

Title: PSI 2018/2019 - Relativity - Lecture 8

Date: Sep 12, 2018 09:00 AM

URL: <http://pirsa.org/18090007>

Abstract:

a) PARTICLE IN CURVED SPACE

GUESS: MOTION MAXIMIZES PROPER TIME

$$S = -m \int d\tau = -m \int d\lambda \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \quad \frac{d}{d\lambda} = \bullet$$

$$L(x, \dot{x}) = -m \sqrt{-g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu}$$

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0$$

$$\frac{dx^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^{\mu} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0, \quad G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}$$

(FOLLOWING EINSTEIN'S NOSE - EINSTEIN WAY)

$$\frac{\partial L}{\partial x^\alpha} = \frac{m g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu}{2 \sqrt{\quad}}$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{m g_{\alpha\mu} \dot{x}^\mu}{\sqrt{\quad}}$$

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{d}{d\lambda} \left(\frac{1}{\sqrt{\quad}} \right) m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{m}{\sqrt{\quad}} \left(g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu + g_{\mu\alpha} \ddot{x}^\mu \right)$$

$$\left(g_{\mu\nu} \dot{x}^\nu \dot{x}^\mu + g_{\mu\alpha} \ddot{x}^\mu \right), \quad f = -\sqrt{\frac{d}{d\lambda} \left(\frac{1}{f} \right)}$$

$$\frac{\partial L}{\partial x^\alpha} = \frac{m g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu}{2 \sqrt{\quad}}$$

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{m g_{\alpha\mu} \dot{x}^\mu}{\sqrt{\quad}}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{d}{dt} \left(\frac{1}{\sqrt{\quad}} \right) m g_{\mu\alpha} \dot{x}^\mu + \frac{m}{\sqrt{\quad}} (g_{\alpha\mu,\nu} \dot{x}^\mu \dot{x}^\nu)$$

$$g_{\mu\alpha} \ddot{x}^\mu + \frac{1}{2} g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\alpha\nu,\mu} \dot{x}^\nu \dot{x}^\mu$$

(FOLLOWING EINSTEIN'S NOSE - EINSTEIN WAY)

$$\frac{L}{x^2} = \frac{m g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu}{2 \sqrt{\quad}}, \quad \frac{\partial L}{\partial \dot{x}^\alpha} = \frac{m g_{\alpha\mu} \dot{x}^\mu}{\sqrt{\quad}}$$

$$\frac{d}{ds} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{d}{ds} \left(\frac{1}{\sqrt{\quad}} \right) m g_{\mu\alpha} \dot{x}^\mu + \frac{m}{\sqrt{\quad}} \left(g_{\alpha\mu, \nu} \dot{x}^\nu \dot{x}^\mu + g_{\mu\alpha} \right)$$

$$g_{\mu\alpha} \ddot{x}^\mu + \frac{1}{2} g_{\alpha\mu, \nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\alpha\nu, \mu} \dot{x}^\nu \dot{x}^\mu - \frac{1}{2} g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu = f$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{m g_{\alpha\mu} \dot{x}^\mu}{\sqrt{\dots}}$$

$$\frac{d}{d\lambda} \left(\frac{1}{\sqrt{\dots}} \right) m g_{\mu\alpha} \dot{x}^\alpha + \frac{m}{\sqrt{\dots}} \left(g_{\alpha\mu, \nu} \dot{x}^\nu \dot{x}^\mu + g_{\mu\alpha} \ddot{x}^\mu \right), \quad f = -\sqrt{\dots} \frac{d}{d\lambda} \left(\frac{1}{\sqrt{\dots}} \right)$$

$$\frac{1}{2} g_{\alpha\mu, \nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\alpha\nu, \mu} \dot{x}^\nu \dot{x}^\mu - \frac{1}{2} g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu = f g_{\mu\alpha} \dot{x}^\mu$$

$$\frac{1}{2} \dot{x}^\nu \dot{x}^\mu (g_{\alpha\mu, \nu} + g_{\alpha\nu, \mu} - g_{\mu\nu, \alpha})$$

$g_{\mu\nu} - g_{\nu\mu}$

x^μ

β

AFFINE PARAMETER. FOR EXAMPLE $\lambda = \tau$.. $f = 0$

$$u^\alpha \nabla_\alpha u^\beta = 0 \quad u^\alpha = \frac{dx^\alpha}{d\tau}$$

$$\left(g_{\mu\nu} \dot{x}^\nu \dot{x}^\mu + g_{\mu\alpha} \ddot{x}^\mu \right), \quad f = -\sqrt{\frac{d}{dx} \left(\frac{1}{f} \right)}$$

$$-\frac{1}{2} g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu = f g_{\mu\alpha} \dot{x}^\mu \quad T_{\alpha\beta} = T(\alpha\beta) + T[\alpha\beta]$$

01721
AFFINE PARAMETER. FOR EXAMPLE $\boxed{\lambda = \tau}$.. $f = 0$

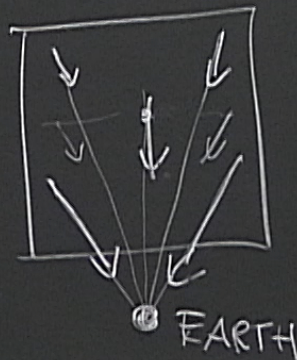
$$\boxed{u^\alpha \nabla_\alpha u^\beta = 0 \quad u^\alpha = \frac{dx^\alpha}{d\tau}}$$

$$\tau \rightarrow a\tau + b$$

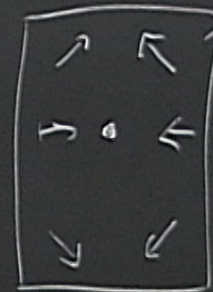
- GEODESIC DEVIATION EQUATION

- HOW TO QUANTIFY STRENGTH OF GRAVITY?

A: BY TIDAL FORCES



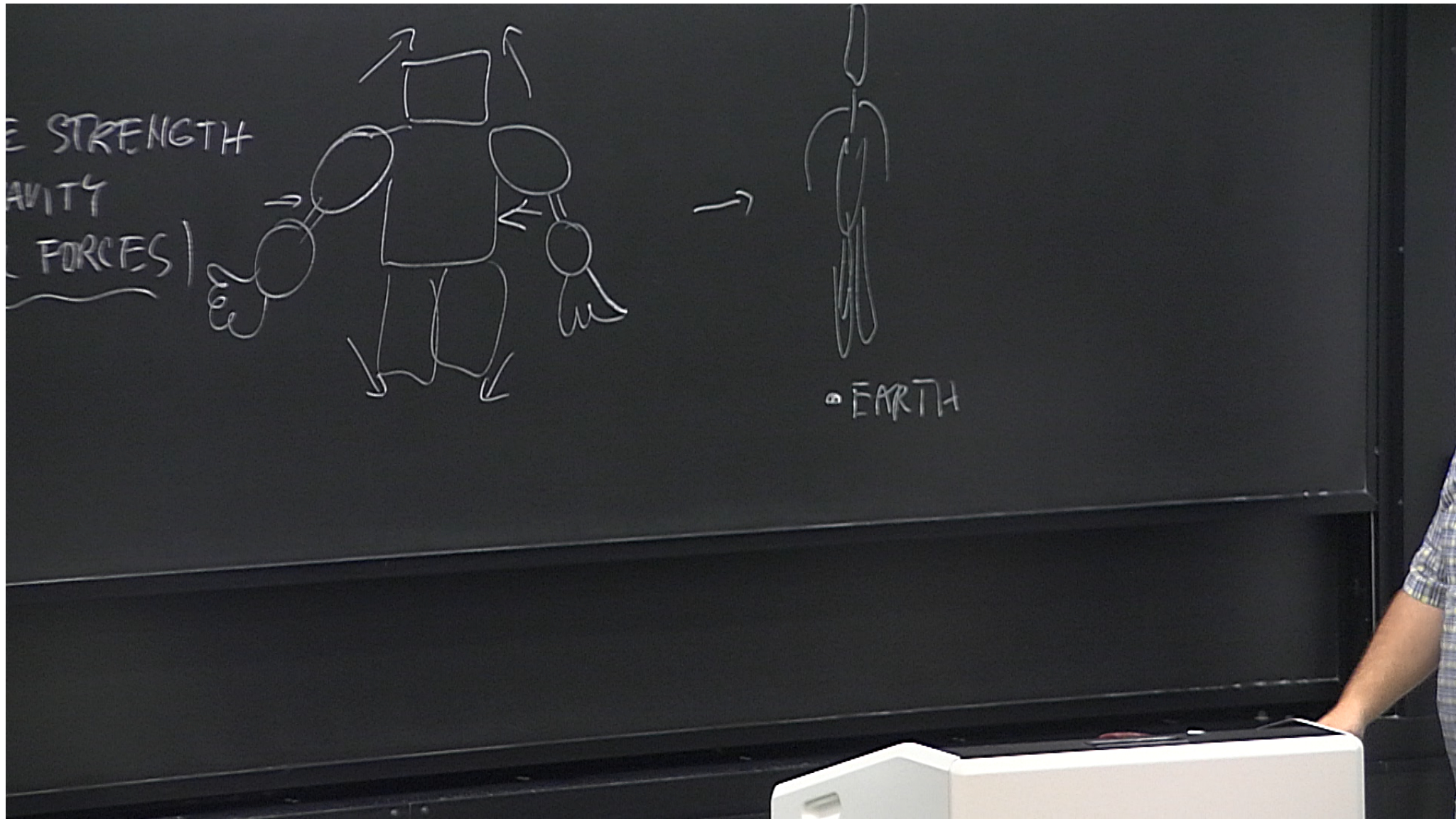
GOING
TO
INERTIAL
FRAME



THIS IS THE STRENGTH
OF GRAVITY
(TIDAL FORCES)

FREELY
FALLING

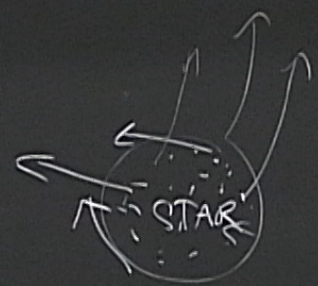
• EARTH



EARTH

EARTH

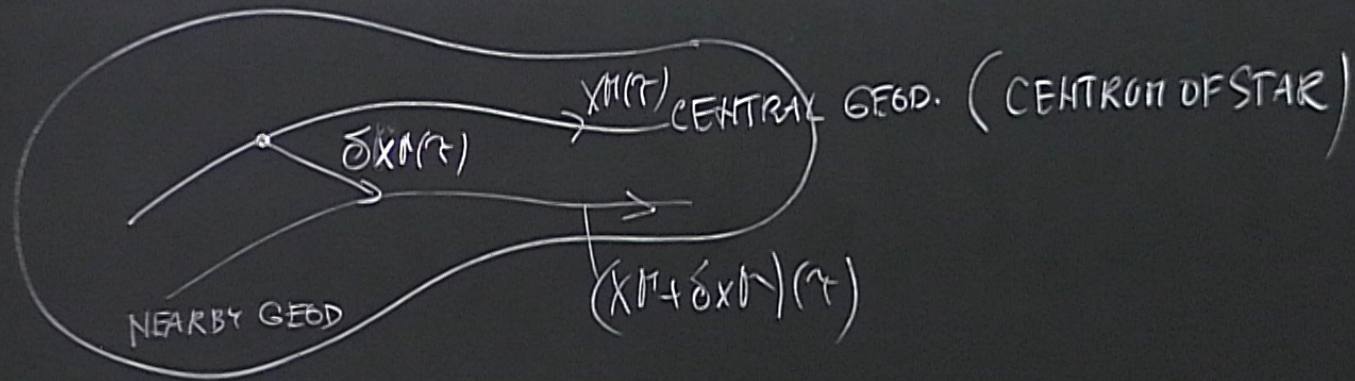
EX:
BH



STAR GETS DEFORMED

MODEL: "SWARM OF PARTICLES"
MOVING ON GEODESICS.

GEODESICS START DEVIATING BECAUSE OF TIDAL FORCES.



$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \quad (\text{CENTRAL})$$

$$0 = \frac{d^2 (x^\mu + \delta x^\mu)}{d\tau^2} + \Gamma^\mu_{\alpha\beta} (x + \delta x) \frac{d(x^\alpha + \delta x^\alpha)}{d\tau} \frac{d(x^\beta + \delta x^\beta)}{d\tau} \quad (\text{NEARBY})$$

SUBTRACTING

$$\frac{d^2 \delta x^\mu}{d\tau^2} + \delta x^\sigma \partial_\sigma \Gamma^\mu_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta + 2 \Gamma^\mu_{\alpha\beta}(x) \dot{x}^\alpha \frac{d}{d\tau} \delta x^\beta = 0$$

TO MAKE THIS A TENSOR EQ.

COVARIANT ACCELERATION

$$\frac{D^2 \delta x^\mu}{d\gamma^2} = \frac{d u^\mu}{d\gamma} + \sum^\mu_{\alpha\beta} \Gamma_{\alpha\beta}^\mu u^\alpha \dot{x}^\beta, \quad u^\alpha = \frac{D \delta x^\alpha}{d\gamma}$$

COVARIANT ACCELERATION

$$\frac{D^2 \delta x^\mu}{d\tau^2} = \frac{d u^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha \dot{x}^\beta, \quad u^\alpha = \frac{D \delta x^\alpha}{d\tau}$$

SCHEMATICALLY:

$$\frac{D^2 \delta x}{d\tau^2} = \frac{d^2 \delta x}{d\tau^2} + \underbrace{2 \Gamma \dot{x} \frac{d}{d\tau} \delta x}_{- \delta x \partial_x \Gamma \dot{x} \dot{x}} + (\partial \Gamma) \dot{x} \delta x + (\Gamma \Gamma - \Gamma \Gamma) \dot{x} \delta x$$

$$\frac{D^2 \delta x}{d\tau^2} = \underbrace{(\partial_\tau - \partial_\tau + \Gamma^\tau_\tau - \Gamma^\tau_\tau)}_R x \dot{x} \delta x$$

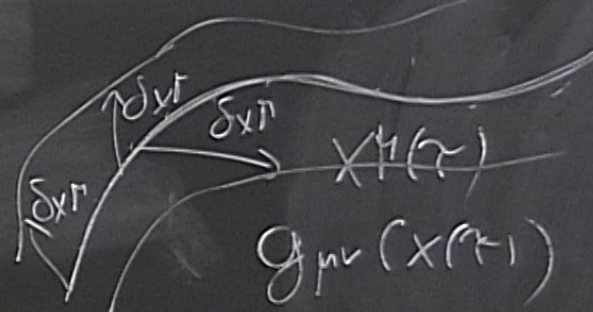
$$\boxed{\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} \delta x^\alpha \dot{x}^\beta \dot{x}^\gamma}$$

GEOD. DEV. EQ.

EVALUATED
ON CENTRAL
GEOD.

$x^\mu(\tau)$
 $g_{\mu\nu}(x(\tau))$

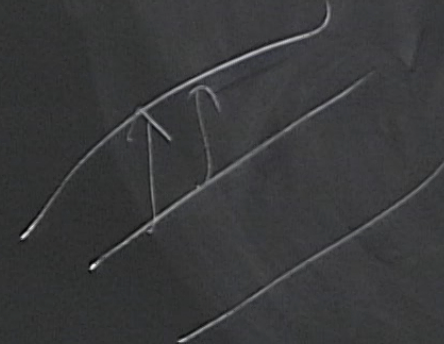
$$\frac{D^2 \delta x}{d\tau^2} = \underbrace{(\partial \Gamma - \partial \Gamma + \Gamma \Gamma - \Gamma \Gamma)}_R x \dot{x} \delta x$$



$$\boxed{\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu{}_{\alpha\beta\gamma} \delta x^\alpha \dot{x}^\beta \dot{x}^\gamma}$$

GEOD. DEV. EQ.

EVALUATED
ON CENTRAL
GEOD.



STAR GETS DEFORMED

b) MATTER IN CURVED SPACE

MINIMAL COUPLING PRINCIPLE (VAGUE)

LAWS IN SR $\rightarrow$ LAWS IN GR

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \quad | \quad \partial_\mu \rightarrow D_\mu$$

I COULD AS WELL ADD SOME CURVATURE TERM
 $+ R\phi^2$

• ACTION PRINCIPLE

$$S_m = \int \underbrace{d^d x \sqrt{-g}}_{\text{INVARIANT VOLUME ELEMENT}} \underbrace{\mathcal{L}_m(\Phi, \nabla\Phi, g)}_{\text{SCALAR}}$$

ACTION PRINCIPLE

$$S_m = \int d^d x \sqrt{-g} \mathcal{L}_m(\Phi, \nabla\Phi, g)$$

$\uparrow$ SCALAR
 INVARIANT VOLUME ELEMENT

$$\delta_\Phi S_m = \int d^d x \frac{\delta S_m}{\delta \Phi} \delta \Phi = \int d^d x \sqrt{-g} \left(\frac{\partial \mathcal{L}_m}{\partial \Phi} \delta \Phi + \frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \Phi)} \delta (\nabla_\mu \Phi) \right)$$

• ACTION PRINCIPLE

$$S_m = \int d^d x \sqrt{-g} \mathcal{L}_m(\Phi, \nabla\Phi, g)$$

$\uparrow$ SCALAR
 INVARIANT VOLUME ELEMENT

$$\delta_\Phi S_m = \int d^d x \frac{\delta S_m}{\delta \Phi} \delta \Phi = \int d^d x \sqrt{-g} \left(\frac{\partial \mathcal{L}_m}{\partial \Phi} \delta \Phi + \frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \Phi)} \delta (\nabla_\mu \Phi) \right)$$

$\delta \nabla_\mu \Phi = \nabla_\mu \delta \Phi$

STOKES: $\int \sqrt{g} D_\mu V^\mu d^d x = \int_{\partial \Sigma} m^2 V^\mu \sqrt{g} d^{d-1} x$

$\int \sqrt{g} D_\mu B d^d x = - \int (\partial_\mu A^\mu) B \sqrt{g} d^d x + \text{SURFACE TERM.}$

$\delta_\phi S_m = \int d^d x \sqrt{g} \left(\underbrace{\frac{\partial \mathcal{L}_m}{\partial \phi} - D_\mu \left(\frac{\partial \mathcal{L}_m}{\partial (D_\mu \phi)} \right)}_{\phi} \right) \delta \phi$

SURFACE TERM.

$$\left[\frac{\partial \mathcal{L}_m}{\partial \phi} - \nabla_\mu \left(\frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \phi)} \right) \right] = 0$$

EULER-LAGRANGE.

• ENERGY-MOMENTUM TENSOR

$$\delta_g S_m = -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$$

(ROSENFELD'S) - ENERGY-MOMENTUM TENSOR

THEOREM:

CONSERVATION
LAW

$\nabla_\mu T^{\mu\nu} = 0$ IF FIELD EQS FOR MATTER HOLD

WHY IS THIS CONSERVED : CONSIDER A DIFFEOMORPHISM

$$x^\mu \rightarrow x^\mu + \epsilon \xi^\mu$$

$$\text{INDUCES } \delta\phi, \delta g^{\mu\nu} = \epsilon \nabla^{(\mu} \xi^{\nu)}$$

BUT ACTION IS A SCALAR

$$+ \dots / x x \partial x$$

$$-\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}$$

$2\epsilon D^{\mu} \xi^{\nu}$

$$d^4x \sqrt{-g} \underbrace{D^{\mu} T_{\mu\nu}}_{=0} \xi^{\nu}$$

EX: SCALAR FIELD

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} (D_{\mu} \phi)(D_{\nu} \phi) - V(\phi)$$

(E-L) $\boxed{D^{\mu} D_{\mu} \phi - \frac{dV}{d\phi} = 0}$

$$\boxed{T_{\mu\nu} = D_{\mu} \phi D_{\nu} \phi + g_{\mu\nu} \mathcal{L}}$$

USING $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}$$

MAXWELL:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$\frac{\partial L}{\partial x^\alpha} = \frac{m g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu}{2\sqrt{\dots}}$$

$$\frac{\partial L}{\partial \dot{x}^\alpha}$$

$$\nabla_\mu F^{\mu\nu} = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) = \frac{d}{dt} \left(\frac{1}{\sqrt{\dots}} \right) m g_\mu$$

$$\frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{,\mu} = 0$$

MAX

$$g_{\mu\alpha} \ddot{x}^\mu + \frac{1}{2} g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu$$

$$\frac{1}{2} \dot{x}^\nu \dot{x}^\mu (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu})$$

$$F^{\mu\nu})_{;\mu} = 0$$

USING $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$

$$\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}$$

MAXWELL:

$$S = -\frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} g^{\mu\alpha} g^{\nu\beta}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$\frac{m_1 \alpha \dot{x}^\mu \dot{x}^\nu}{\sqrt{\dots}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}$$

$$\nabla_\mu F^{\mu\nu} = 0$$

USING $\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}$

$$= \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{-g}} \right) m g_{\mu\nu} + \frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{,\mu} = 0$$

$$+ \frac{1}{2} g_{\alpha\mu,\nu} \dot{x}^\nu \dot{x}^\mu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

MAXWELL:

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha} F_{\nu}{}^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right)$$

$$S = -\frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$