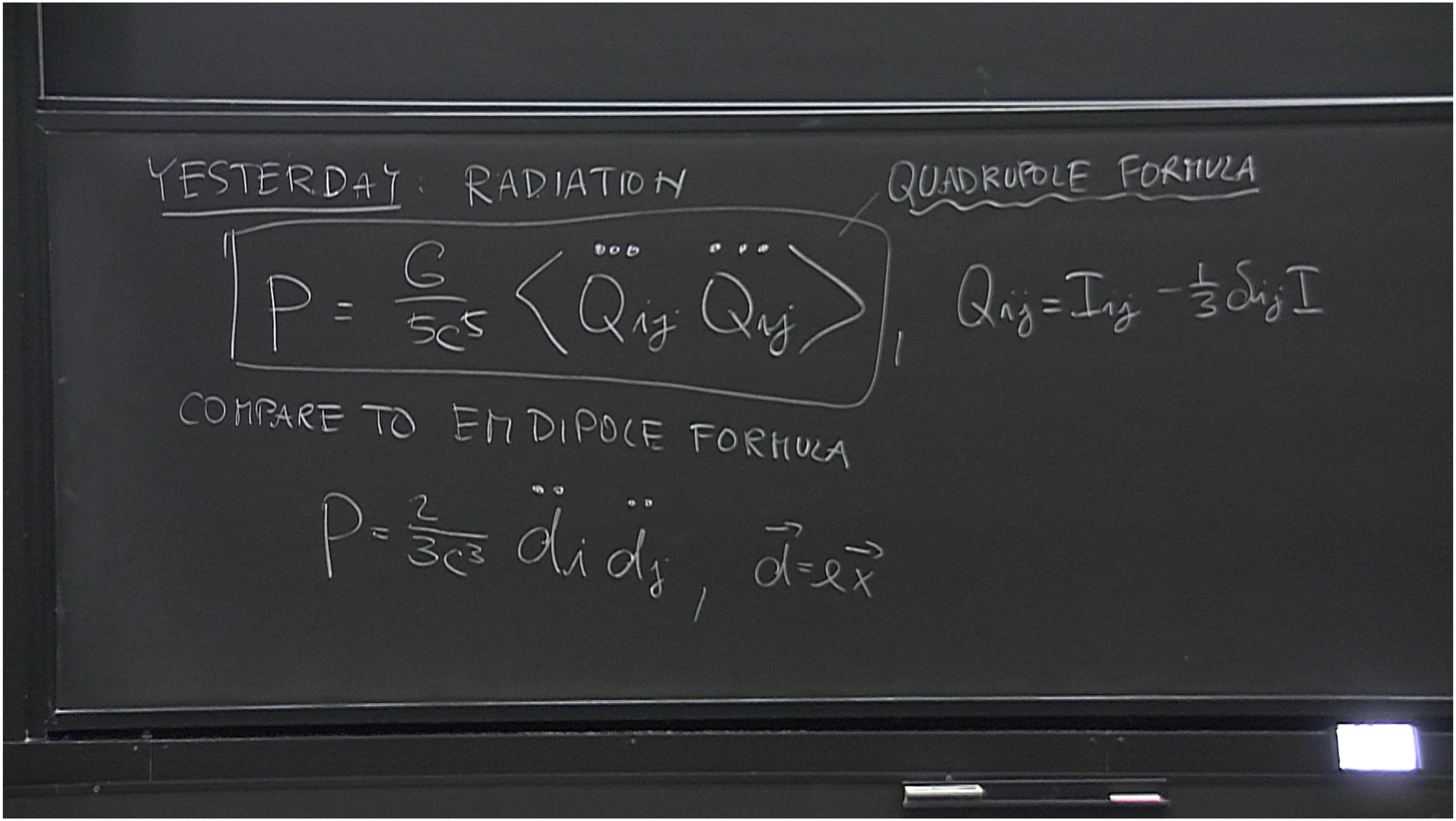
Title: PSI 2018/2019 - Relativity - Lecture 13

Date: Sep 19, 2018 09:00 AM

URL: http://pirsa.org/18090014

Abstract:



YESTERDAY: RADIATION

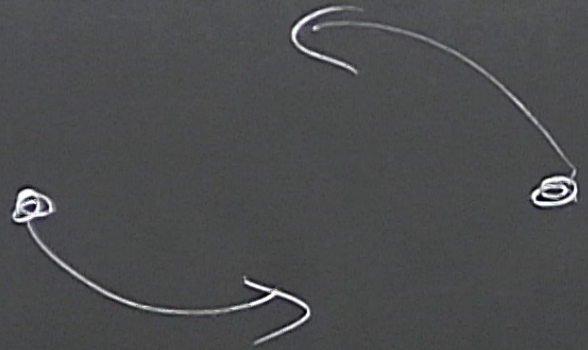
QUADRUPOLE FORMULA

$$\boxed{P = \frac{G}{5c^5}\left\langle \dddot{Q}_{ij}\,\dddot{Q}_{ij}\right\rangle}, \qquad Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I$$

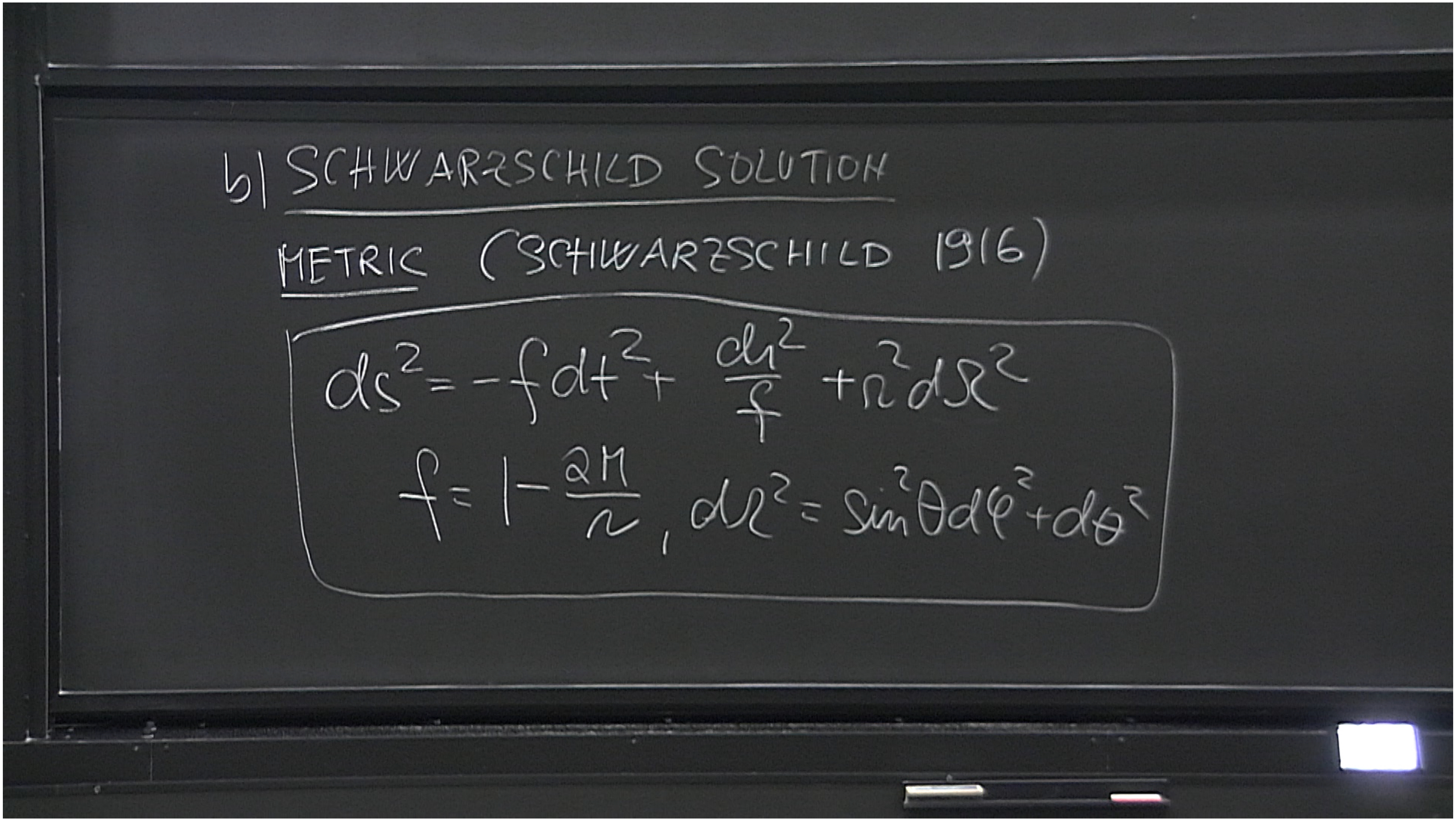
COMPARE TO EM DIPOLE FORMULA

$$P = \frac{2}{3c^3}\ddot{d}_i\,\ddot{d}_j, \qquad \vec{d} = e\vec{x}$$

1974 HULSE & TAYLOR



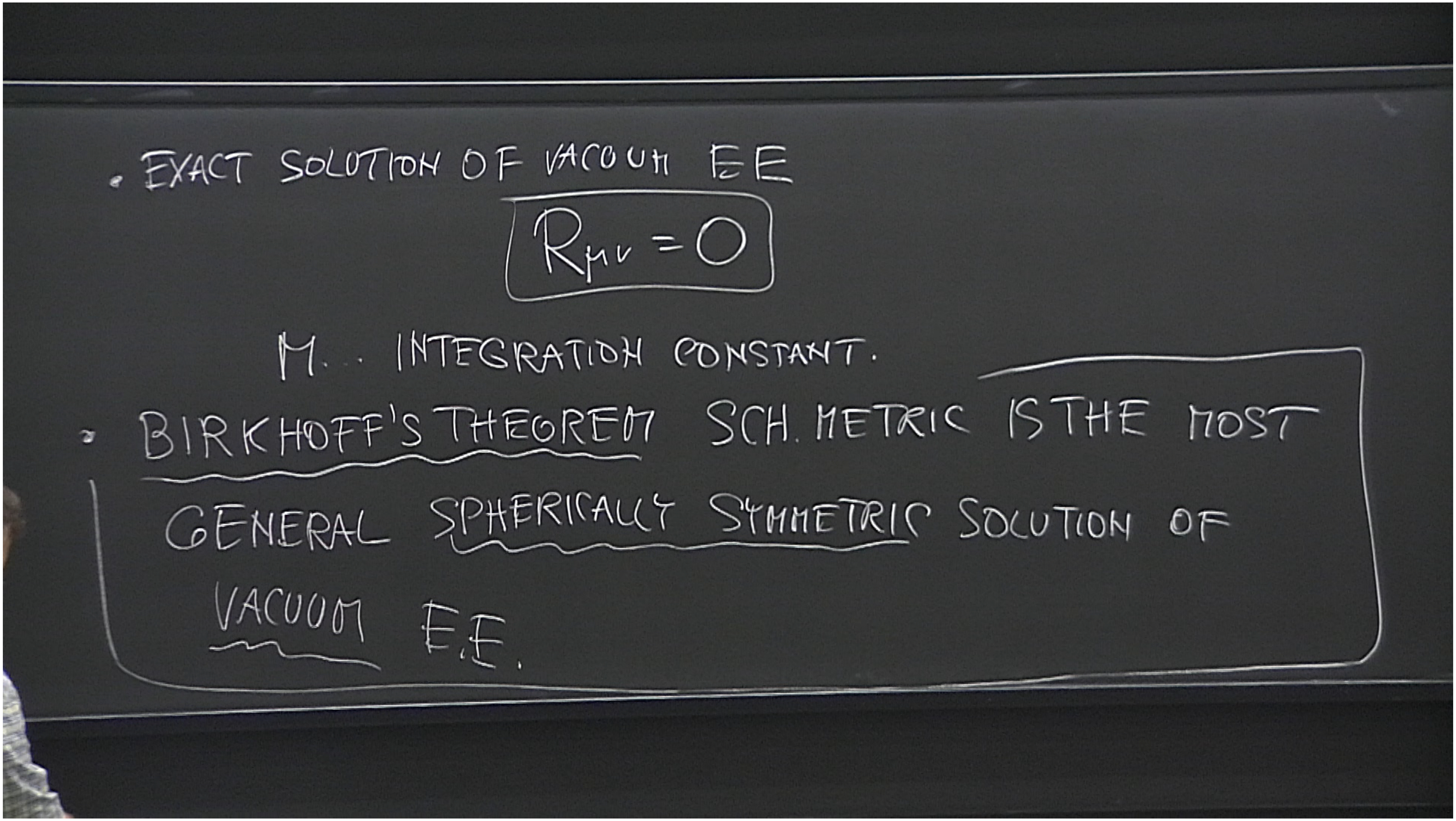
NP



b) SCHWARZSCHILD SOLUTION

METRIC (SCHWARZSCHILD 1916)

$$\boxed{\begin{gathered} ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2 \\ f = 1 - \frac{2M}{r}, \; d\Omega^2 = \sin^2\theta\, d\varphi^2 + d\theta^2 \end{gathered}}$$

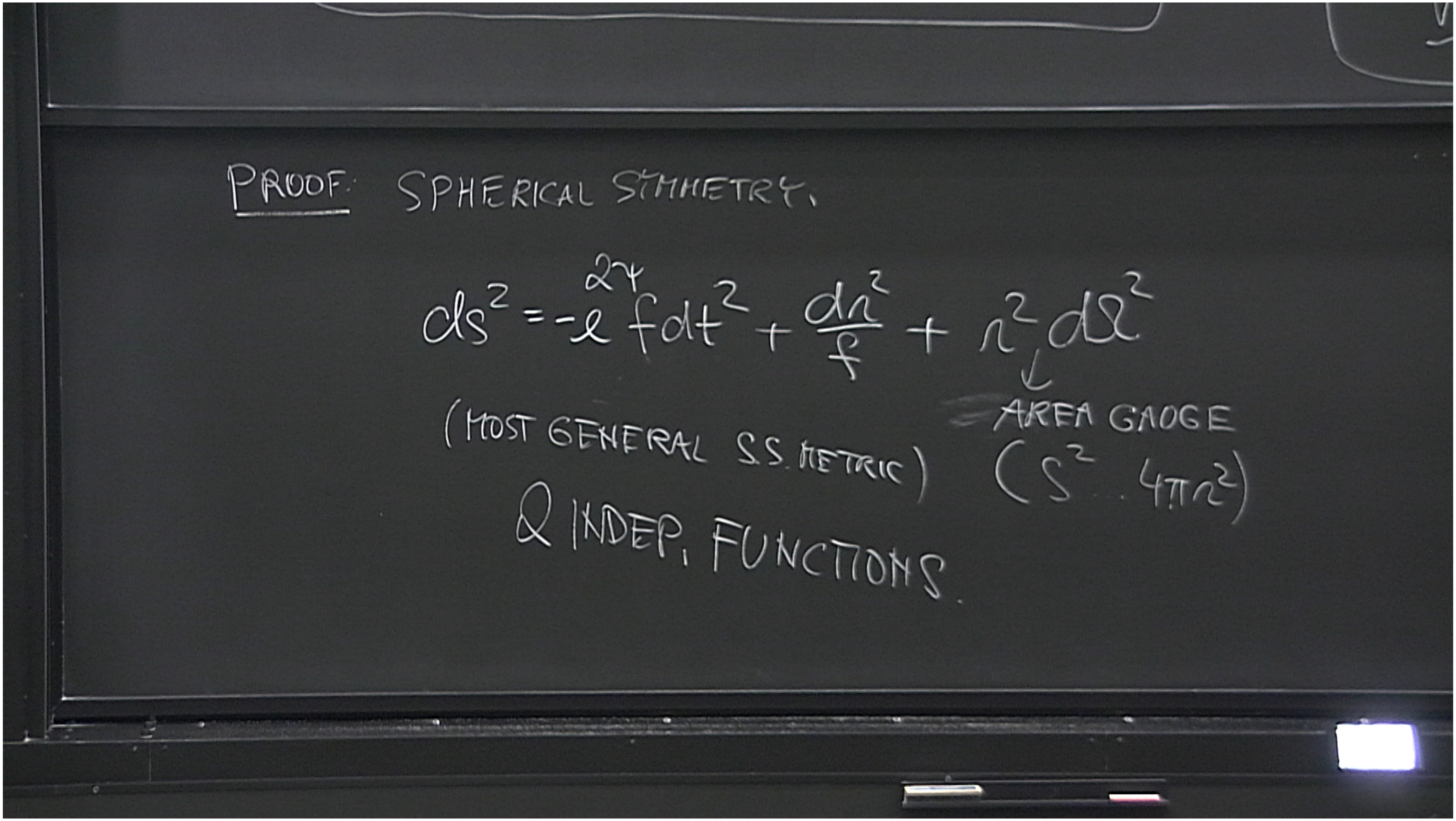


- EXACT SOLUTION OF VACUUM EE

$$\boxed{R_{\mu\nu} = 0}$$

M... INTEGRATION CONSTANT.

- BIRKHOFF'S THEOREM SCH. METRIC IS THE MOST GENERAL SPHERICALLY SYMMETRIC SOLUTION OF VACUUM E.E.



PROOF: SPHERICAL SYMMETRY.

$$ds^2 = -e^{2\psi} f\, dt^2 + \frac{dr^2}{f} + \underset{\downarrow}{r^2}\, d\Omega^2$$

AREA GAUGE

$(S^2 \ldots 4\pi r^2)$

(MOST GENERAL S.S. METRIC)

2 INDEP. FUNCTIONS.

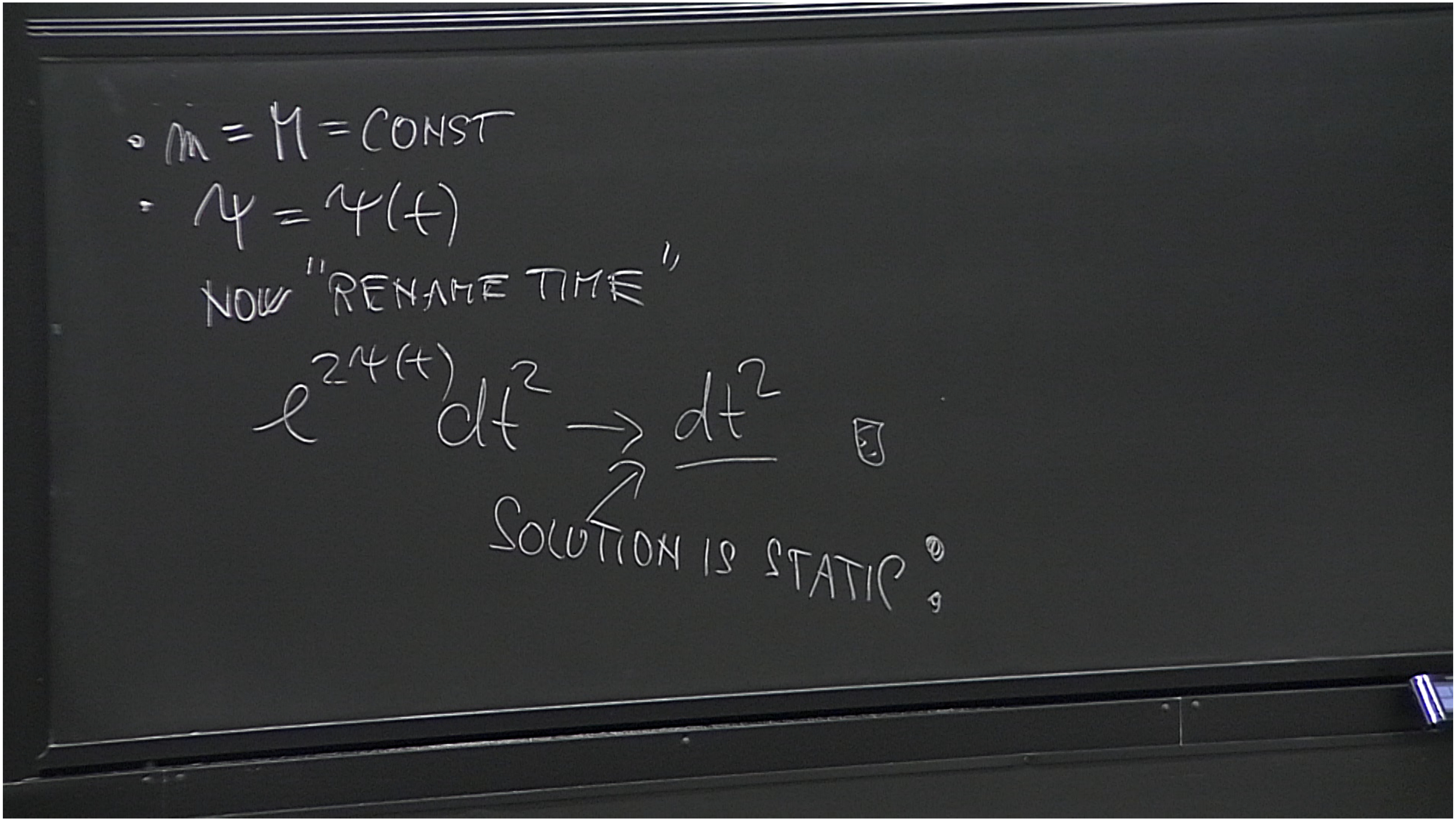
VACUUM E.E.

$$f = 1 - \frac{2m(r,t)}{r}\,, \quad \psi = \psi(r,t)$$

- **EE:** $\frac{\partial m}{\partial r} = 4\pi r^2\left(-T^t{}_t\right)\,, \quad \frac{\partial m}{\partial t} = -4\pi r^2\left(-T^t{}_r\right)$

$$\frac{\partial \psi}{\partial r} = \frac{4\pi r}{f}\left(-T^t{}_t + T^r{}_r\right)$$

VACUUM: $\underline{T^{\mu\nu} = 0}$

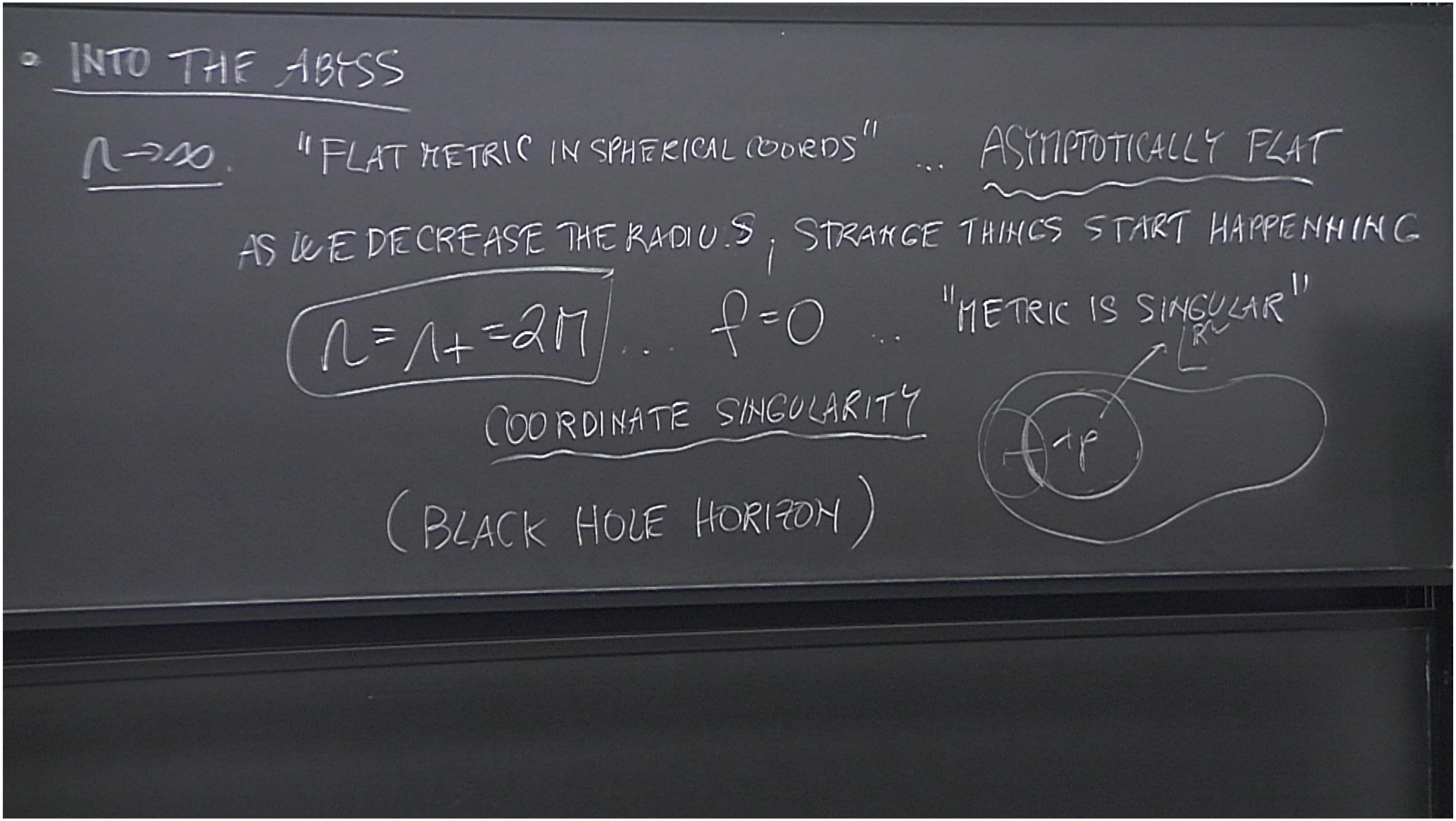


- $m = M = \text{CONST}$
- $\psi = \psi(t)$

NOW "RENAME TIME"

$$e^{2\psi(t)} dt^2 \longrightarrow \underline{dt^2}$$

SOLUTION IS STATIC



- INTO THE ABYSS

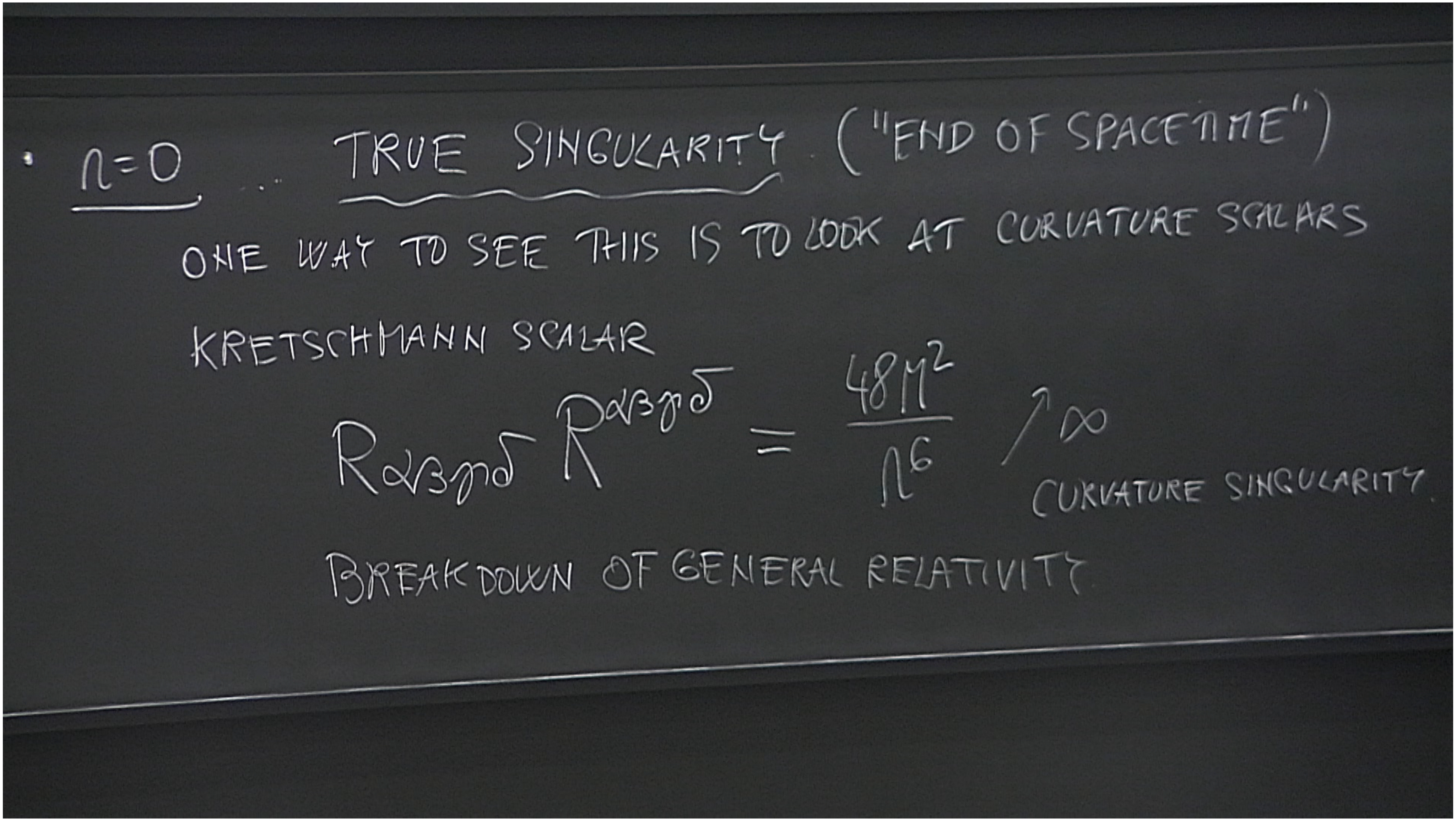
$r \to \infty$: "FLAT METRIC IN SPHERICAL COORDS" ... ASYMPTOTICALLY FLAT

AS WE DECREASE THE RADIUS, STRANGE THINGS START HAPPENNING

$\boxed{r = r_+ = 2M}$ ... $f = 0$ ... "METRIC IS SINGULAR"

COORDINATE SINGULARITY

(BLACK HOLE HORIZON)



- $\underline{r=0}$ ... TRUE SINGULARITY. ("END OF SPACETIME")

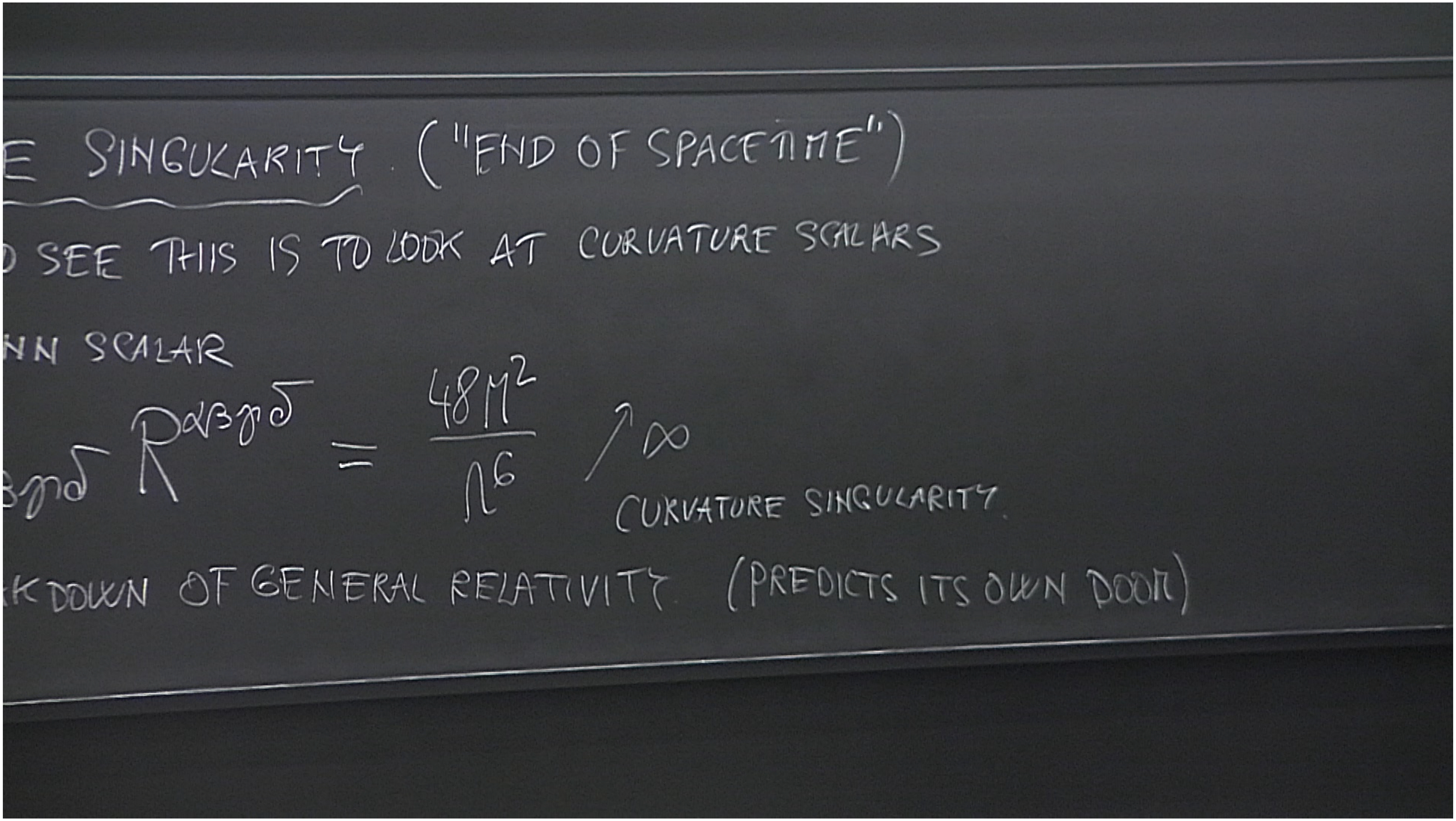
ONE WAY TO SEE THIS IS TO LOOK AT CURVATURE SCALARS

KRETSCHMANN SCALAR

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} \nearrow \infty$$

CURVATURE SINGULARITY.

BREAKDOWN OF GENERAL RELATIVITY.



E SINGULARITY ("END OF SPACETIME")

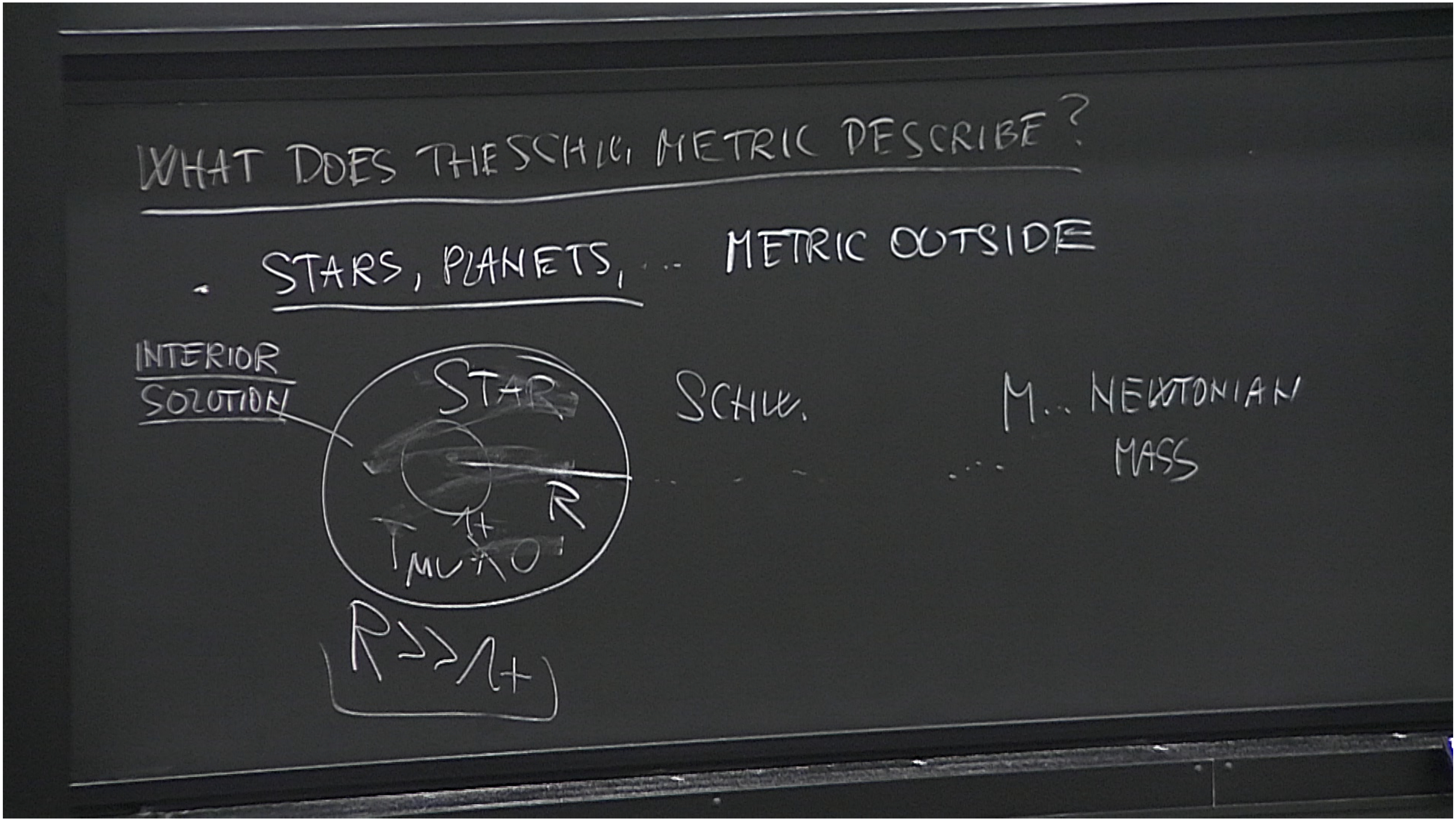
SEE THIS IS TO LOOK AT CURVATURE SCALARS

NN SCALAR

$$_{\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} \nearrow \infty$$

CURVATURE SINGULARITY.

KDOWN OF GENERAL RELATIVITY. (PREDICTS ITS OWN DOOM)



## WHAT DOES THE SCHW. METRIC DESCRIBE?

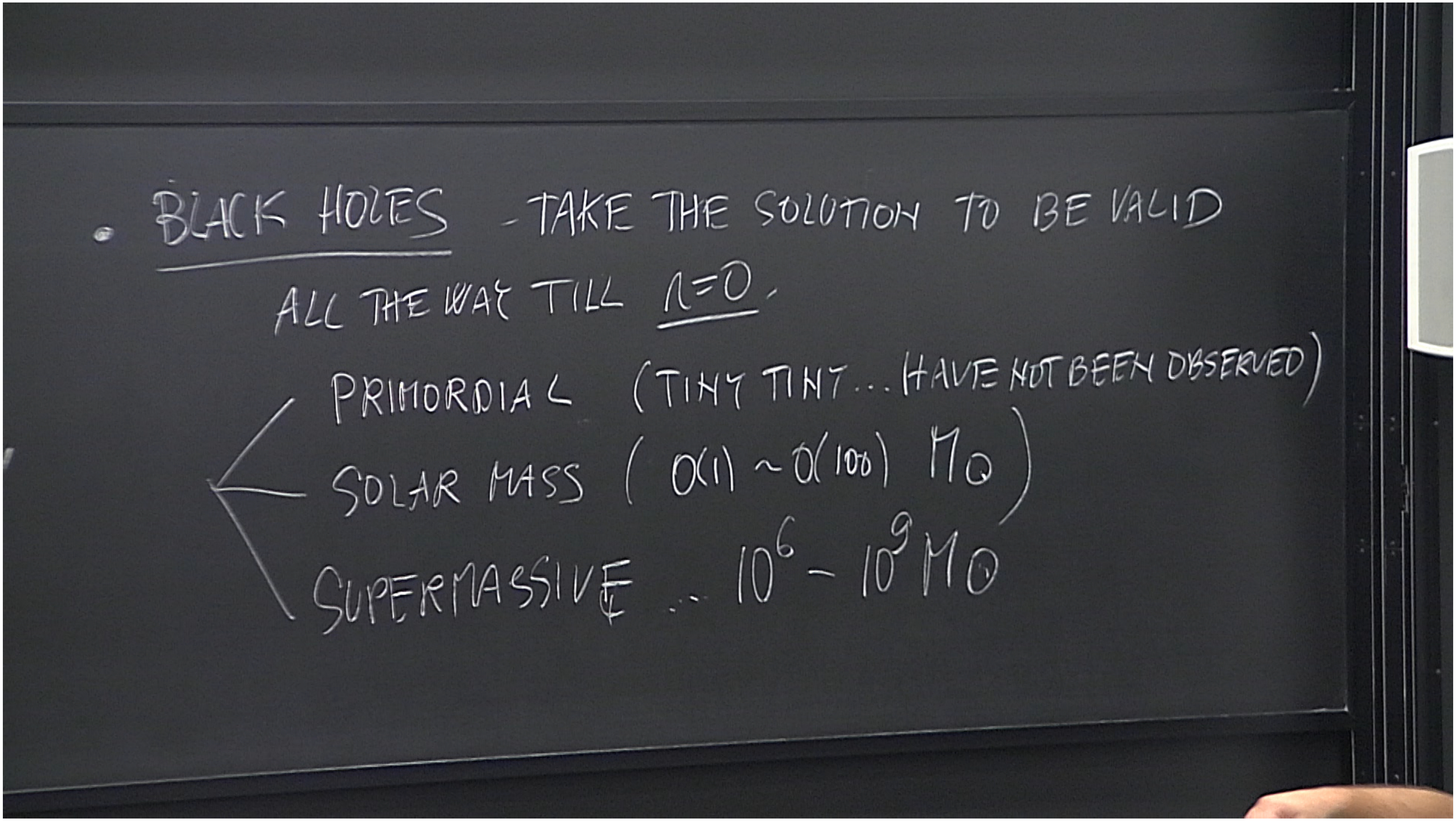
- STARS, PLANETS, ... METRIC OUTSIDE

INTERIOR SOLUTION

SCHW.

M... NEWTONIAN MASS

$$\boxed{R >> r_+}$$

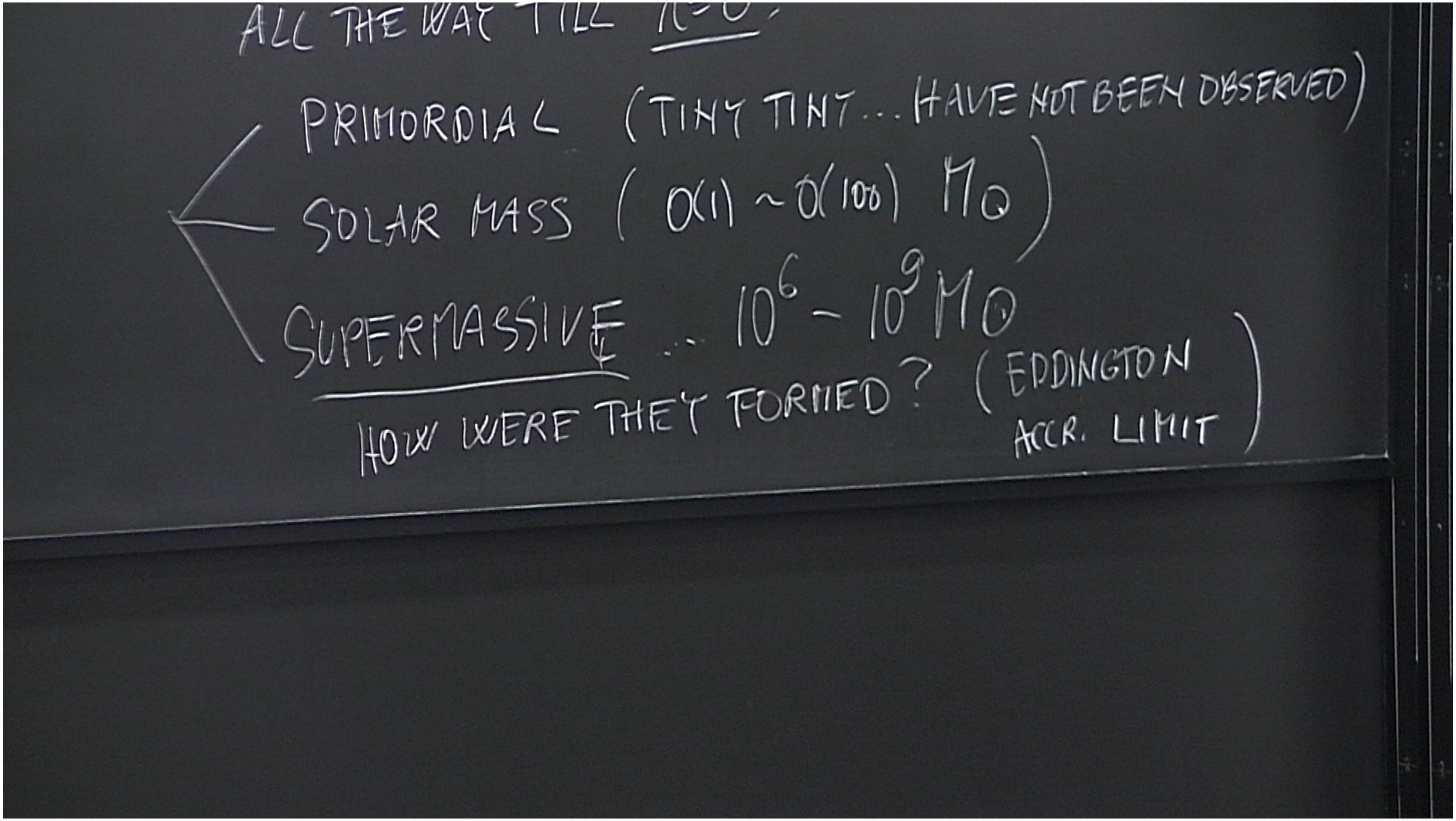


- BLACK HOLES - TAKE THE SOLUTION TO BE VALID ALL THE WAY TILL $\underline{r=0}$.

PRIMORDIAL (TINY TINY... HAVE NOT BEEN OBSERVED)

SOLAR MASS ( $O(1) \sim O(100)$ $M_\odot$ )

SUPERMASSIVE ... $10^6 - 10^9 M_\odot$



ALL THE WAY TILL

PRIMORDIAL (TINY TINY... HAVE NOT BEEN OBSERVED)

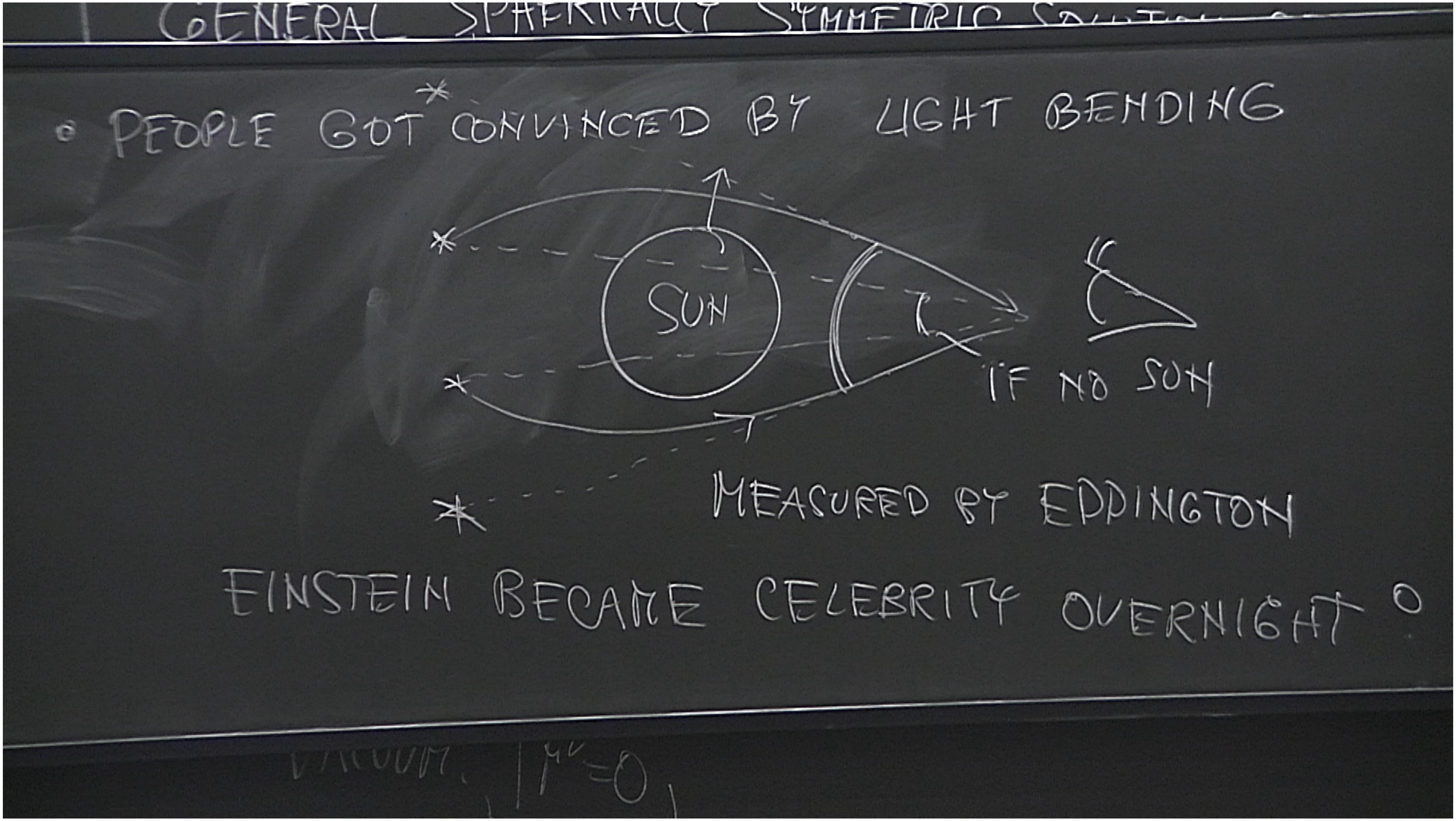
SOLAR MASS ( $O(1) \sim O(100)$ $M_\odot$ )

SUPERMASSIVE ... $10^6 - 10^9 M_\odot$

HOW WERE THEY FORMED? (EDDINGTON ACCR. LIMIT)

FIT.

- PEOPLE GOT CONVINCED BY LIGHT BENDING



GENERAL SPHERICALLY SYMMETRIC

- PEOPLE GOT CONVINCED BY LIGHT BENDING

MEASURED BY EDDINGTON

EINSTEIN BECAME CELEBRITY OVERNIGHT

VACUUM: $T^{\mu\nu}=0$,

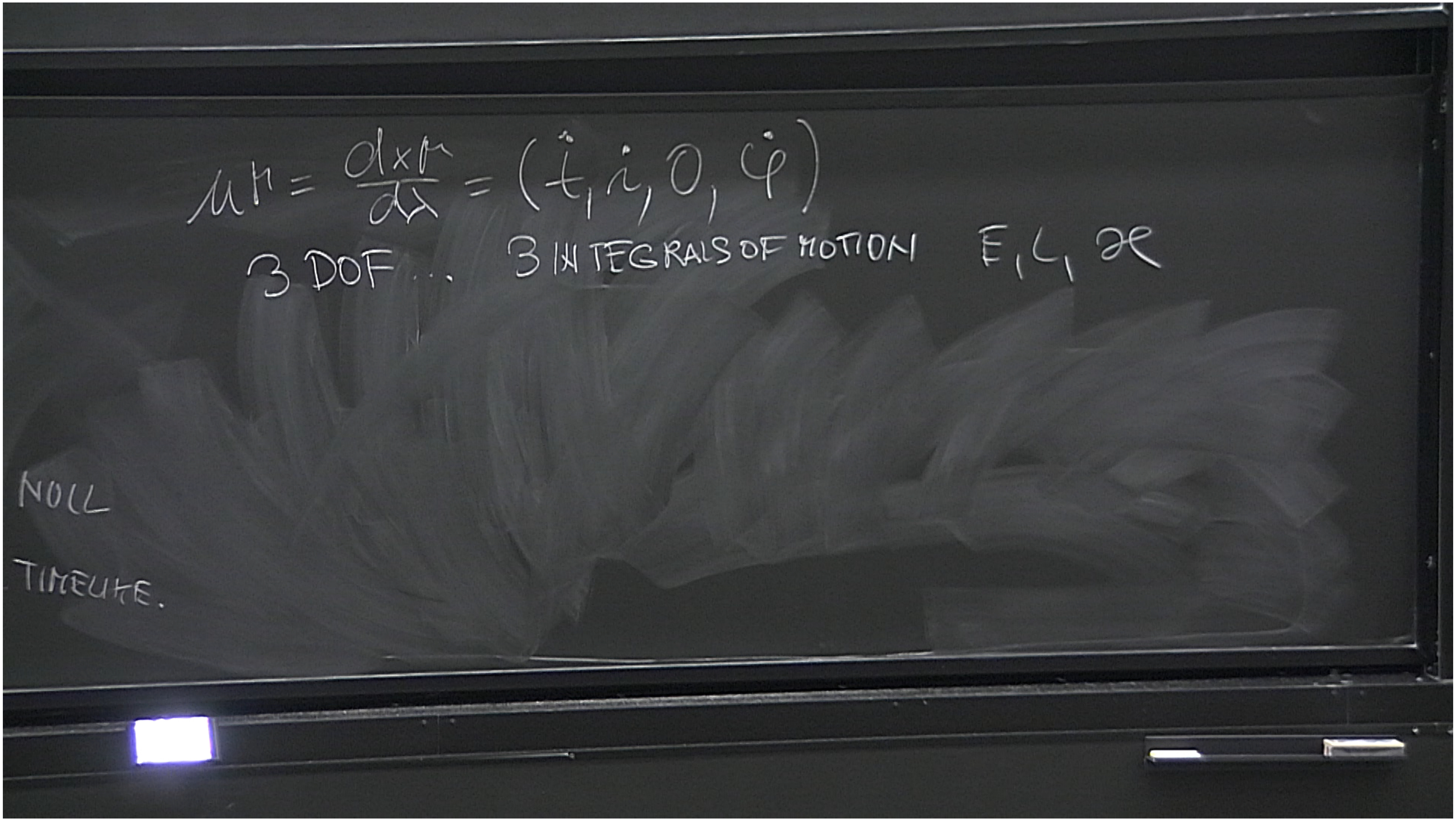
LET'S STUDY GEODESICS AROUND THE SUN
(IN THE SCHWARZSCHILD SPACETIME)

- S.S $\Rightarrow$ MOTION IS PLANAR

$$\boxed{\theta = \pi/2} \quad \text{WLOG}$$

- STATIC... $\partial_t = k$ ... KV

AXISYMM. $\partial_\varphi = m$ ... KV , $u^2 = -\varkappa \begin{cases} 0 & \text{... NULL} \\ 1 & \text{... TIMELIKE.} \end{cases}$

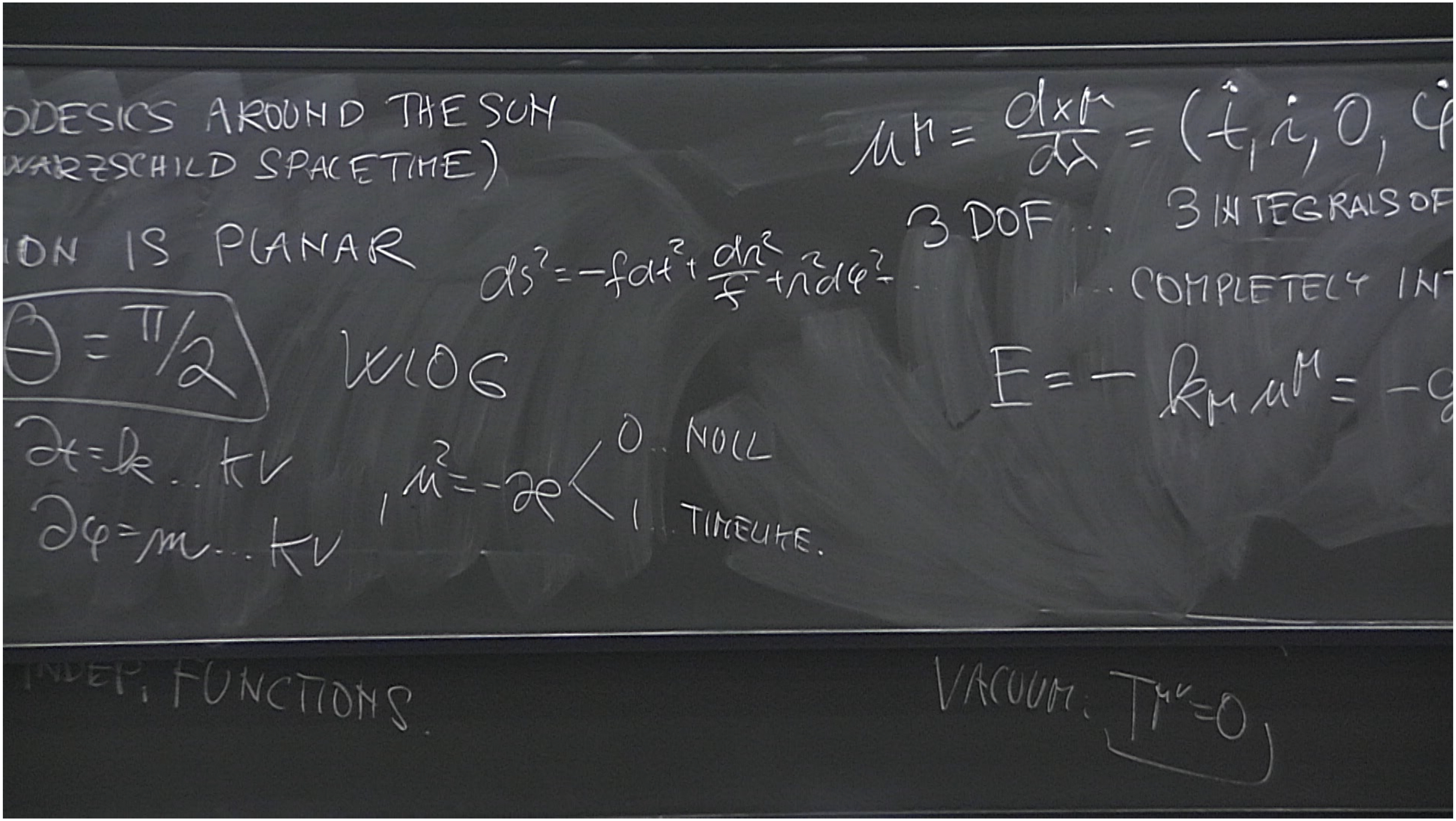


$$u^\mu = \frac{dx^\mu}{d\lambda} = (\dot{t}, \dot{r}, 0, \dot{\varphi})$$

3 DOF … 3 INTEGRALS OF MOTION $E, L, \varkappa$

NULL

TIMELIKE.



ODESICS AROUND THE SUN
WARZSCHILD SPACETIME)

ION IS PLANAR

$\boxed{\Theta = \pi/2}$ WLOG

$\partial_t = k$ ... KV

$\partial_\varphi = m$ ... KV

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\varphi^2 + \ldots$$

$$, u^2 = -\varkappa \begin{cases} 0 \ldots \text{NULL} \\ 1 \ldots \text{TIMELIKE.} \end{cases}$$

$$u^\mu = \frac{dx^\mu}{d\lambda} = (\dot{t}, \dot{r}, 0, \dot{\varphi}$$

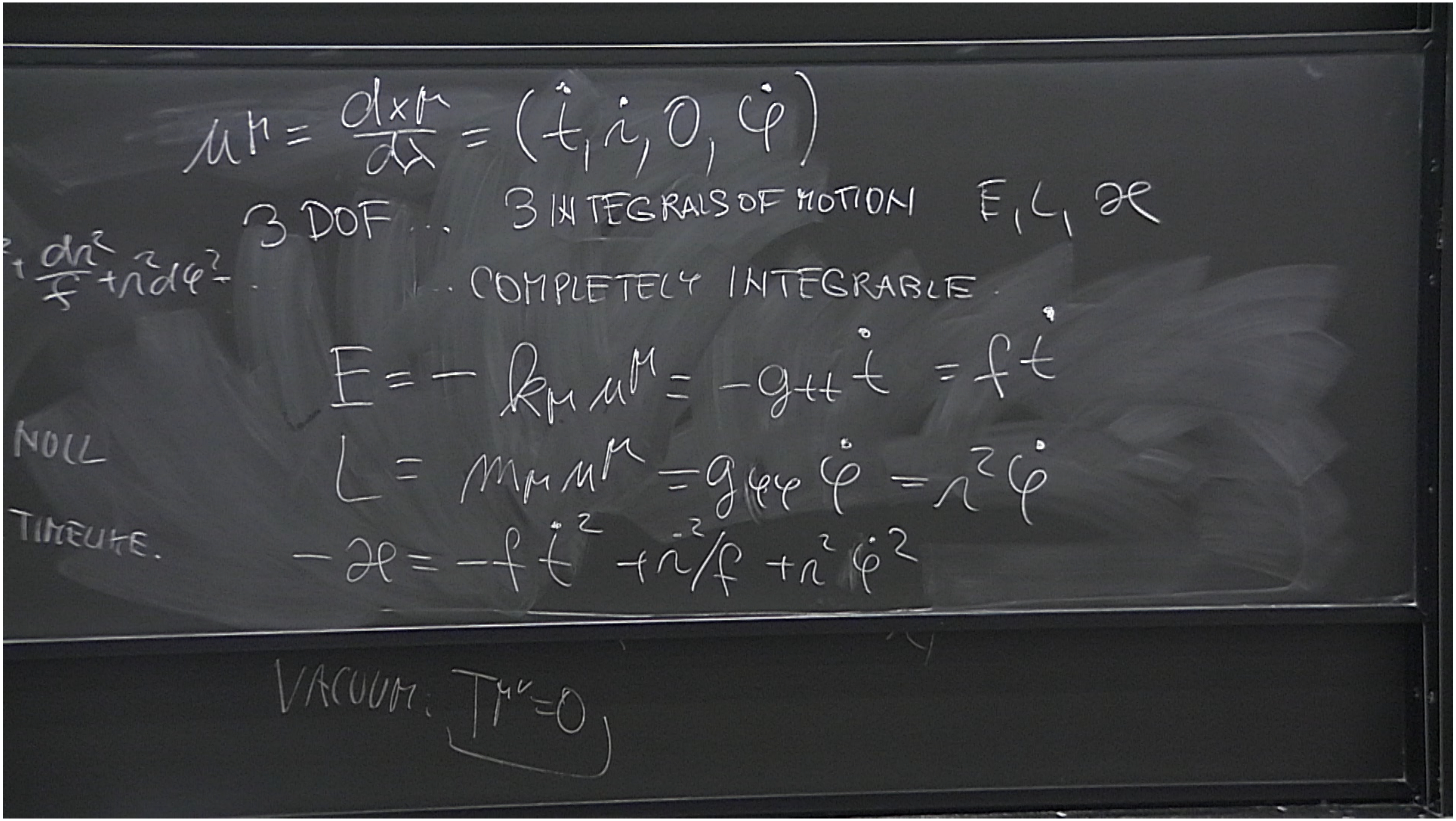
3 DOF ... 3 INTEGRALS OF

... COMPLETELY INT

$$E = -k_\mu u^\mu = -g$$

NDEP. FUNCTIONS.

VACUUM: $\underline{T^{\mu\nu} = 0}$,



$$u^\mu = \frac{dx^\mu}{d\lambda} = (\dot t, \dot r, 0, \dot\varphi)$$

3 DOF ... 3 INTEGRALS OF MOTION $E, L, \varkappa$

$+ \frac{dr^2}{f} + r^2 d\varphi^2 = \ldots$

... COMPLETELY INTEGRABLE.

$$E = -k_\mu u^\mu = -g_{tt}\dot t = f\dot t$$

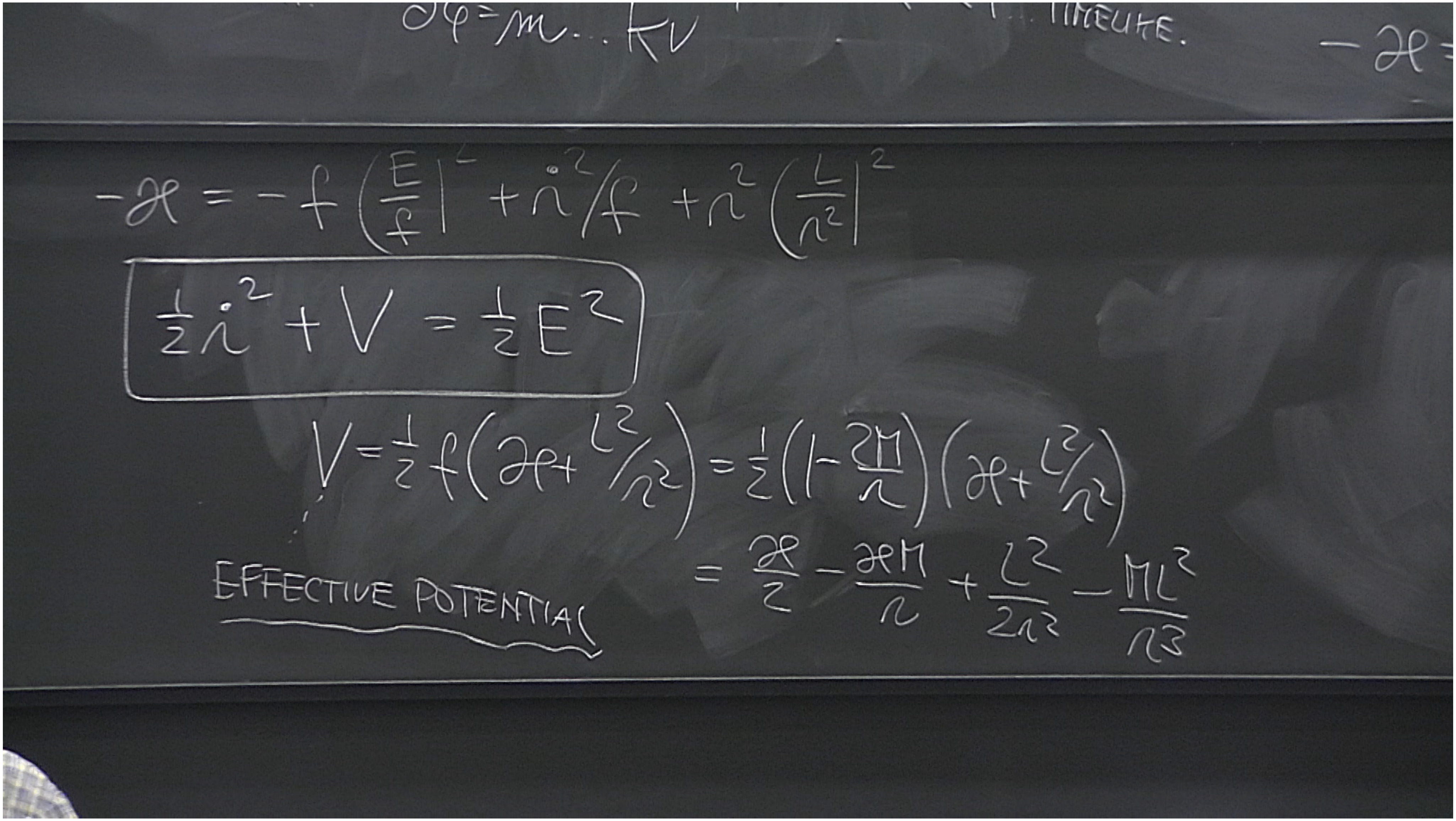
NULL

$$L = m_\mu u^\mu = g_{\varphi\varphi}\dot\varphi = r^2\dot\varphi$$

TIMELIKE.

$$-\varkappa = -f\dot t^2 + \dot r^2/f + r^2\dot\varphi^2$$

VACUUM: $\underline{T^{\mu\nu}=0}$



$\partial\varphi = m \ldots k v$ … TIMELIKE. $-\mathcal{H} =$

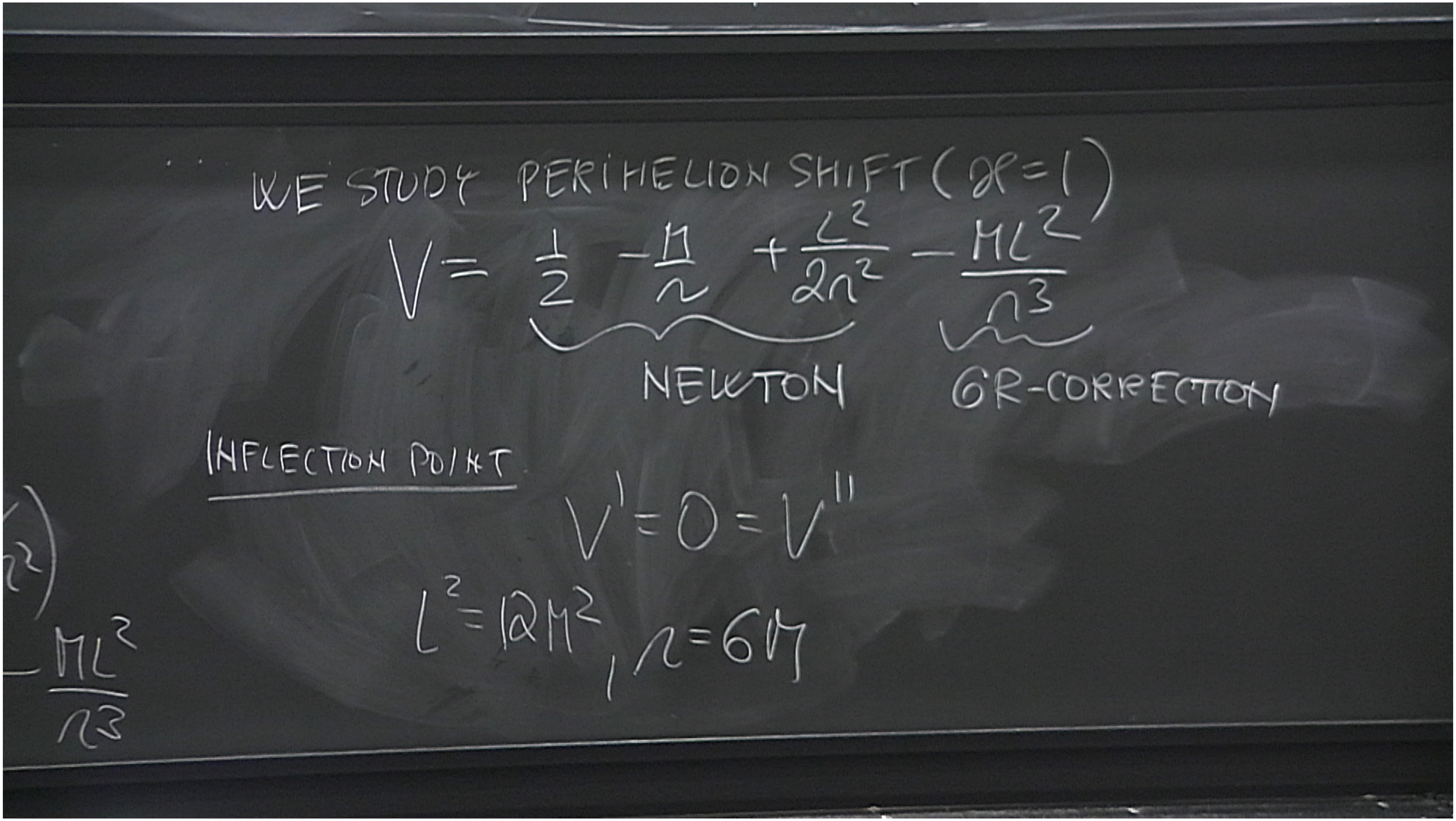
$$-\mathcal{H} = -f\left(\frac{E}{f}\right)^2 + \dot{r}^2/f + r^2\left(\frac{L}{r^2}\right)^2$$

$$\boxed{\frac{1}{2}\dot{r}^2 + V = \frac{1}{2}E^2}$$

$$V = \frac{1}{2}f\left(\mathcal{H} + L^2/r^2\right) = \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\mathcal{H} + L^2/r^2\right)$$

$$= \frac{\mathcal{H}}{2} - \frac{\mathcal{H}M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$\underline{\text{EFFECTIVE POTENTIAL}}$



WE STUDY PERIHELION SHIFT ($\varkappa = 1$)

$$V = \underbrace{\frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2}}_{\text{NEWTON}} - \underbrace{\frac{ML^2}{r^3}}_{\text{GR-CORRECTION}}$$

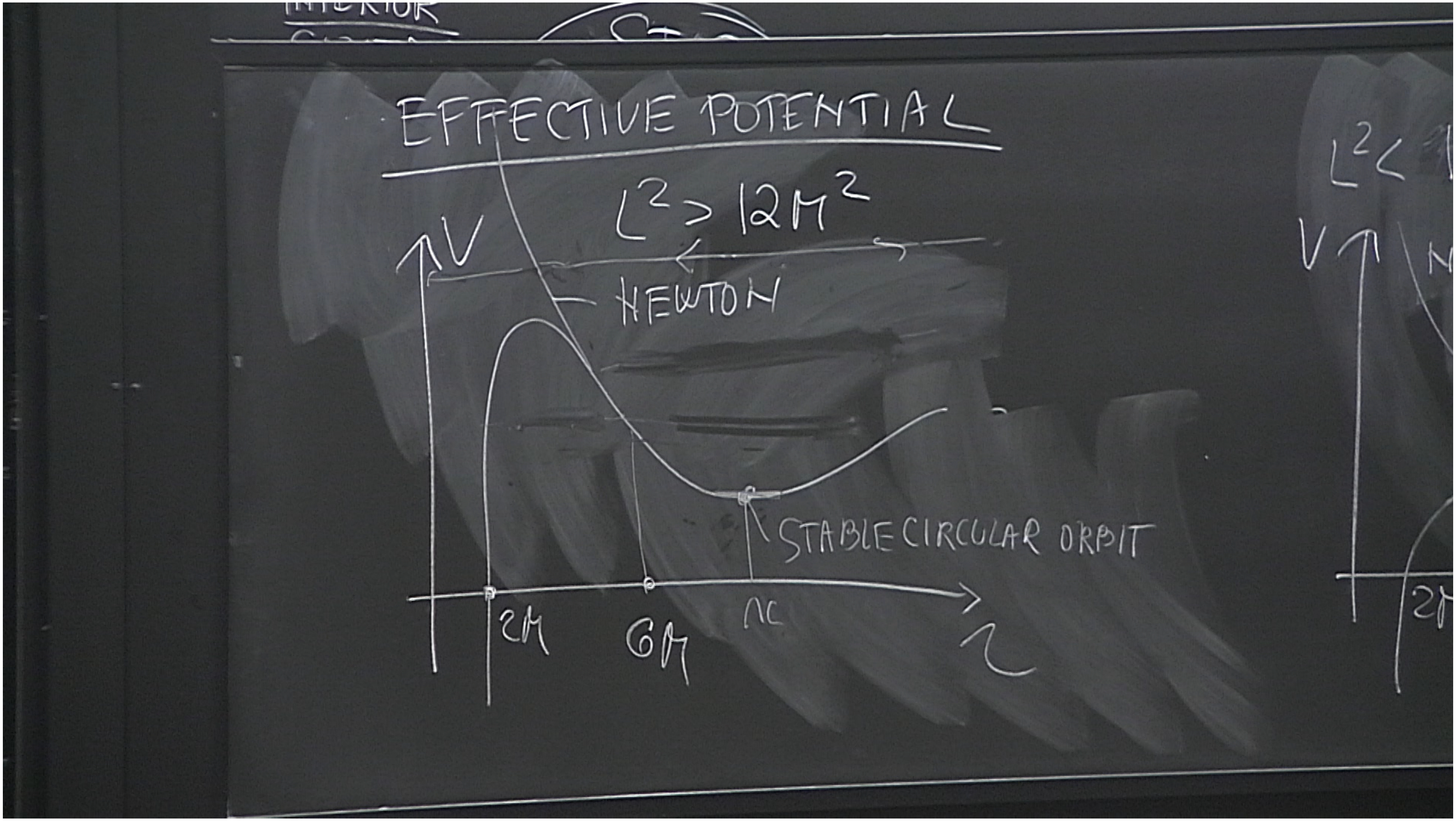
INFLECTION POINT

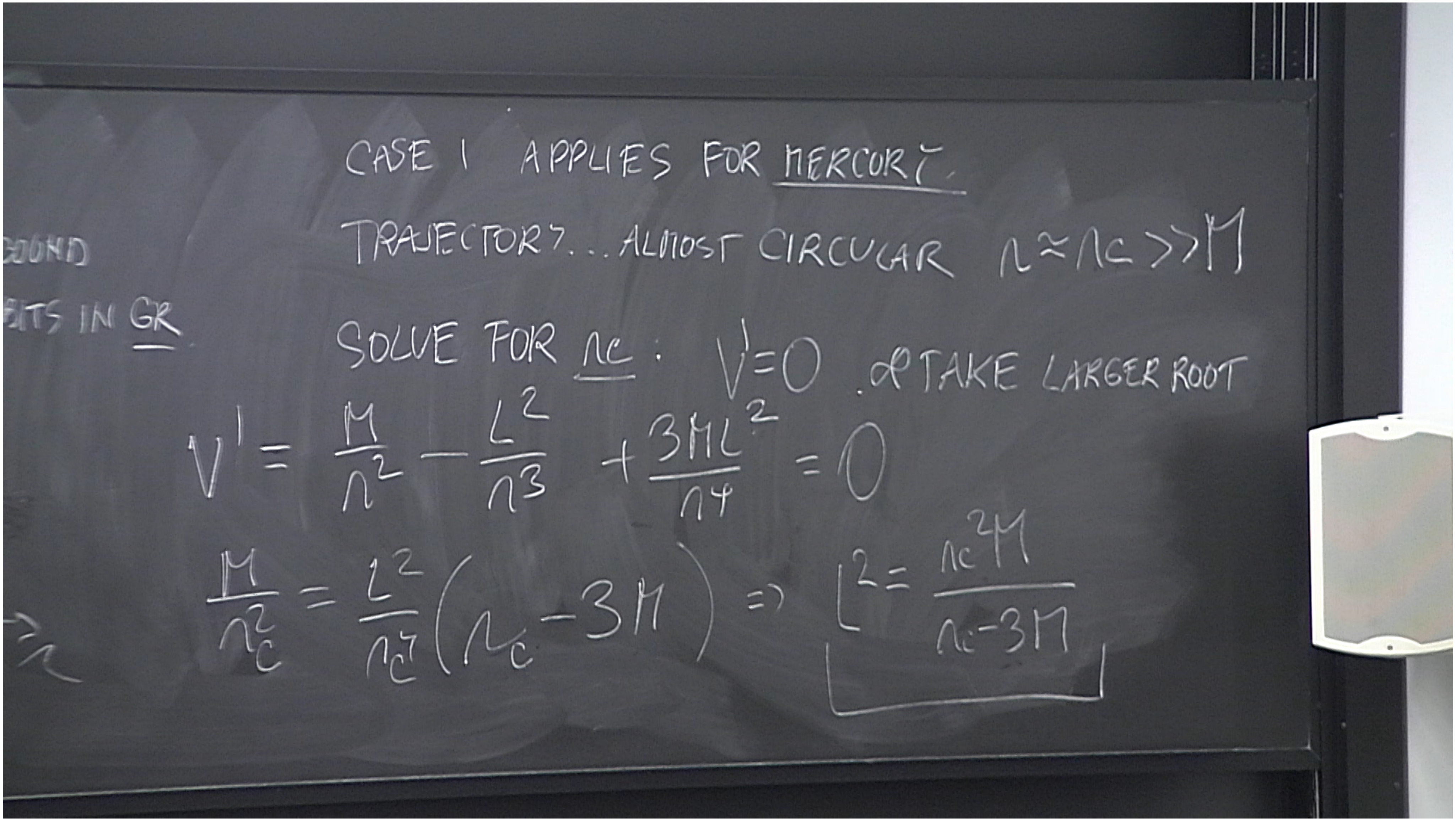
$$V' = 0 = V''$$

$$L^2 = 12M^2, \ r = 6M$$

$r^2)$

$-\frac{ML^2}{r^3}$


INTERIOR
EFFECTIVE POTENTIAL
$L^2 > 12M^2$
V
NEWTON
STABLE CIRCULAR ORBIT
2M
6M
$r_c$
r
$L^2 <$
V
2M



OUND

BITS IN GR

$\rightarrow r$

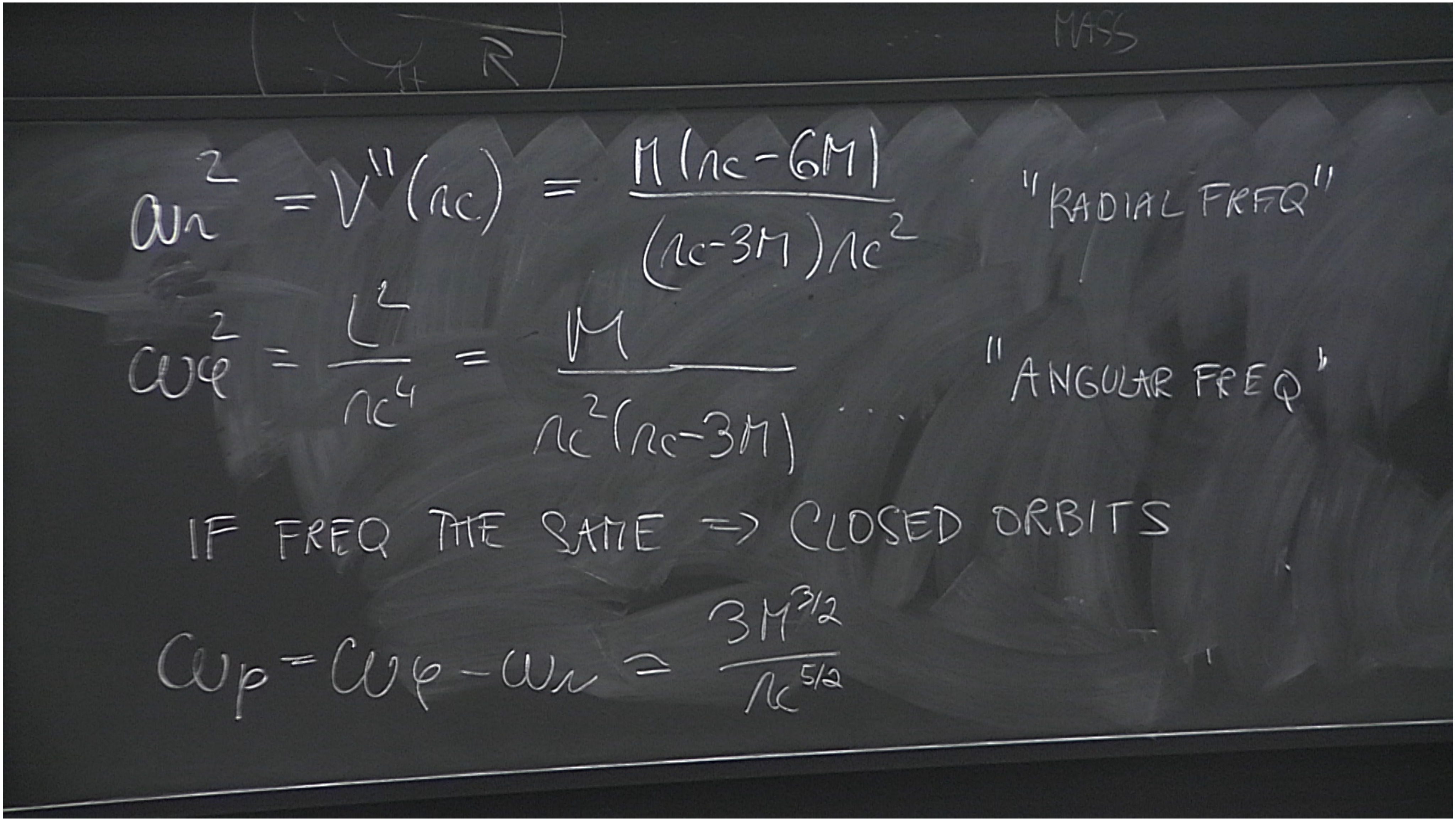
CASE 1 APPLIES FOR MERCURY.

TRAJECTORY... ALMOST CIRCULAR $r \approx r_c >> M$

SOLVE FOR $r_c$: $V'=0$ & TAKE LARGER ROOT

$$V' = \frac{M}{r^2} - \frac{L^2}{r^3} + \frac{3ML^2}{r^4} = 0$$

$$\frac{M}{r_c^2} = \frac{L^2}{r_c^4}\left(r_c - 3M\right) \Rightarrow \boxed{L^2 = \frac{r_c^2 M}{r_c - 3M}}$$



R

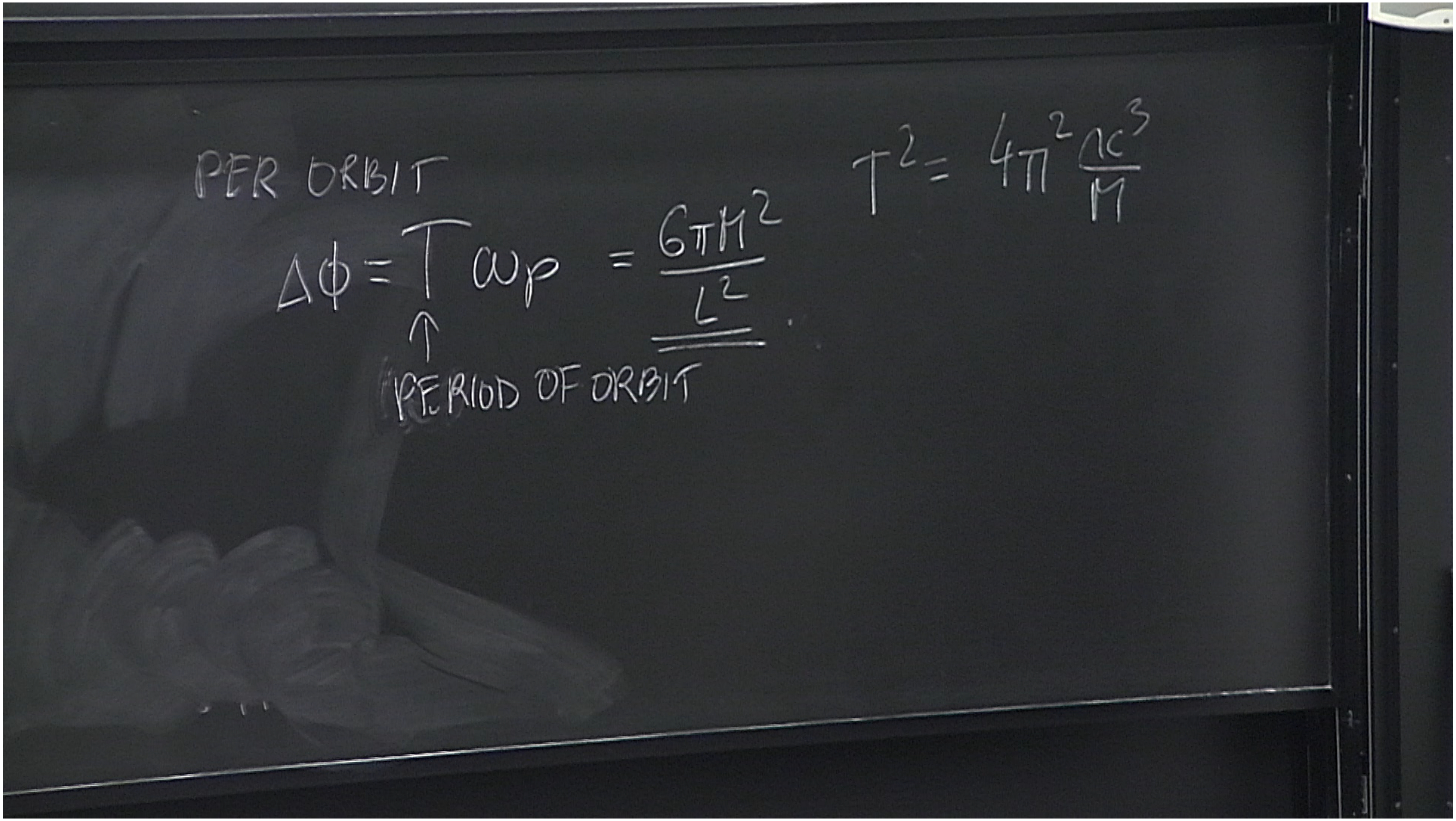
MASS

$$\omega_r^2 = V''(r_c) = \frac{M(r_c - 6M)}{(r_c - 3M)r_c^2}$$ "RADIAL FREQ"

$$\omega_\varphi^2 = \frac{L^2}{r_c^4} = \frac{M}{r_c^2(r_c - 3M)}$$ ... "ANGULAR FREQ"

IF FREQ THE SAME ⇒ CLOSED ORBITS

$$\omega_p = \omega_\varphi - \omega_r = \frac{3M^{3/2}}{r_c^{5/2}}$$



PER ORBIT

$$T^2 = 4\pi^2 \frac{r^3}{M}$$

$$\Delta\phi = \underset{\uparrow}{T}\,\omega_p = \underline{\underline{\frac{6\pi M^2}{L^2}}}$$

PERIOD OF ORBIT

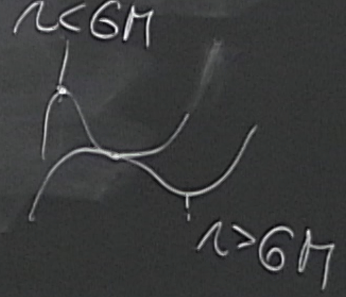
WE STUDY PERIHELION SHIFT ($\mathcal{E}=1$)

$$V = \underbrace{\frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2}}_{\text{NEWTON}} - \underbrace{\frac{ML^2}{r^3}}_{\text{GR-CORRECTION}}$$

INFLECTION POINT

$$V' = 0 = V''$$

$$\boxed{L^2 = 12M^2, \; r = 6M}$$



$\frac{ML^2}{r^3}$