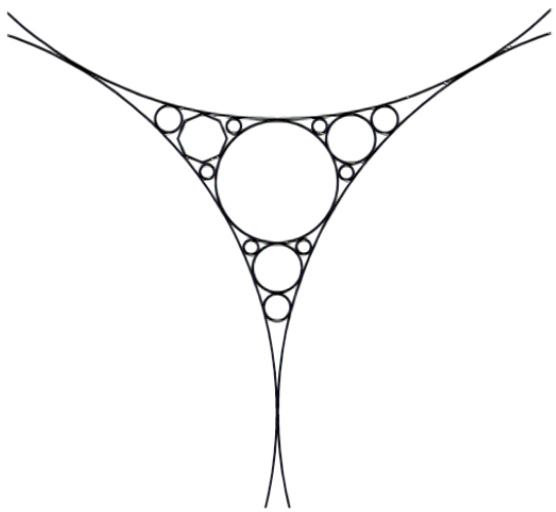
Critical temperature

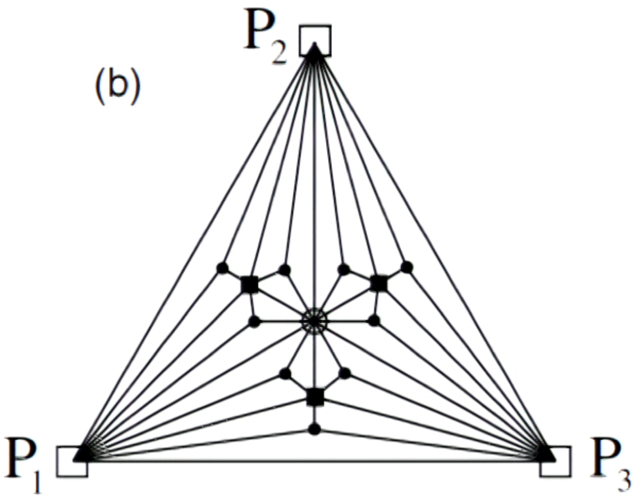
$$T_c \approx J \frac{\langle k^2 \rangle}{\langle k \rangle} \to \infty$$

The Ising model on scale-free networks
is always in the ferromagnetic phase

Apollonian networks

Apollonian networks are formed by linking the centers of an Apollonian sphere packing
They are scale-free and are described by the Apollonian group





[Andrade et al. PRL 2005]
[Soderberg PRA 1992]

Growth by uniform attachment of links

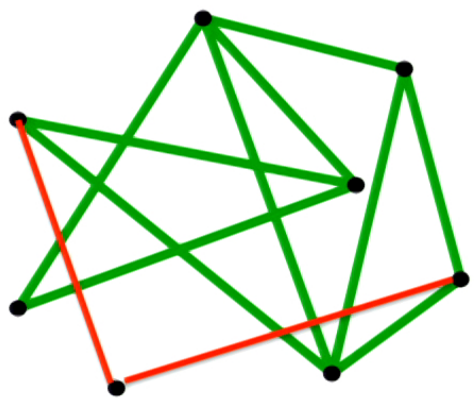
GROWTH :

At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

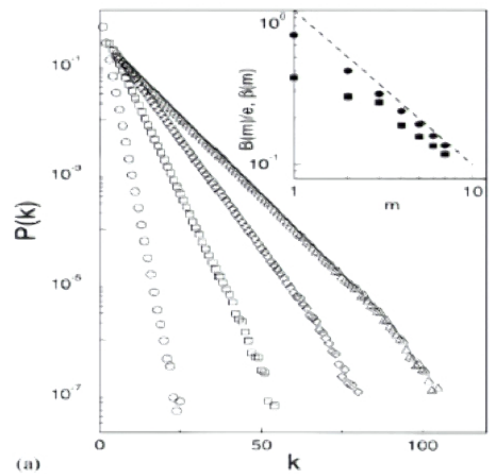
UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$



Exponential



[Barabási & Albert, Physica A (1999)]

Barabasi-Albert model

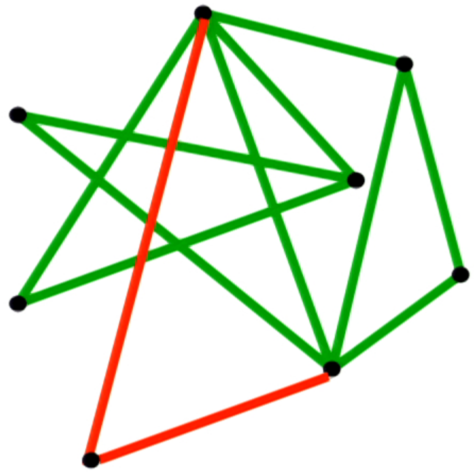
GROWTH :

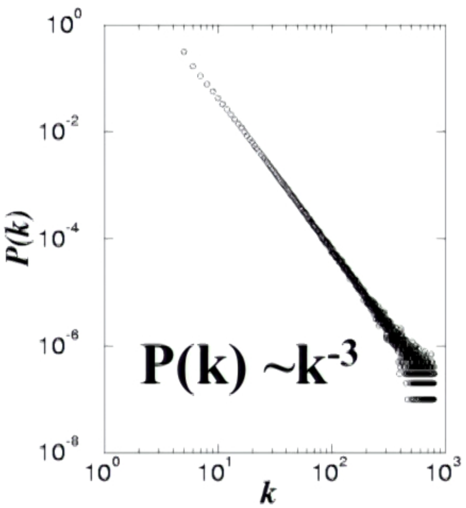
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

PREFERENTIAL ATTACHMENT :

The probability $\Pi(k_i)$ that a new node will be connected to node *i* depends on the connectivity k_i of that node

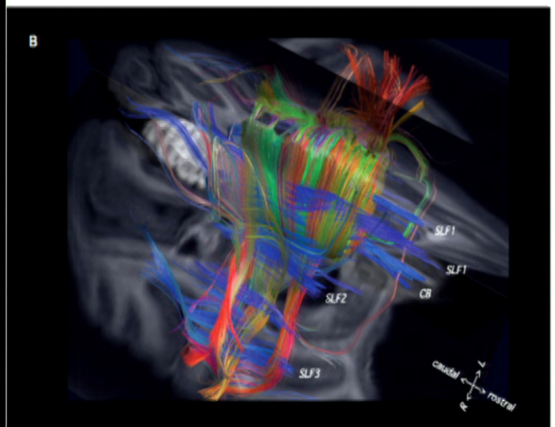
$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

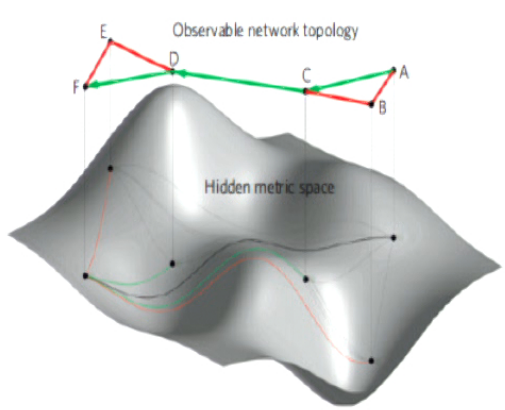


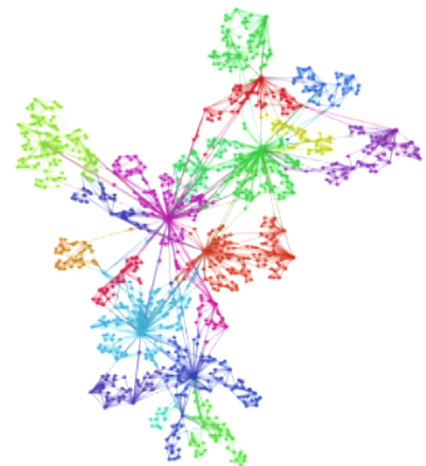


[Barabási et al. Science (1999)]

Network Topology and Network Geometry







are expected to have impact in a variety of applications,

ranging from

brain research to routing protocols in the Internet

Hyperbolicity & Complex Networks

- Open debate over best measures of curvature for discrete networks

  [see for instance recent works of Ollivier,Jost,Yau,Loll]

- Hyperbolic networks have been claimed to allow better navigability of the Internet

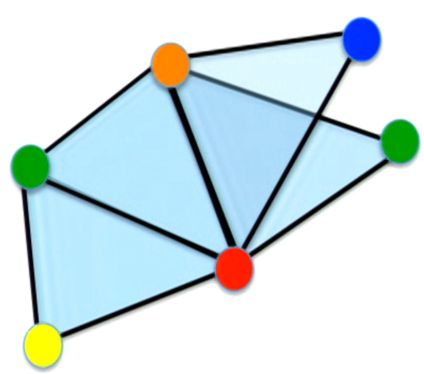
  [Kleinberg 2007, Boguna et al. 2009]

- Hidden hyperbolic metric is considered often as latent space for network evolution

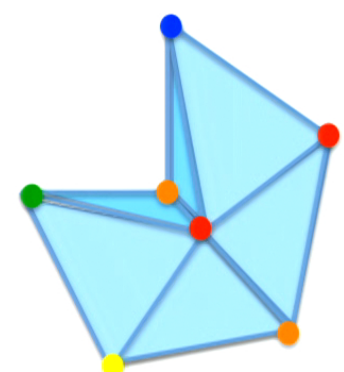
  [Krioukov et al. 2010,Osat,Radicchi 2018]

Simplicial Complexes

Simplicial complexes are characterizing the interaction between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.



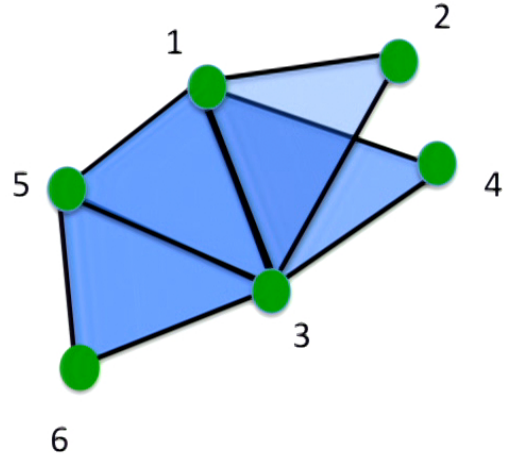
d=2 simplicial complex



d=3 simplicial complex

Generalized degrees

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.



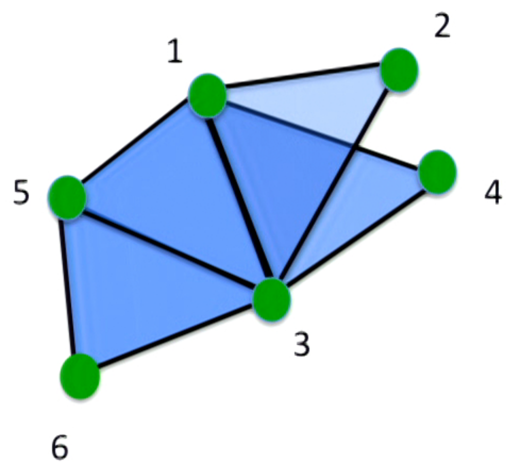
$k_{2,0}(\mu)$ Number of triangles incident to the node μ

$k_{2,1}(\mu)$ Number of triangles incident to the link μ

[Bianconi & Rahmede (2016)]

Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.



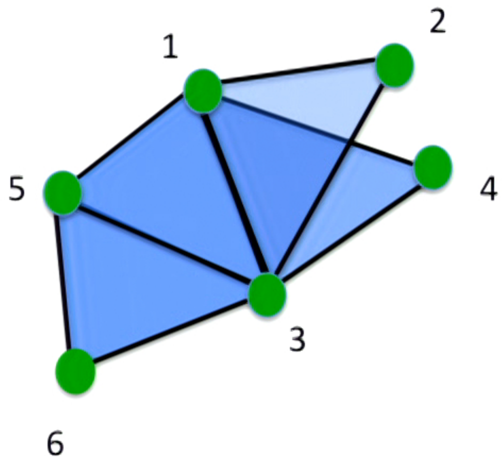
i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

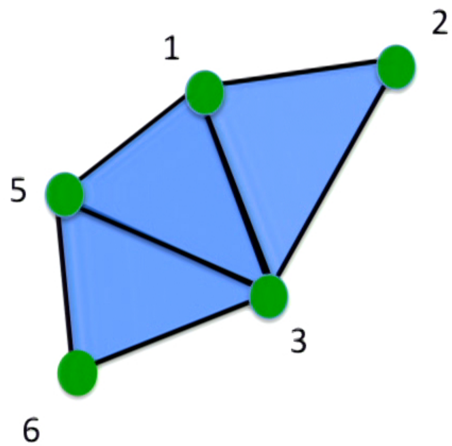
Manifolds

If n_μ takes only values $n_\mu=0,1$ each (d-1)-face is incident at most to two d-dimensional simplices.

In this case the simplicial complex is a discrete manifold.



NOT A MANIFOLD



MANIFOLD

Network Geometry with Flavor

Starting from a single d-dimensional simplex

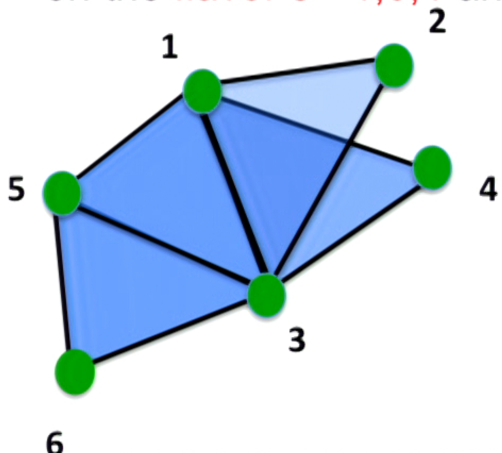
GROWTH :

At every timestep we add a new d simplex

(formed by one new node and an existing (d-1)-face).

ATTACHMENT:

The probability that a new node will be connected to a face μ depends on the flavor s=-1,0,1 and is given by



$$\Pi_{\mu}^{[s]} = \frac{1 + s n_{\mu}}{\sum_{\mu'} (1 + s n_{\mu'})}$$

[Bianconi & Rahmede (2016)]

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Attachment probability

$$\Pi_{\mu}^{[s]} = \frac{(1 + s\, n_{\mu})}{\sum_{\mu' \in Q_{d,d-1}} (1 + s\, n_{\mu'})} = \begin{cases} \dfrac{(1 - n_{\mu})}{Z^{[-1]}}, & s = -1 \\ \dfrac{1}{Z^{[0]}}, & s = 0 \\ \dfrac{k_{\mu}}{Z^{[1]}}, & s = 1 \end{cases}$$

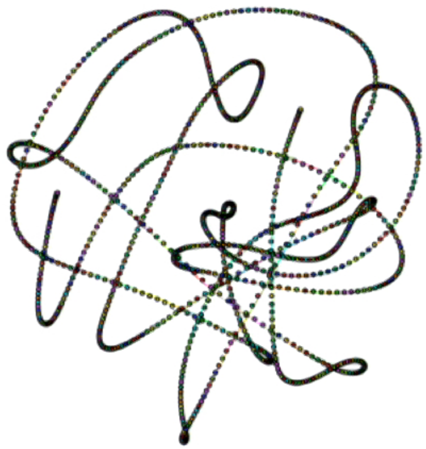
s=-1 Manifold n_{μ}=0,1

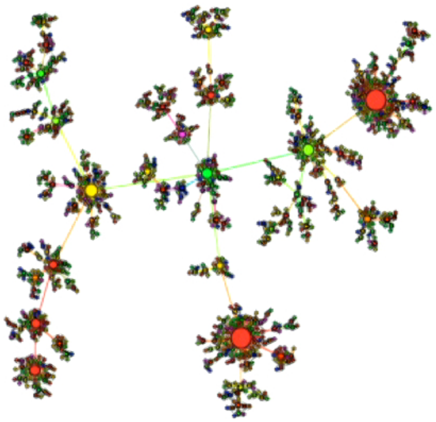
s=0 Uniform attachment n_{μ}=0,1,2,3,4…

s=1 Preferential attachment n_{μ}=0,1,2,3,4…

Dimension d=1

Manifold | **Uniform attachment** | **Preferential attachment**

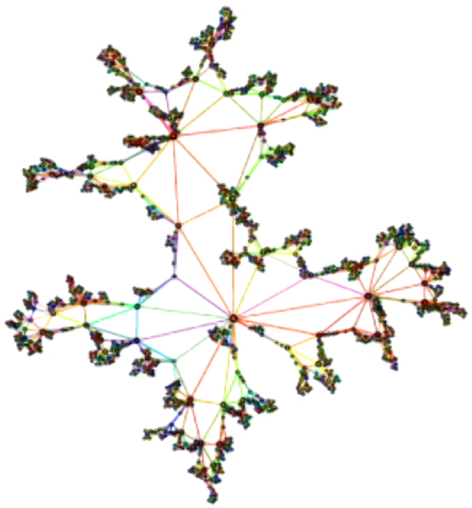


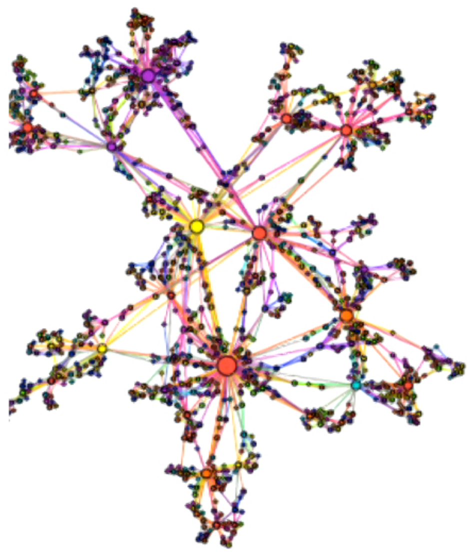


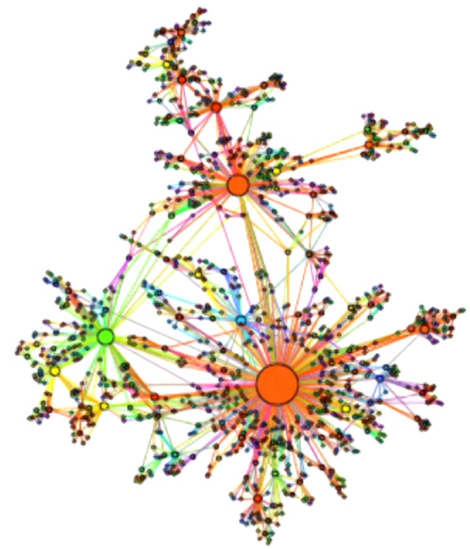
Chain | **Exponential** | **Scale-free BA model**

Dimension d=2

Manifold | **Uniform attachment** | **Preferential attachment**

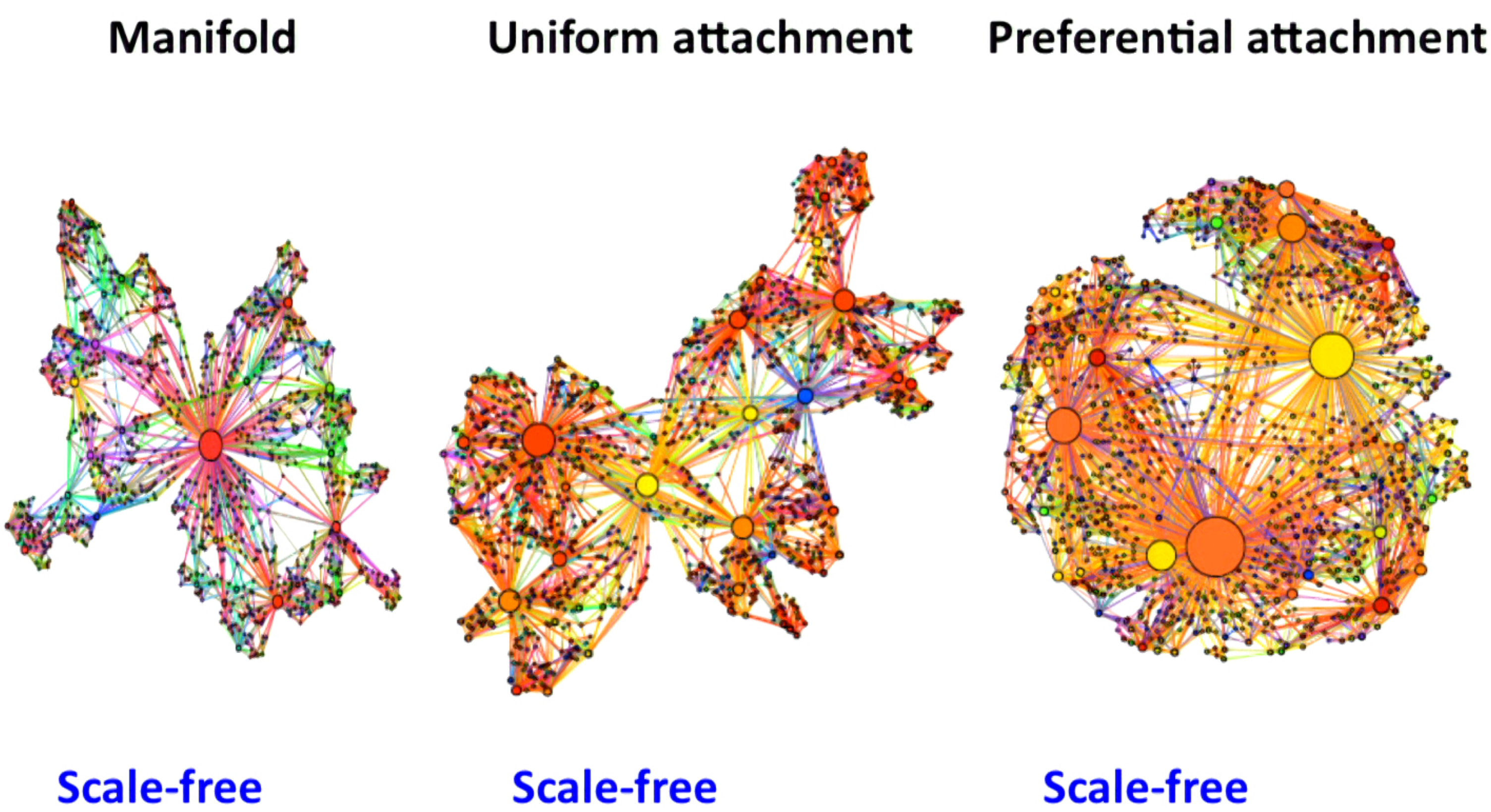




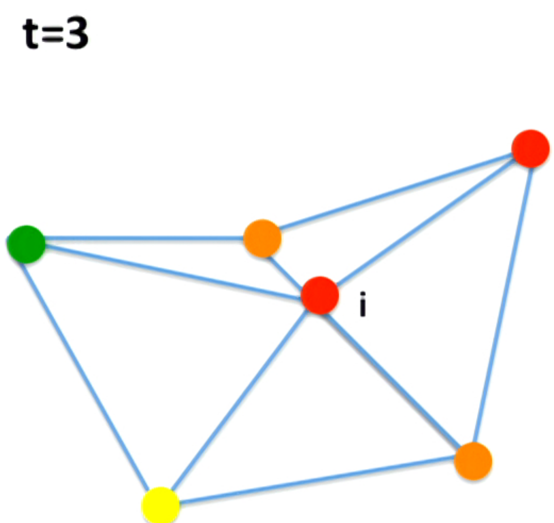


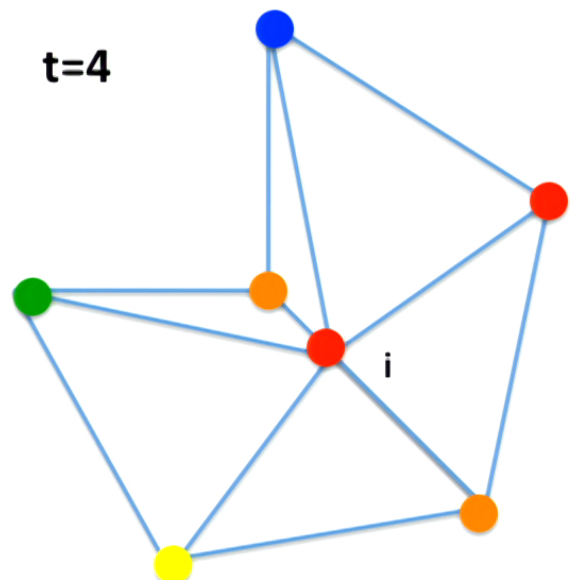
Exponential | **Scale-free** | **Scale-free**

Dimension d=3



Effective preferential attachment in d=3





Node i has generalized degree 3

Node i is incident to 5 unsaturated faces

Node i has generalized degree 4

Node i is incident to 6 unsaturated faces

Degree distribution

For d+s=1

$$P_d(k) = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

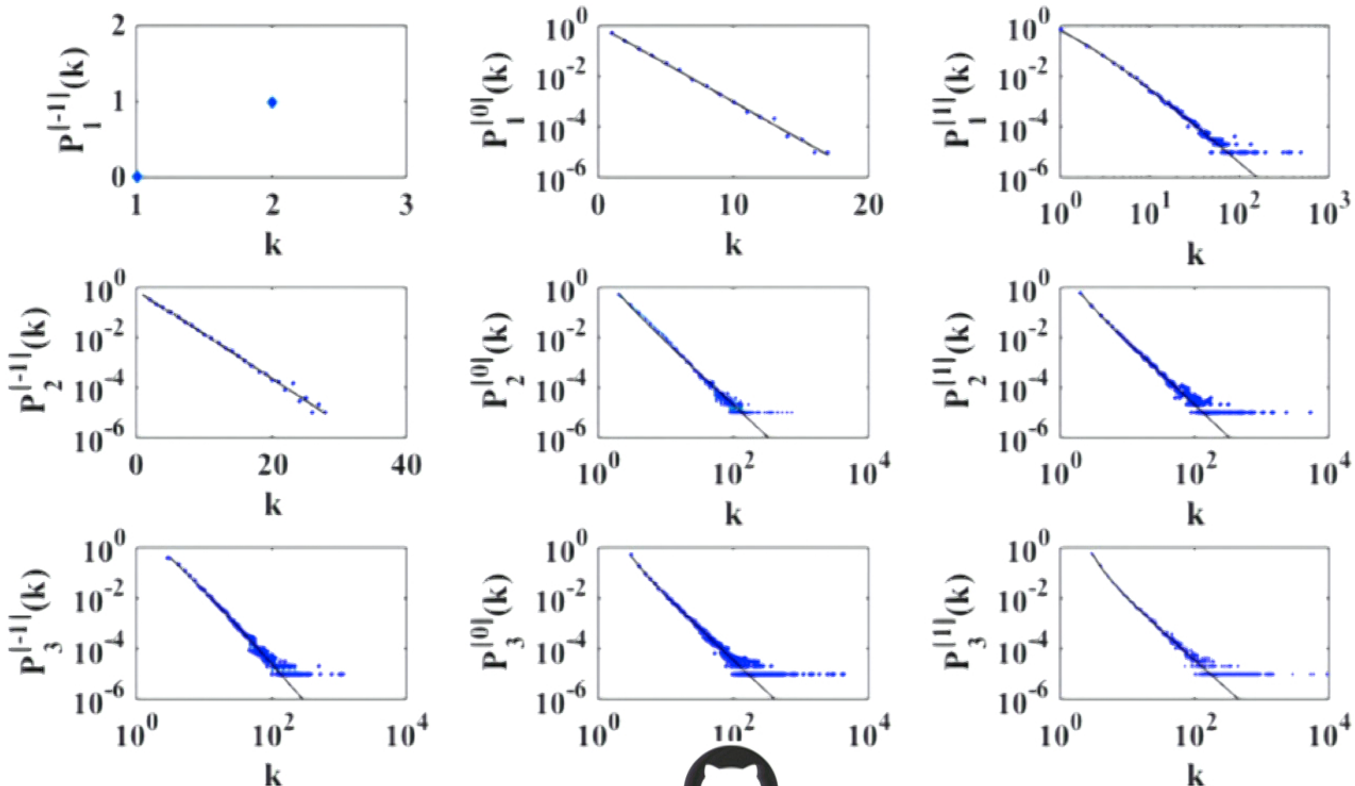
For d+s>1

$$P_d(k) = \frac{d+s}{2d+s} \frac{\Gamma(1+(2s+s)(d+s-1))}{\Gamma(d/(d+s-1))} \frac{\Gamma(k-d+d/(d+s-1))}{\Gamma(k-d+(2d+s)(d+s-1))}$$

NGF are always scale-free for d>1-s

- **For s=1 NGF are always scale free**
- **For s=0 and d>1 the NGF are scale-free**
- **For s=-1 and d>2 the NGF are scale-free**

Degree distribution of NGF



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Generalized degree distributions

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Bimodal	Exponential	Power-law
$\delta = d-2$	Exponential	Power-law	Power-law
$\delta \leq d-3$	Power-law	Power-law	Power-law

The power-law

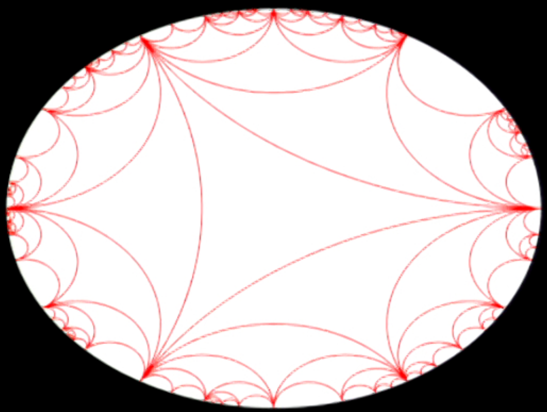
generalized degree distribution

are scale-free for

$$d \geq d_c^{[\delta,s]} = 2(\delta+1)+s$$

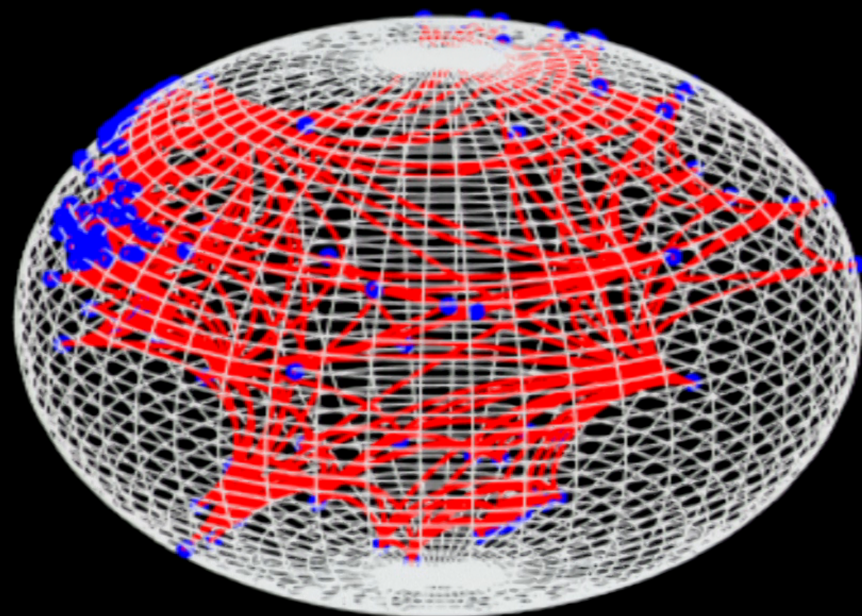
Emergent Hyperbolic geometry

The emergent hidden geometry is the hyperbolic H^d space
Here all the links have equal length



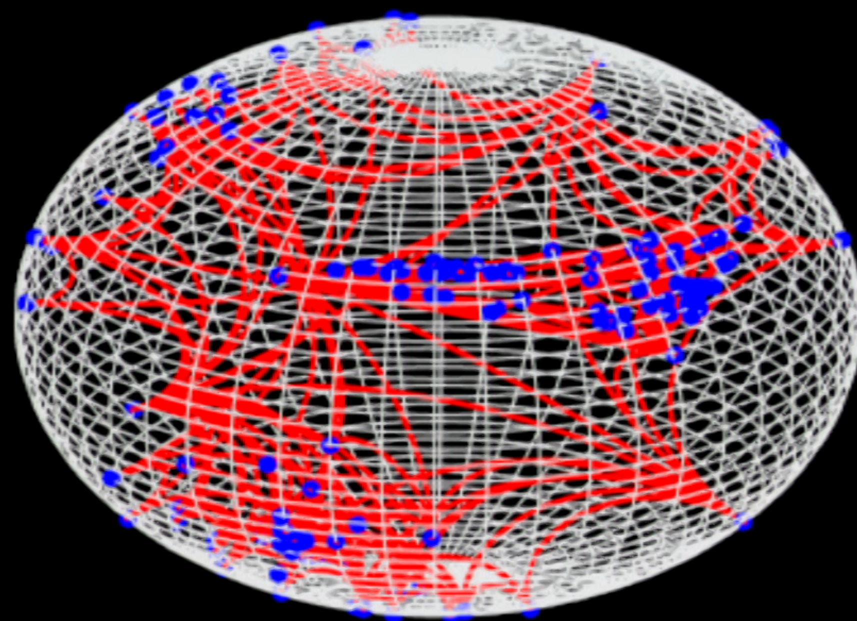
d=2

Emergent hyperbolic geometry



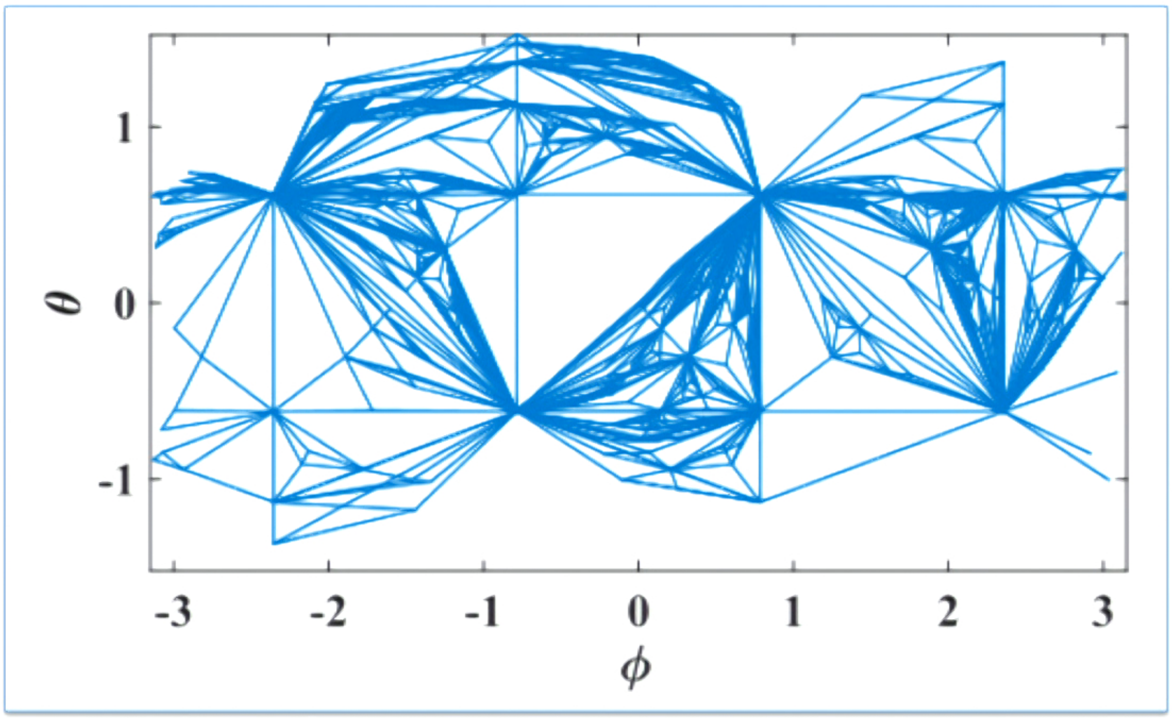
d=3

Emergent hyperbolic geometry

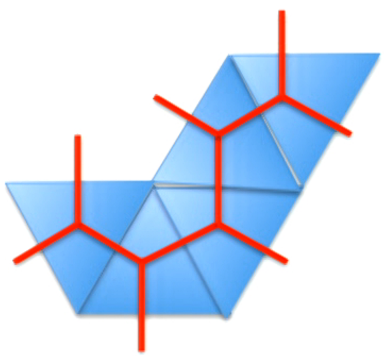


d=3

Connection with the Apollonian network

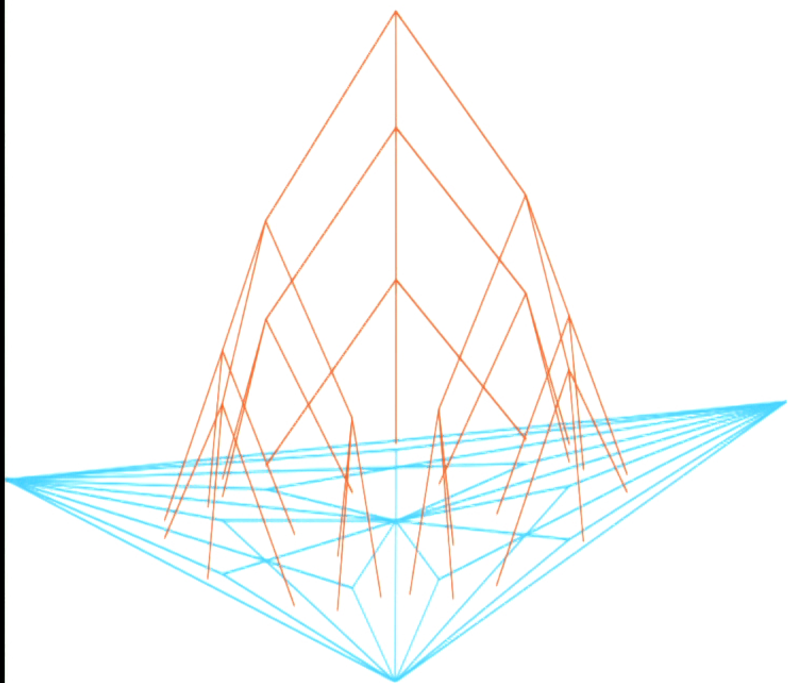


The Dual of the NGF with s=-1 is a tree of branching ratio d

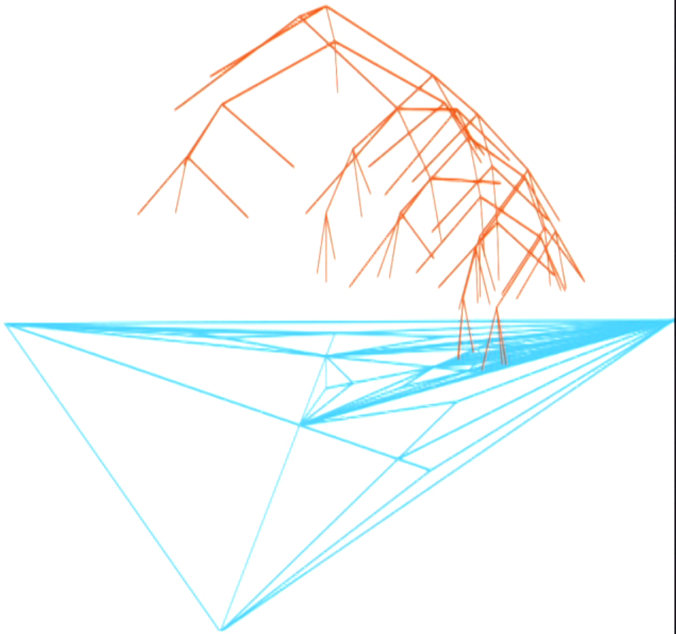


- To every d-simplex we associate a node of the dual
- To every (d-1)-face shared by two d-simplicies we associate a link

The relation to Trees



Line graph of the Apollonian network



Line graph of the NGF

Non-equilibrium

Topological phase transitions

can occur if (d-1)-faces are not choosen randomly

G. Bianconi Rahmede Scientific Reports (2017)

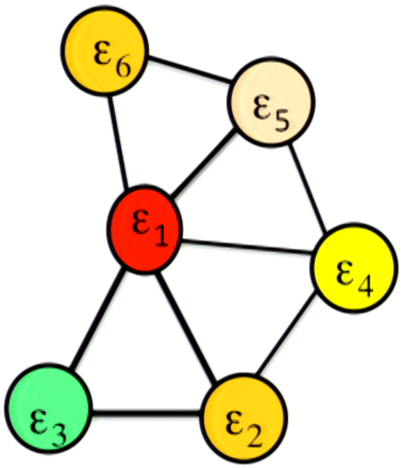
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

$g(\varepsilon)$ *distribution*



Network Geometry with Flavor

Starting from a single d-dimensional simplex

GROWTH :

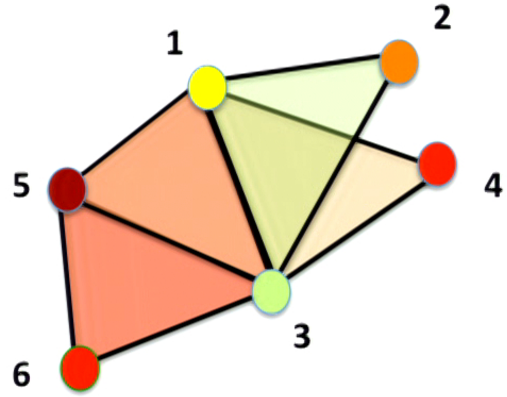
At every timestep we add a new node d simplex

(formed by one new node and an existing (d-1)-face).

The new node has energy ε drawn from the distribution $g(\varepsilon)$

ATTACHMENT:

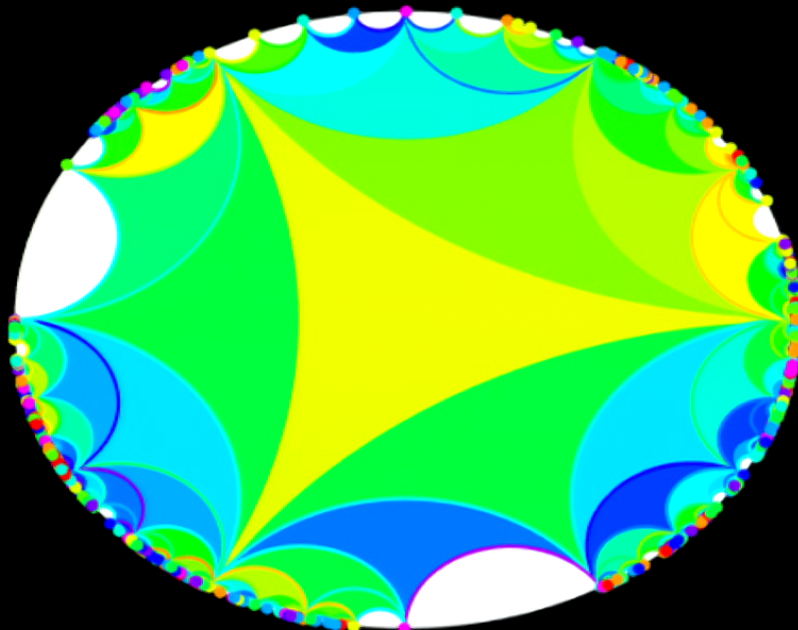
The probability that a new node will be connected to a face μ depends on the flavor s=-1,0,1 and is given by



$$\Pi_{\mu}^{[s]} = \frac{e^{-\beta\varepsilon_{\mu}}(1+sn_{\mu})}{\sum_{\mu'} e^{-\beta\varepsilon_{\mu'}}(1+sn_{\mu'})}$$

Bianconi & Rahmede (2016)

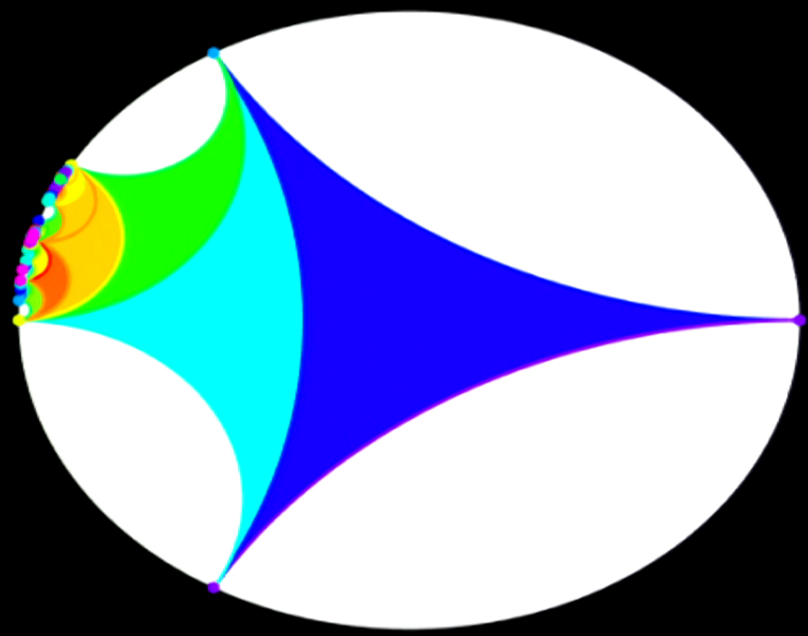
Emergent geometry at high temperature



d=2

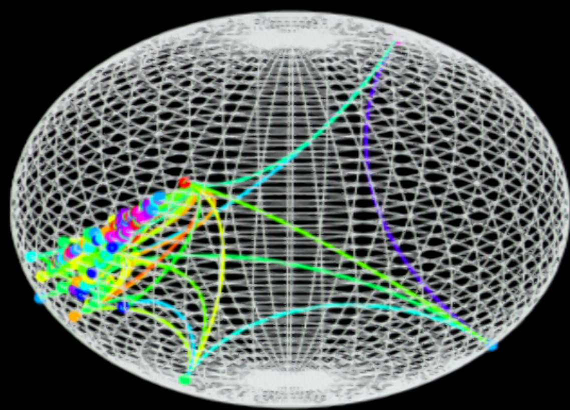
β=0.01

Emergent geometry at low temperature



d=2
β=5

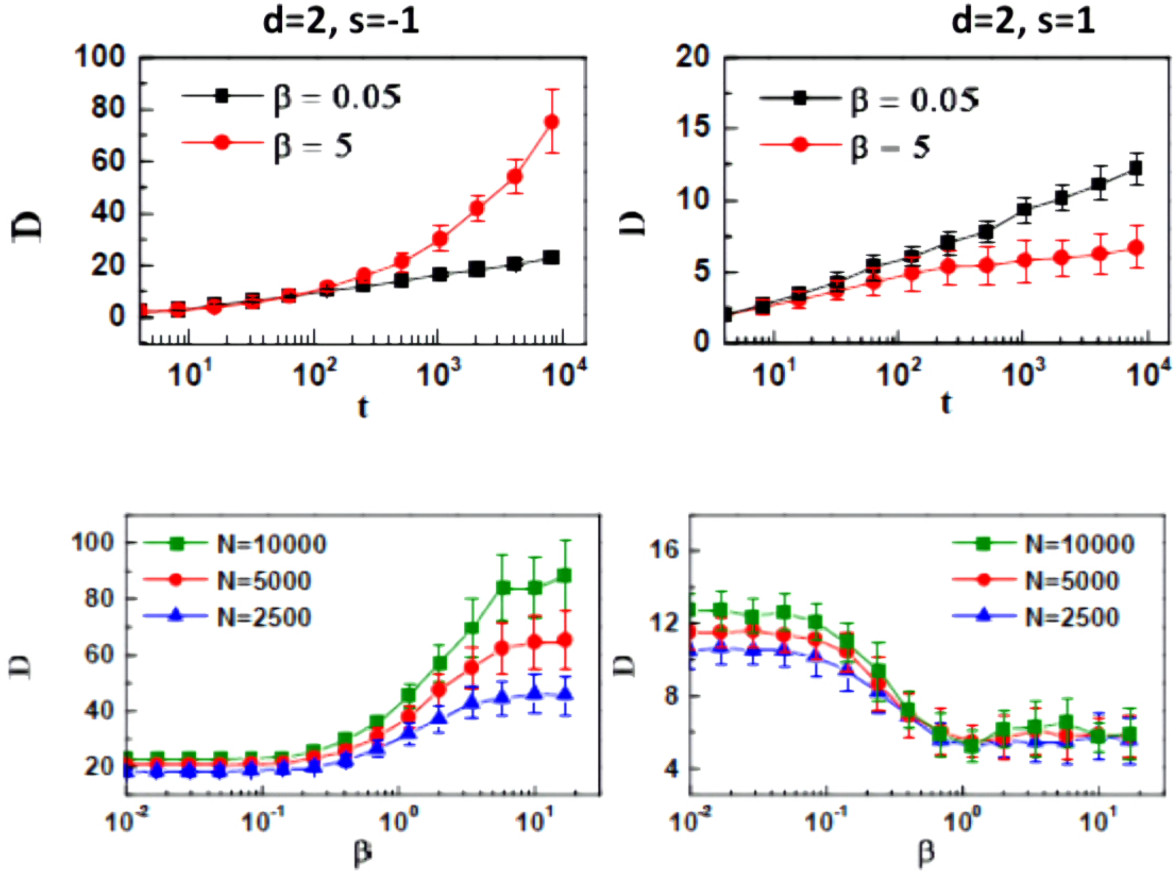
Emergent geometry at low temperature



d=3

β=5

Transition: Diameter



[Bianconi, Rahmede, Wu (2015)]

Topological percolation

Topological damage

On networks

damage can occur only

on nodes or on links.

On simplicial complexes

topological damage can be directed also

to higher dimensional simplicies,

such as

triangles, tetrahedra etc.

Topological percolation

On d=2 simplicial complexes we distinguish
4 types of topological percolation problems:

Link percolation: Links are removed with probability q=1-p.
Nodes are connected to nodes through intact links
Triangle percolation: Triangles are removed with probability q=1-p.
Links are connected to links through intact triangles.
Node percolation: Nodess are removed with probability q=1-p.
Links are connected to links through intact nodes
Upper link percolation: Links are removed with probability q=1-p.
Triangles are connected to triangles though intact links

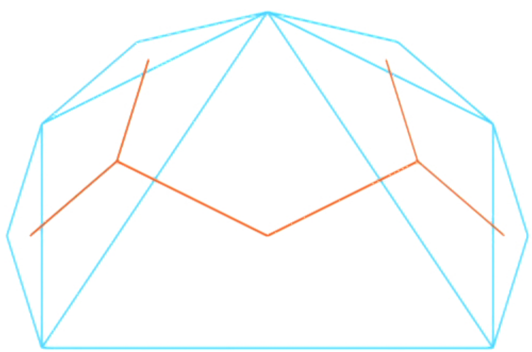
On d=3 simplicial complexes we distinguish
6 types of topological percolation problems:
Link percolation, Triangle percolation, tetrahedron percolation
Node percolation, upper link percolation, Upper triangle percolation

[Bianconi and Ziff 2018]

Hyperbolic Simplicial complexes

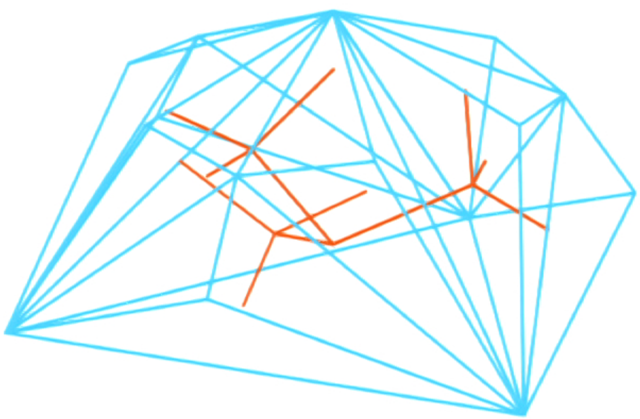
d=2 HYPERBOLIC SIMPLICIAL COMPLEX

We start from a link.
At each iteration we glue a triangle to any link added at the previous iteration



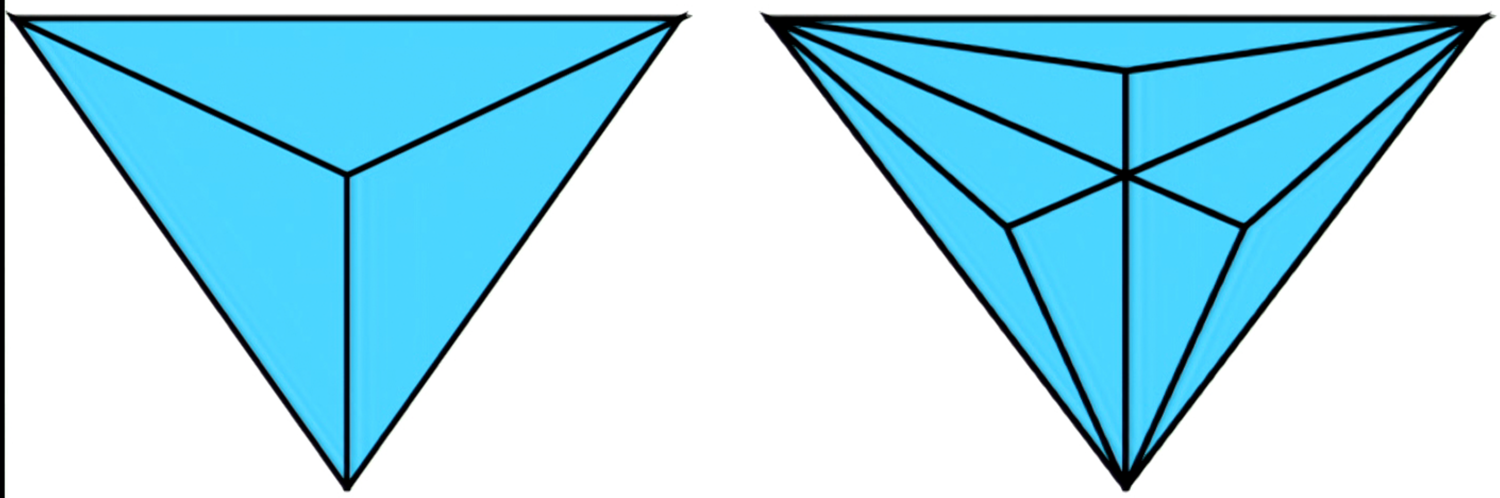
d=3 HYPERBOLIC SIMPLICIAL COMPLEX

We start from a triangle.
At each iteration we glue a tetrahedron to any triangle added at the previous iteration

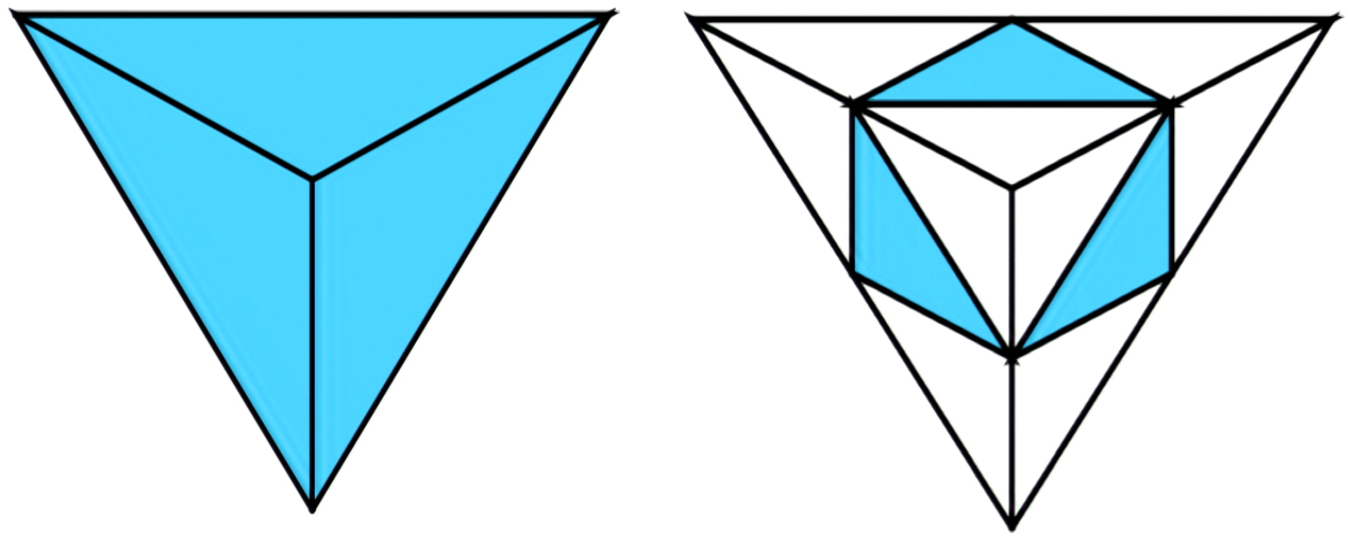


The d=3 Hyperbolic Simplical Complex

At the level of the network skeleton
the d=3 Hyperbolic Simplicial Complex
reduces to the Apollonian network

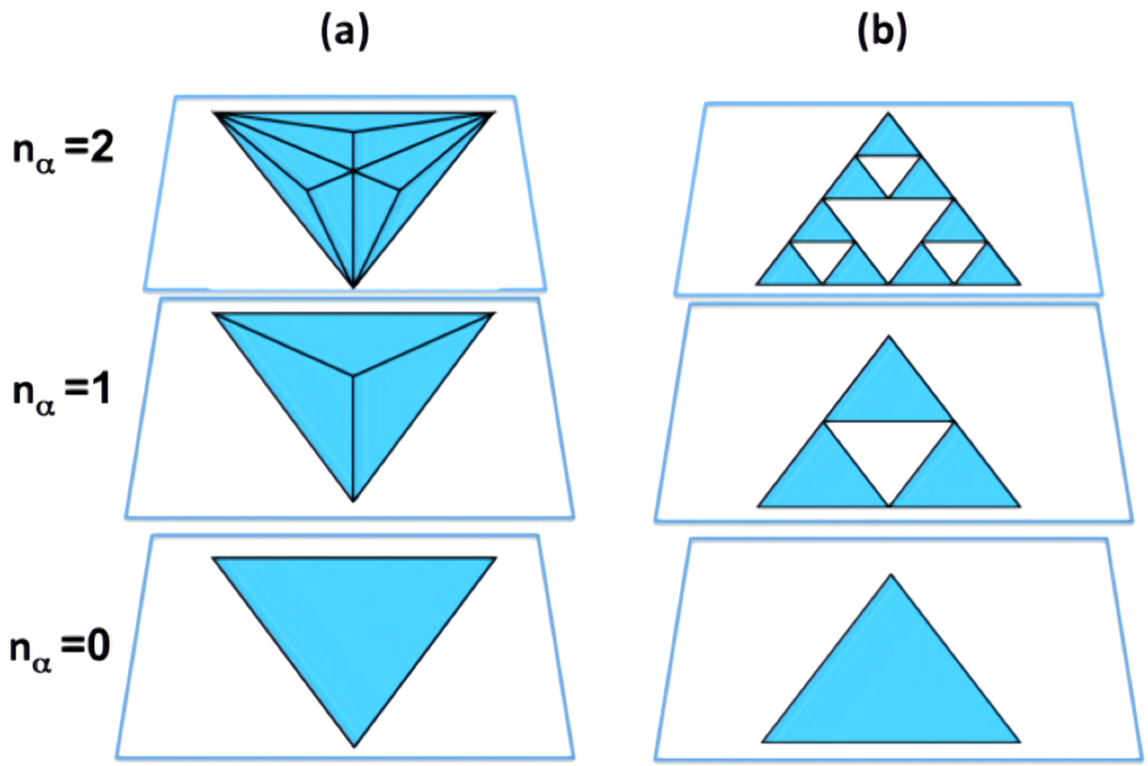


The line graph of the Apollonian network is the Sierpinski gasket



[Bianconi and Ziff 2018]

The line graph of the d=3 Hyperbolic Simplicial Complex is the multiplex Sierpinski gasket



Percolation in hyperbolic networks

Percolation in hyperbolic networks is known to have two percolation thresholds p^l and p^u.

- For $p<p^l$ no infinite cluster exist
- For $p^l<p<p^u$ the maximum cluster is infinite but sub-extensive
- For $p>p^u$ the maximum cluster is extensive

Topological percolation for d=2 hyperbolic simplicial complex

d=2	p^l	p^u	
Link percolation	0	1/2	Discontinuous: non trivial
Triangle percolation	1/2	1	Discontinuous: trivial
Node percolation	0	1	Discontinuous: trivial
Upper link percolation	1/2	1	Discontinuous: trivial

All transitions are discontinuous. Only link percolation is non-trivial

[Bianconi and Ziff 2018]

Topological percolation for d=3 hyperbolic simplicial complex

d=3	p^l	p^u	
Link percolation	N/A	0	**Continuous: Typical Scale-free network scaling**
Triangle percolation	0	0.307981...	Continuous: **BKT transition**
Tetrahedron percolation	1/3	1	Discontinuous: Trivial
Node percolation	0	1	Discontinuous: Trivial
Upper link percolation	0	1	Discontinuous: Trivial
Upper triangle percolation	1/3	1	Discontinuous: Trivial

[Bianconi and Ziff 2018]

Comments

Nodes and link percolation
cannot be used to predict
the other topological percolation problems

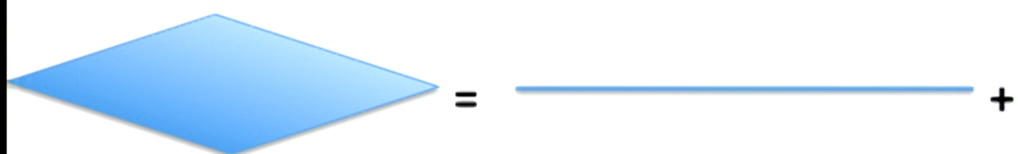
- *In d=2 Hyperbolic simplicial complex all transitions are discontinuous while in d=3 link and triangle percolation are continuous*
- *Link percolation in d=2 displays a non trivial discontinuous transition while no such transition is observed in d=3*
- *Triangle percolation in d=3 is a BKT transition while no such transition is observed in d=2*

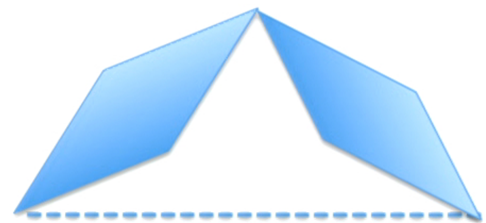
Link percolation in d=2 hyperbolic simplicial complex

The probability T_{n+1} that the two initial nodes are connected at iteration n+1 is given by the RG equation

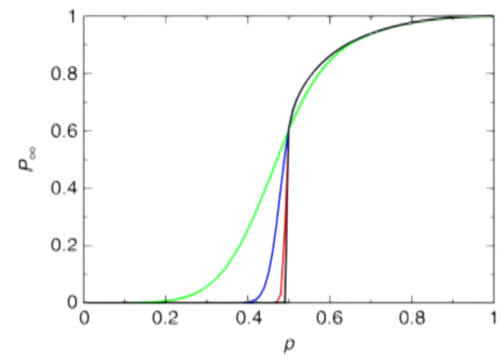
[Boettcher,Singh, Ziff 2012]

$$T_{n+1} = p + (1-p)T_n^2$$



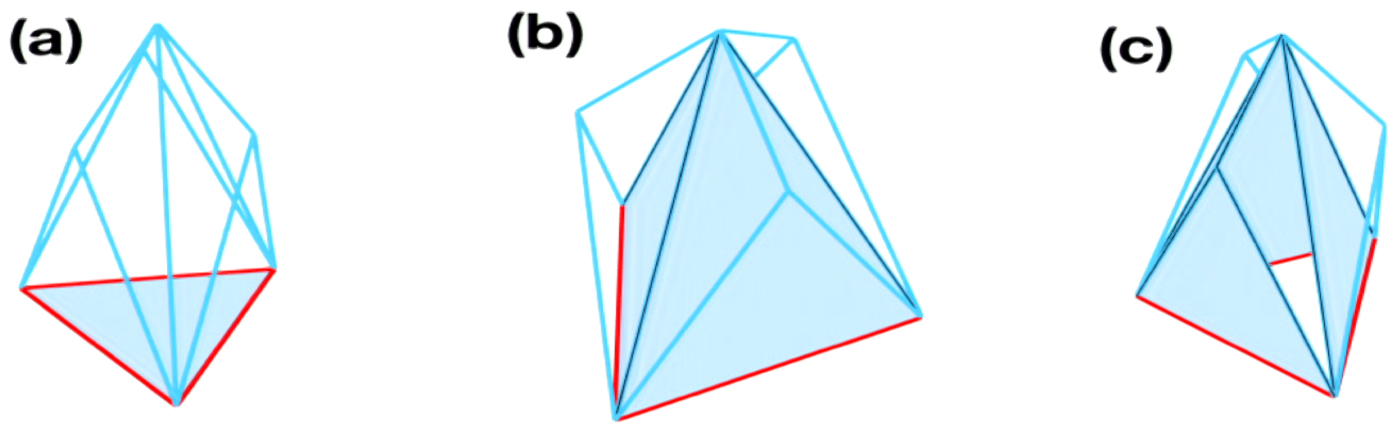


A RG study of the generating functions show that the upper percolation transition Is discontinuous at p=0.5 and non-trivial.



Triangle percolation for the d=3 hyperbolic simplicial complex

The order parameter is the fraction of links
connected to the initial three links through intact triangles

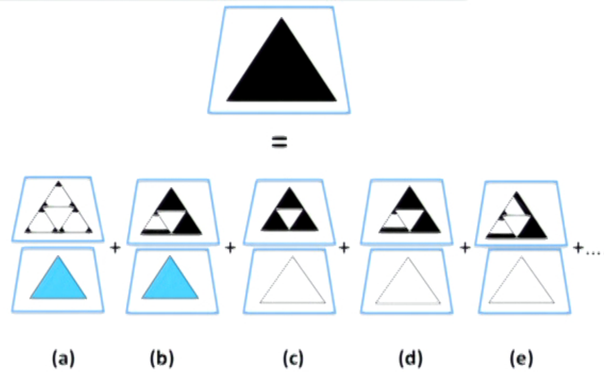


The RG equations

The probability T_{n+1}, S_{n+1}, W_{n+1} that three, two or none of the initial links are connected at iteration n+1 is given by the RG equation

$$T_{n+1} = p + (1-p)(T_n^3 + 6T_n^2 S_n + 3T_n S_n^2)$$
$$S_{n+1} = (1-p)[T_n^2(S_n + W_n) + T_n S_n(7S_n + 2W_n) + S_n^2(4S_n + W_n)]$$
$$W_{n+1} = 1 - T_{n+1} - 3S_{n+1}$$

The RG equations can be written down diagramatically using the multiplex Sierpinski gasket

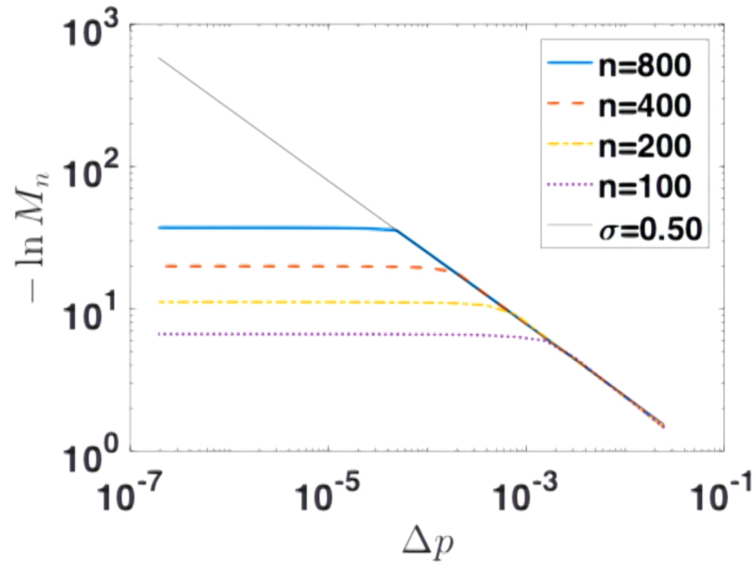


[Bianconi and Ziff 2018]

Berezinskii-Kosterlitz-Thouless transition

Triangle percolation on the d=3 hyperbolic simplicial complex undergoes a BKT transition
with the order parameter scaling like

$$M_\infty \propto \mathbf{exp}\left(-A / |\Delta p|^\sigma\right) \qquad \sigma = 1/2$$



[Bianconi and Ziff 2018]

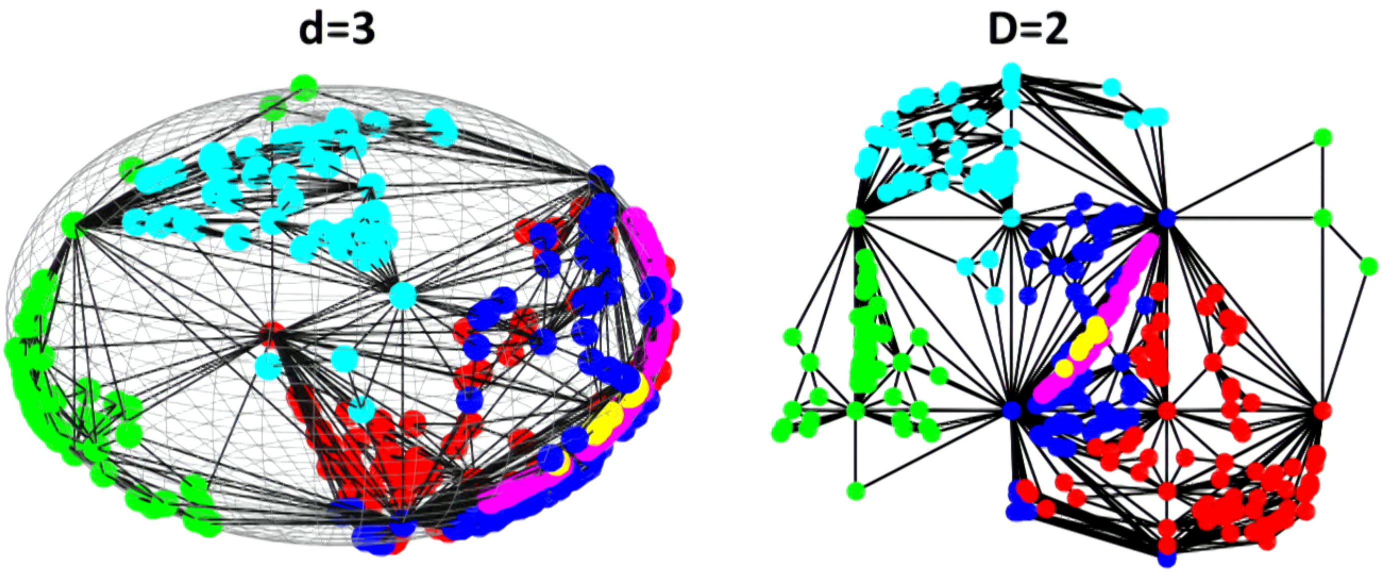
Complex Network Manifolds

(NGF with s=-1)

and

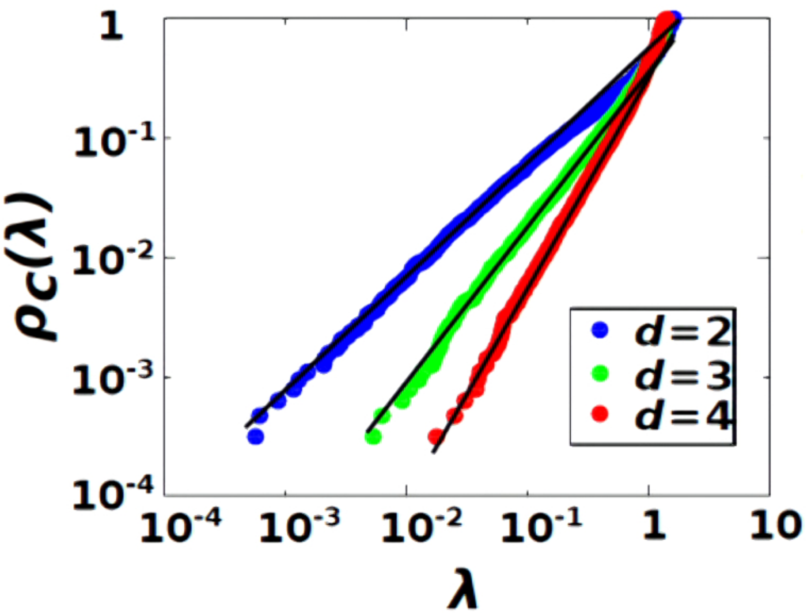
Frustrated Synchronization

Holography of Complex Network Manifolds



d-dimensional Complex Network Manifolds can be interpreted as D-dimensional manifolds with D=d-1

Spectral dimensions of Complex Network Manifolds



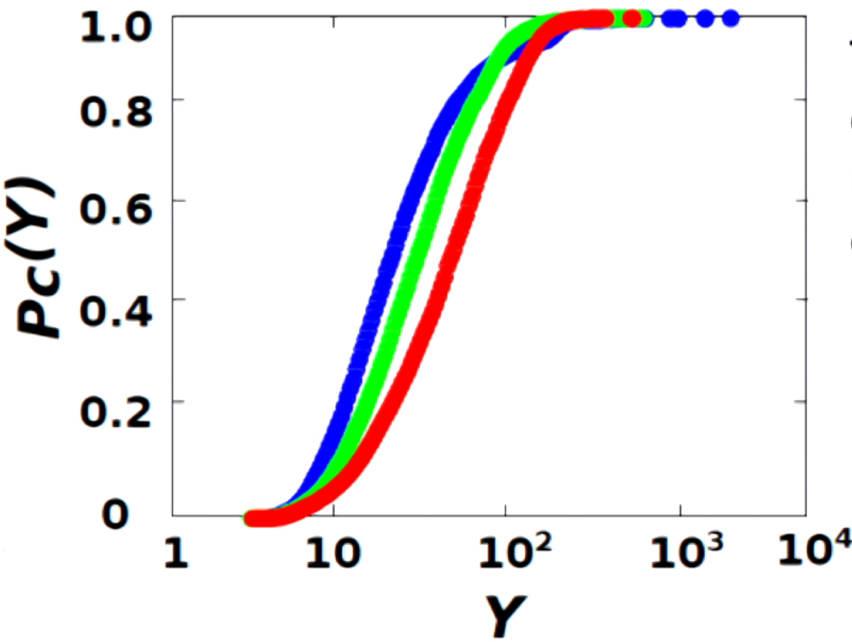
$$L_{ij} = \delta_{ij} - \frac{a_{ij}}{k_i}$$

$$\rho_c(\lambda) \approx \lambda^{-d_s/2}$$

Complex Network Manifolds have finite spectral dimension with

$$d_s \approx d \;\; for \; d = 2,3,4$$

Localization of the eigenvectors



The participation ratio evaluates the effective number of nodes on which an eigenmode is localized

$$Y_\lambda = \left[\sum_{i=1}^{N} \left(u_i^\lambda v_i^\lambda \right)^2 \right]^{-1}$$

A large number of eigenmodes are localized

The Kuramoto model

We consider the Kuramoto model

$$\frac{d\vartheta_i}{dt} = \omega_i + \sigma \sum_{j=1}^{N} \frac{a_{ij}}{k_i} \sin(\vartheta_j - \vartheta_i)$$

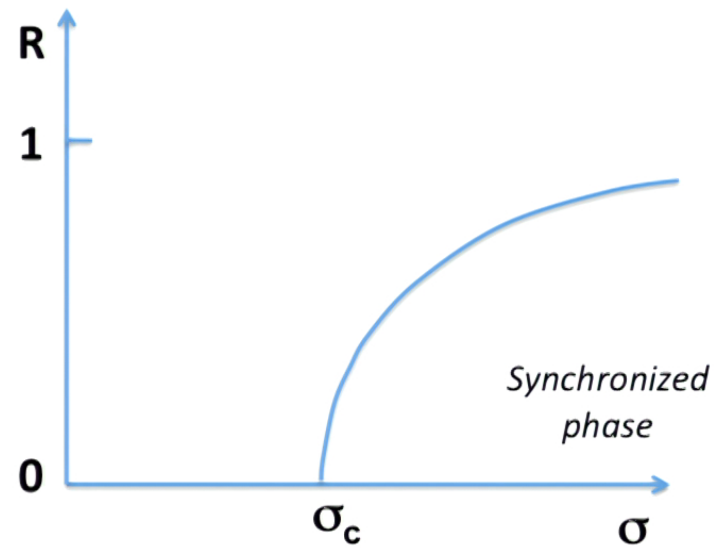
where ω_i is the internal frequency of node i drawn randomly from a Gaussian distribution

The global order parameter is

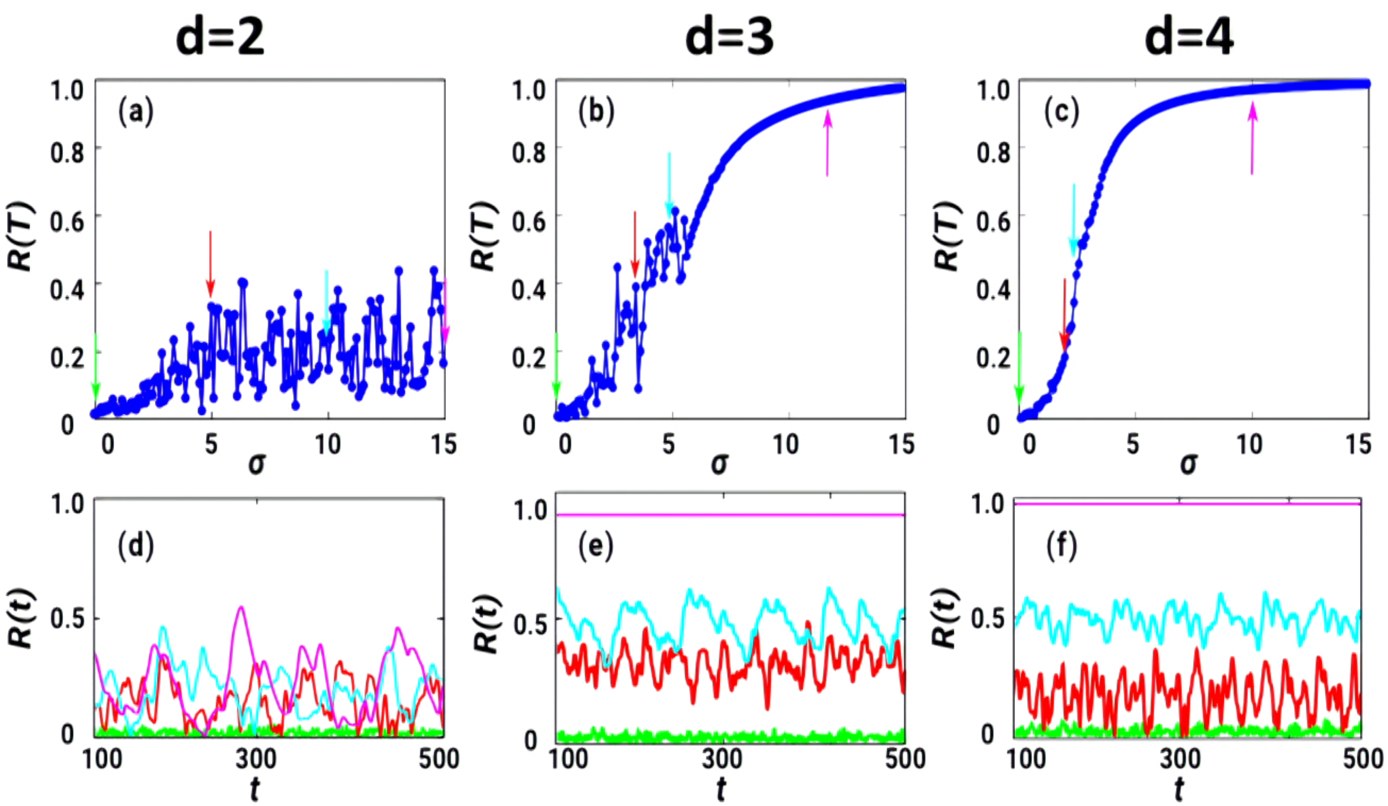
$$R = \frac{1}{N} \sum_{j=1}^{N} e^{i\vartheta_j}$$

Kuramoto Model

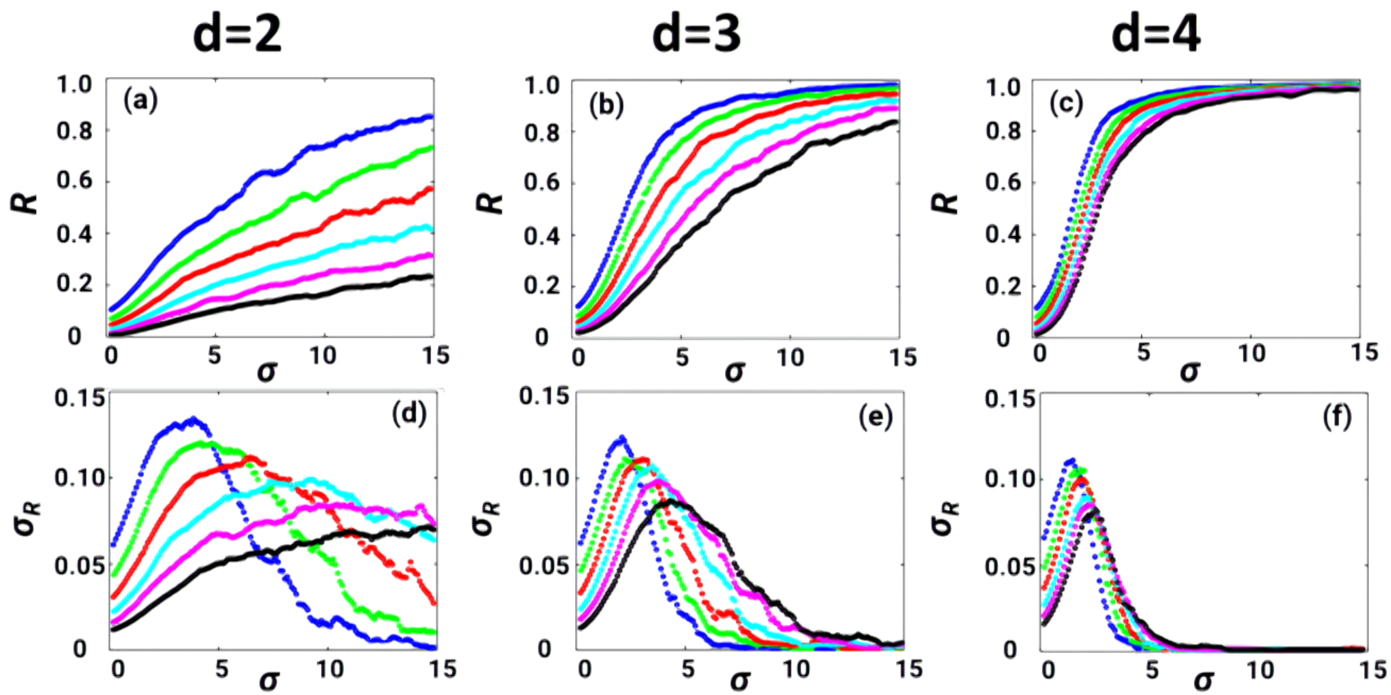
In an infinite fully connected network we have



Frustrated synchronization



Finite size effects



N=100,200,400,800,1600,3200

The finite size effects are less pronounced in larger dimensions

Fully synchronized phase and the spectral dimension

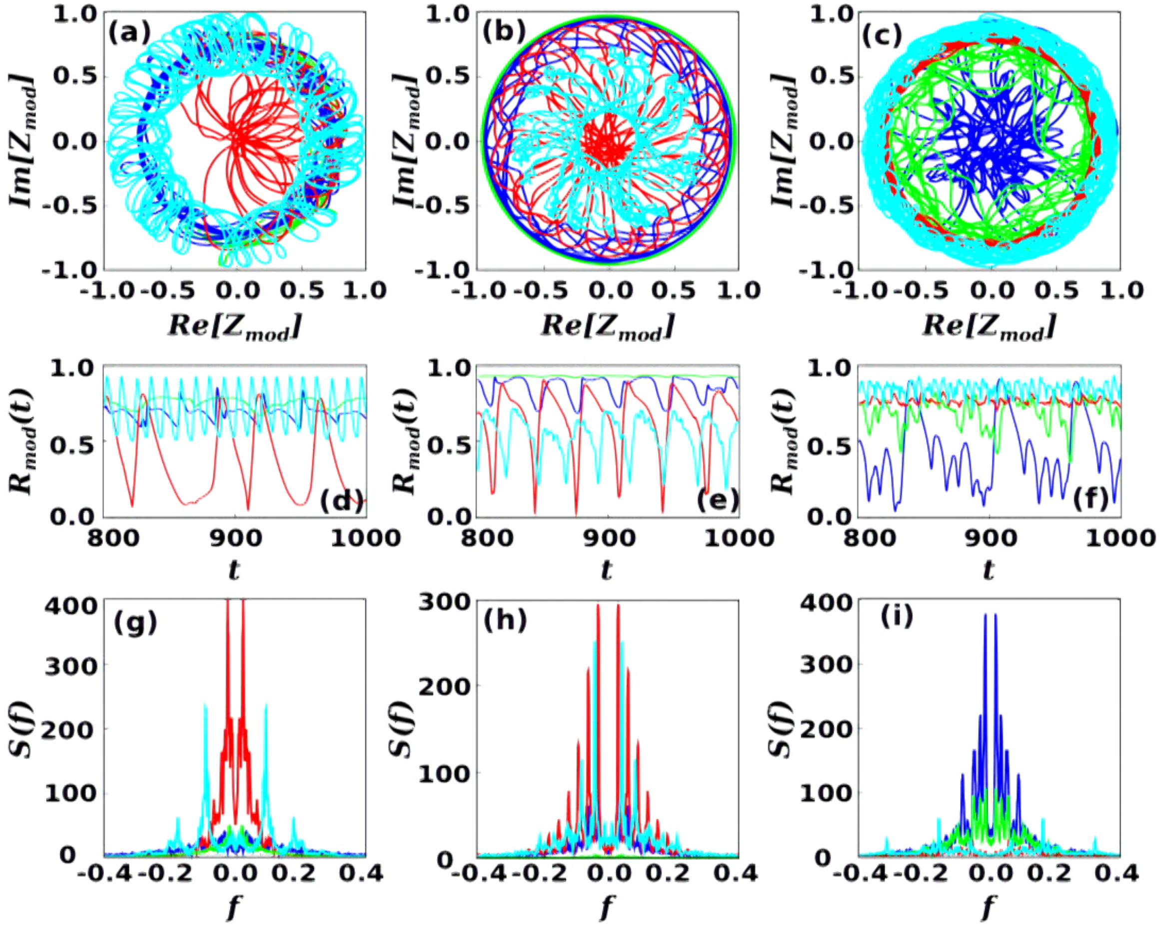
The fully synchronized phase is not thermodynamically achieved

for networks with spectral dimension

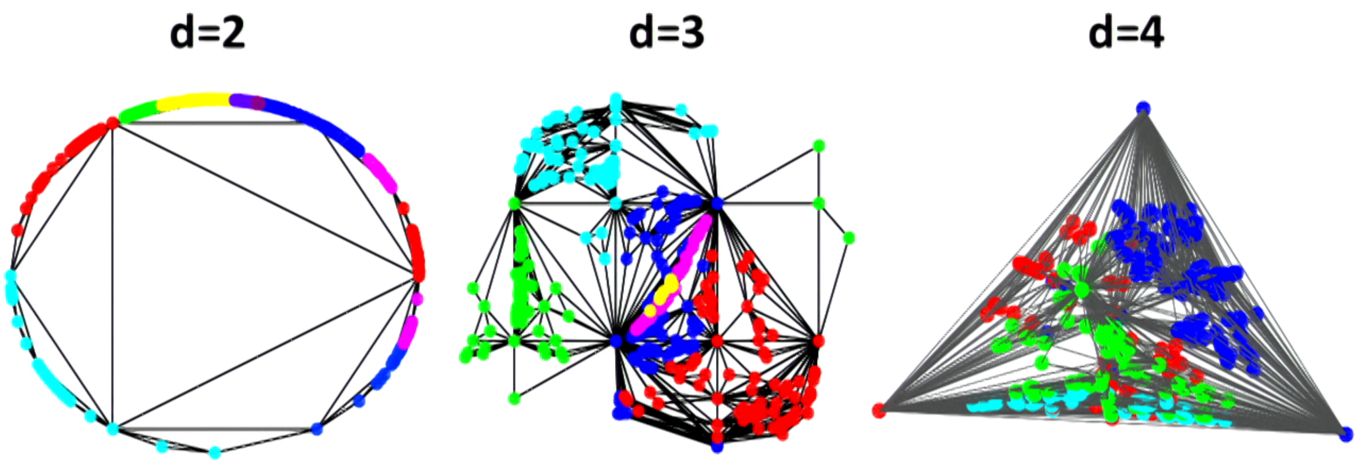
$$d_s \leq 4$$

In Complex Network Manifolds with D=3 the fully synchronized state is marginally stable

Communities and Frustrated Synchronization



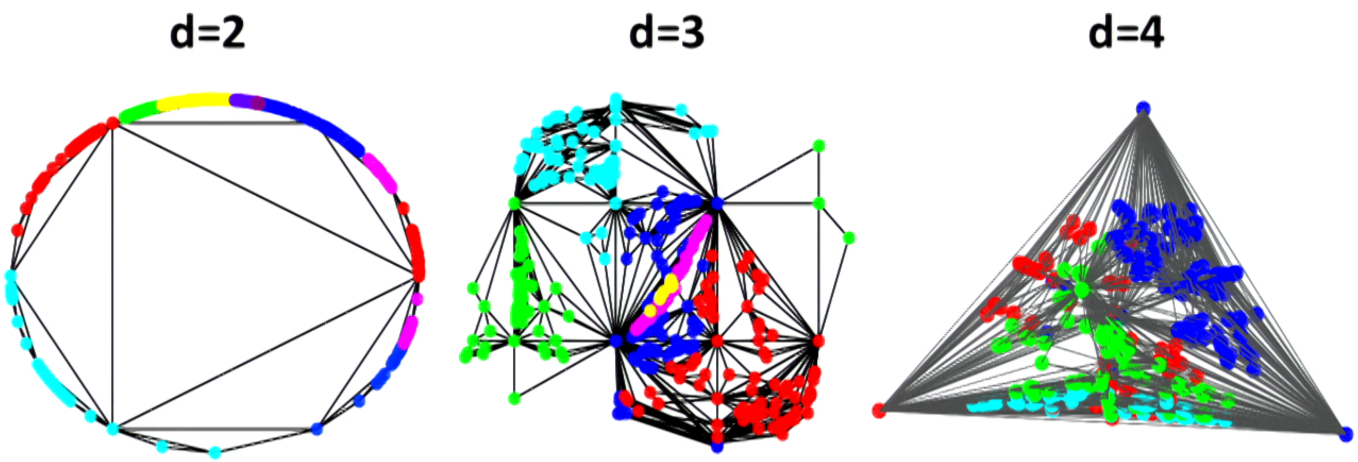
Frustrated synchronization and community structure



For every community with n_C nodes we can define the local order parameter

$$Z_{\mathbf{mod}} = R_{\mathbf{mod}} e^{i\psi_{\mathbf{mod}}} = \frac{1}{n_C} \sum_{j \in C}^{N} e^{i\vartheta_j}$$

Frustrated synchronization and community structure



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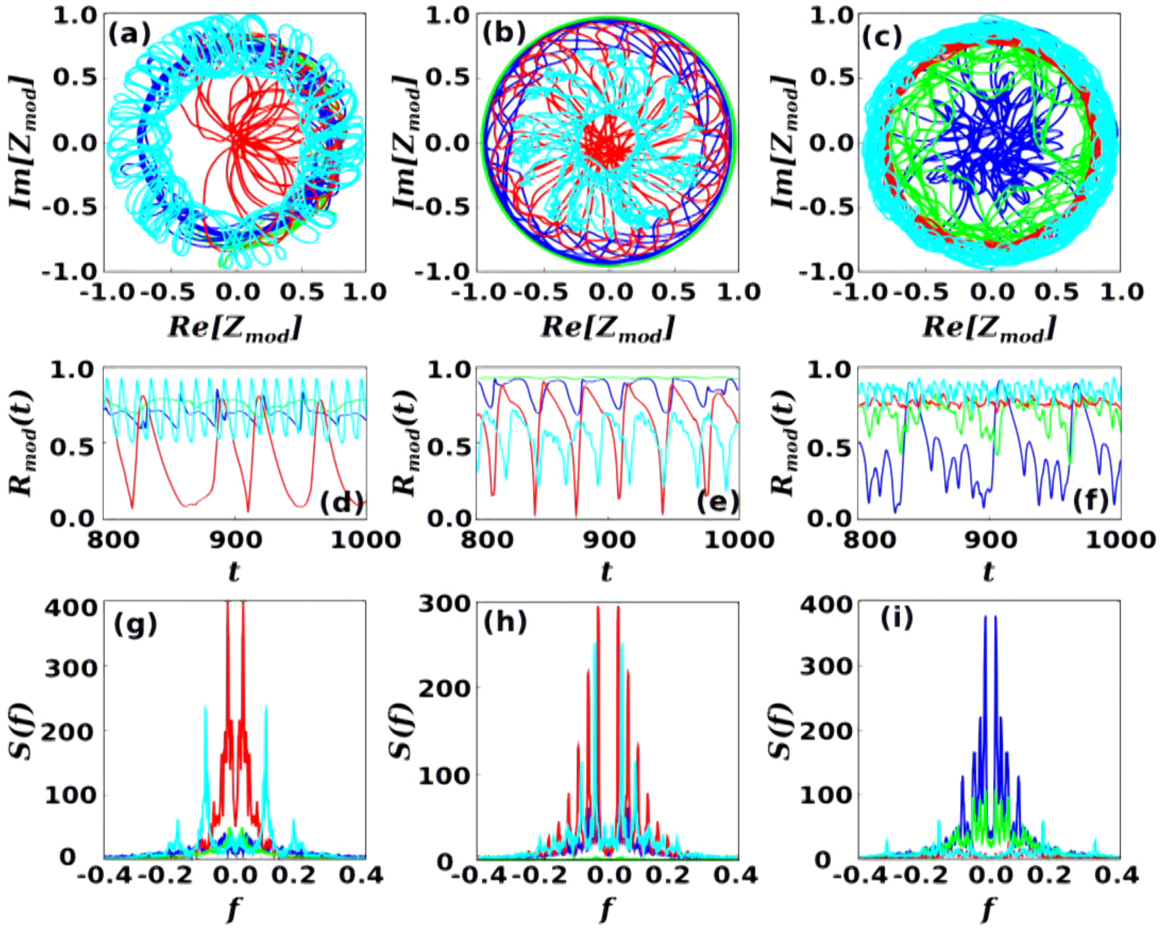
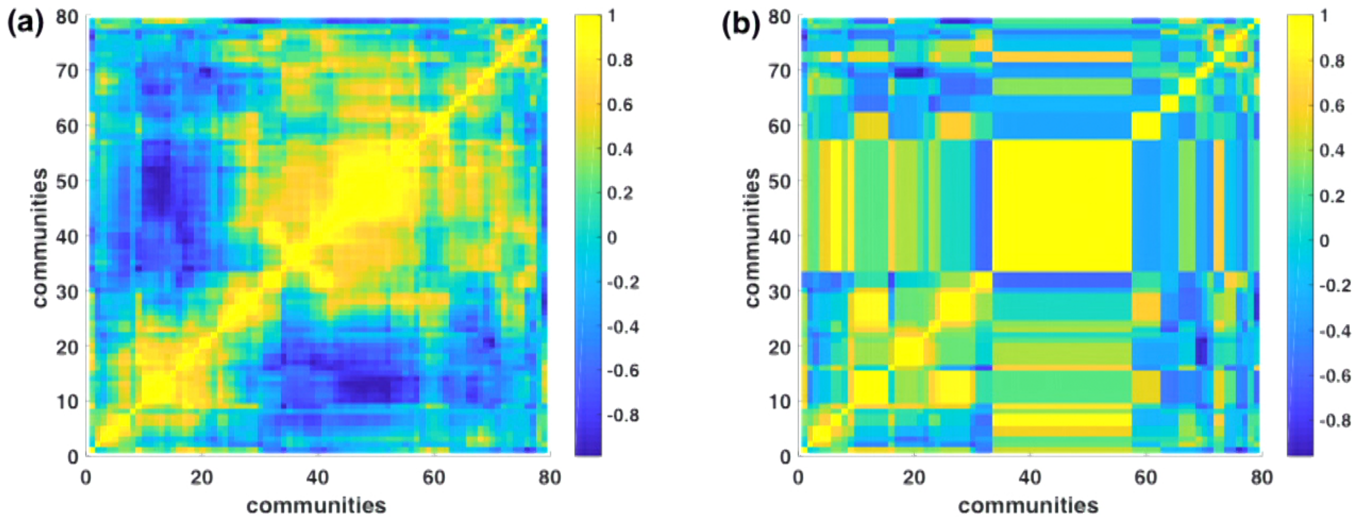
$$Z_{\mathbf{mod}} = R_{\mathbf{mod}} e^{i\psi_{\mathbf{mod}}} = \frac{1}{n_C} \sum_{j \in C}^{N} e^{i\vartheta_j}$$

Communities and Frustrated Synchronization



Correlations among communities and network coarse graining

n_c=70 **n_c=30**



N=1000, d=3, σ=5

Conclusions

Network Geometry with Flavor
provides a fundamental mechanism
for emergent hyperbolic network geometry
and and ideal framework to investigate the relation
between
Network Geometry and Dynamics

We have found
significant effects of the
dimensionality and the hyperbolicity of of the simplicial complexes

- ***degree distribution of the NGF***
- ***critical properties of topological percolation***
- ***stability of the synchronized state***

Collaborators and References

Emergent network geometry

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G. Bianconi and R. M. Ziff arxiv:1808.05836 (2018)

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