Title: PSI 2018/2019 - Relativity - Lecture 6

Date: Sep 10, 2018 09:00 AM

URL: http://pirsa.org/18090005

Abstract:

LAST TIME: TENSORS = "COORDINATE INVARIANT OBJECTS" - COVARIANT

EX. (1,2) TENSOR

$$T = T^{\alpha}{}_{\beta\gamma} \; \frac{\partial}{\partial x^{\alpha}} \otimes dx^{\beta} \otimes dx^{\gamma}$$

COMPTS. BASIS

i)

ii)

iii)

iv)

- COVARIANT DERIVATIVE ∇: (k,ℓ) TENSORS $\to$ $(k,\ell+1)$ TENSORS

i) DERIVATIVE: LINEAR & OBEYS LEIBNITZ

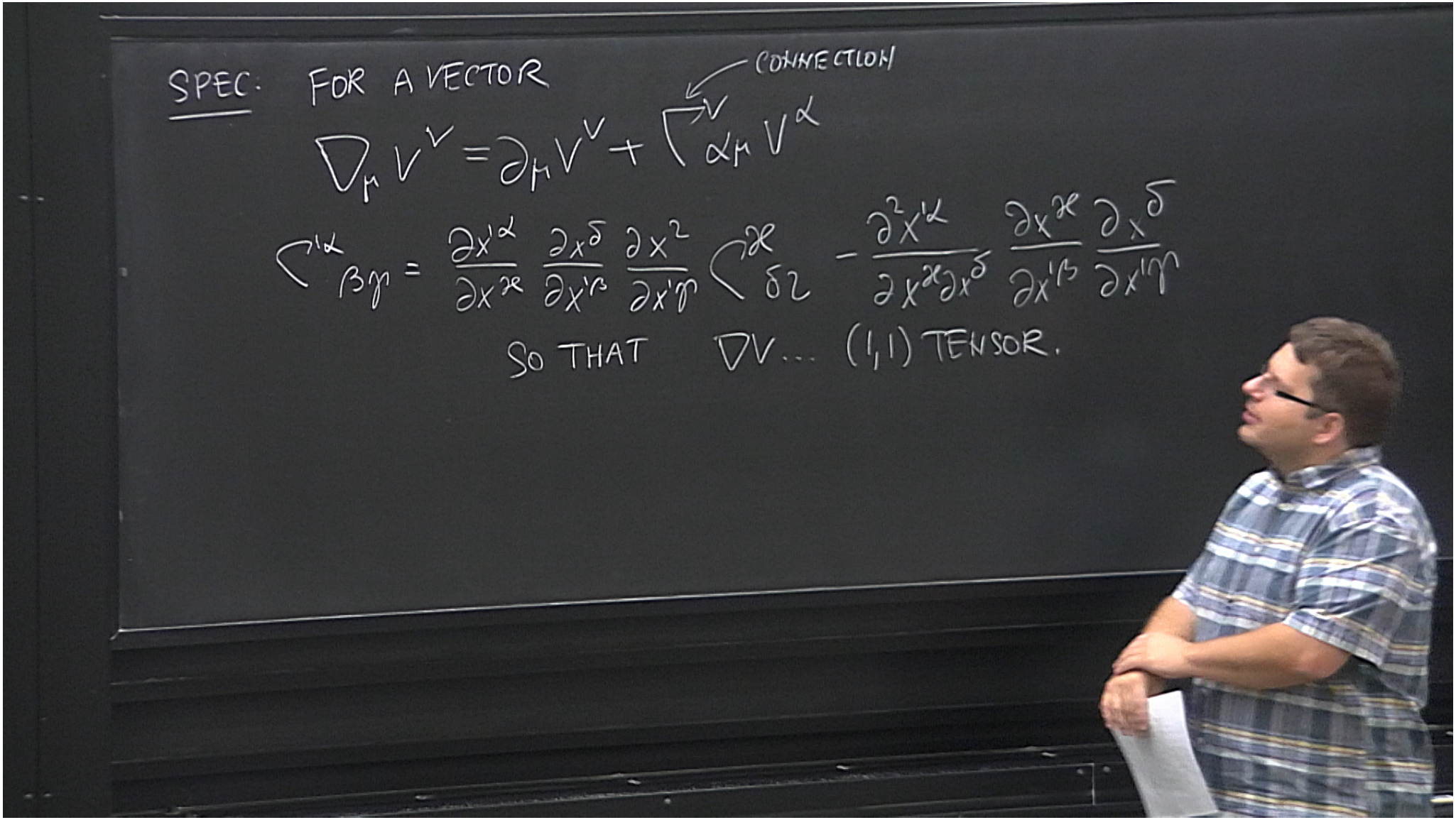
ii) REDUCES TO ∂ ON SCALARS: $\nabla_\mu f = \partial_\mu f$

iii) COMMUTES WITH CONTRACTION $\nabla_\mu (T^{\alpha \cdots}{}_{\alpha \cdots}) = \nabla_\mu T_{\text{CONTR} \cdots}$

iv) $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = -T^{\partial\epsilon}{}_{\mu\nu} \nabla_{\partial\epsilon} f$

$\Gamma^{\partial\epsilon}{}_{\nu\mu} - \Gamma^{\partial\epsilon}{}_{\mu\nu}$

TORSION


SPEC: FOR A VECTOR
CONNECTION
$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha$
$\Gamma'^\alpha_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\varepsilon}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^\zeta}{\partial x'^\gamma}\Gamma^\varepsilon_{\delta\zeta} - \frac{\partial^2 x'^\alpha}{\partial x^\varepsilon \partial x^\delta}\frac{\partial x^\varepsilon}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}$
SO THAT $\nabla V \ldots$ (1,1) TENSOR.

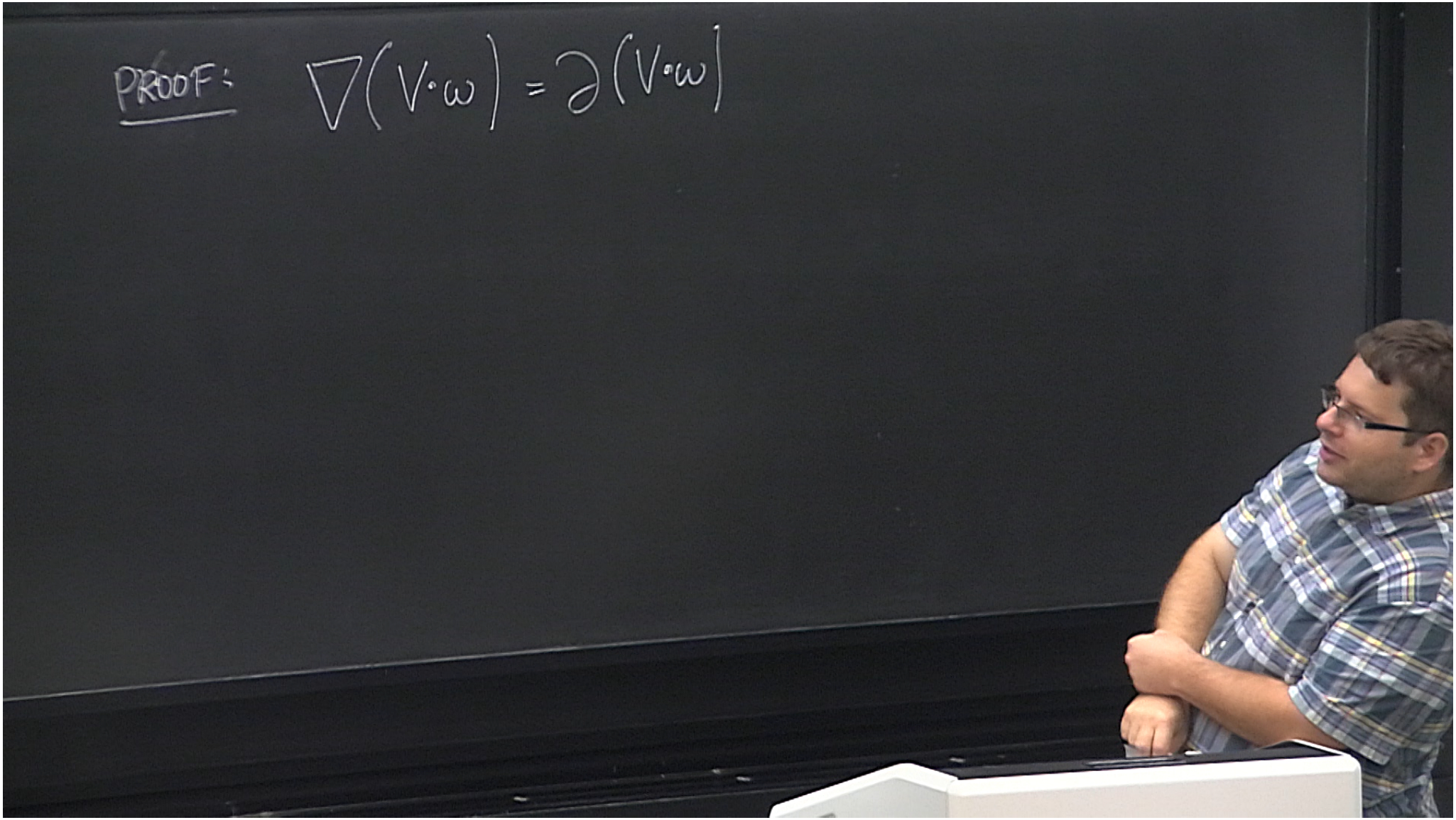
$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\alpha\mu} V^\alpha$$

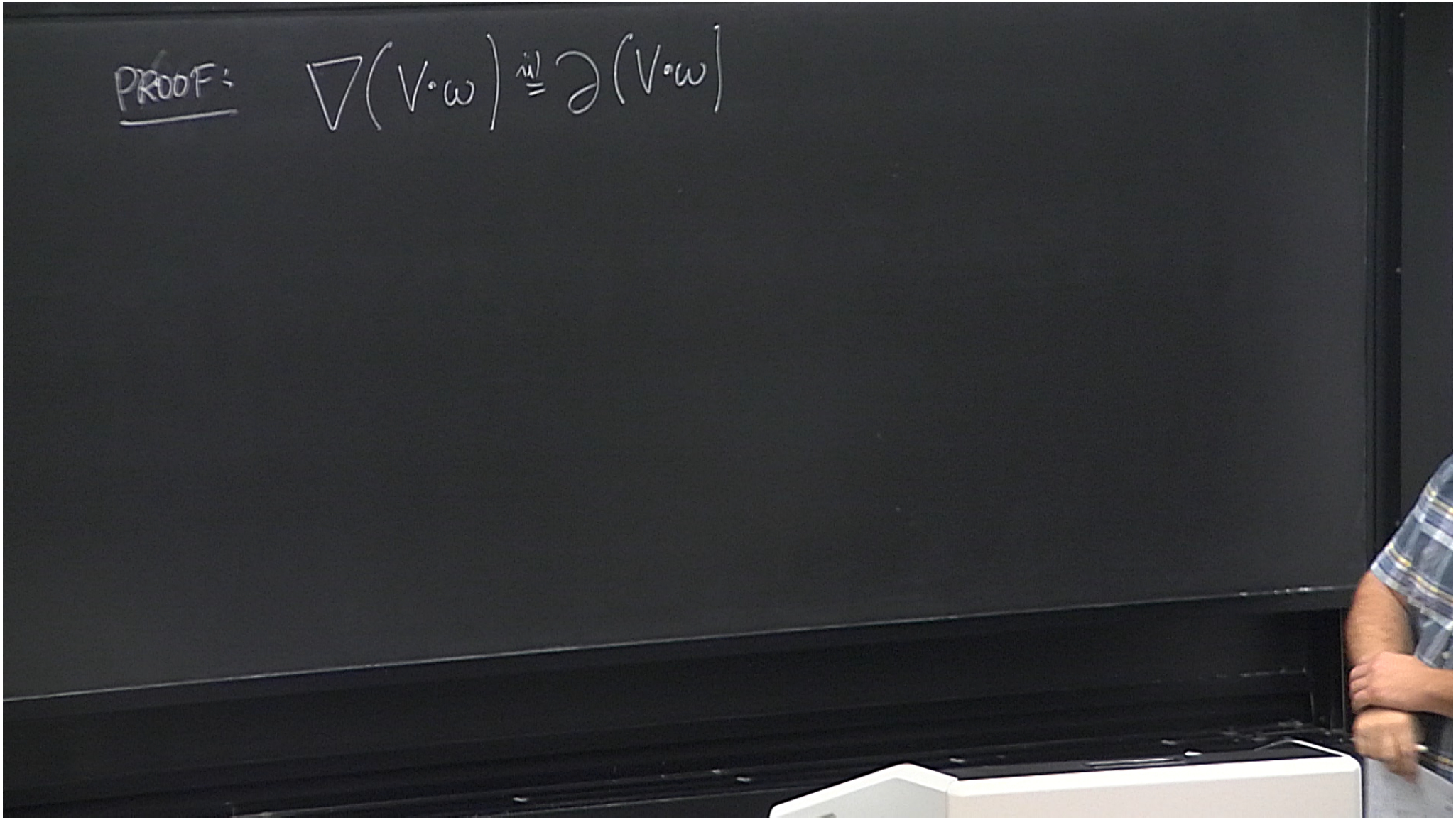
$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial x'^\alpha}{\partial x^\varepsilon}\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x^\zeta}{\partial x'^\gamma}\Gamma^{\varepsilon}_{\delta\zeta} - \frac{\partial^2 x'^\alpha}{\partial x^\varepsilon \partial x^\delta}\frac{\partial x^\varepsilon}{\partial x'^\beta}\frac{\partial x^\delta}{\partial x'^\gamma}$$

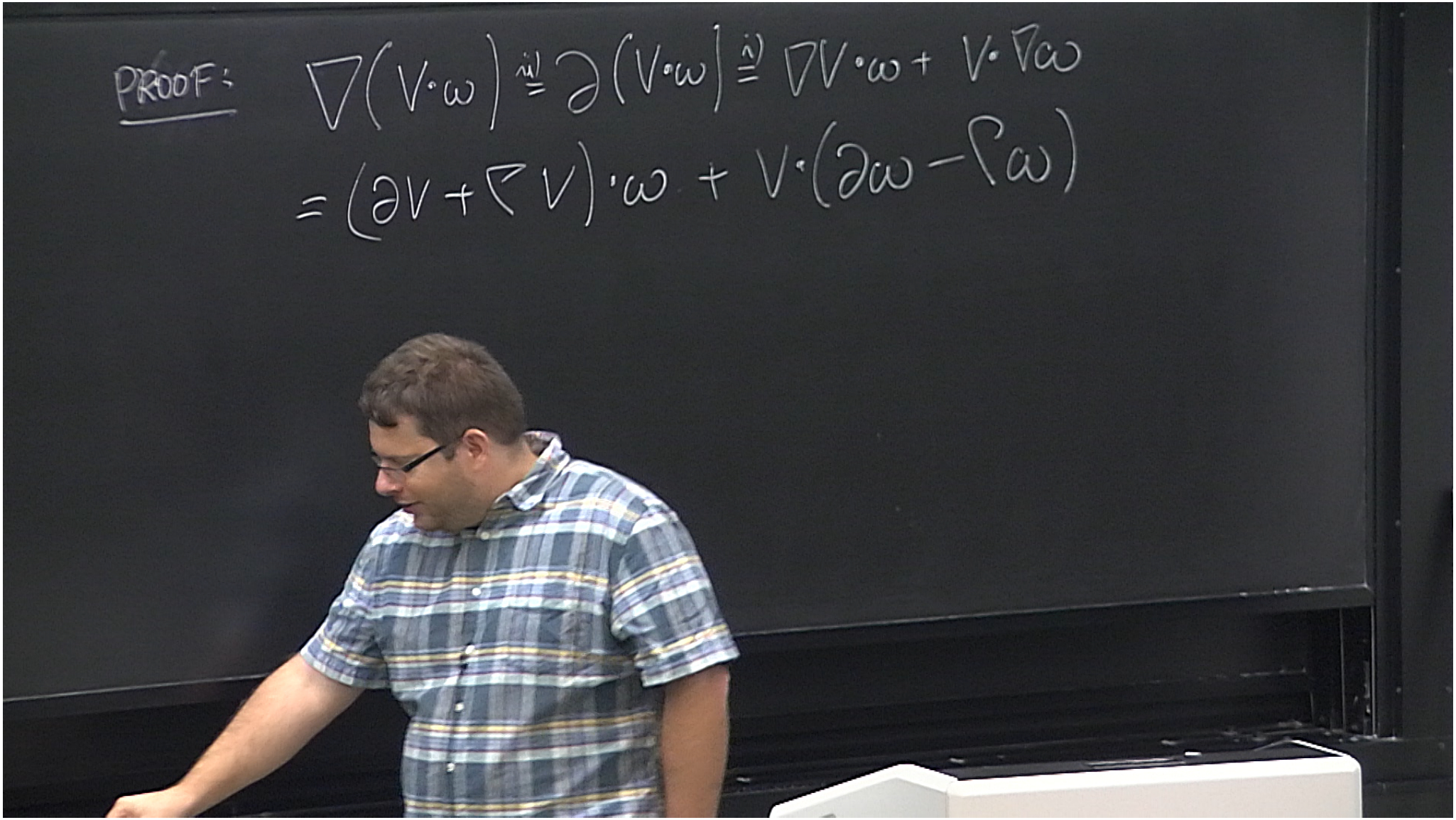
SO THAT $\nabla V \ldots$ (1,1) TENSOR.

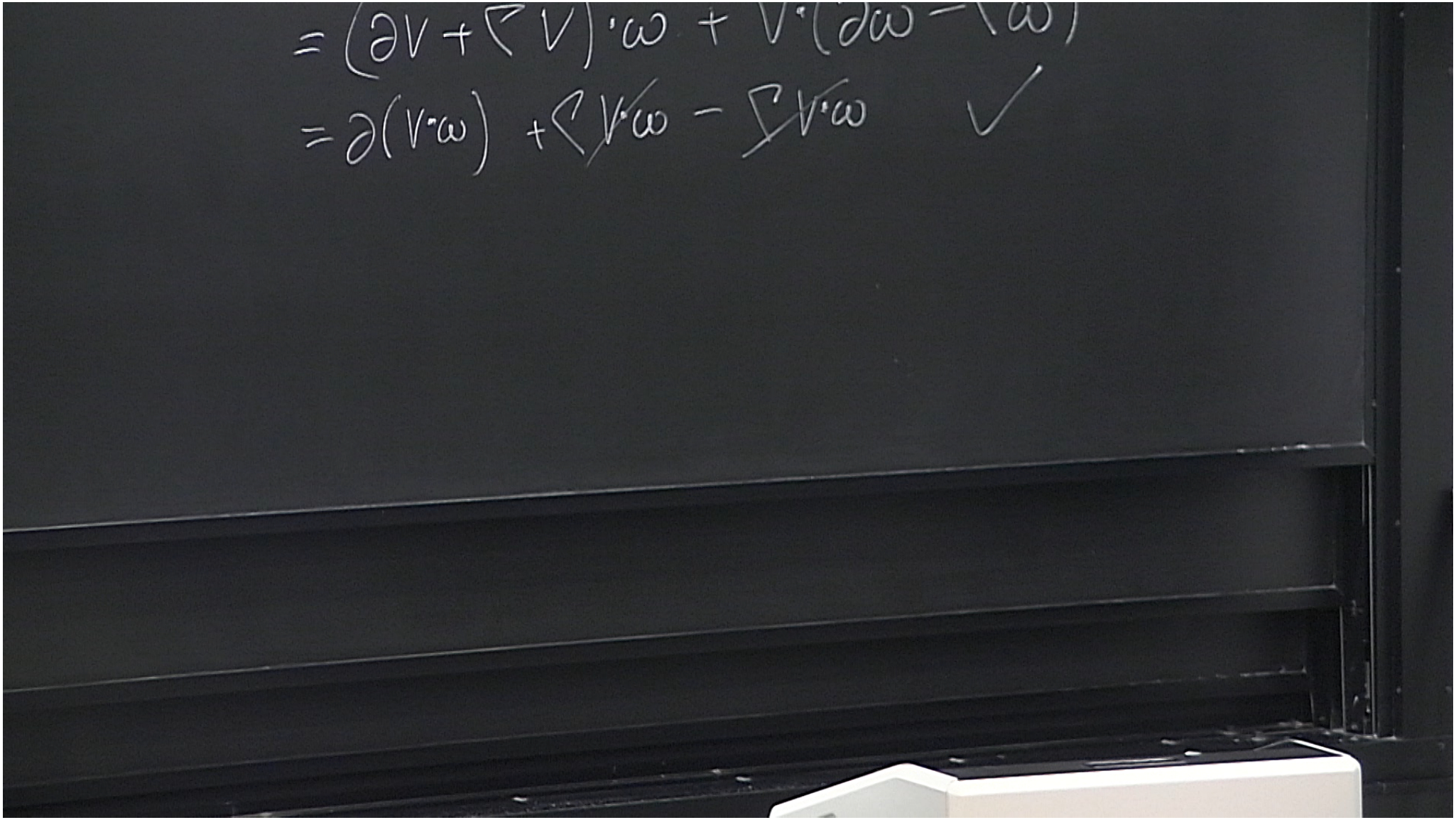
- COMMUTATION WITH CONTRACTION

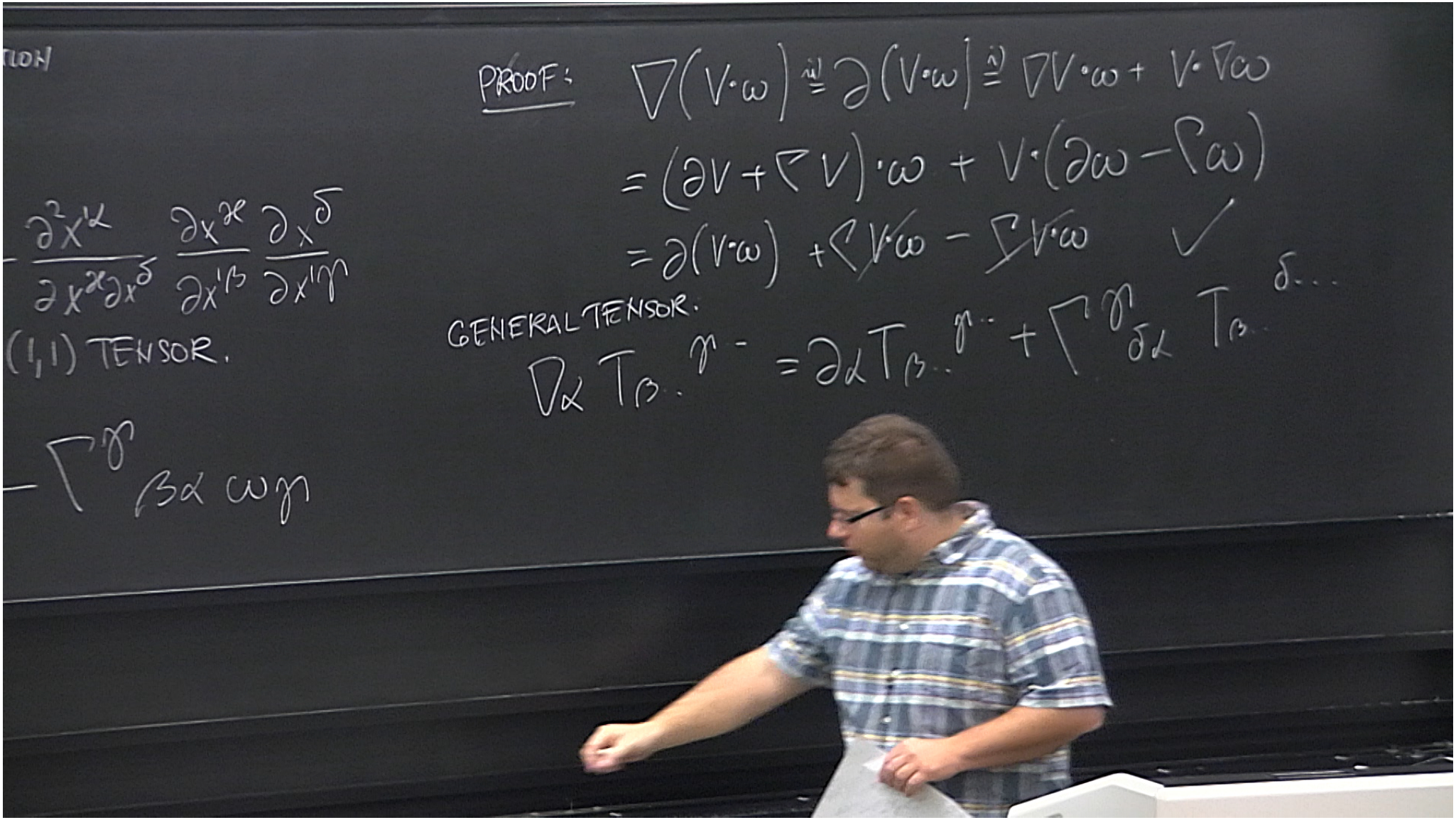
$$\nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^{\gamma}_{\beta\alpha}\omega_\gamma$$

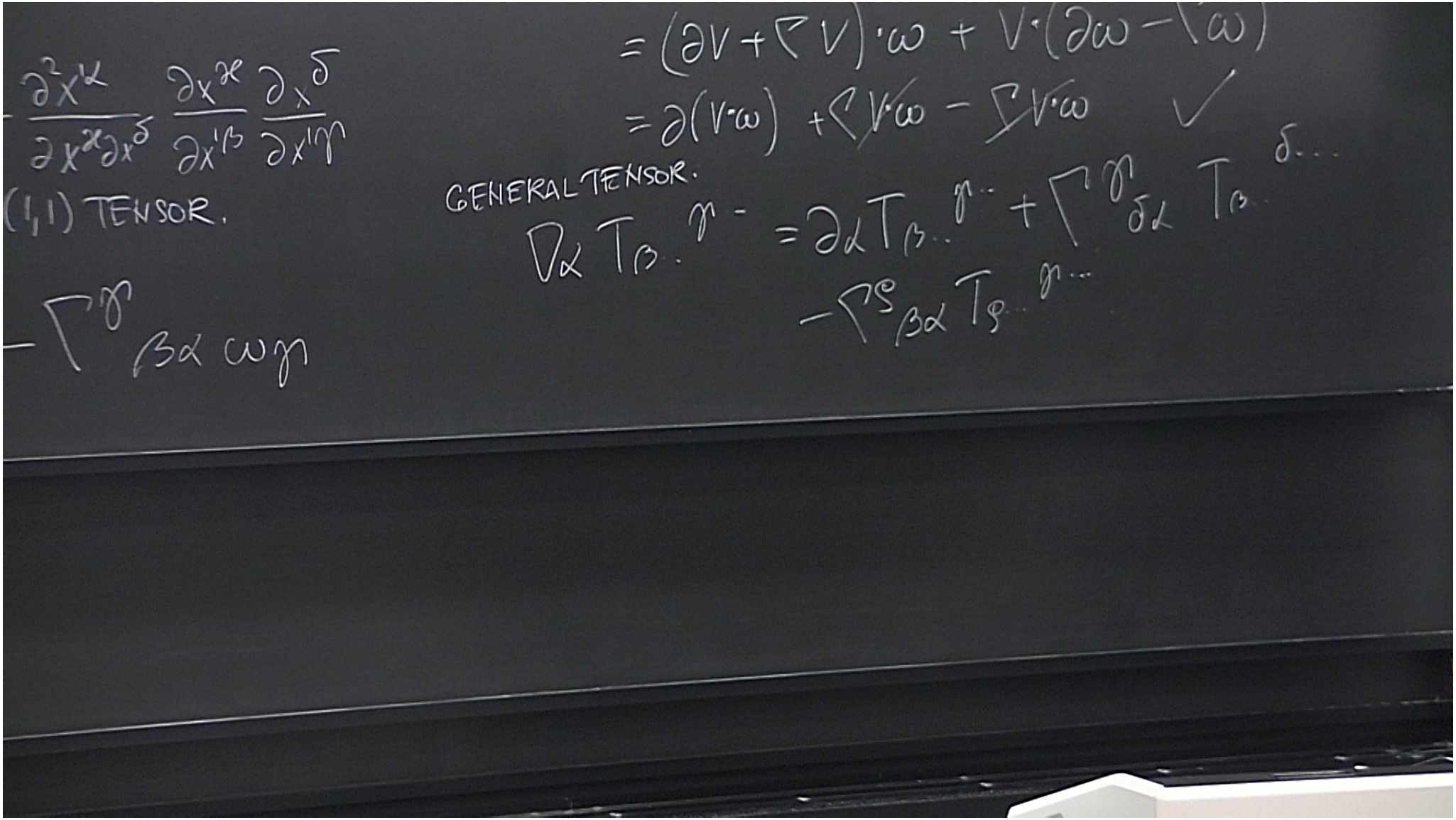

PROOF:
$\nabla(V \cdot \omega) = \partial(V \cdot \omega)$

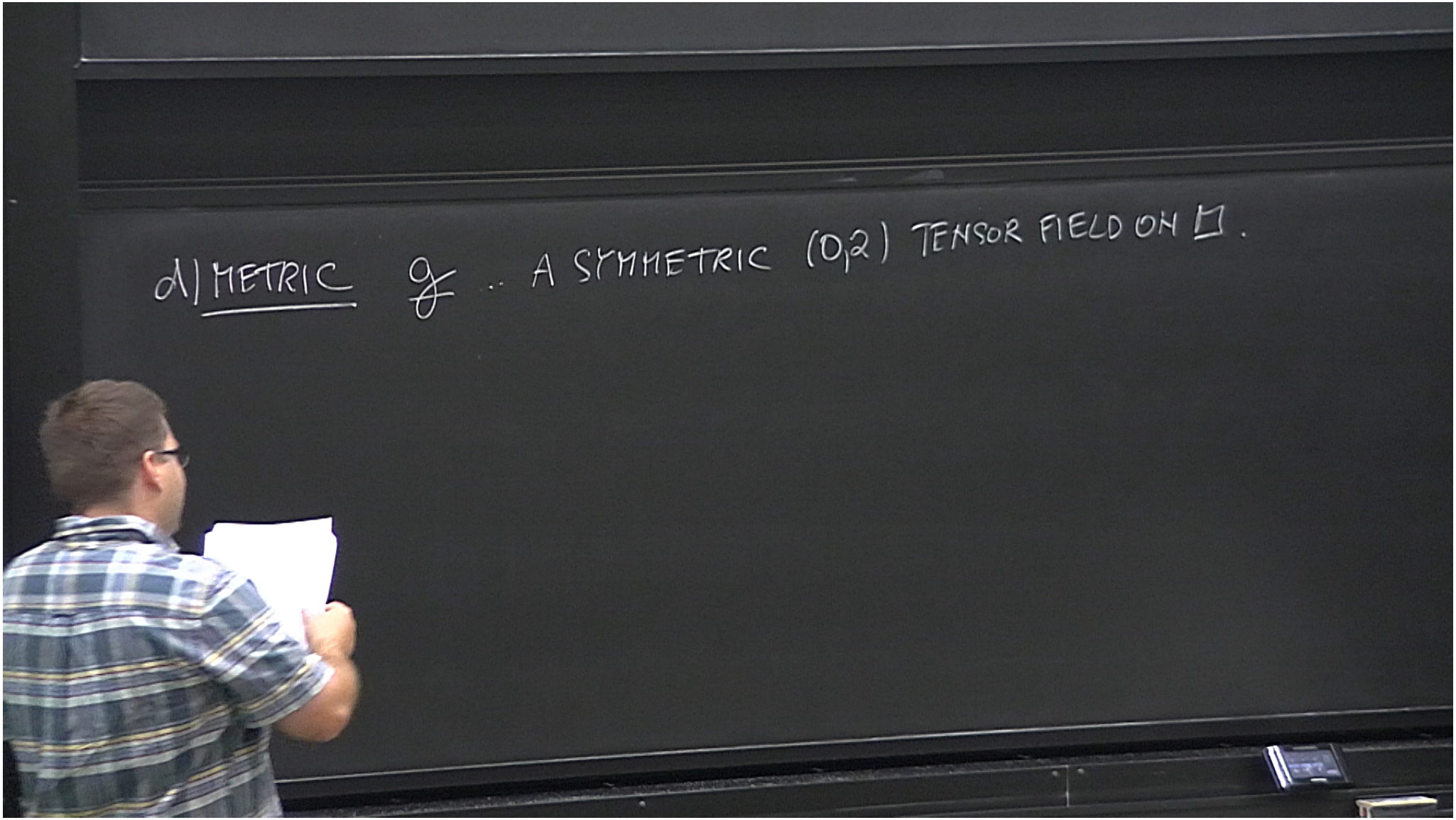

PROOF: $\nabla(V\cdot\omega) \overset{(1)}{=} \partial(V\cdot\omega)$


PROOF:
$\nabla(V\cdot\omega) \overset{ii)}{=} \partial(V\cdot\omega) \overset{i)}{=} \nabla V\cdot\omega + V\cdot\nabla\omega$
$= (\partial V + \nabla V)\cdot\omega + V\cdot(\partial\omega - \Gamma\omega)$


$= (\partial V + \Gamma V) \cdot \omega + V \cdot (\partial \omega - \Gamma \omega)$
$= \partial(V \cdot \omega) + \Gamma V \cdot \omega - \Gamma V \cdot \omega$

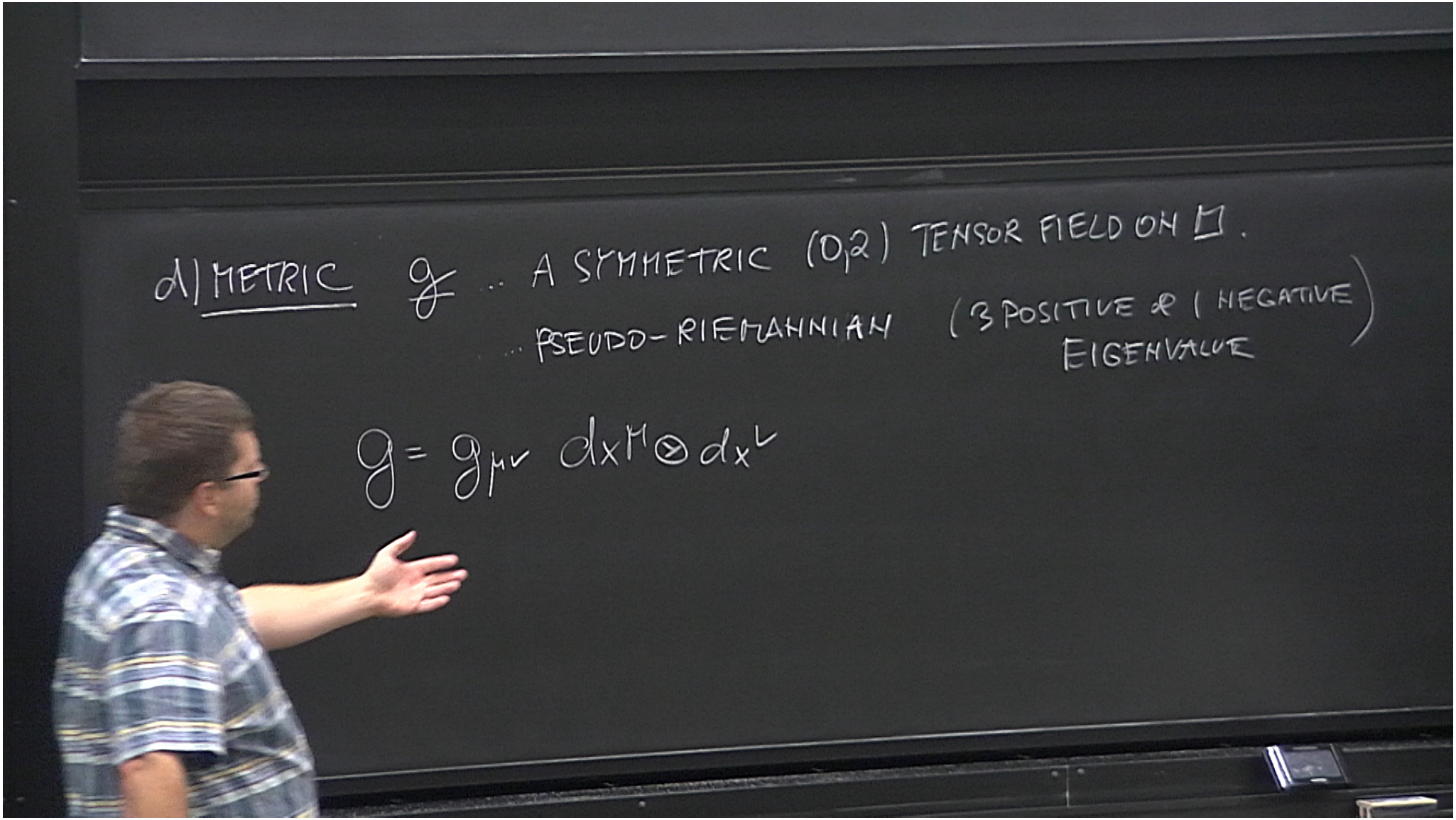

PROOF:
$\nabla(V\cdot\omega) \overset{(1)}{=} \partial(V\cdot\omega) \overset{(2)}{=} \nabla V\cdot\omega + V\cdot\nabla\omega$
$= (\partial V + \Gamma V)\cdot\omega + V\cdot(\partial\omega - \Gamma\omega)$
$= \partial(V\cdot\omega) + \Gamma V\cdot\omega - \Gamma V\cdot\omega$ ✓
GENERAL TENSOR.
$\nabla_\alpha T_{\beta\ldots}{}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}{}^{\gamma\ldots} + \Gamma^{\gamma}_{\delta\alpha} T_{\beta\ldots}{}^{\delta\ldots}$
$\frac{\partial^2 x'^{\alpha}}{\partial x^{\alpha}\partial x^{\delta}} \frac{\partial x^{\alpha}}{\partial x'^{\beta}} \frac{\partial x^{\delta}}{\partial x'^{\gamma}}$
(1,1) TENSOR.
$-\Gamma^{\gamma}_{\beta\alpha}\omega_\gamma$


$= (\partial V + \Gamma V)\cdot\omega + V\cdot(\partial\omega - \Gamma\omega)$
$= \partial(V\cdot\omega) + \Gamma V\cdot\omega - \Gamma V\cdot\omega$ ✓
$\frac{\partial^2 x'^{\alpha}}{\partial x^{\beta}\partial x^{\delta}} \frac{\partial x^{\beta}}{\partial x'^{\beta}} \frac{\partial x^{\delta}}{\partial x'^{\eta}}$
(1,1) TENSOR.
GENERAL TENSOR.
$D_\alpha T_{\beta\ldots}^{\gamma\ldots} = \partial_\alpha T_{\beta\ldots}^{\gamma\ldots} + \Gamma^{\gamma}_{\delta\alpha} T_{\beta\ldots}^{\delta\ldots}$
$- \Gamma^{\rho}_{\beta\alpha} T_{\rho\ldots}^{\gamma\ldots}$
$- \Gamma^{\gamma}_{\beta\alpha}\omega_\eta$


d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON □ .

1) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)


d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON □.
.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)
$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

... PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu$$



d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

... PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu = ds^2$$

d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

... PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = \underbrace{g_{\mu\nu}}_{\text{COMPTS}} \underbrace{dx^\mu \otimes dx^\nu}_{\text{BASIS}} = g_{\mu\nu} dx^\mu dx^\nu = ds^2$$



d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu = ds^2$$

$g_{\mu\nu}$: COMPTS; $dx^\mu \otimes dx^\nu$: BASIS

$$g(V, W)$$

d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = \underbrace{g_{\mu\nu}}_{\text{COMPTS}} \underbrace{dx^\mu \otimes dx^\nu}_{\text{BASIS}} = g_{\mu\nu} dx^\mu dx^\nu = ds^2$$

$$g(V, W) = g_{\mu\nu} V^\mu W^\nu \quad .. \text{ NUMBER}$$

d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu = ds^2$$

$g_{\mu\nu}$ COMPTS $dx^\mu \otimes dx^\nu$ BASIS

$$g(V, W) = g_{\mu\nu} V^\mu W^\nu \quad \text{.. NUMBER}$$

SPEC $V = W$ $g(V, V)$ .. HOLE BIG V IS

d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON $\square$.

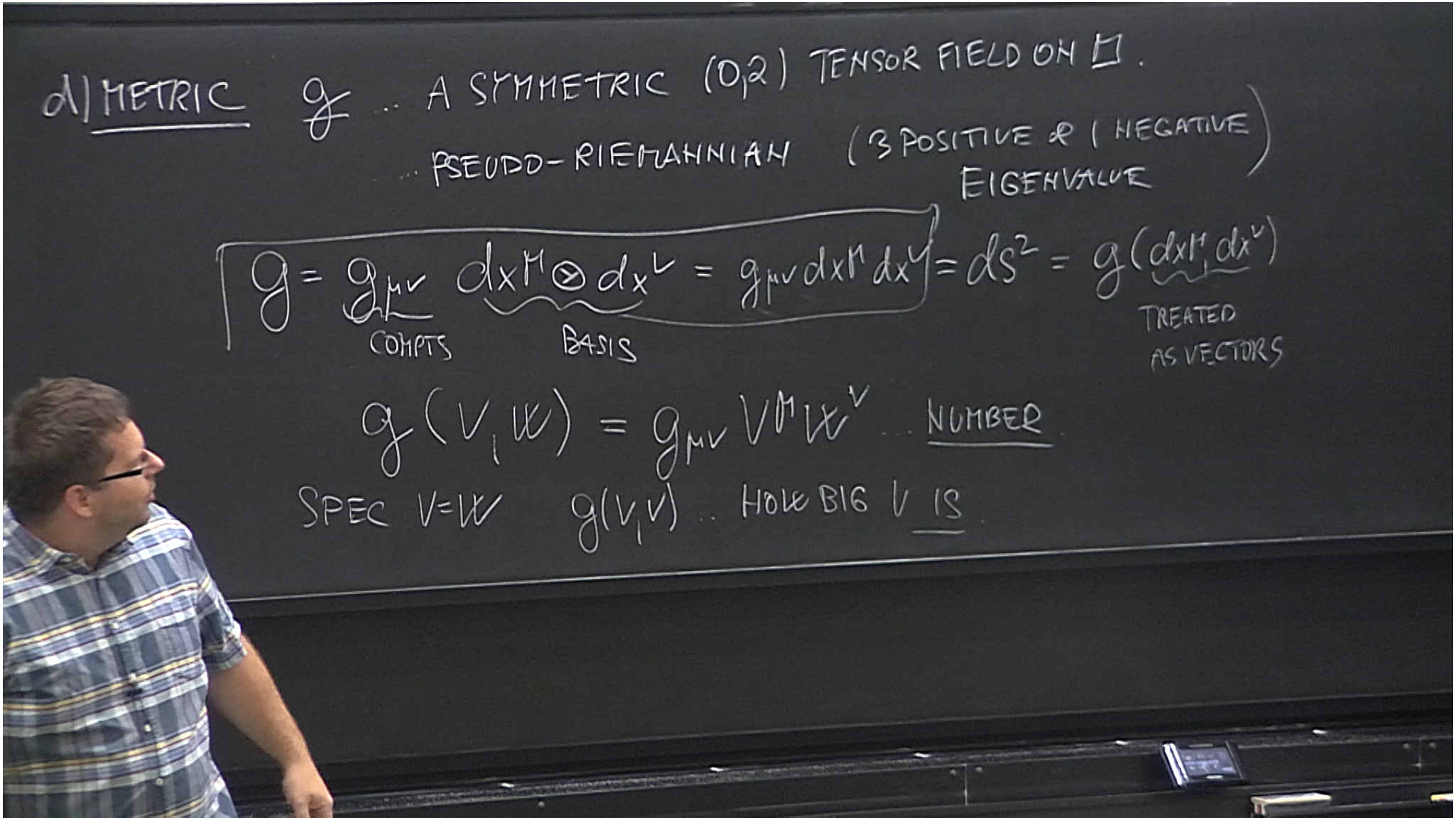
.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu = ds^2 = g(dx^\mu, dx^\nu)$$

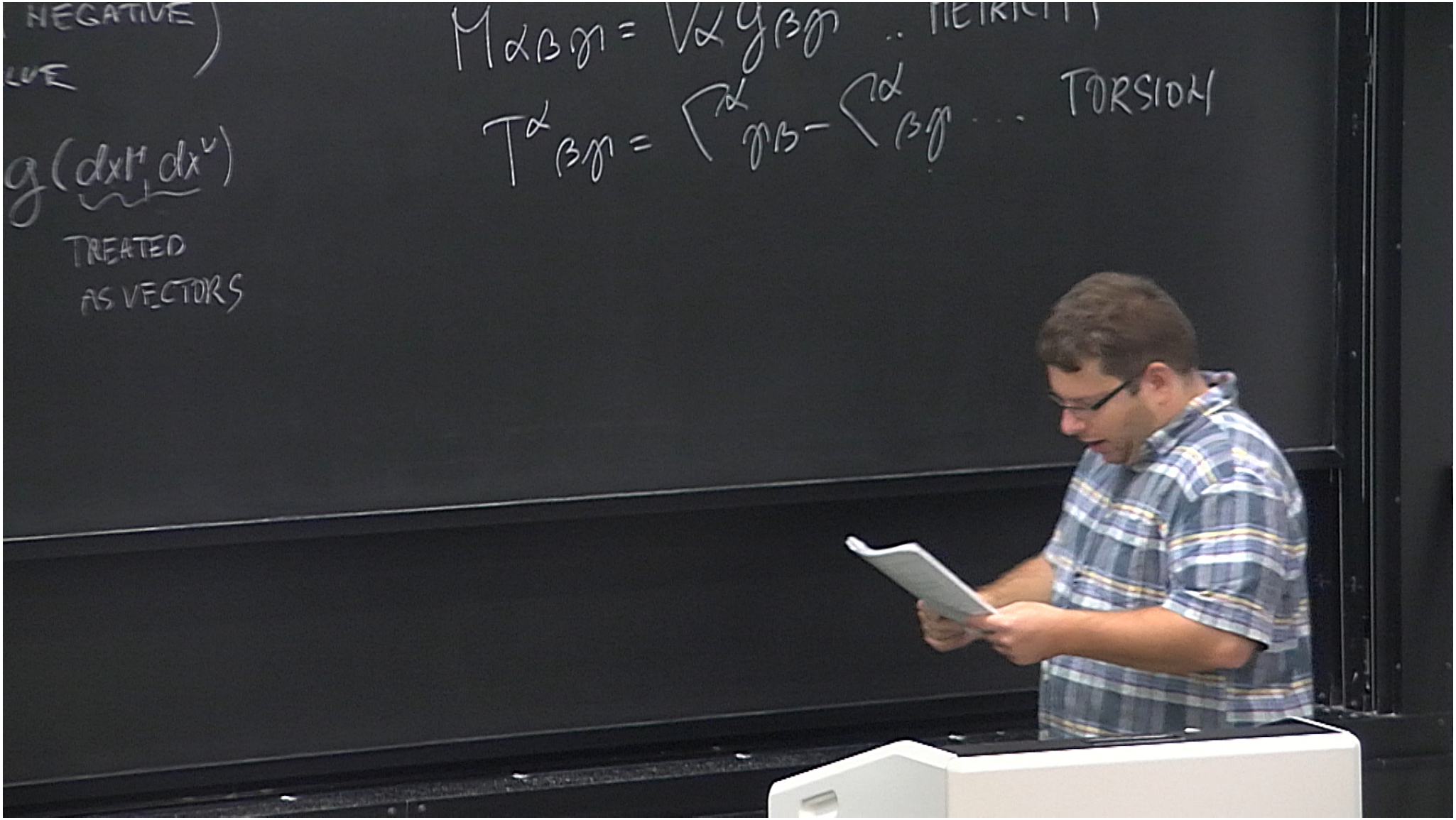
$g_{\mu\nu}$: COMPTS; $dx^\mu \otimes dx^\nu$: BASIS; dx^μ, dx^ν: TREATED AS VECTORS

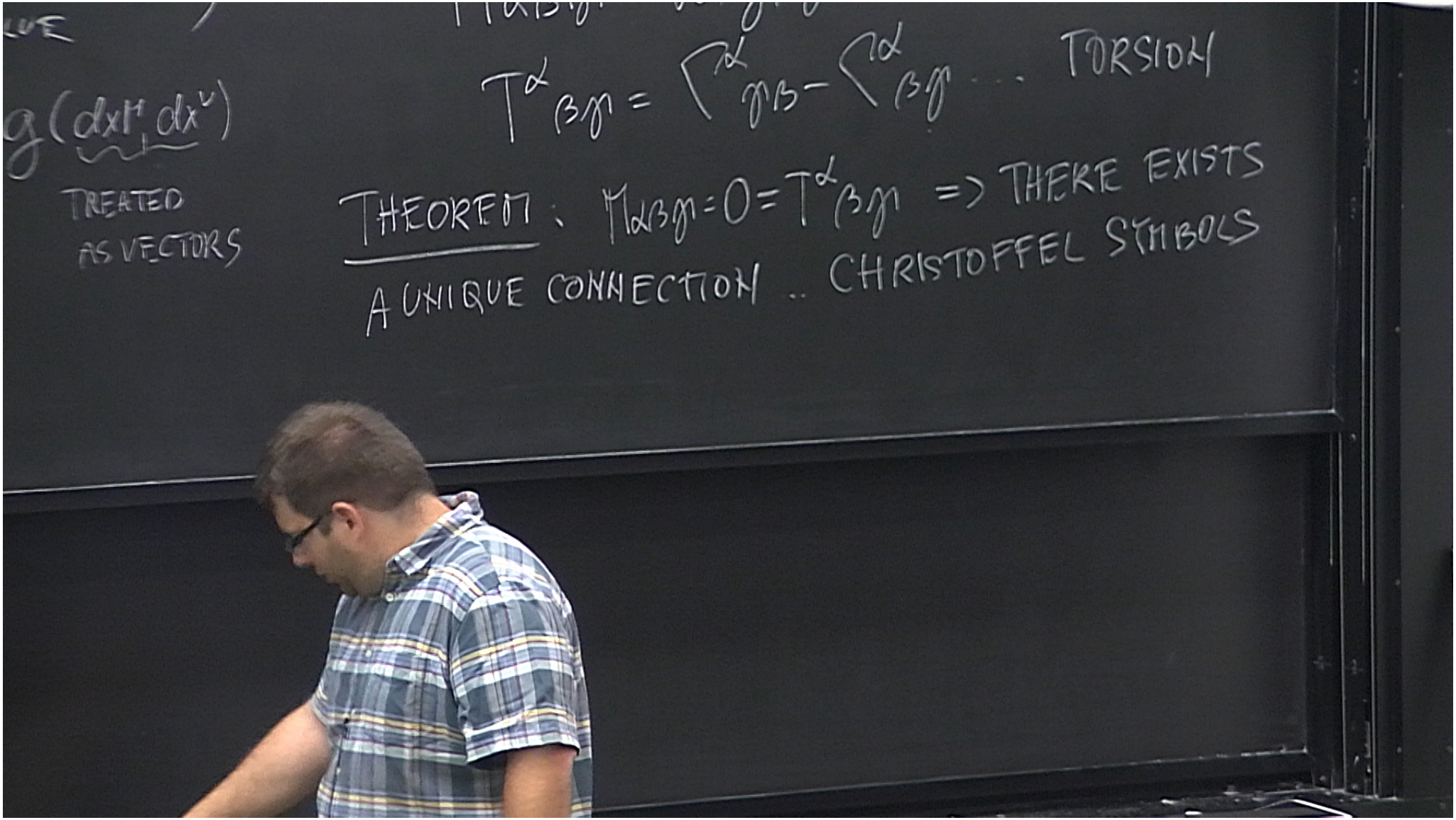
$$g(V, W) = g_{\mu\nu} V^\mu W^\nu \quad \text{.. NUMBER}$$

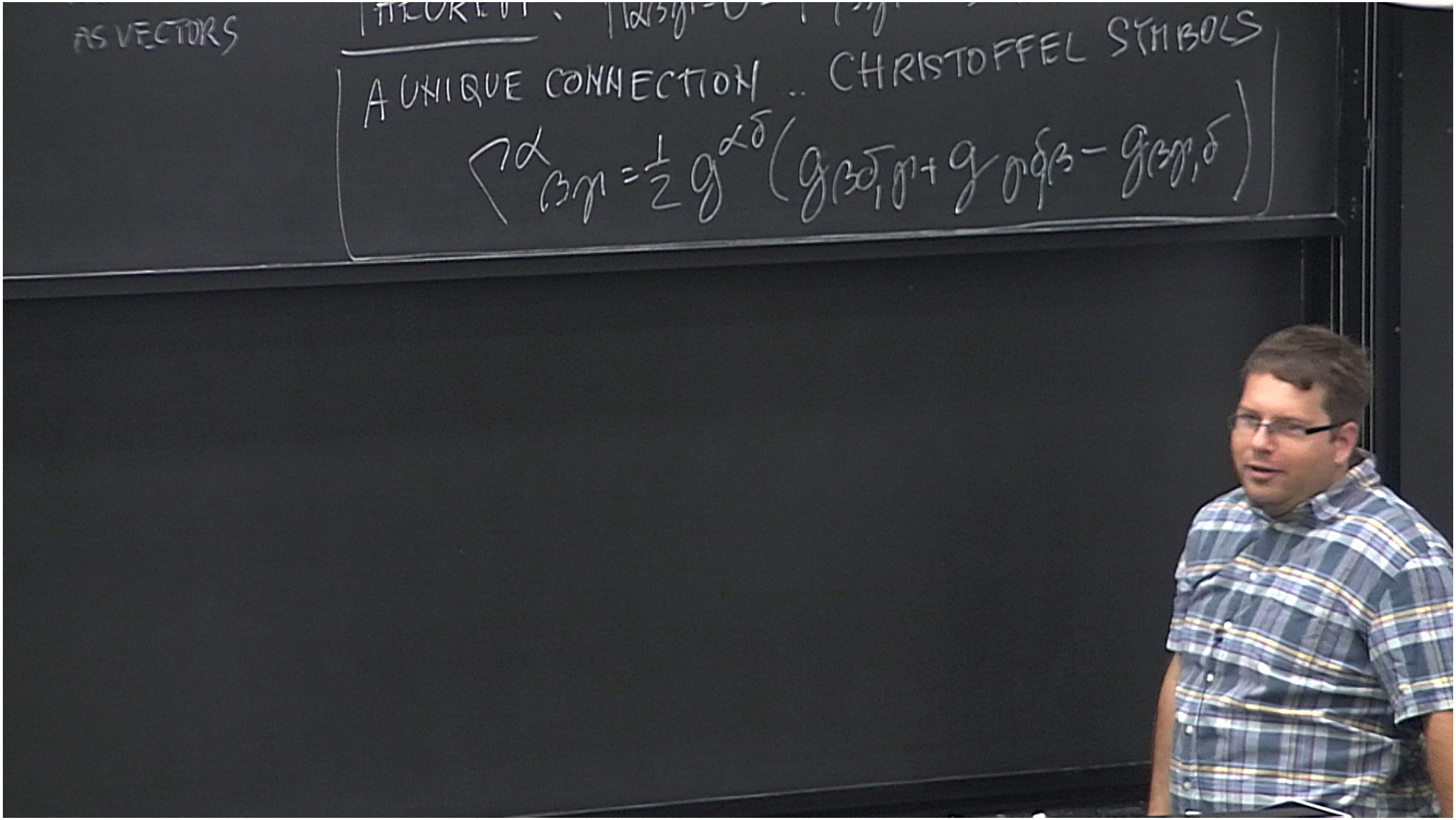
SPEC $V = W$ $\quad g(V, V)$ .. HOLE BIG V IS.


d) METRIC g .. A SYMMETRIC (0,2) TENSOR FIELD ON ☐.
.. PSEUDO-RIEMANNIAN (3 POSITIVE & 1 NEGATIVE EIGENVALUE)
g = g_μν dx^μ ⊗ dx^ν = g_μν dx^μ dx^ν = ds² = g(dx^μ, dx^ν)
COMPTS
BASIS
TREATED AS VECTORS
g(V, W) = g_μν V^μ W^ν .. NUMBER
SPEC V=W g(V,V) .. HOLE BIG V IS.


CHRISTOFFEL SYMBOLS
$M_{\alpha\beta\gamma} = D_\alpha g_{\beta\gamma}$ .. METRICITY
HEGATIVE
$g(dx^\mu, dx^\nu)$
TREAT
AS VI


NEGATIVE
$g(dx^\mu, dx^\nu)$
TREATED
AS VECTORS
$M_{\alpha\beta\gamma} = \nabla_\alpha g_{\beta\gamma}$
$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma}$
TORSION


$T^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\gamma\beta} - \Gamma^{\alpha}{}_{\beta\gamma}$ ... TORSION
$g(dx^{\mu}, dx^{\nu})$
TREATED AS VECTORS
THEOREM: $\nabla_{\alpha}\beta\gamma = 0 = T^{\alpha}{}_{\beta\gamma}$ => THERE EXISTS A UNIQUE CONNECTION .. CHRISTOFFEL SYMBOLS


AS VECTORS
A UNIQUE CONNECTION .. CHRISTOFFEL SYMBOLS

SPEC $V=W$ $g(v,v)$ .. HOLE BIG V IS.

- EQUIVALENCE PRINCIPLE.
  - AT A POINT $g_{\mu\nu} = \eta_{\mu\nu}$

SPEC $V=V^\vee$ $g(v,v)$ .. HOLE BIG V IS

- EQUIVALENCE PRINCIPLE.
  - AT A POINT $g_{\mu\nu} = \eta_{\mu\nu}$

    GRAM-SCHMIDT DIAGONALIZATION

$$g_{\mu\nu} dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D O\, dx$$

$$D = \begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$$

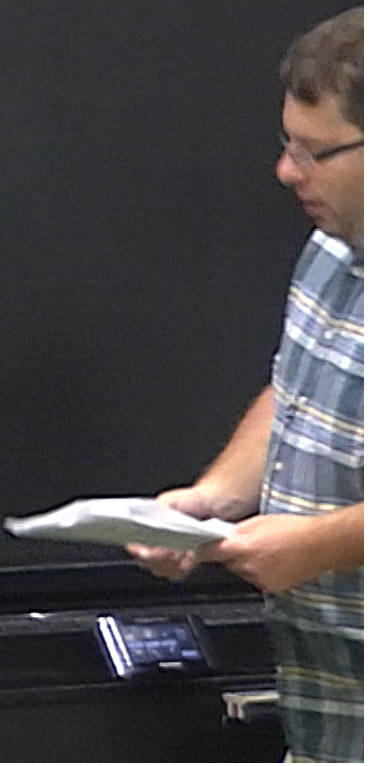
SPEC $V=W$ $g(v,v)$ .. HOLE BIG V IS

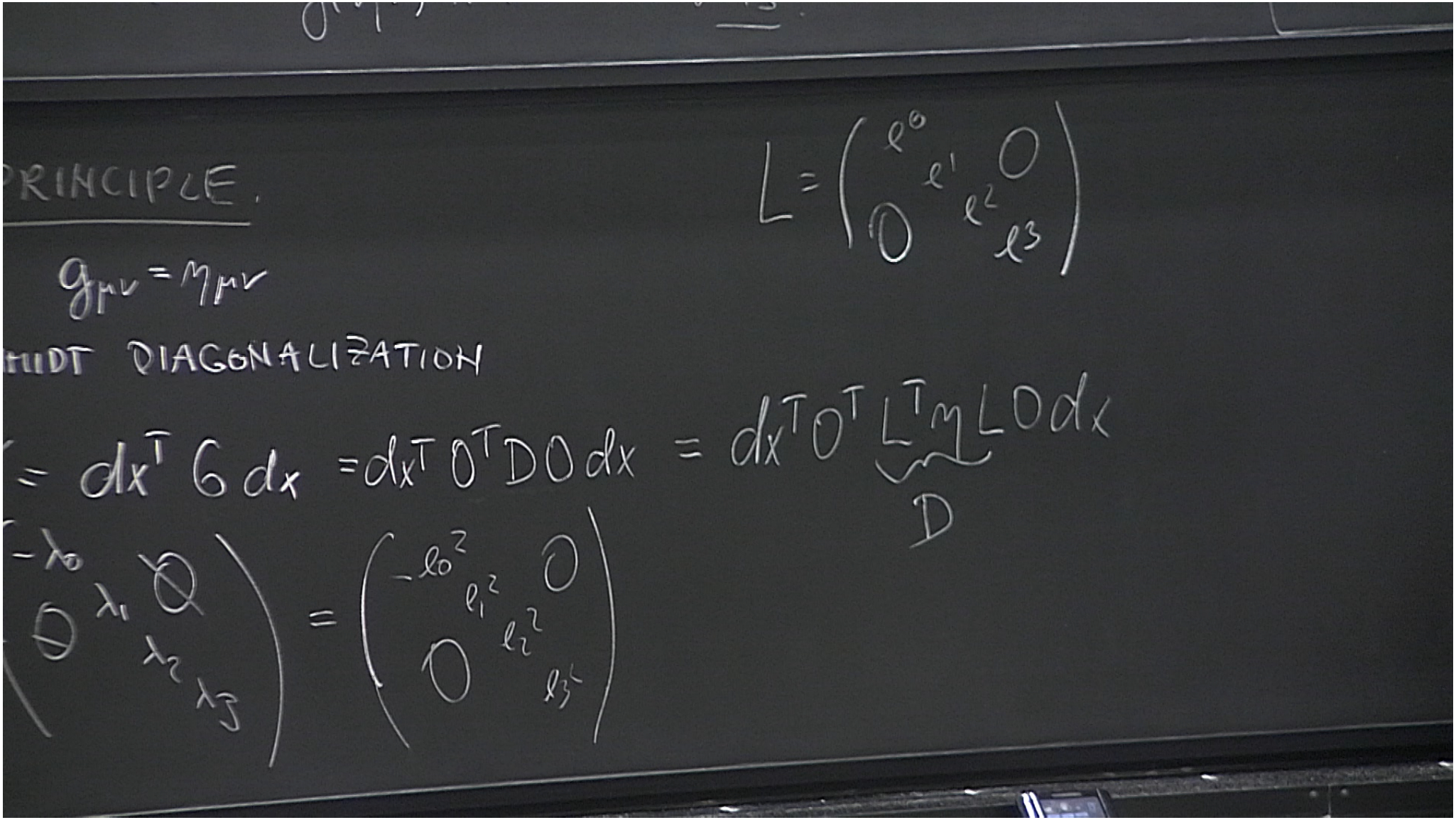
- EQUIVALENCE PRINCIPLE.
  - AT A POINT $g_{\mu\nu} = \eta_{\mu\nu}$

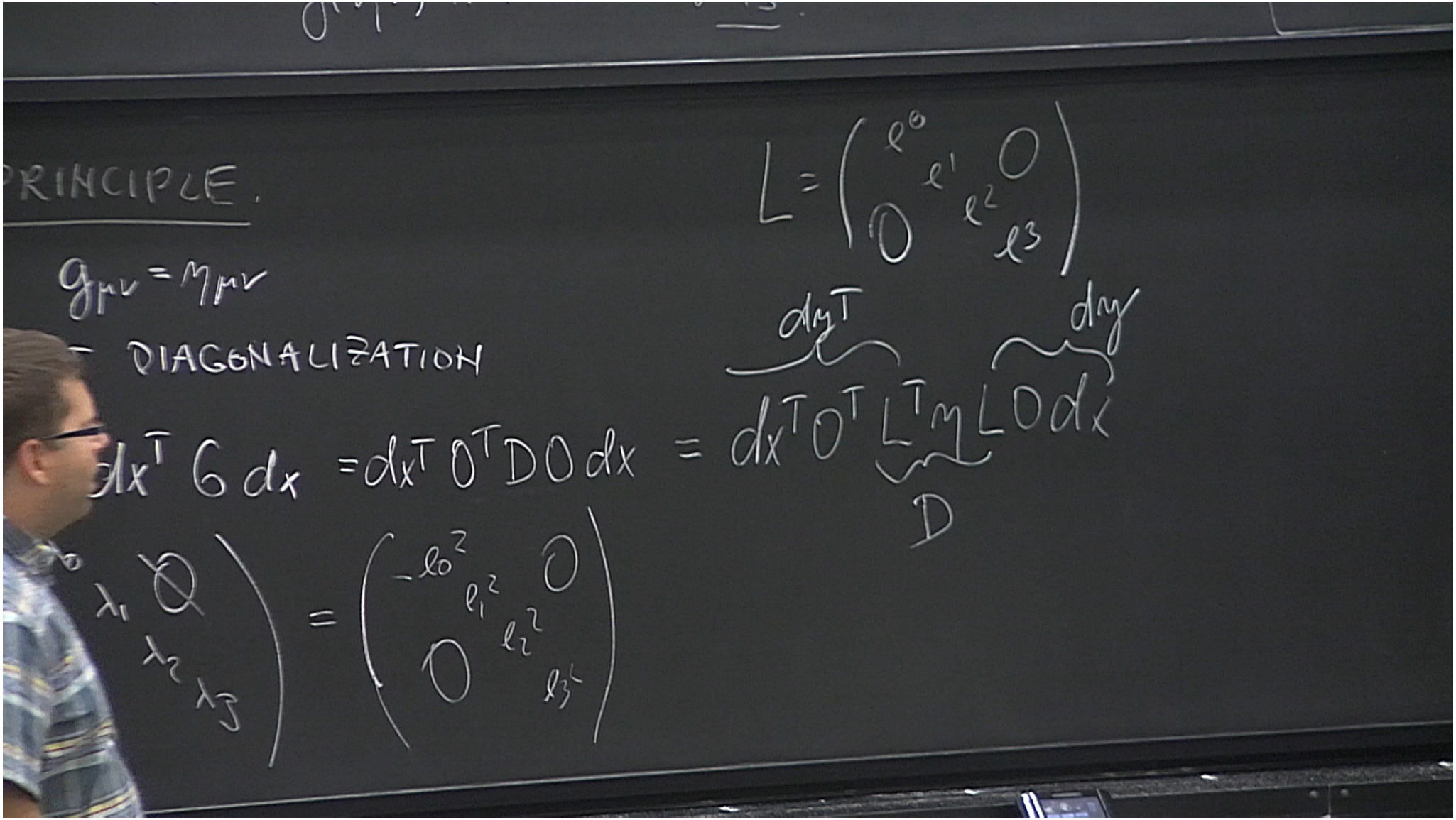
    GRAM-SCHMIDT DIAGONALIZATION

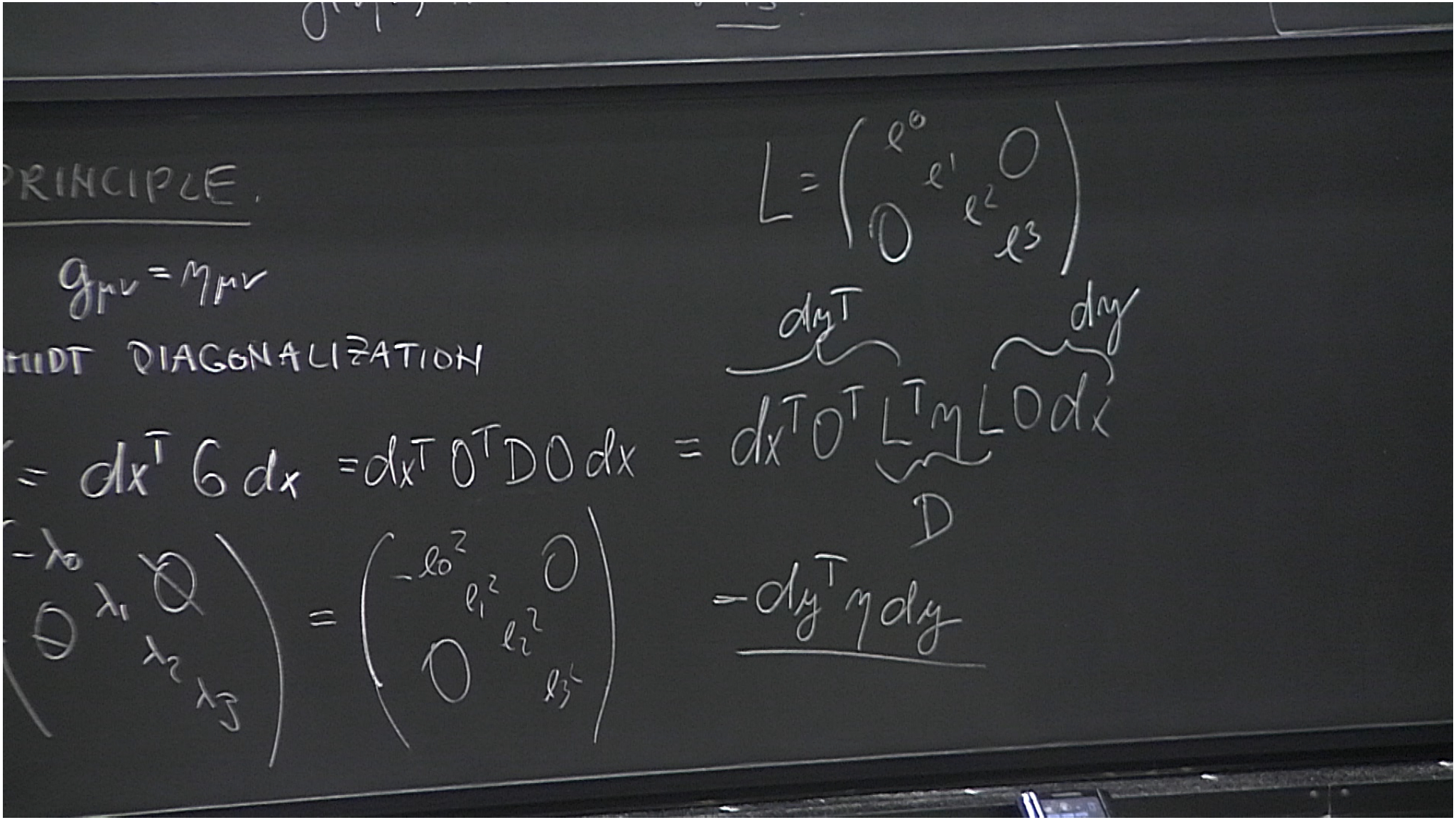
$$g_{\mu\nu} dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D O\, dx$$

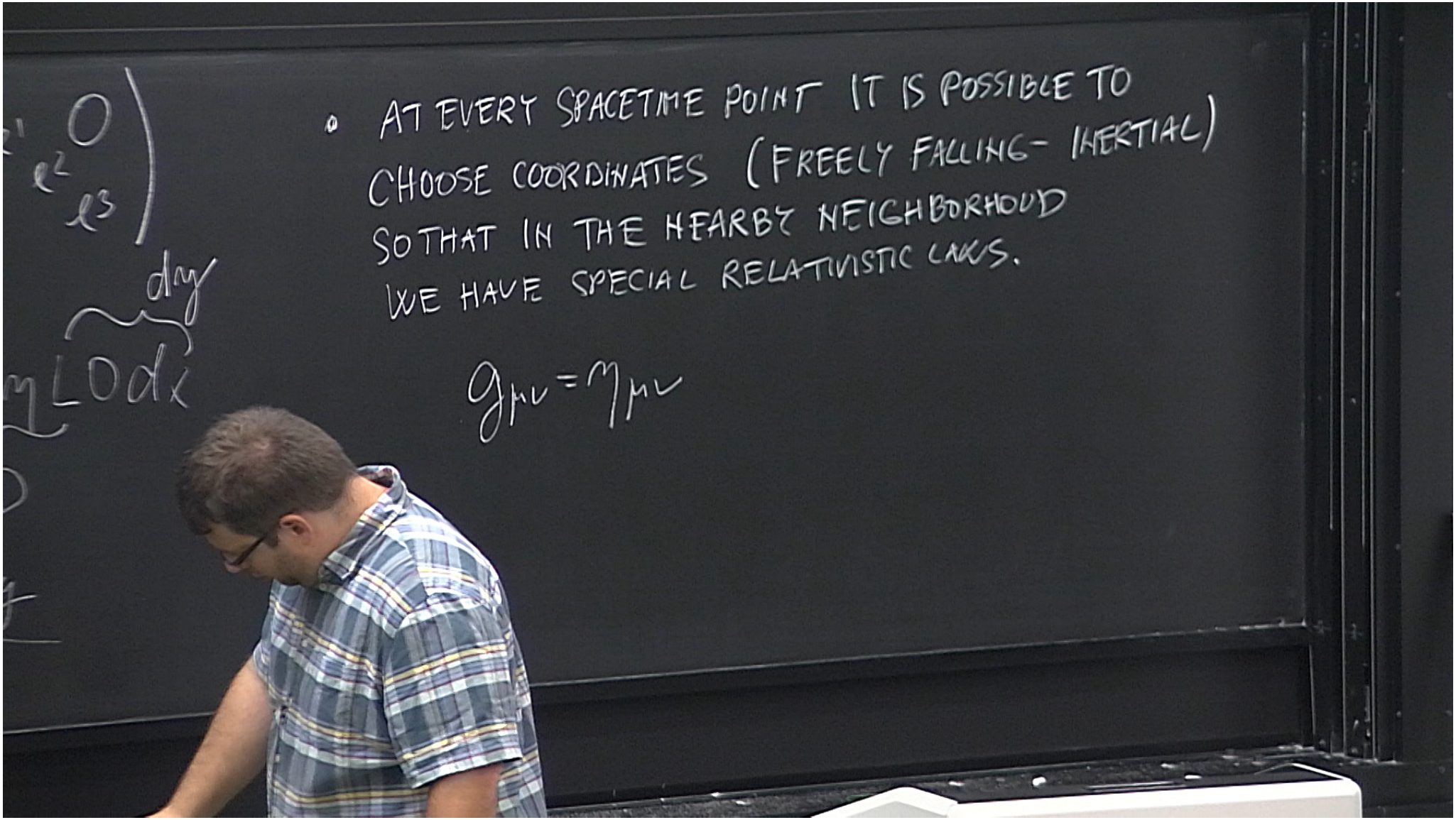
$$D = \begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -\ell_0^2 & & & \\ & \ell_1^2 & & \\ & & \ell_2^2 & \\ & & & \ell_3^2 \end{pmatrix}$$

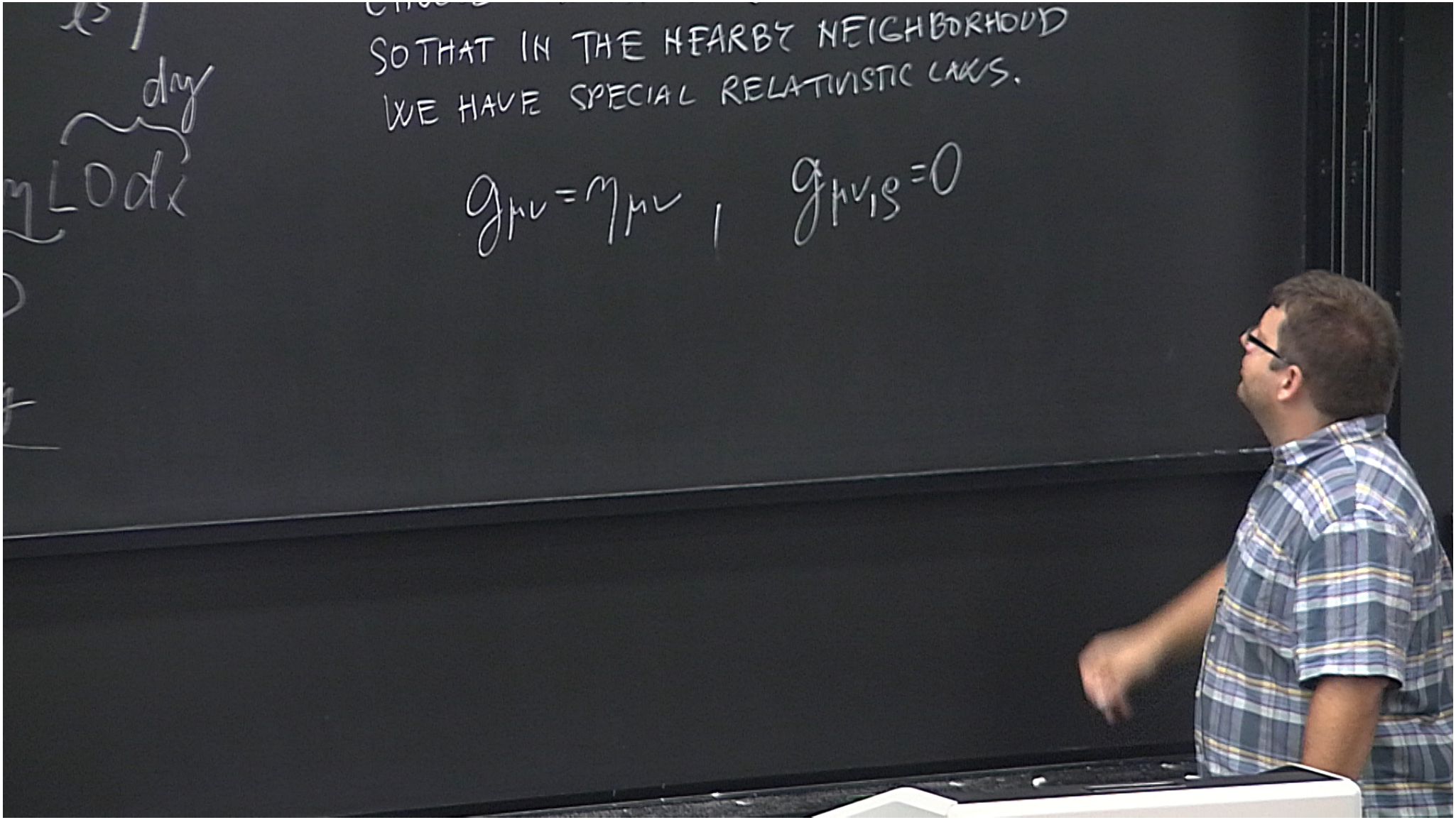


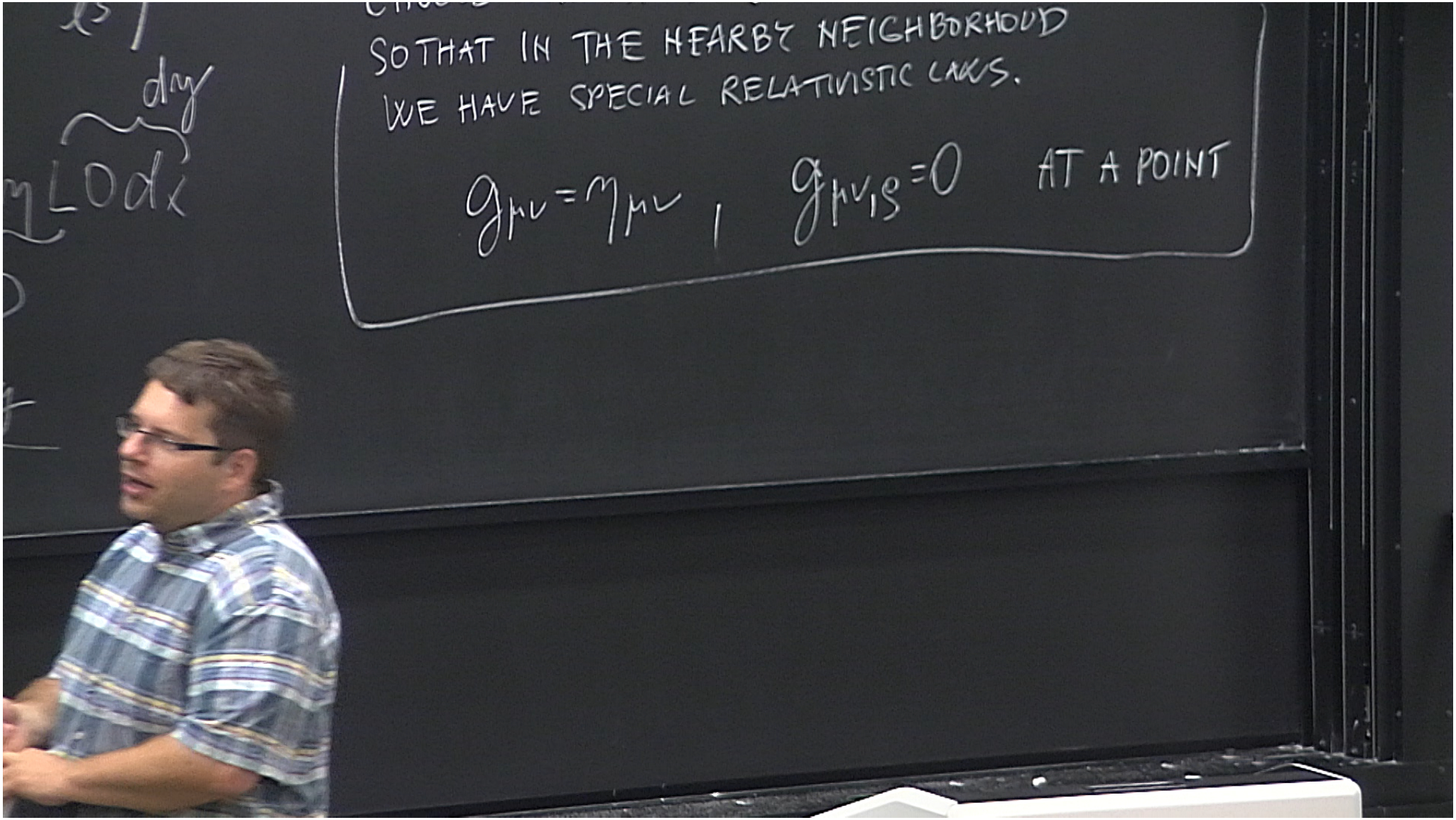

PRINCIPLE.
$g_{\mu\nu} = \eta_{\mu\nu}$
SMIDT DIAGONALIZATION
$L = \begin{pmatrix} \ell^0 & & \\ & \ell^1 & \\ 0 & & \ell^2 \\ & & & \ell^3 \end{pmatrix}$
$= dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T \underbrace{L^T \eta L}_{D} O\, dx$
$\begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ 0 & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -\ell_0^2 & & & \\ & \ell_1^2 & & \\ & & \ell_2^2 & \\ 0 & & & \ell_3^2 \end{pmatrix}$

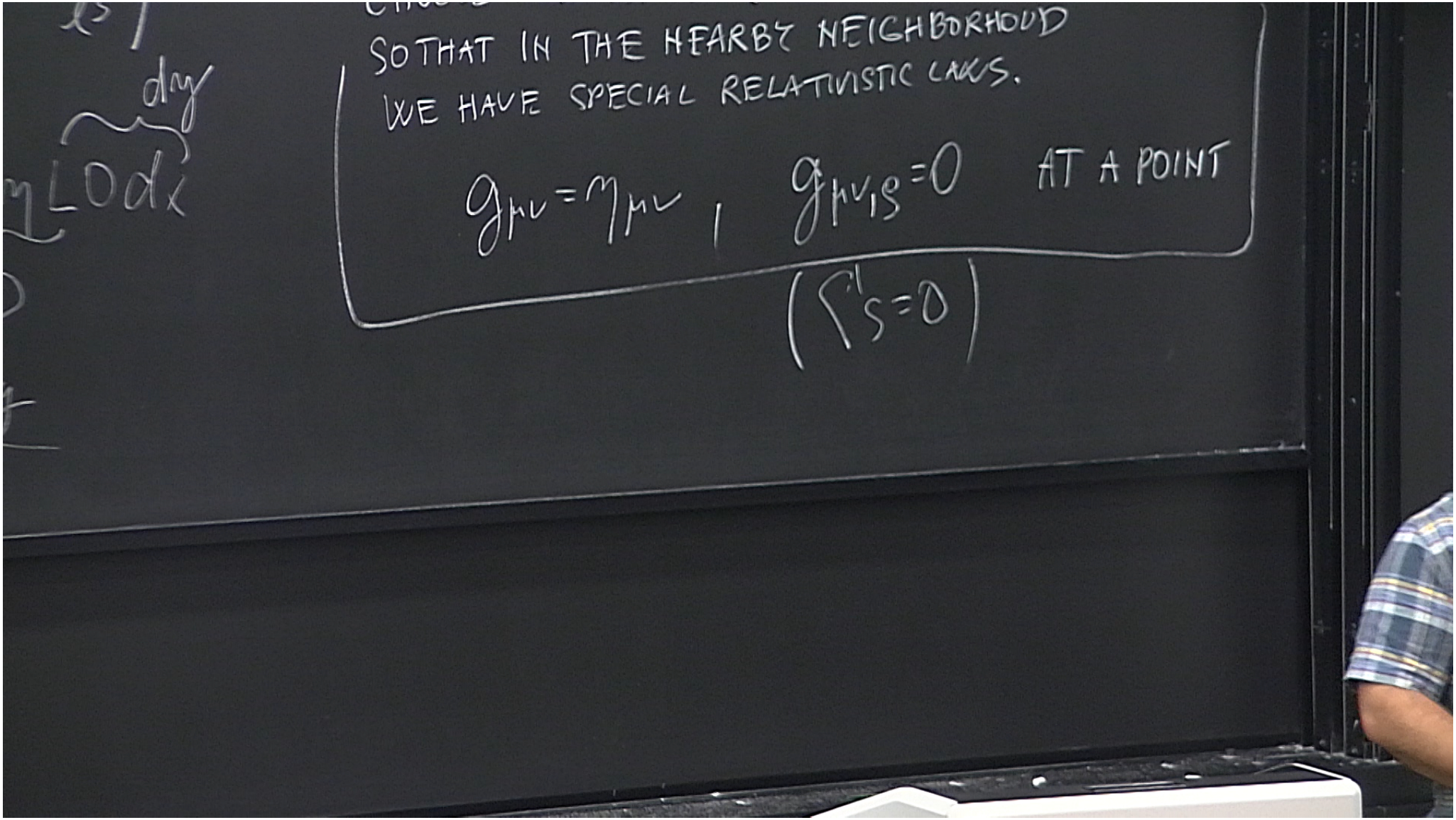

PRINCIPLE.
$g_{\mu\nu} = \eta_{\mu\nu}$
- DIAGONALIZATION
$dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T L^T \eta L O\, dx$
dy^T
dy
D
$L = \begin{pmatrix} e^0 & & \\ & e^1 & \\ 0 & & e^2 \\ & & & e^3 \end{pmatrix}$
$\begin{pmatrix} \lambda_0 & & & \\ & \lambda_1 & & 0 \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -e_0^2 & & & \\ & e_1^2 & & 0 \\ & & e_2^2 & \\ & & & e_3^2 \end{pmatrix}$


PRINCIPLE.
$g_{\mu\nu} = \eta_{\mu\nu}$
SCHMIDT DIAGONALIZATION
$L = \begin{pmatrix} e^0 & & \\ & e^1 & \\ 0 & & e^2 \\ & & & e^3 \end{pmatrix}$
$= dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T L^T \eta L O\, dx$
dy^T
dy
D
$\begin{pmatrix} -\lambda_0 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -e_0^2 & & & \\ & e_1^2 & & \\ & & e_2^2 & \\ & & & e_3^2 \end{pmatrix}$
$= -dy^T \eta\, dy$


• AT EVERY SPACETIME POINT IT IS POSSIBLE TO CHOOSE COORDINATES (FREELY FALLING- INERTIAL) SOTHAT IN THE NEARBY NEIGHBORHOOD WE HAVE SPECIAL RELATIVISTIC LAWS.
$g_{\mu\nu} = \eta_{\mu\nu}$

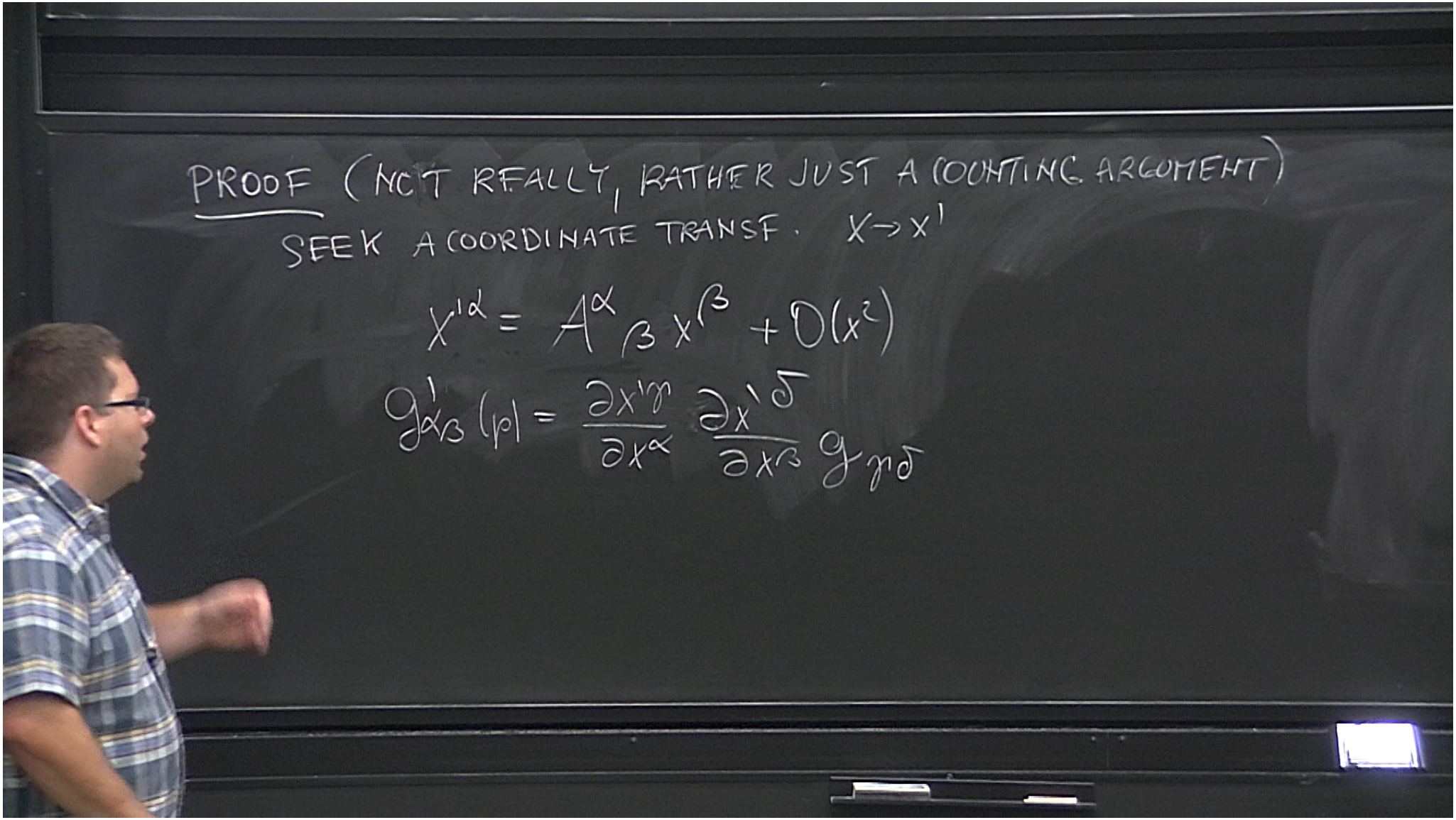

SO THAT IN THE HEARBY NEIGHBORHOUD
WE HAVE SPECIAL RELATIVISTIC LAWS.
$g_{\mu\nu} = \eta_{\mu\nu}$, $g_{\mu\nu,s} = 0$

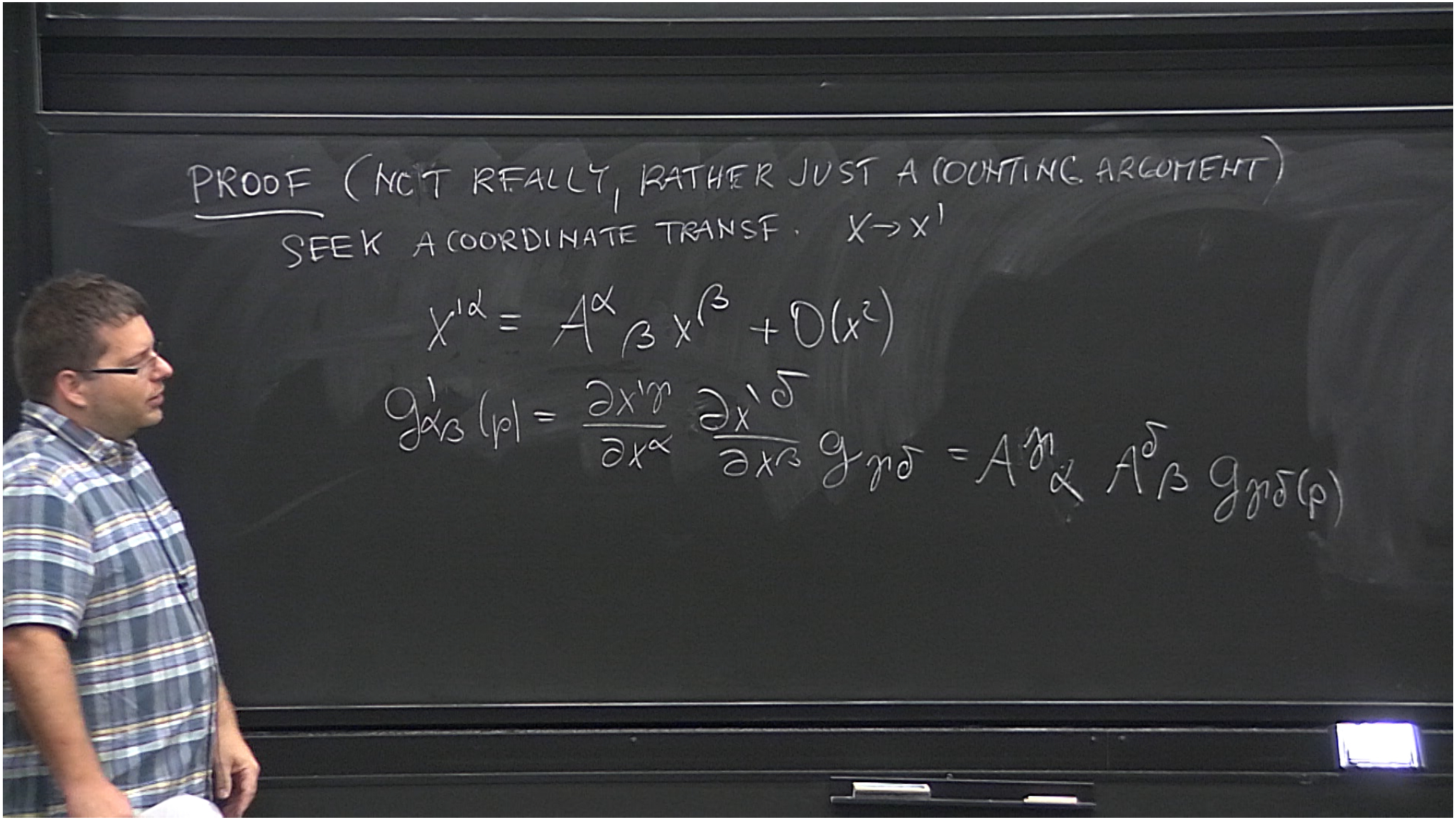

SO THAT IN THE HEARBY NEIGHBORHOOD
WE HAVE SPECIAL RELATIVISTIC LAWS.
$g_{\mu\nu} = \eta_{\mu\nu}$, $g_{\mu\nu,\rho} = 0$ AT A POINT

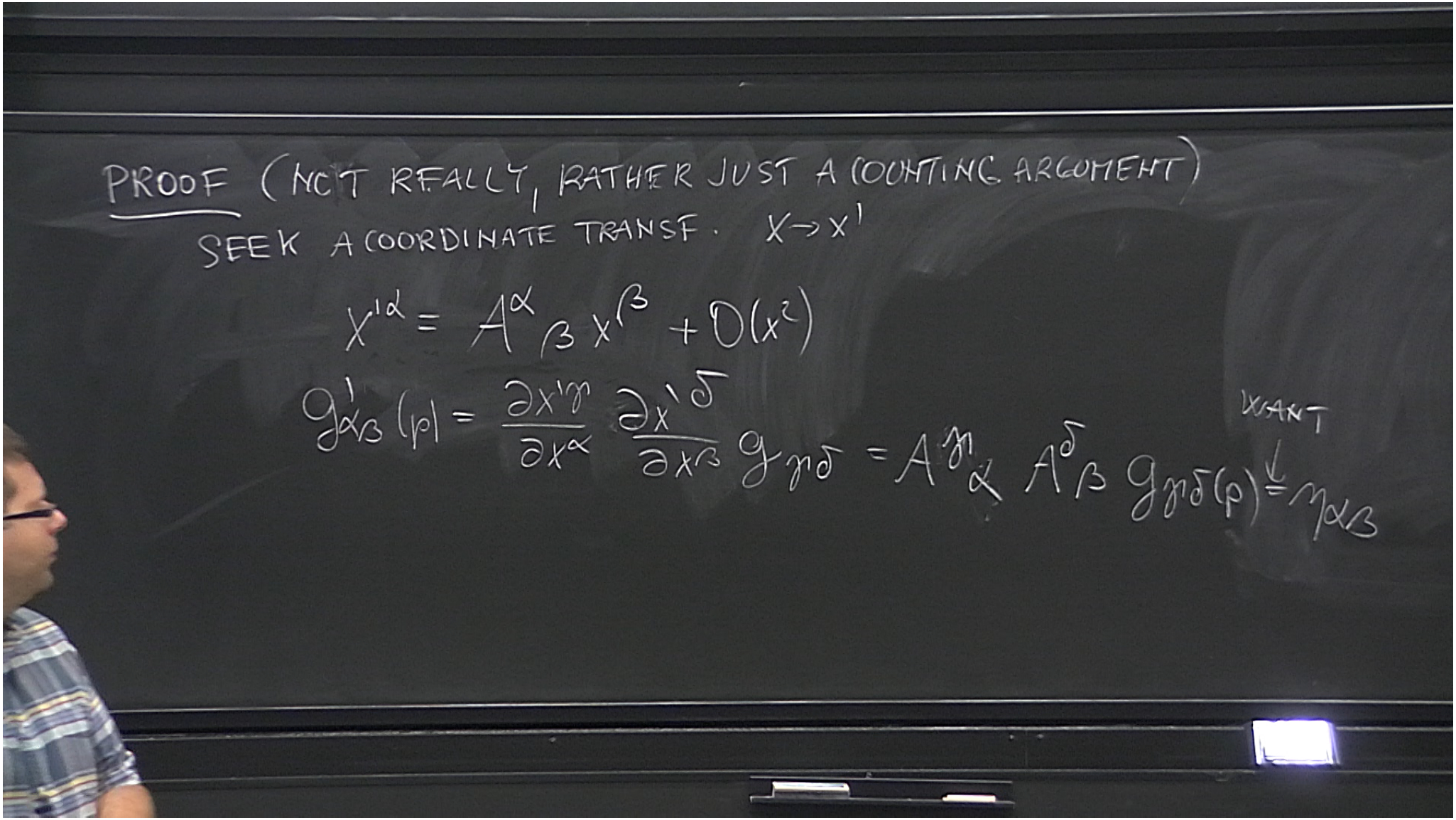

SO THAT IN THE HEARBY NEIGHBORHOOD
WE HAVE SPECIAL RELATIVISTIC LAWS.
$g_{\mu\nu} = \eta_{\mu\nu}$, $g_{\mu\nu,s} = 0$ AT A POINT
$(\Gamma^{l}_{s} = 0)$

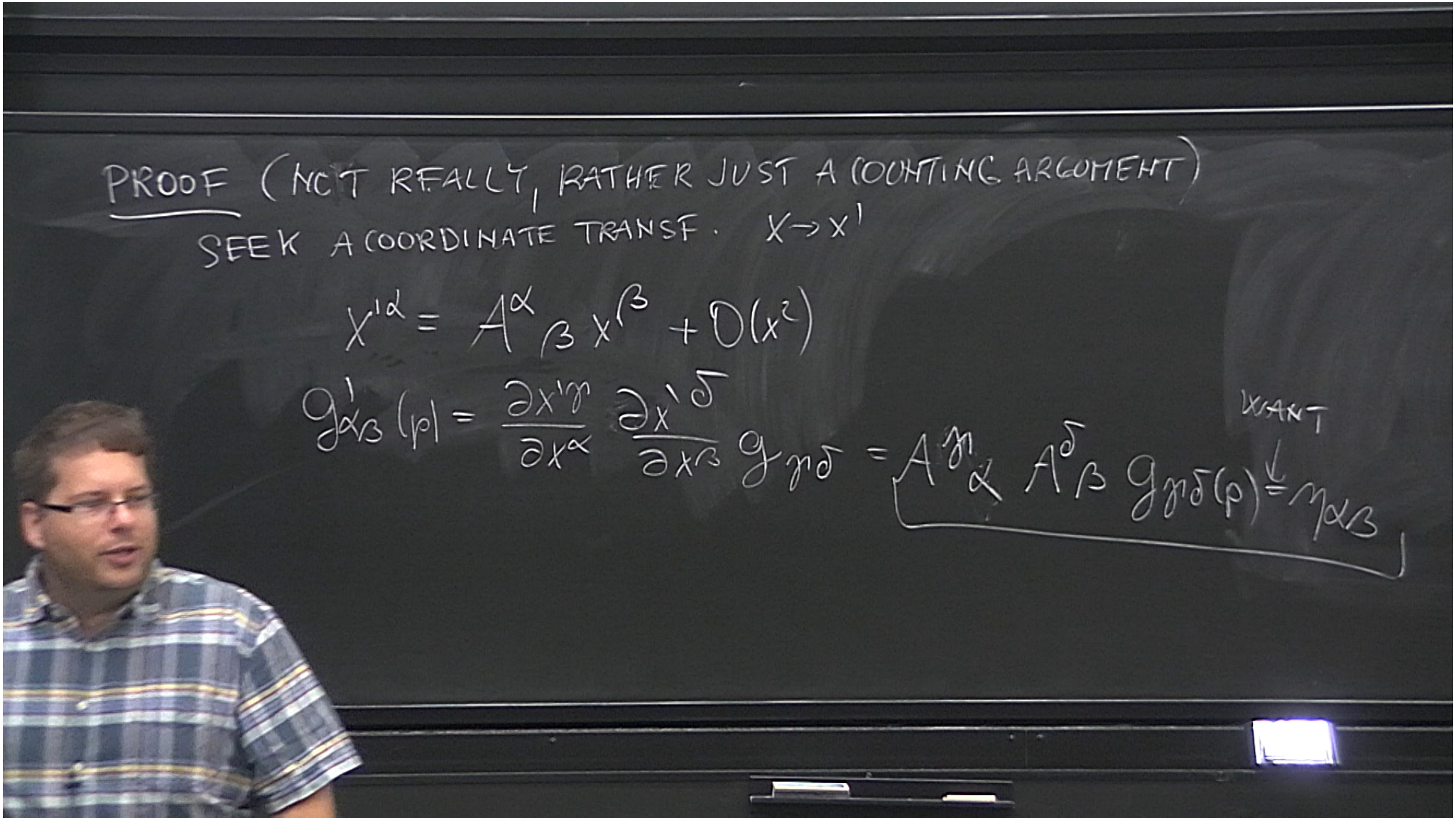
PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)

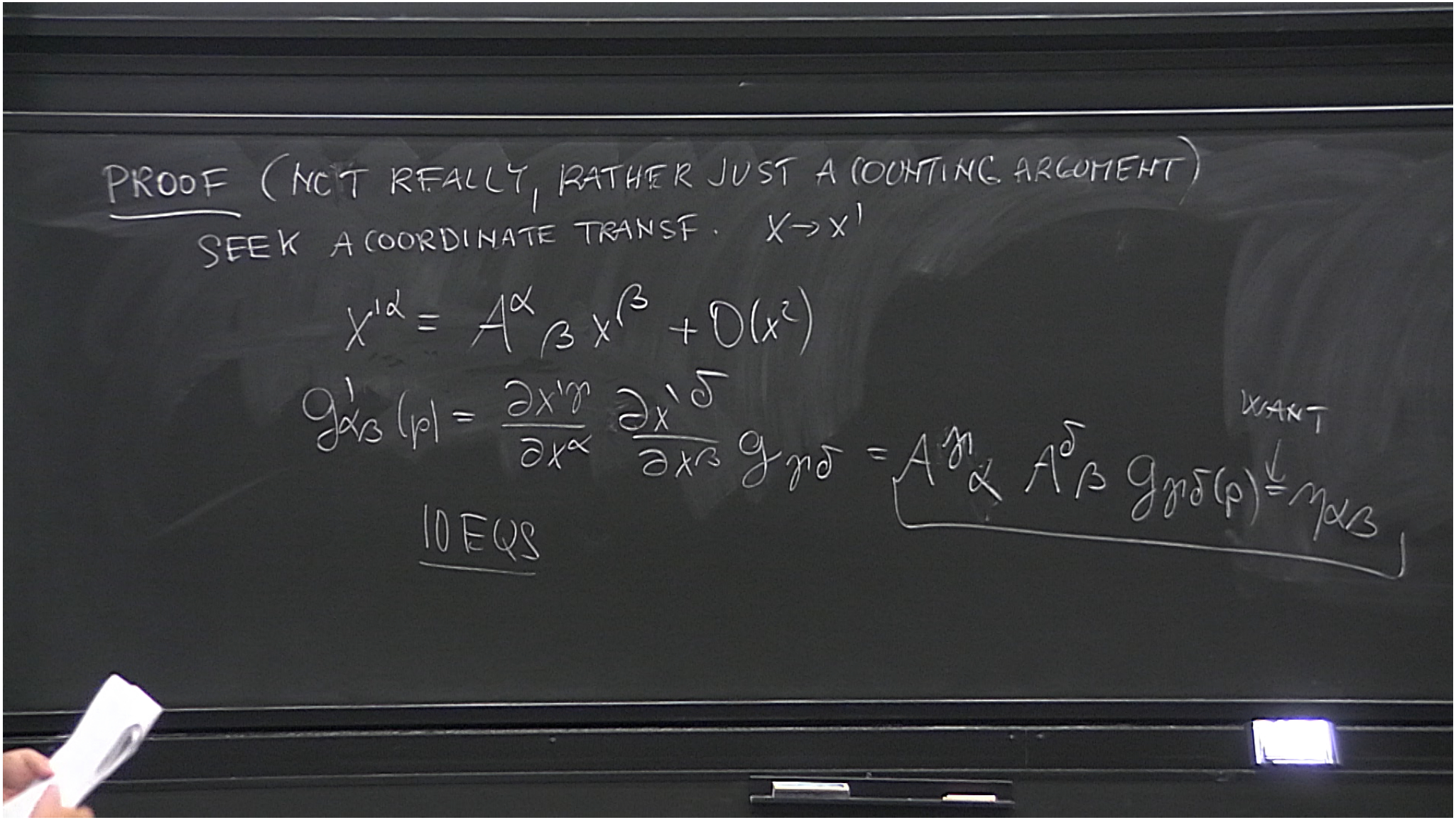
SEEK A COORDINATE TRANSF. $X \to X'$

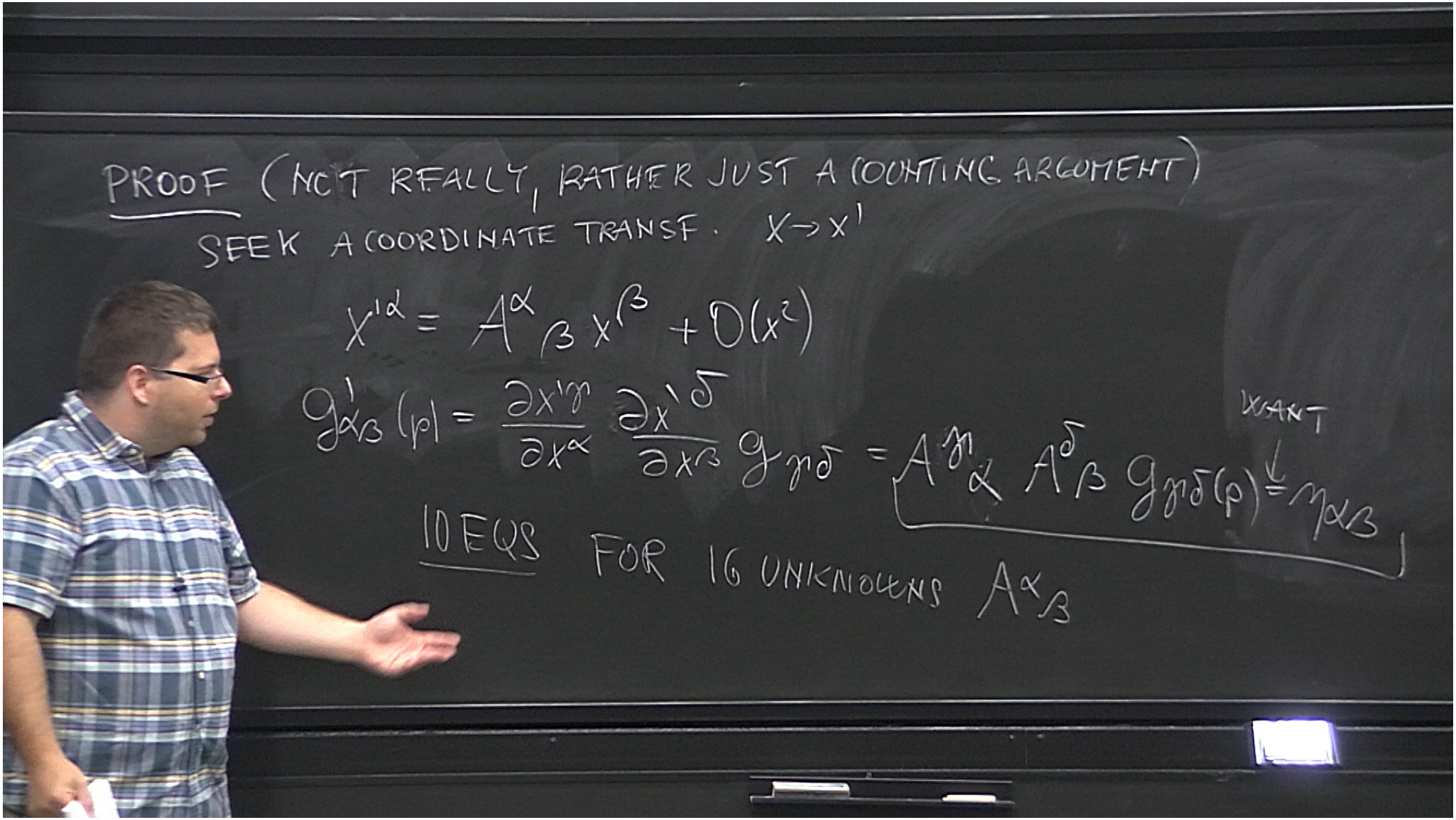

PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta}$


PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p)$


PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{!}{=} \eta_{\alpha\beta}$
WANT


PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{!}{=} \eta_{\alpha\beta}$
WANT


PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{!}{=} \eta_{\alpha\beta}$
WANT
10 EQS


PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)
SEEK A COORDINATE TRANSF. $X \to X'$
$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$
$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{!}{=} \eta_{\alpha\beta}$
WANT
10 EQS FOR 16 UNKNOWNS $A^{\alpha}{}_{\beta}$

PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)

SEEK A COORDINATE TRANSF. $X \to X'$

$$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$$

$$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{\text{WANT}}{=} \eta_{\alpha\beta}$$

10 EQS FOR 16 UNKNOWNS $A^{\alpha}{}_{\beta}$ ✓

PROOF (NOT REALLY, RATHER JUST A COUNTING ARGUMENT)

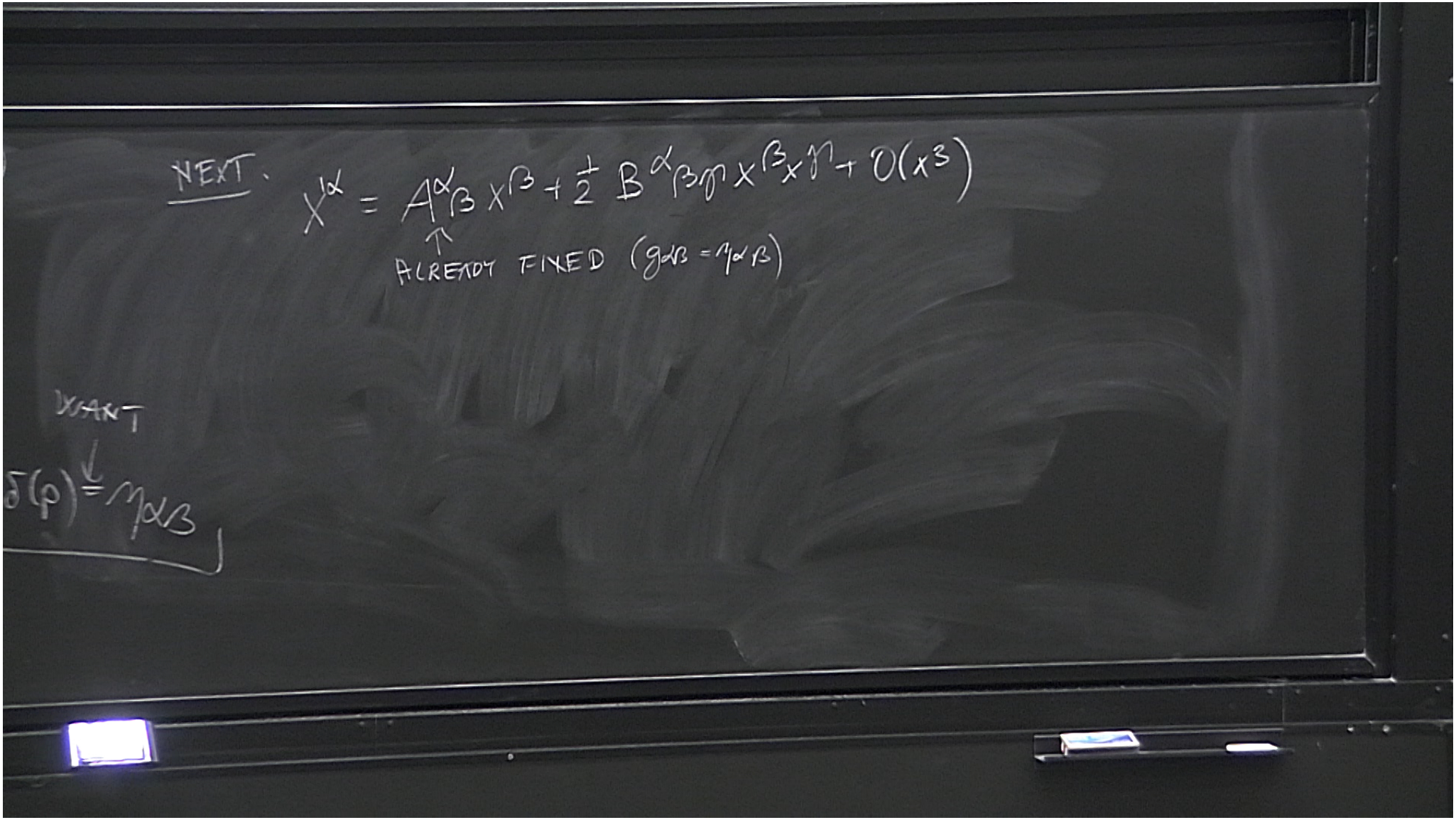
SEEK A COORDINATE TRANSF. $X \to X'$

$$X'^{\alpha} = A^{\alpha}{}_{\beta} X^{\beta} + O(x^2)$$

$$g'_{\alpha\beta}(p) = \frac{\partial x'^{\gamma}}{\partial x^{\alpha}} \frac{\partial x'^{\delta}}{\partial x^{\beta}} g_{\gamma\delta} = A^{\gamma}{}_{\alpha} A^{\delta}{}_{\beta} g_{\gamma\delta}(p) \overset{\text{WANT}}{=} \eta_{\alpha\beta}$$

10 EQS FOR 16 UNKNOWNS $A^{\alpha}{}_{\beta}$ ✓

REMAINING 6 ~ LORENTZ TRANSF.


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{\beta\gamma} x^{\beta} x^{\gamma} + O(x^3)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
WANT
$\delta(p) \stackrel{!}{=} \eta_{\alpha\beta}$

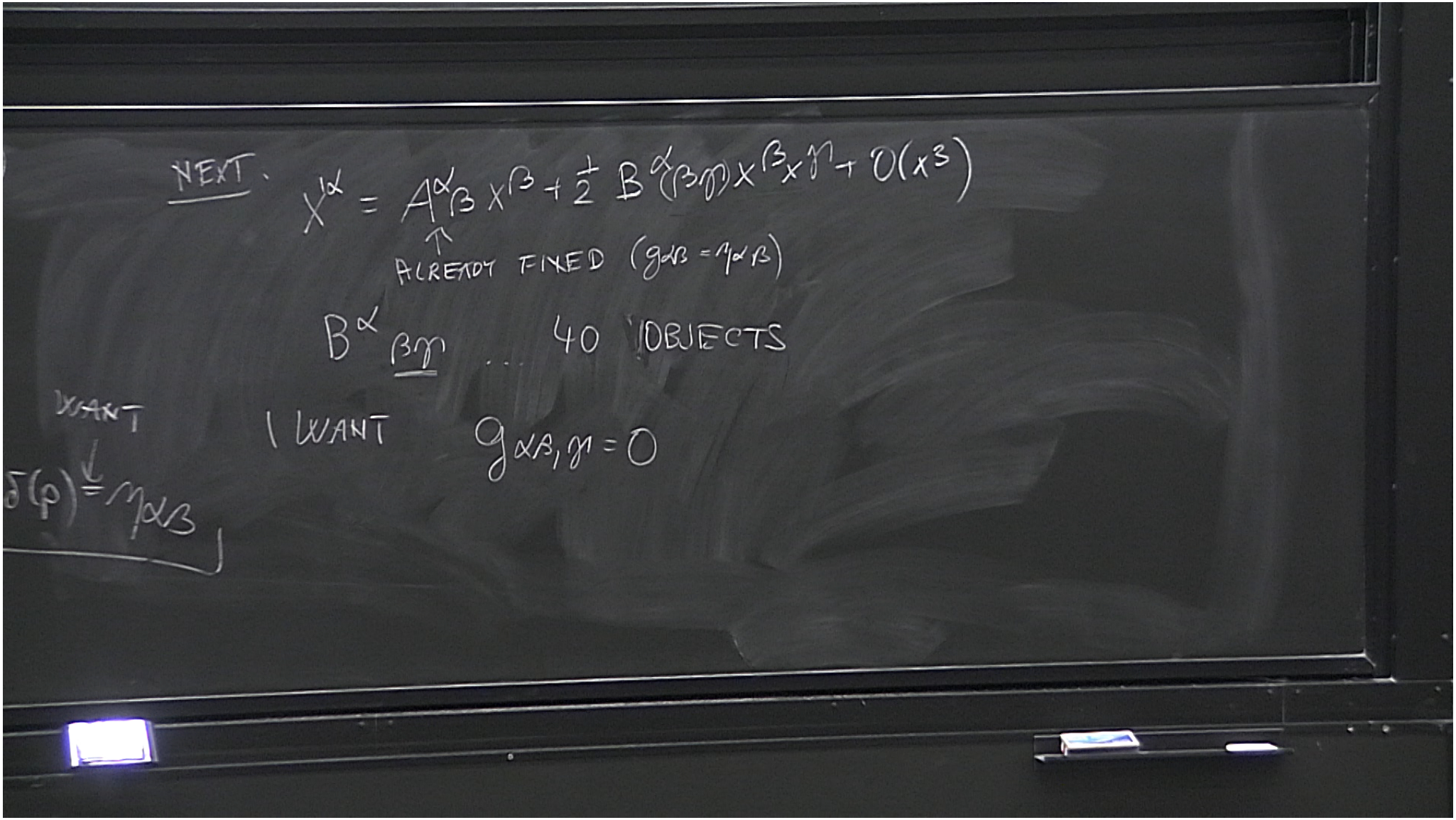
NEXT.

$$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + \mathcal{O}(x^3)$$

ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)

WANT

$\delta(p) \stackrel{!}{=} \eta_{\alpha\beta}$


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + O(x^3)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
I WANT
I WANT $g_{\alpha\beta,\gamma} = 0$
$\delta(p) \stackrel{\downarrow}{=} \eta_{\alpha\beta}$

NEXT.

$$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + O(x^3)$$

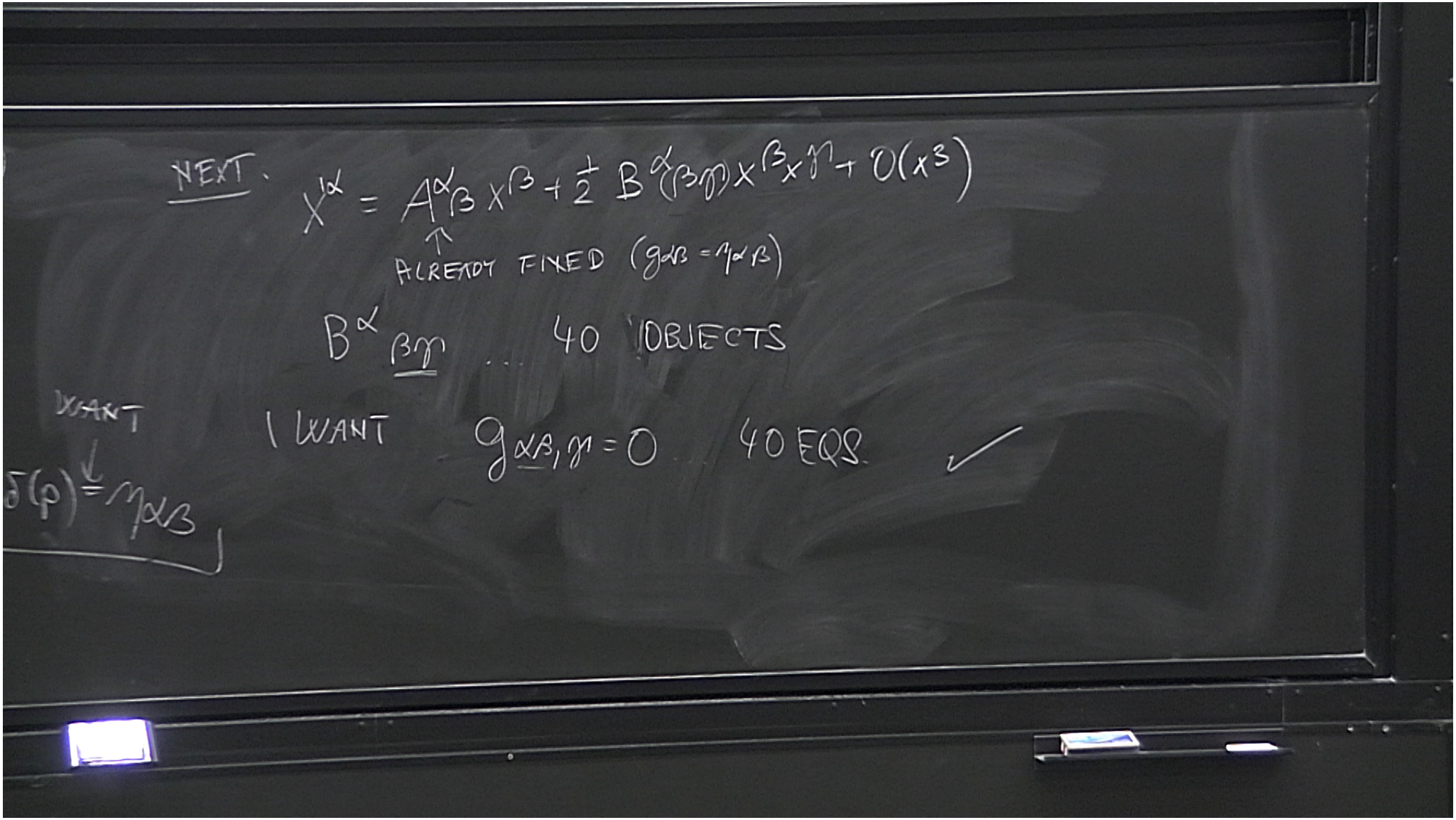
↑ ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)

$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS

I WANT $g_{\alpha\beta,\gamma} = 0$ ... 40 EQS.

WANT

$\delta(\rho) \stackrel{!}{=} \eta_{\alpha\beta}$


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + O(x^3)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
I WANT
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
$g(p) \stackrel{!}{=} \eta_{\alpha\beta}$

NEXT.

$$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + O(x^3)$$

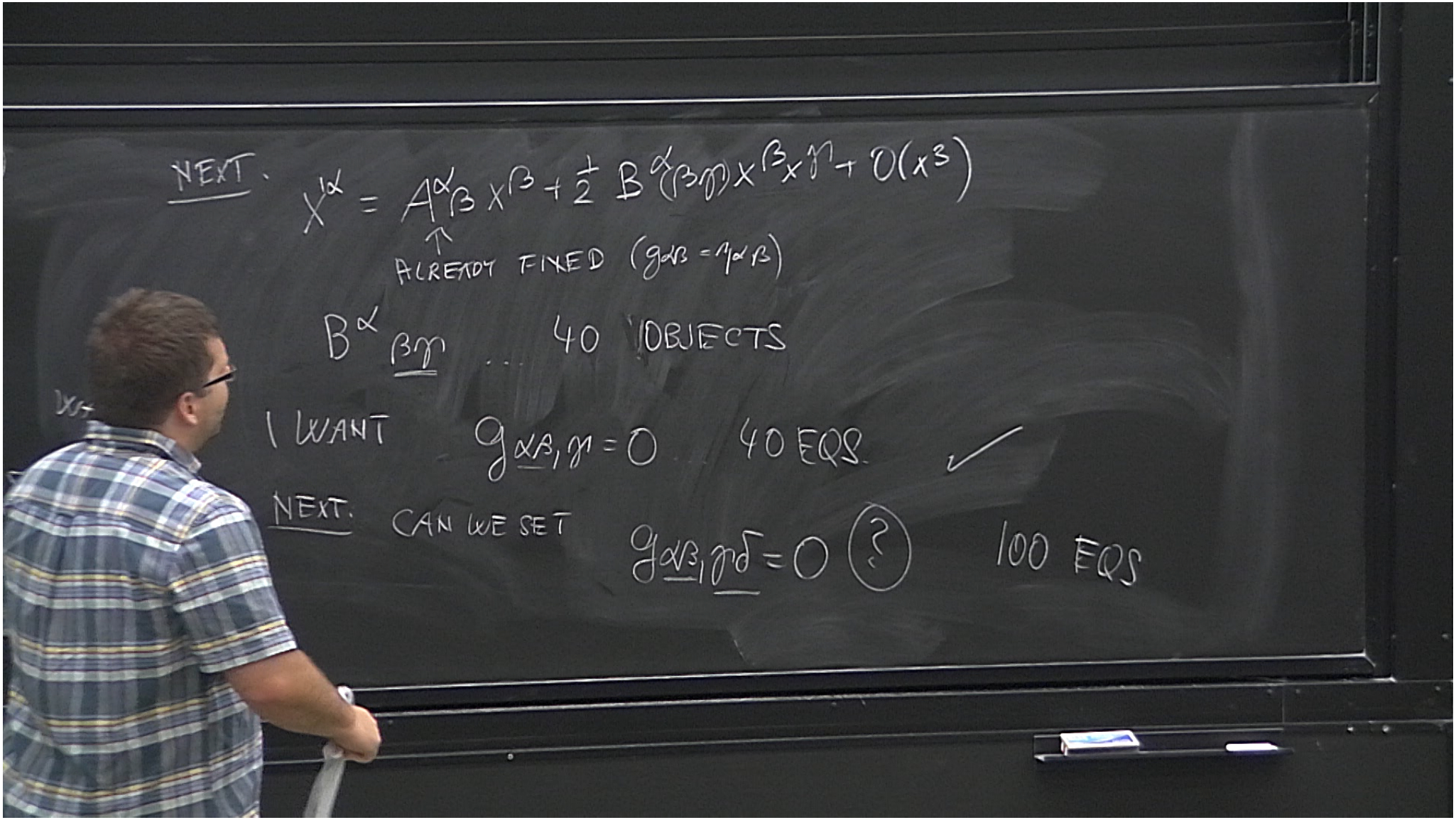
$A^{\alpha}{}_{\beta}$ ↑ ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)

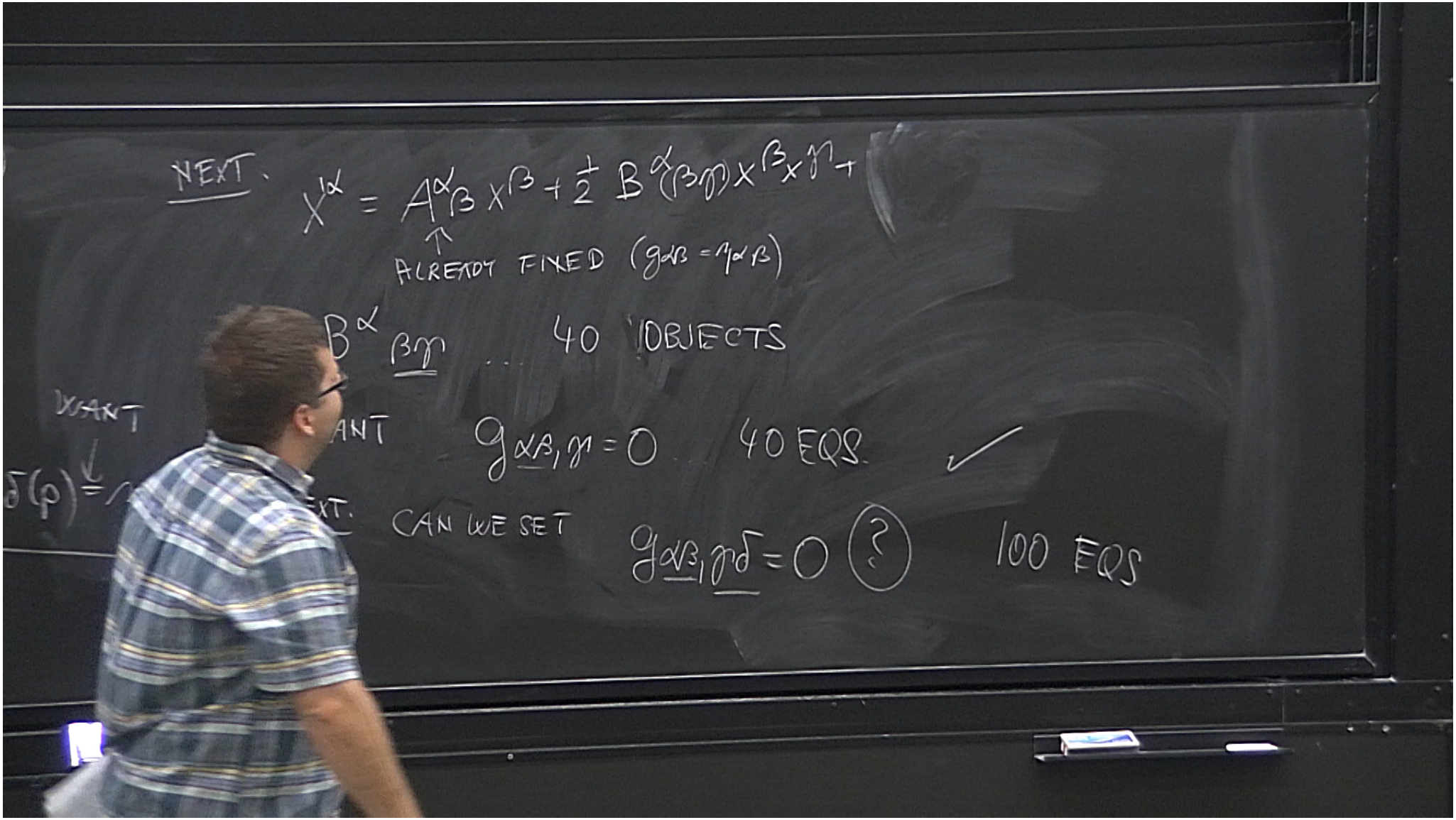
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS

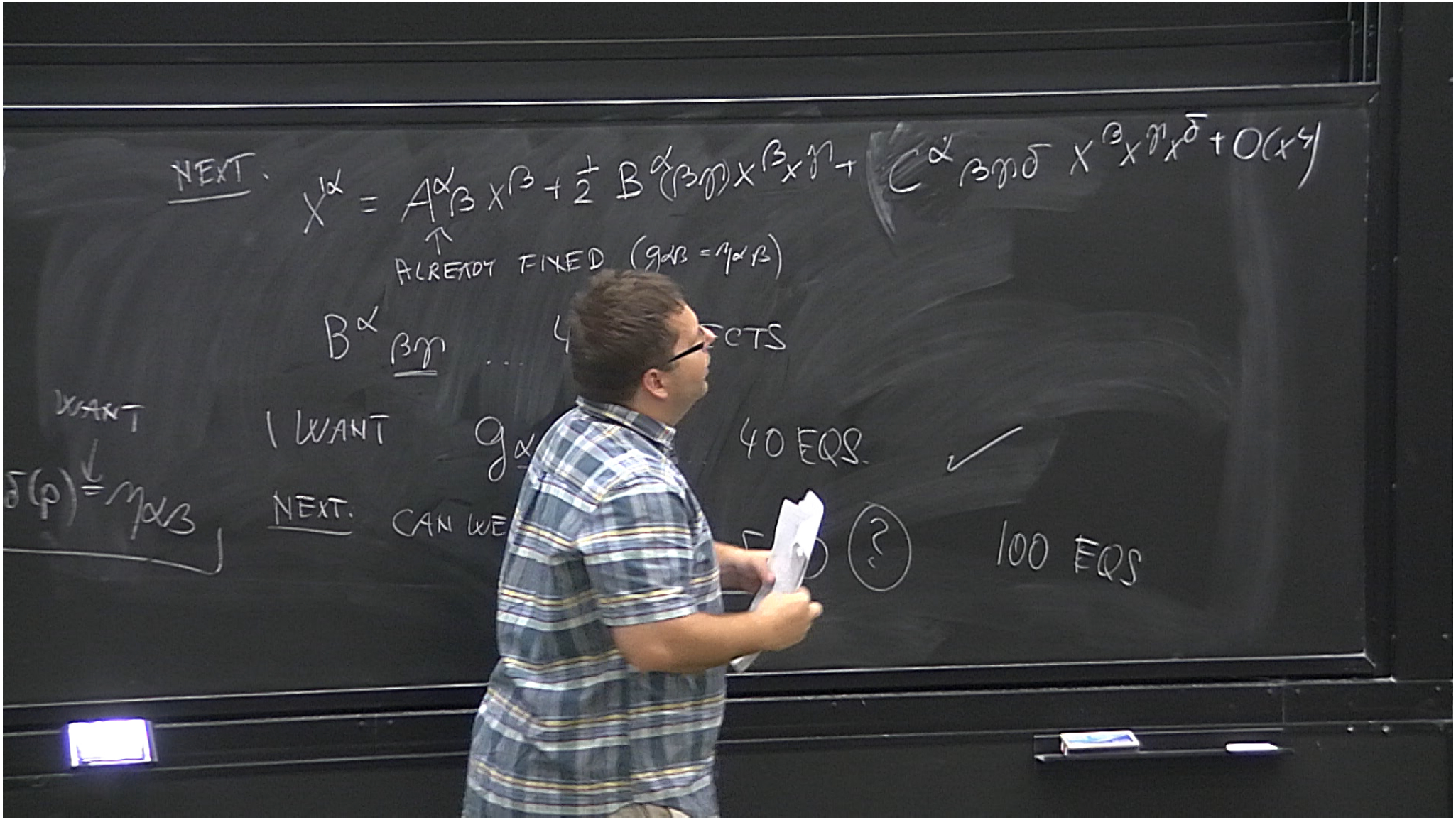
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS. ✓

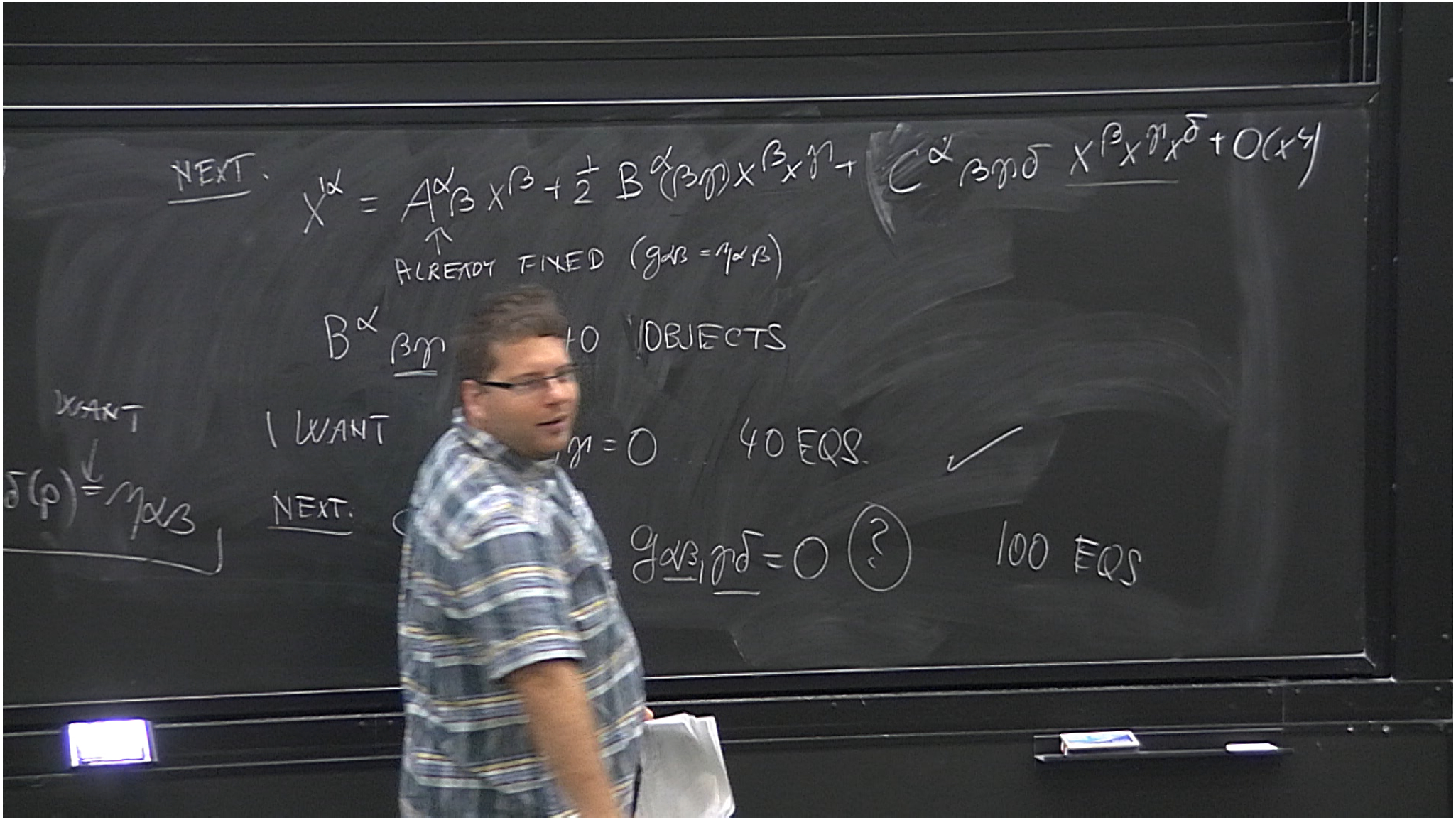
I WANT ↓ $\delta(p) \stackrel{!}{=} \eta_{\alpha\beta}$

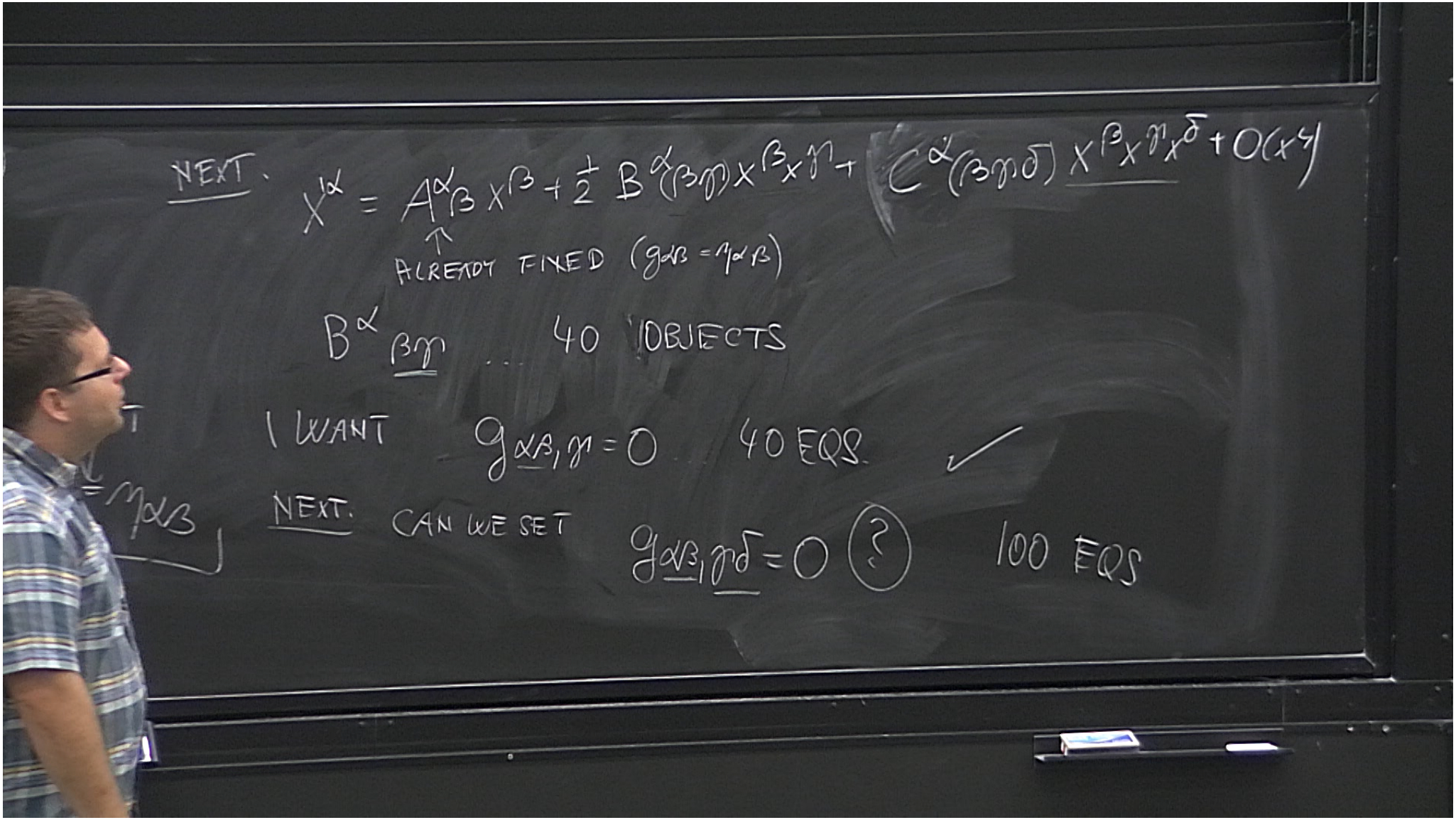
NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$ (?)


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + O(x^3)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} +$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
WANT
$g_{\alpha\beta,\gamma} = 0$ 40 EQS.
CAN WE SET
$g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{\beta\gamma\delta} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 4 ECTS
I WANT
I WANT g_{α}
40 EQS.
$\delta(\rho) \stackrel{!}{=} \eta_{\alpha\beta}$
NEXT. CAN WE
100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{\beta\gamma\delta} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ 40 OBJECTS
I WANT
$\partial(\rho) \stackrel{!}{=} \eta_{\alpha\beta}$
I WANT
$\partial_\gamma = 0$ 40 EQS.
NEXT.
$g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{(\beta\gamma\delta)} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$? 100 EQS

NEXT.

$$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{(\beta\gamma\delta)} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$$

$A^{\alpha}{}_{\beta}$ ↑ ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)

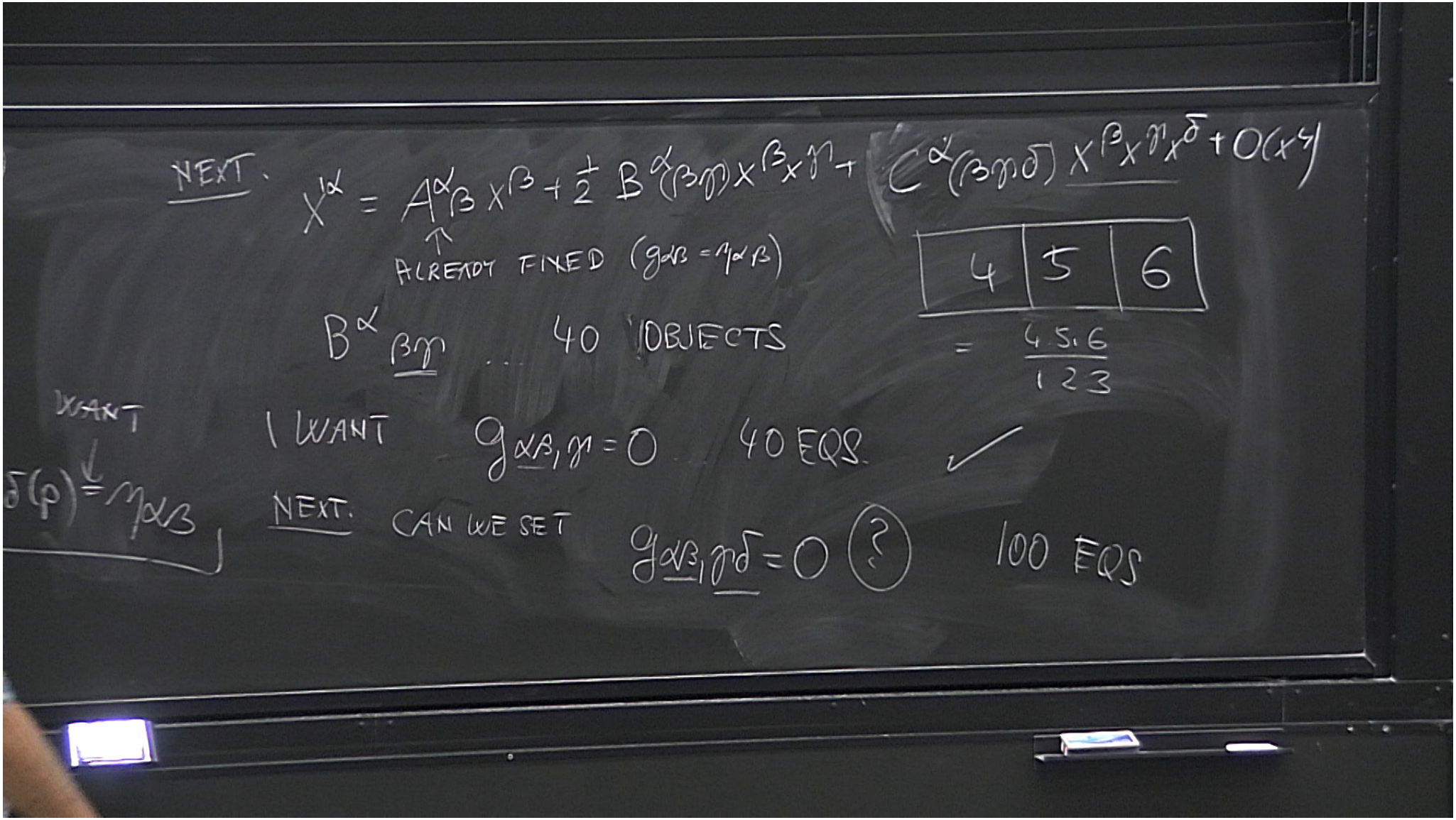
4	5	6

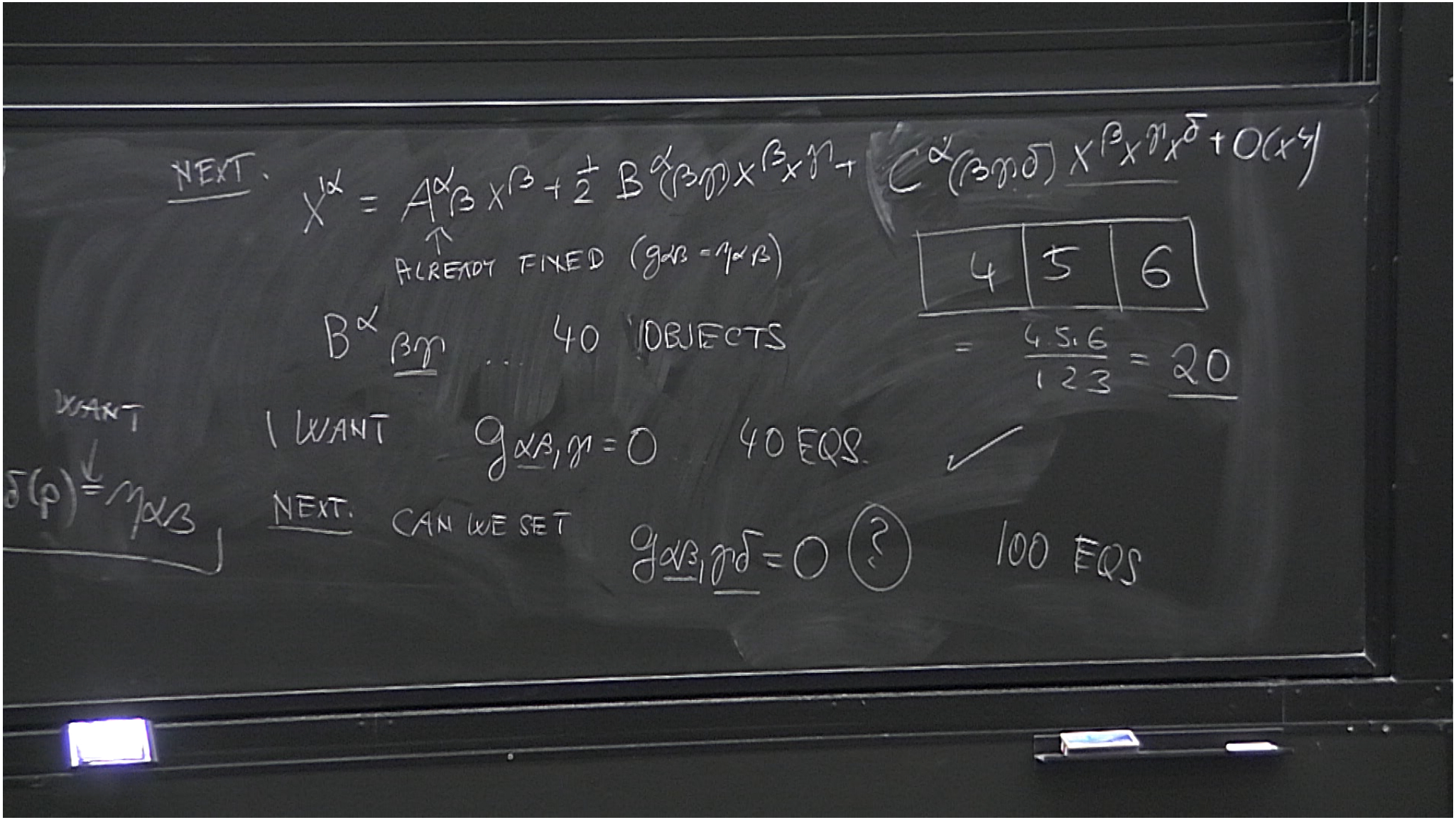
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS

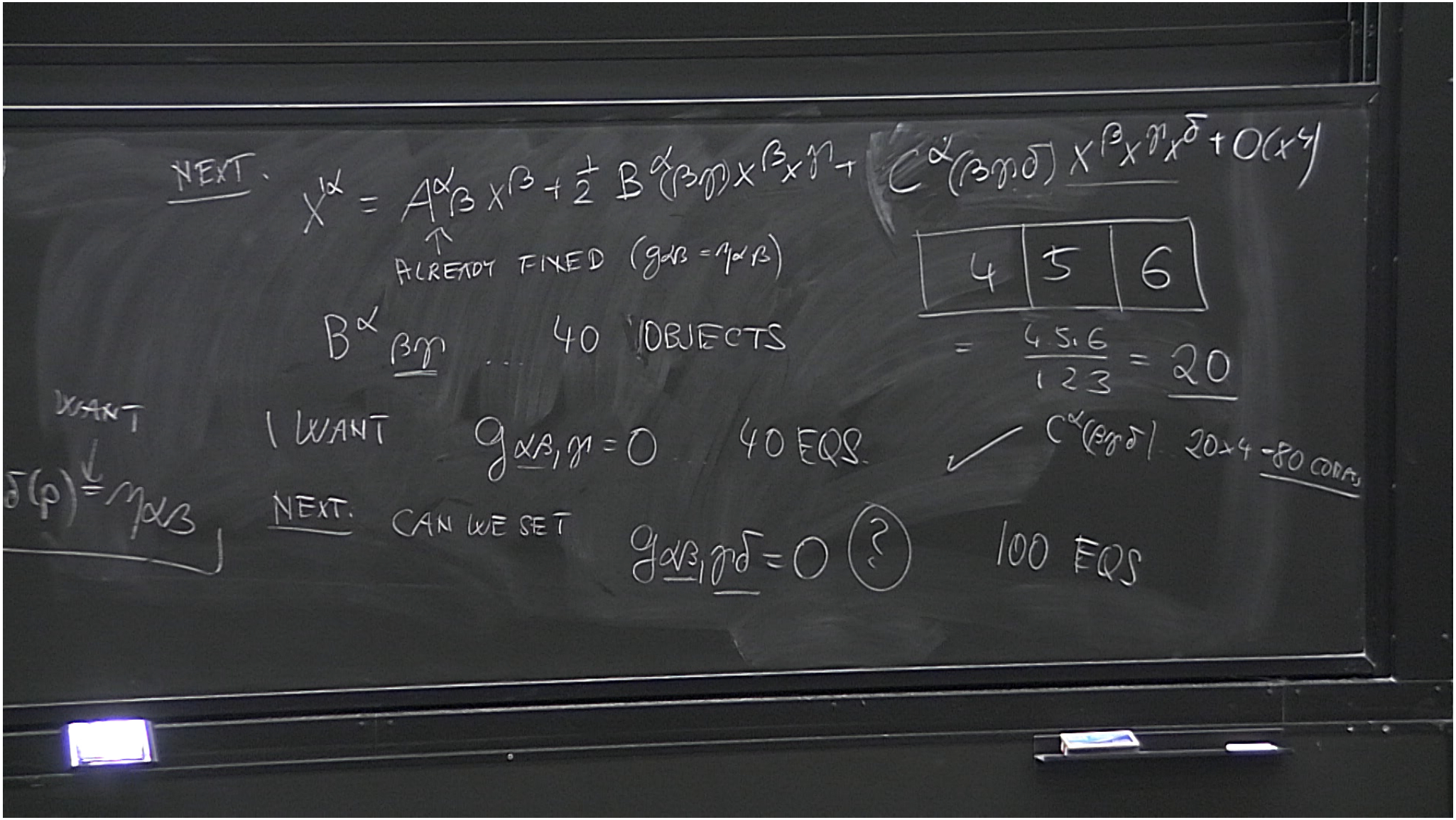
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS. ✓

$\delta(p) \stackrel{\downarrow}{=} \eta_{\alpha\beta}$ I WANT

NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{(\beta\gamma\delta)} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
4 5 6
$= \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3}$
WANT
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
$\delta(\rho) \stackrel{\downarrow}{=} \eta_{\alpha\beta}$
NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$? 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{(\beta\gamma\delta)} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
4 5 6
$= \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$
I WANT
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
$\delta(p) \stackrel{!}{=} \eta_{\alpha\beta}$
NEXT. CAN WE SET
$g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS


NEXT.
$X'^{\alpha} = A^{\alpha}{}_{\beta} x^{\beta} + \frac{1}{2} B^{\alpha}{}_{(\beta\gamma)} x^{\beta} x^{\gamma} + C^{\alpha}{}_{(\beta\gamma\delta)} x^{\beta} x^{\gamma} x^{\delta} + O(x^4)$
ALREADY FIXED ($g_{\alpha\beta} = \eta_{\alpha\beta}$)
$B^{\alpha}{}_{\beta\gamma}$ ... 40 OBJECTS
4 5 6
$= \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20$
$C^{\alpha}{}_{(\beta\gamma\delta)}$ $20 \times 4 = 80$ COEFFS
I WANT $g_{\alpha\beta,\gamma} = 0$ 40 EQS.
I WANT $\delta(p) \to \eta_{\alpha\beta}$
NEXT. CAN WE SET $g_{\alpha\beta,\gamma\delta} = 0$ (?) 100 EQS

GRAM-SCHMIDT DIAGONALIZATION

$$g_{\mu\nu} dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T \tilde{L}^T \eta \tilde{L} O\, dx$$

WE MISS 20 COMPTS... THERE WILL BE 20 2ND-ORDER DERIVATIVES OF THE METRIC LEFT.

GRAM-SCHMIDT DIAGONALIZATION

$$g_{\mu\nu} dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T L^T \eta L O\, dx$$

WE MISS 20 COMPTS ... THERE WILL BE 20 2ND-ORDER DERIVATIVES OF THE METRIC LEFT.

THESE CAN BE COMBINED TO

THE RIEMANN TENSOR

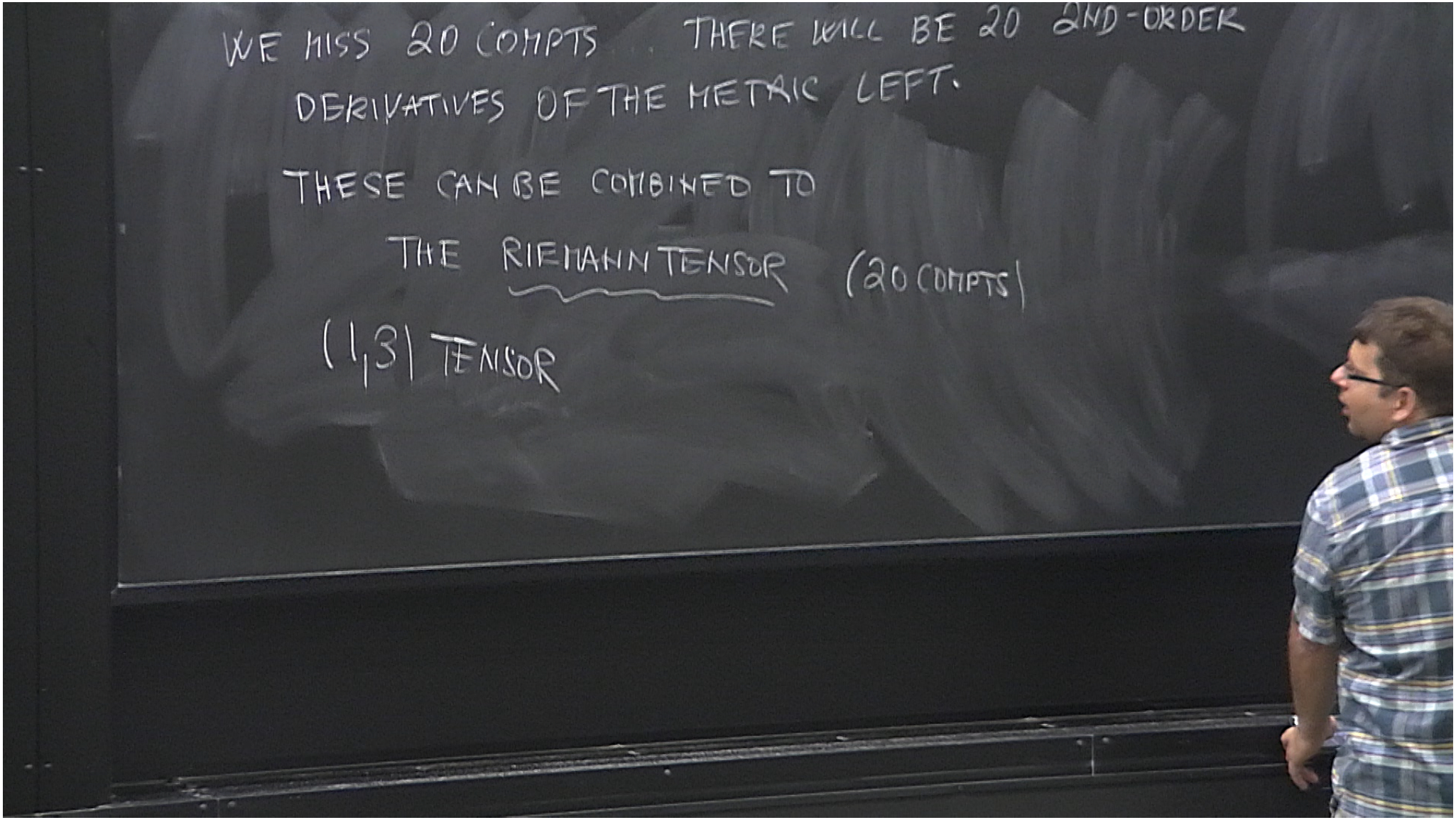
GRAM-SCHMIDT DIAGONALIZATION

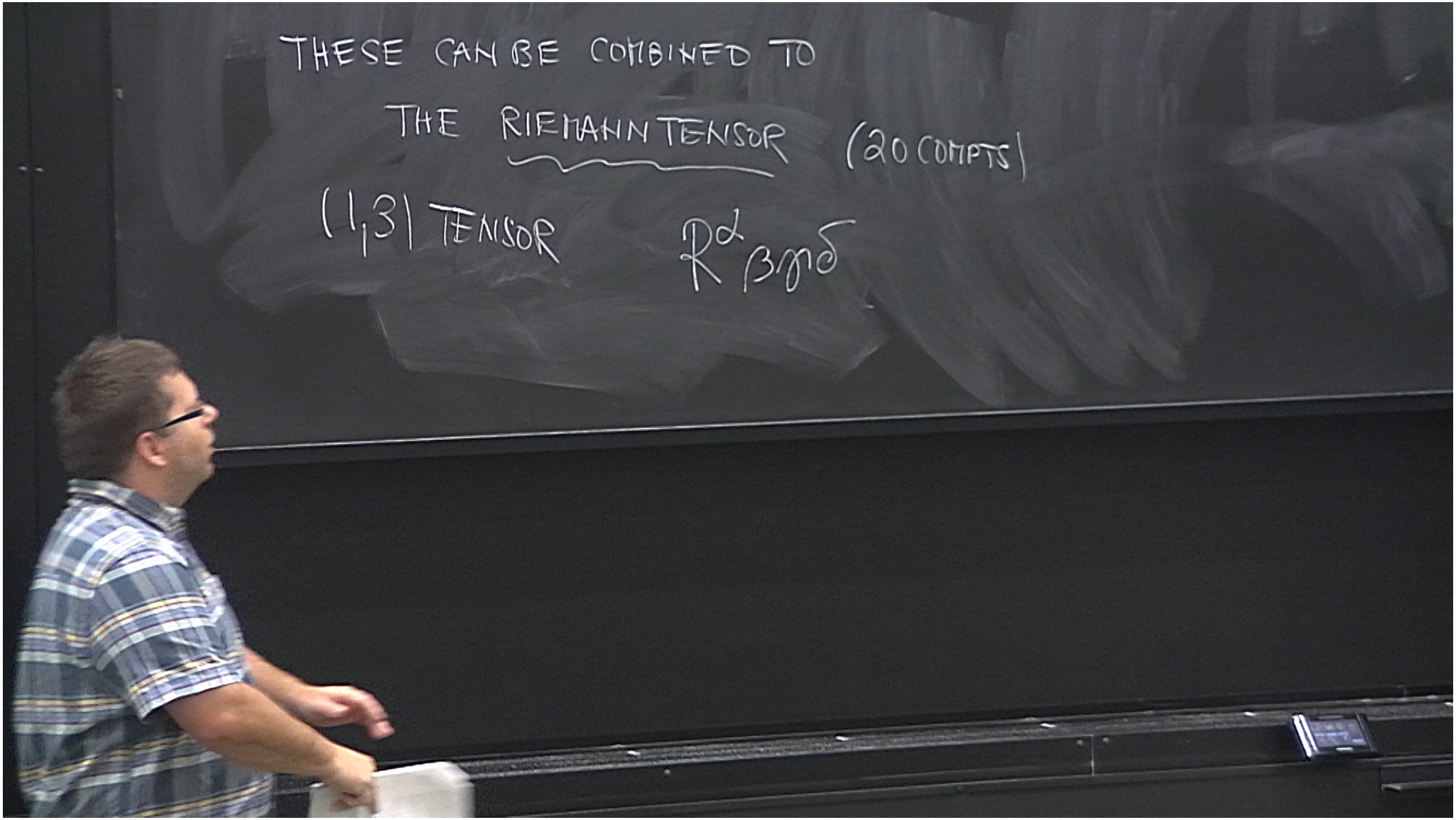
$$g_{\mu\nu} dx^\mu dx^\nu = dx^T G\, dx = dx^T O^T D O\, dx = dx^T O^T L^T \eta L O\, dx$$

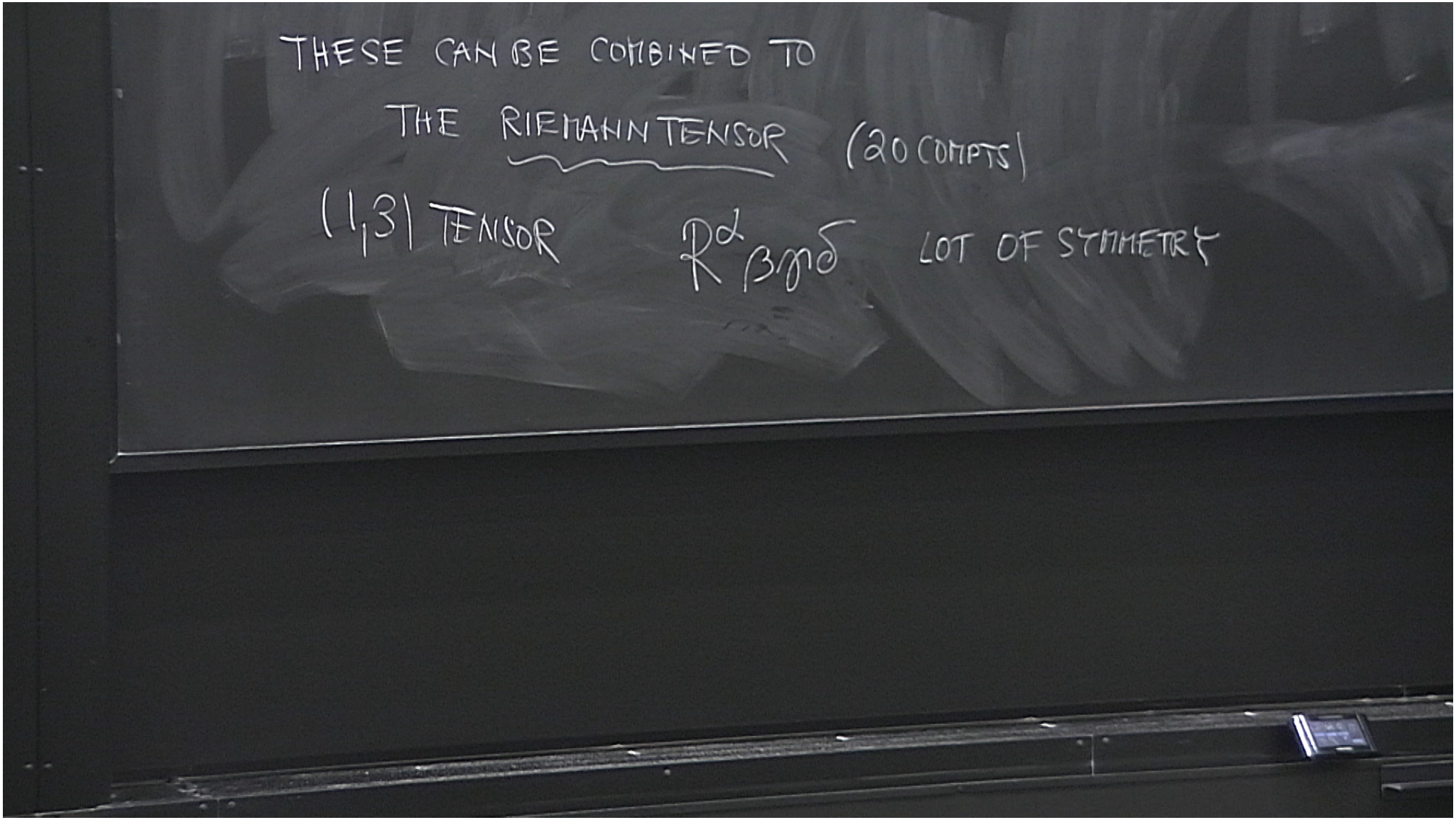
WE MISS 20 COMPTS ... THERE WILL BE 20 2ND-ORDER DERIVATIVES OF THE METRIC LEFT.

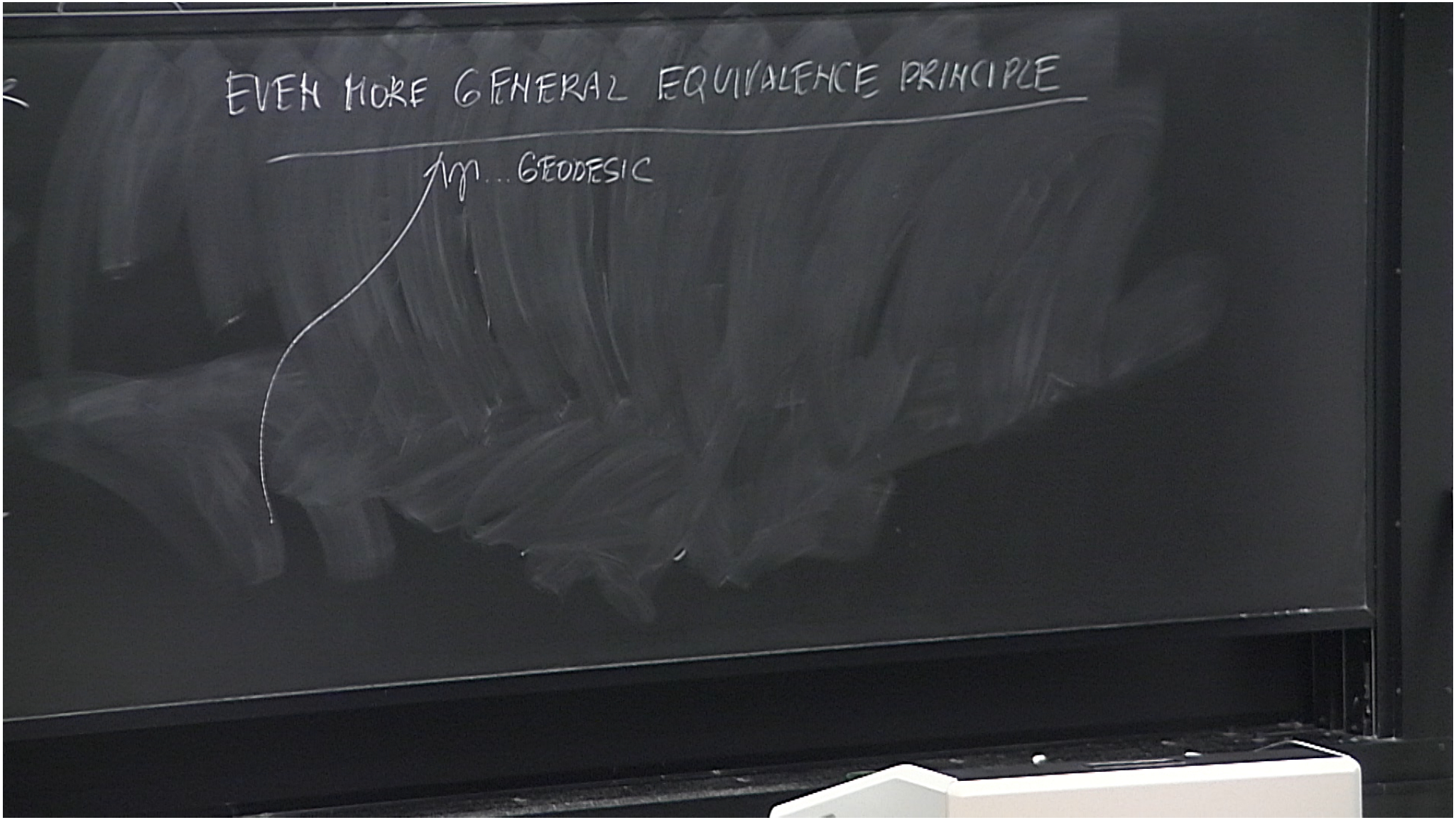
THESE CAN BE COMBINED TO

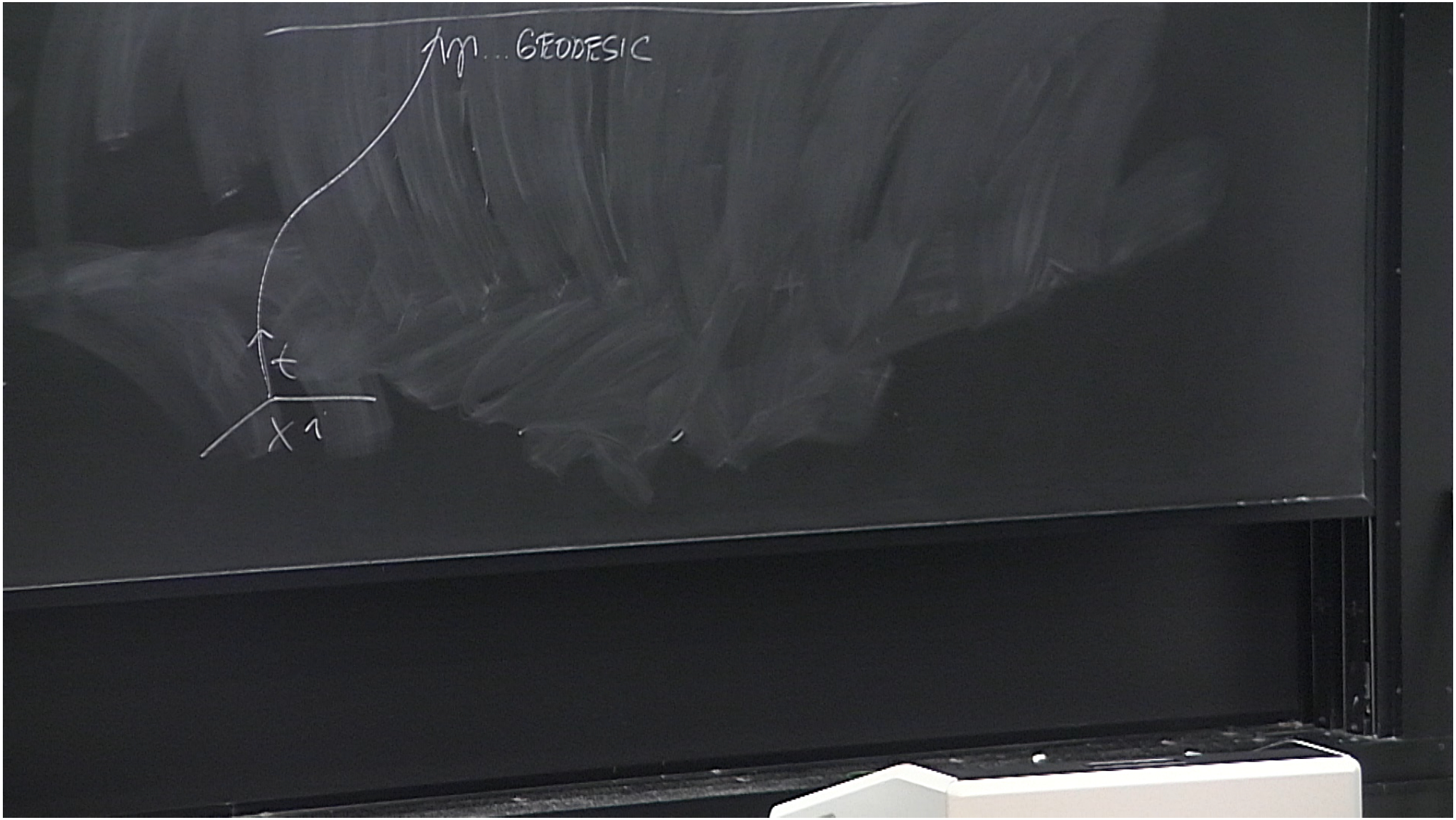
THE RIEMANN TENSOR (20 COMPTS)

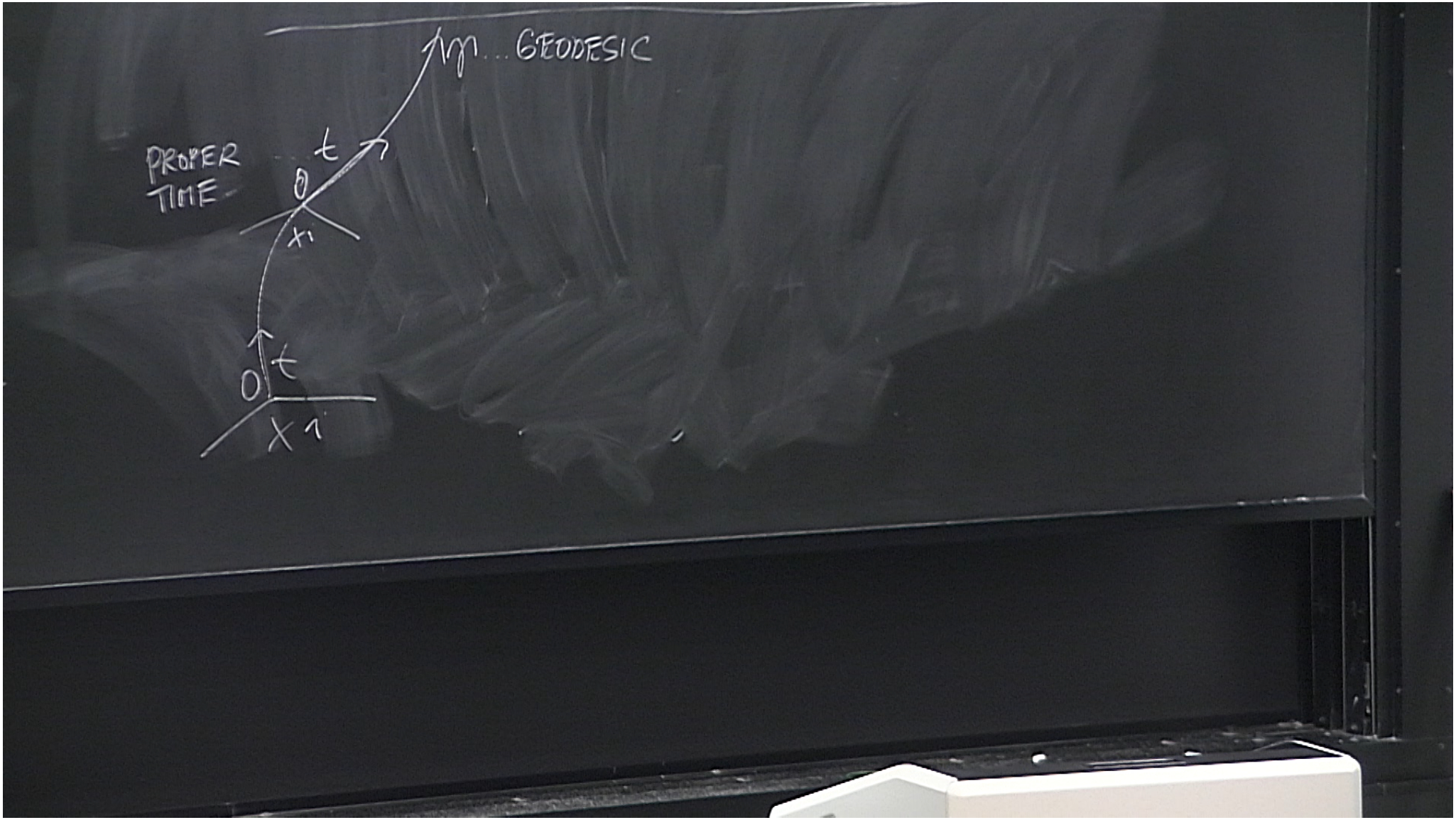

WE MISS 20 COMPTS. THERE WILL BE 20 2ND-ORDER
DERIVATIVES OF THE METRIC LEFT.
THESE CAN BE COMBINED TO
THE RIEMANN TENSOR (20 COMPTS)
(1,3) TENSOR

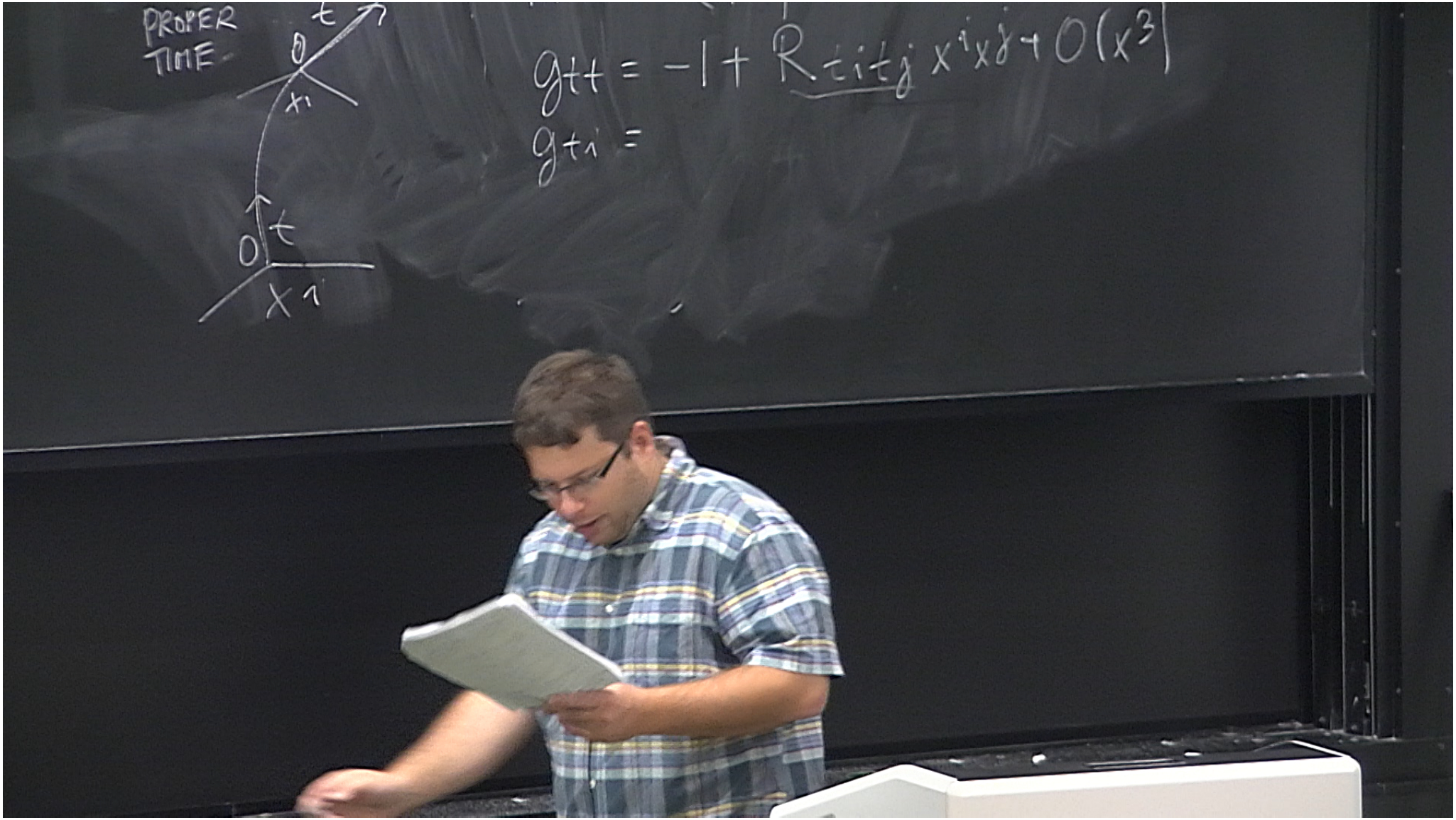

THESE CAN BE COMBINED TO
THE RIEMANN TENSOR (20 COMPTS)
(1,3) TENSOR
$R^{\alpha}{}_{\beta\gamma\delta}$


THESE CAN BE COMBINED TO
THE RIEMANN TENSOR (20 COMPTS)
(1,3) TENSOR
$R^{\alpha}{}_{\beta\gamma\delta}$
LOT OF SYMMETRY

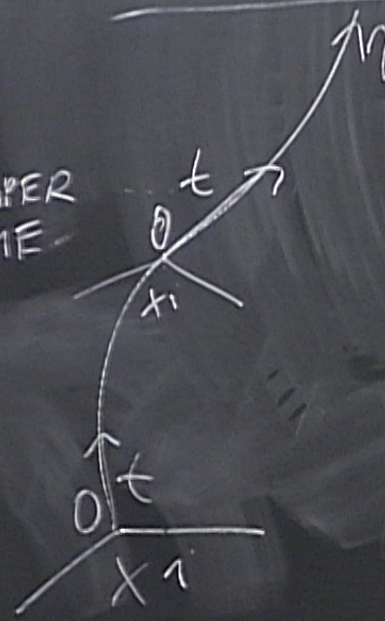

EVEN MORE GENERAL EQUIVALENCE PRINCIPLE
... GEODESIC


... GEODESIC
t
x1


... GEODESIC
PROPER TIME
t
0
x1
0
t
x1


PROPER TIME
t
0
x_1
0
t
x_1
$g_{tt} = -1 + R_{titj} x^i x^j + O(x^3)$
$g_{ti} =$

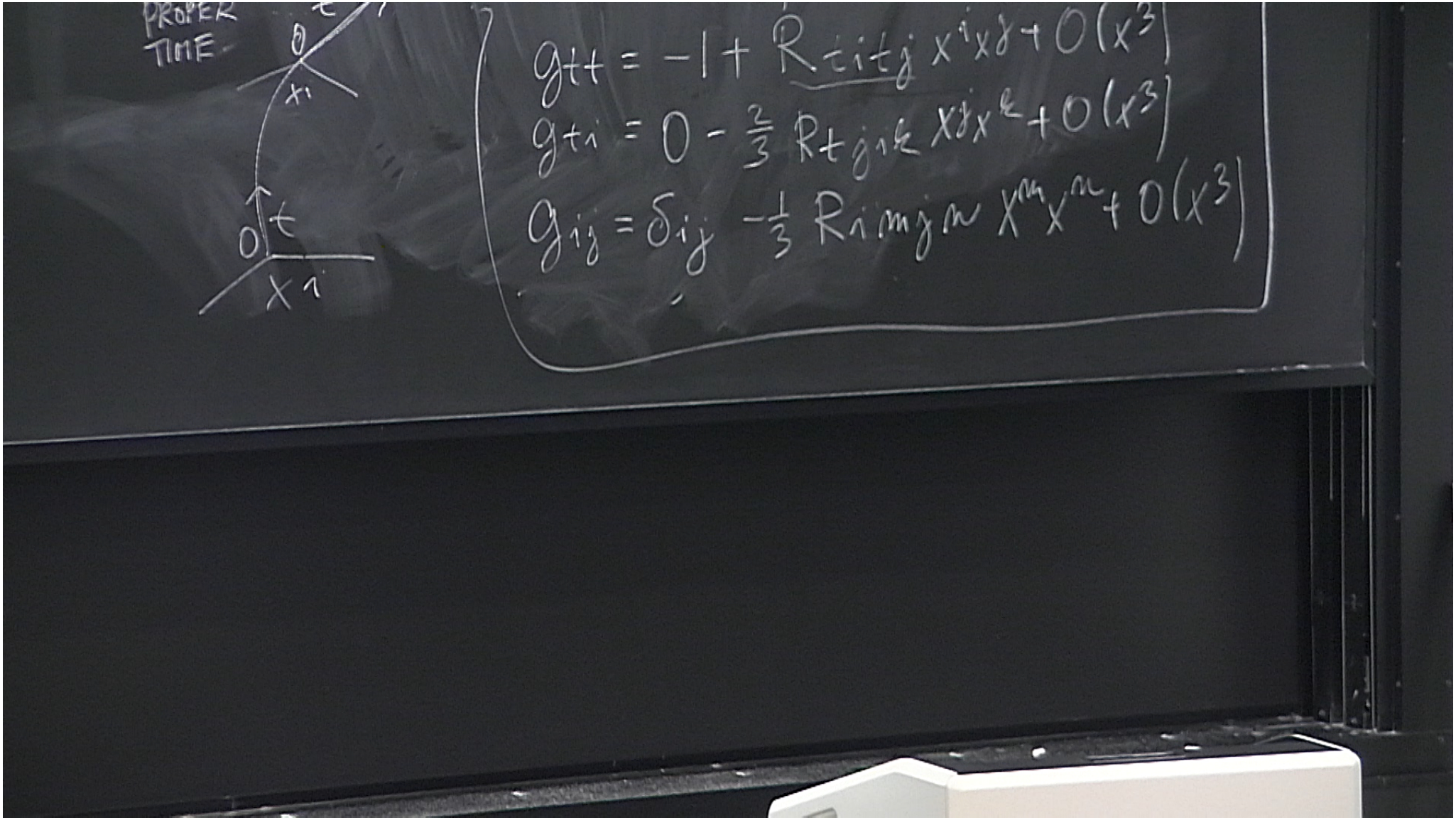
EVEN MORE GENERAL EQUIVALENCE PRINCIPLE



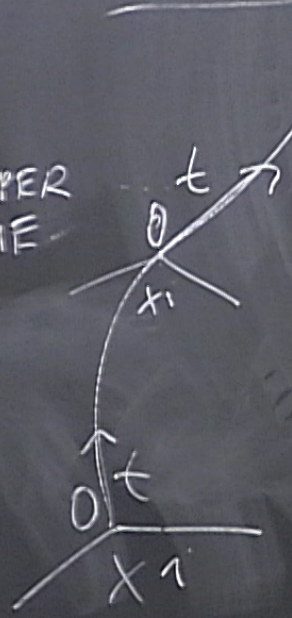
IN COORDS (t, x^i)

$$g_{tt} = -1 + R_{titj} x^i x^j + O(x^3)$$

$$g_{ti} = 0 - \frac{2}{3} R_{tjik} x^j x^k + O(x^3)$$


PROPER TIME
0
t
0
t
x
$g_{tt} = -1 + R_{titj} x^i x^j + O(x^3)$
$g_{ti} = 0 - \frac{2}{3} R_{tjik} x^j x^k + O(x^3)$
$g_{ij} = \delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n + O(x^3)$

EVEN MORE GENERAL EQUIVALENCE PRINCIPLE



γ ... GEODESIC

IN COORDS (t, x^i)

$$g_{tt} = -1 + R_{titj} x^i x^j + O(x^3)$$

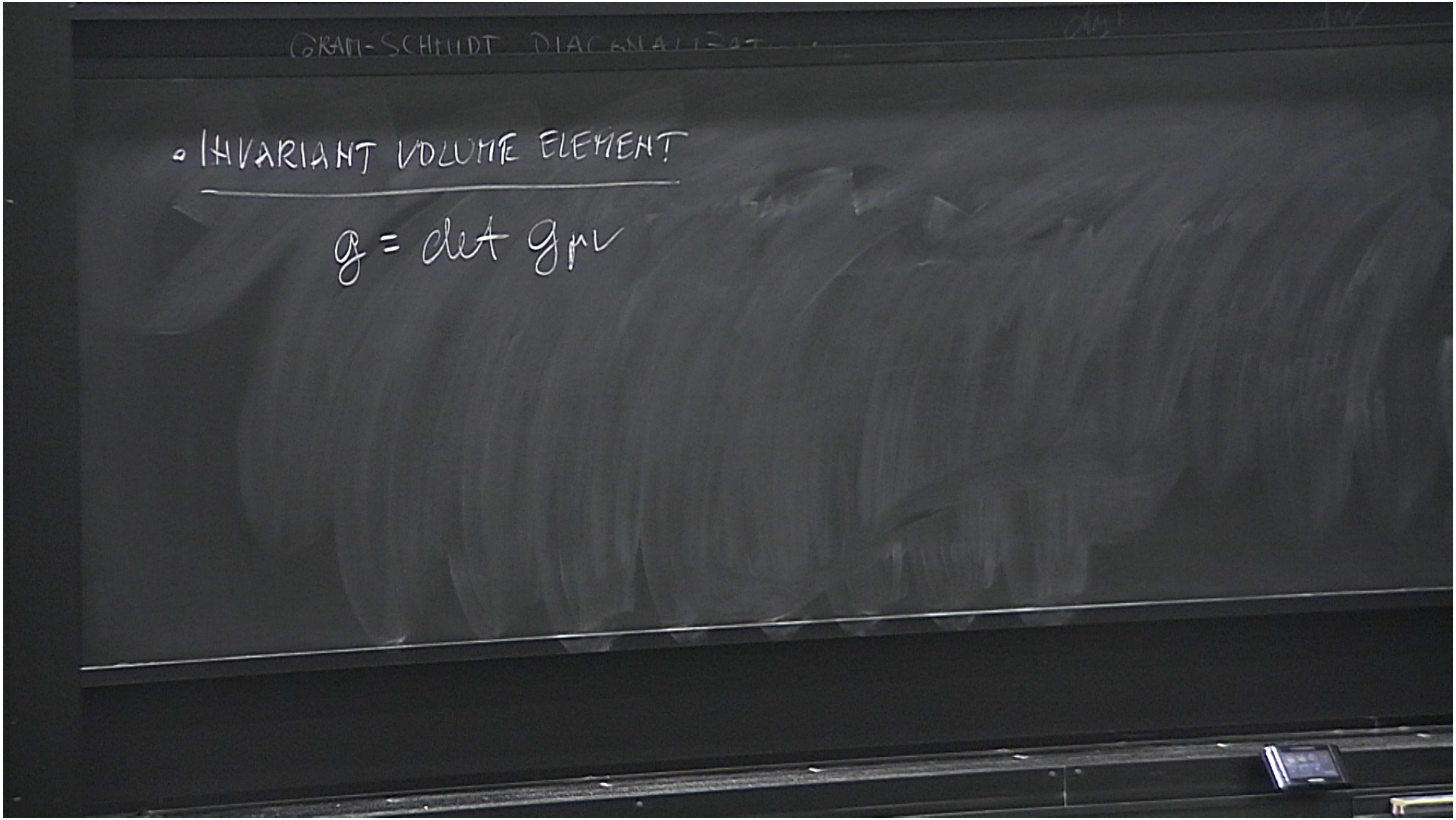
$$g_{ti} = 0 - \frac{2}{3} R_{tjik} x^j x^k + O(x^3)$$

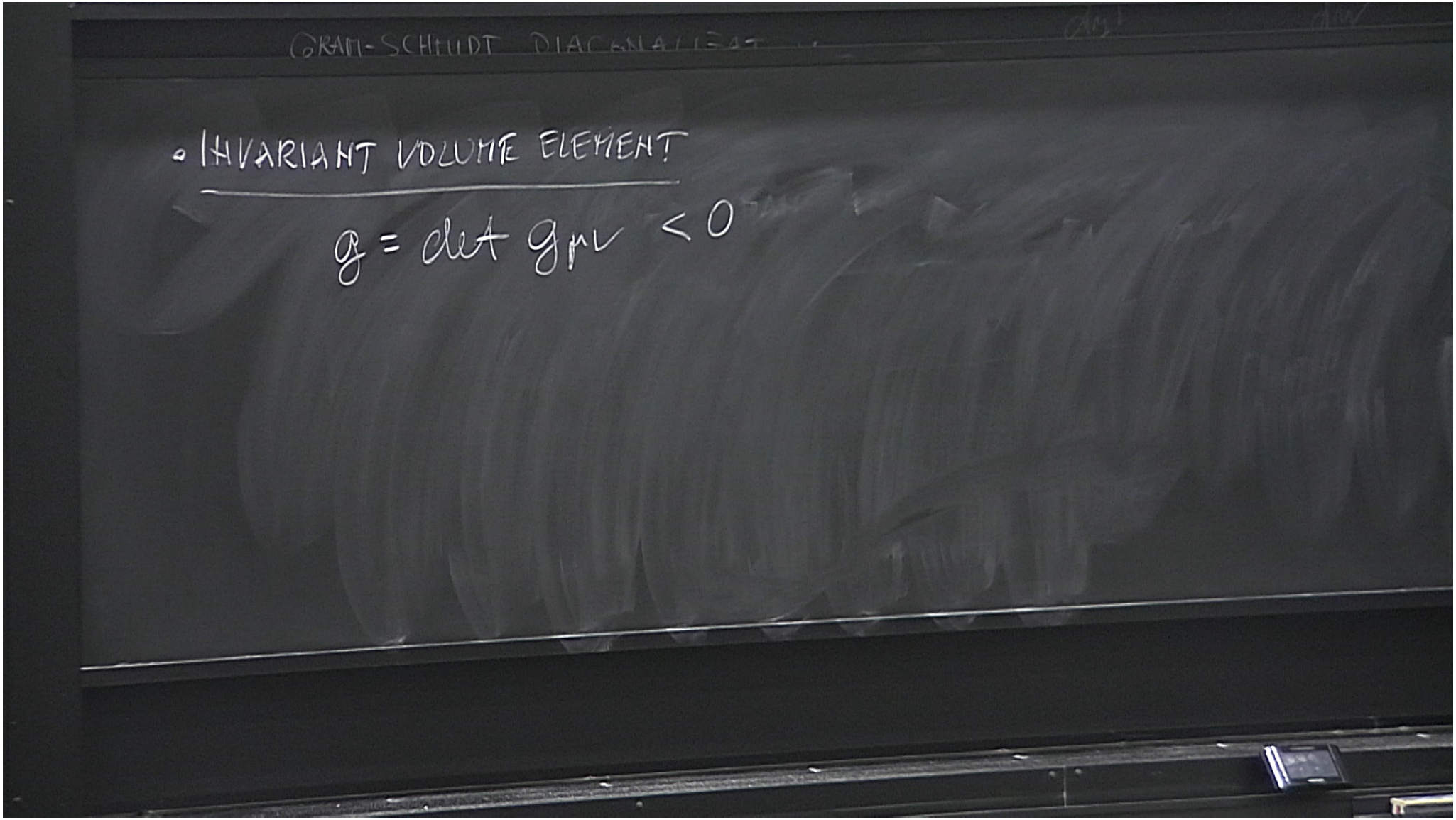
$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{imjn} x^m x^n + O(x^3)$$

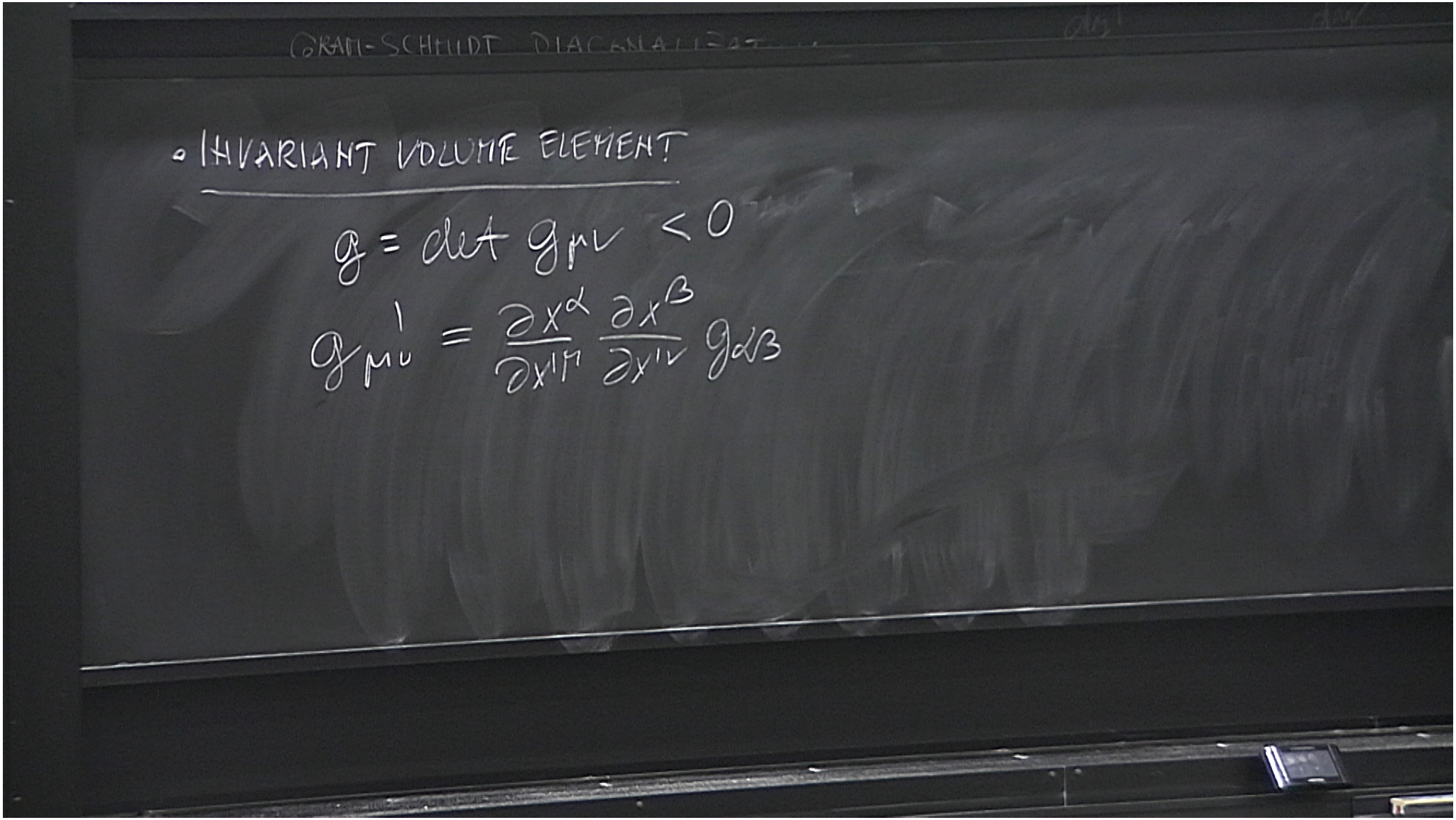
FERMI COORDINATES

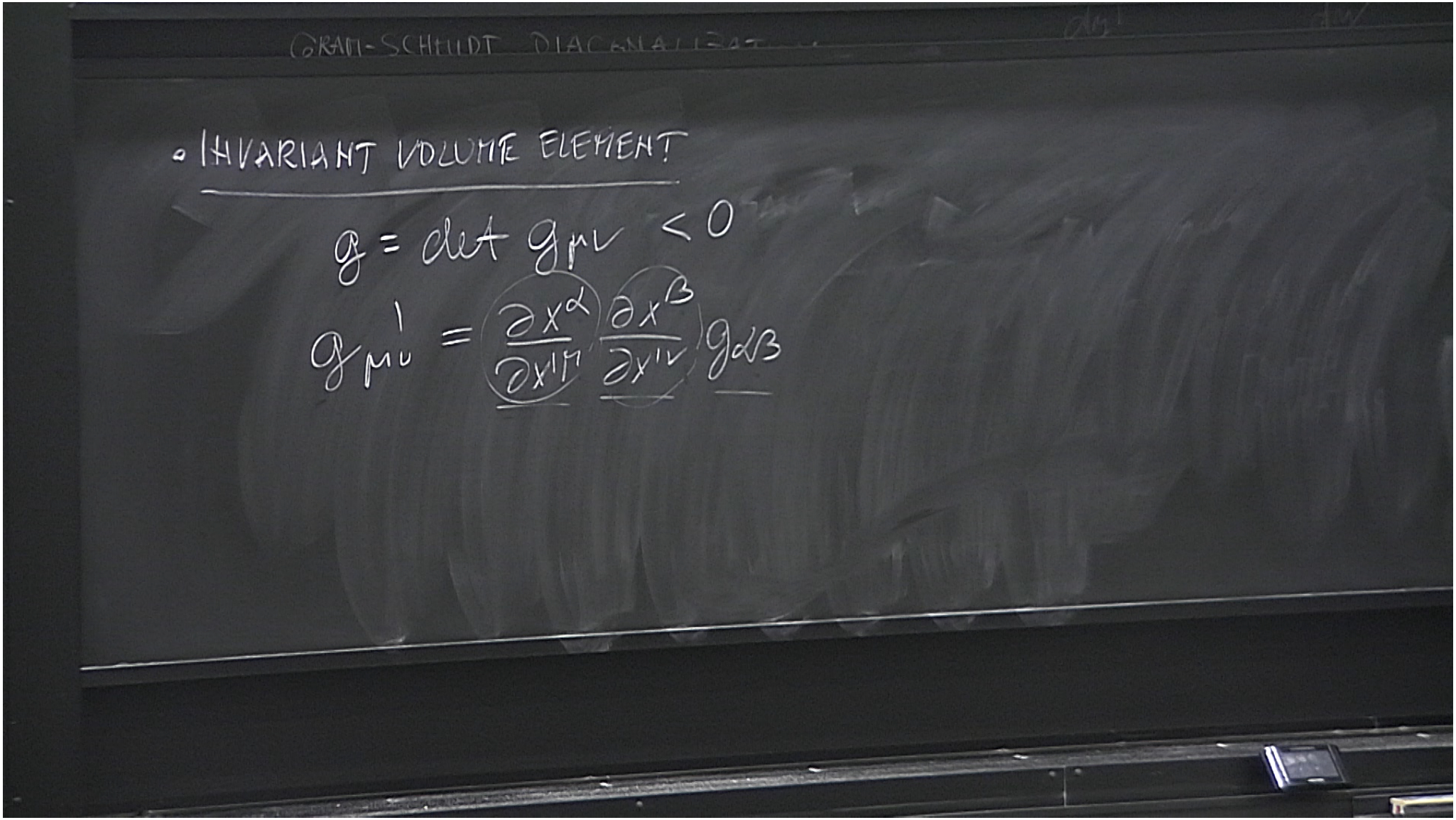
- CHRISTOFFEL SYMBOLS

$\nabla_\alpha g_{\beta\gamma}$ ... METRICITY


GRAM-SCHMIDT
• INVARIANT VOLUME ELEMENT
$g = \det g_{\mu\nu}$


GRAM-SCHMIDT DIAGONALIZAT
• INVARIANT VOLUME ELEMENT
$g = \det g_{\mu\nu} < 0$


GRAM-SCHMIDT DIAGONALIZAT
• INVARIANT VOLUME ELEMENT
$g = \det g_{\mu\nu} < 0$
$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$


GRAM-SCHMIDT DIAGONALIZAT
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GRAM-SCHMIDT DIAGONALIZAT...

- INVARIANT VOLUME ELEMENT

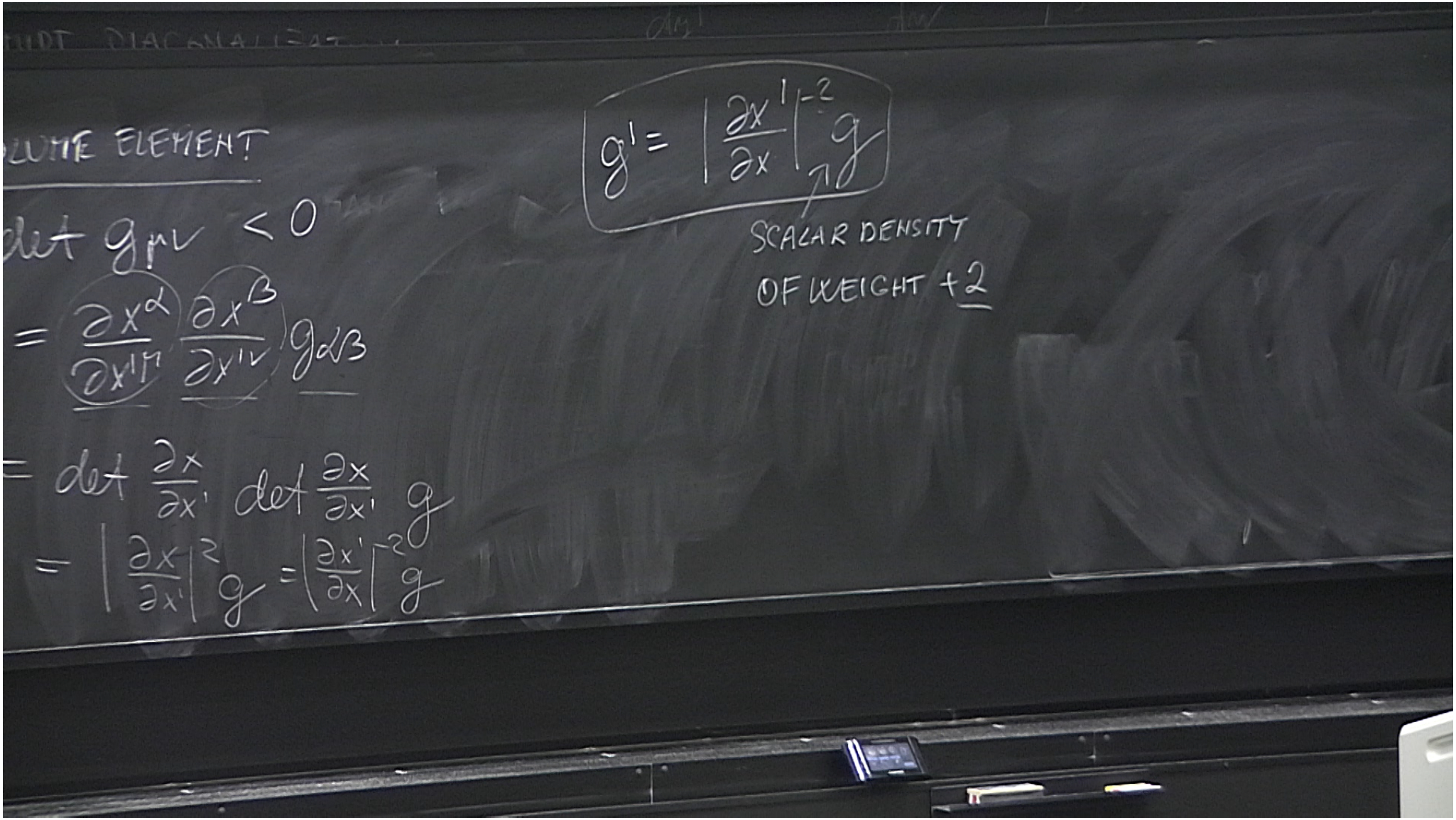
$$g = \det g_{\mu\nu} < 0$$

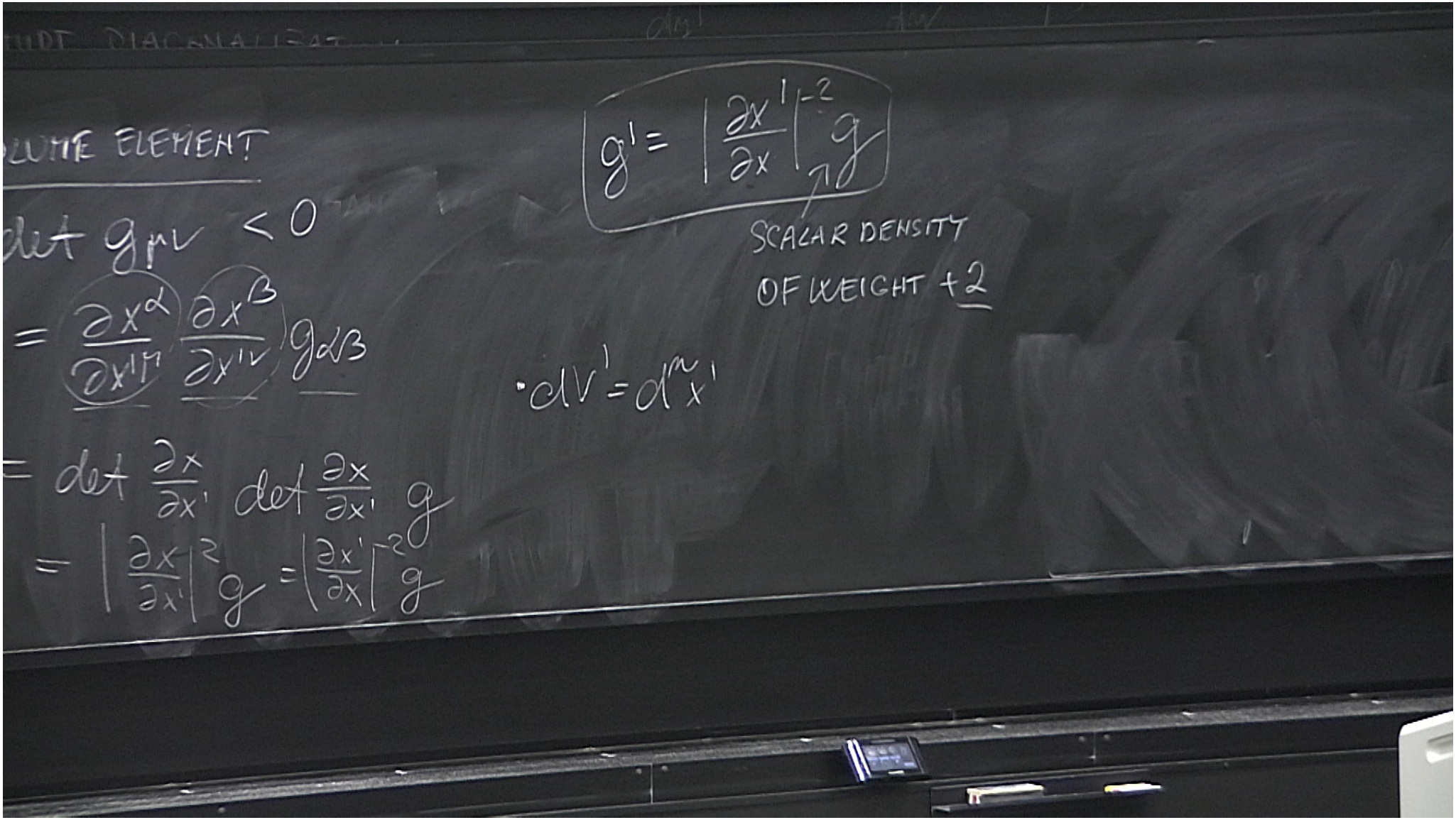
$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

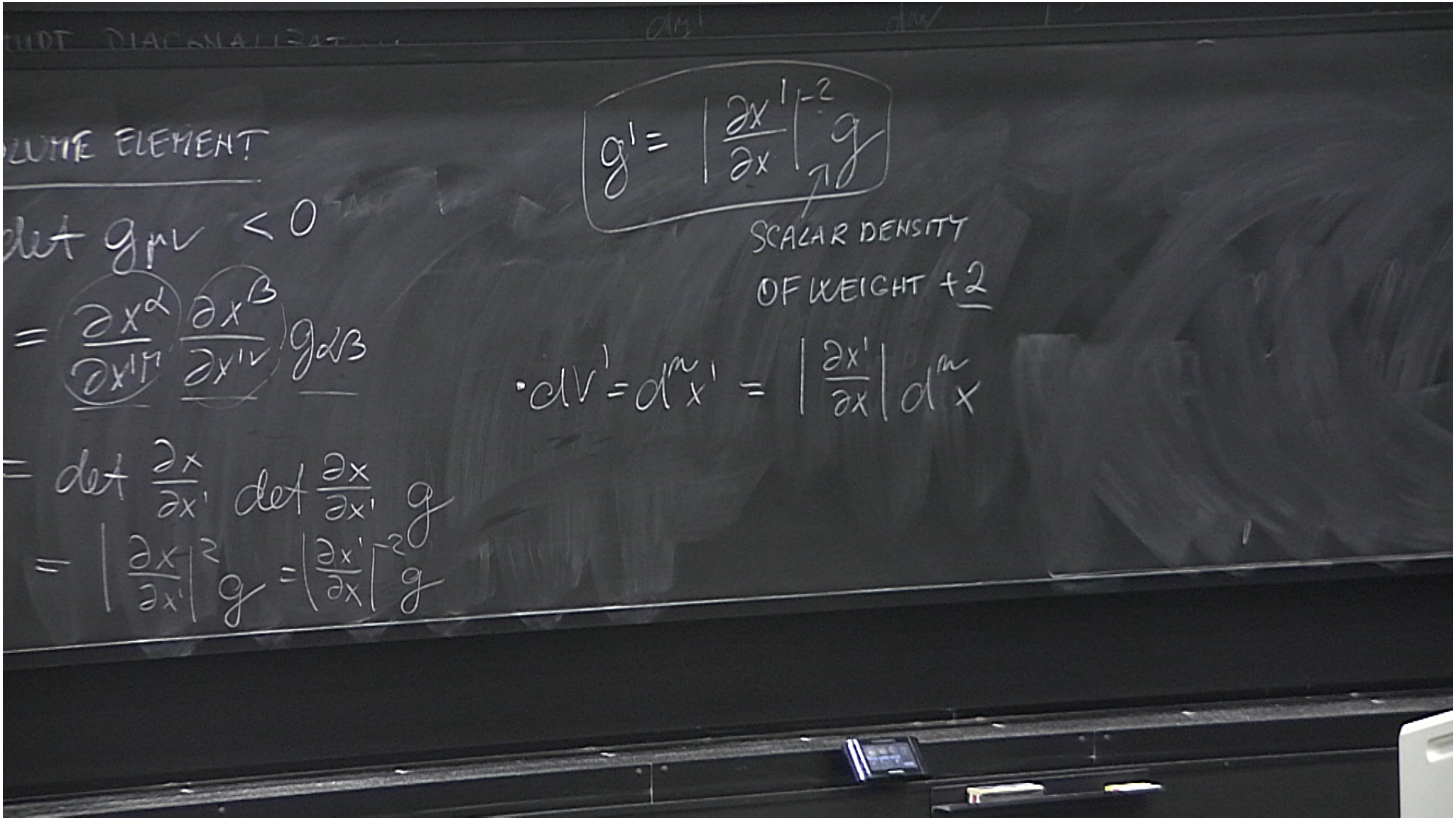
$$g' = \det \frac{\partial x}{\partial x'} \det \frac{\partial x}{\partial x'} g$$

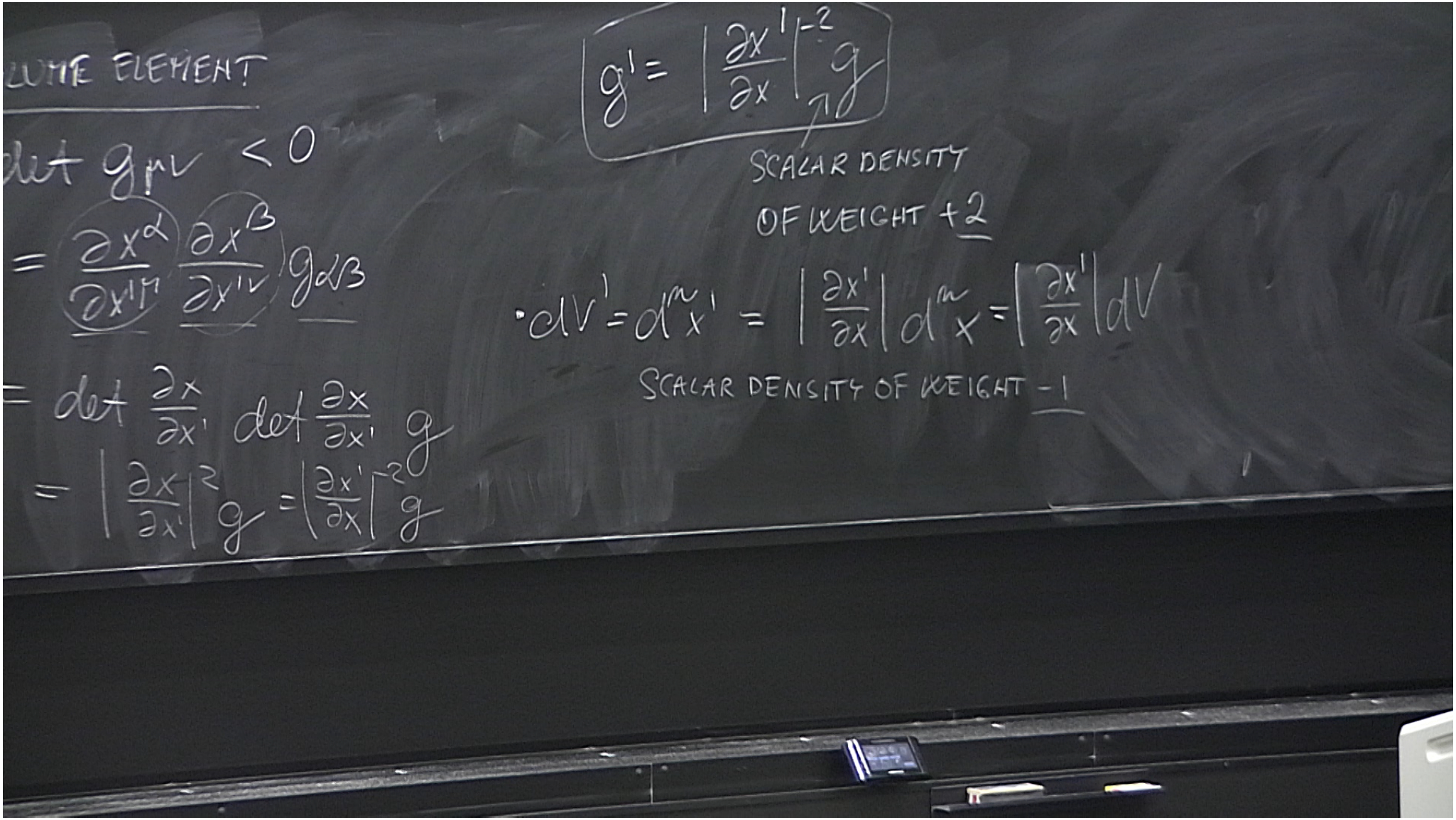
$$= \left| \frac{\partial x}{\partial x'} \right|^2 g = \left| \frac{\partial x'}{\partial x} \right|^{-2} g$$

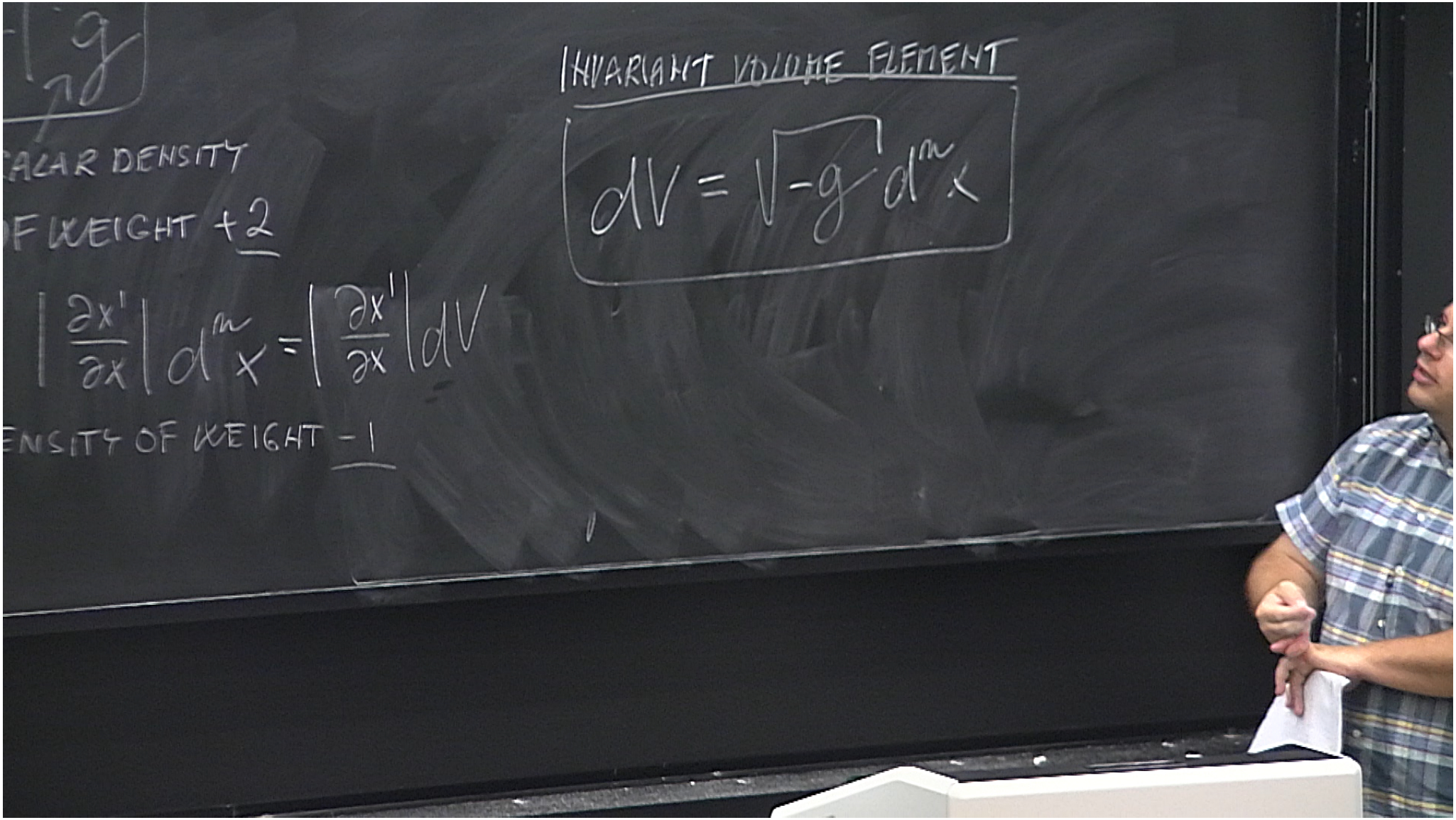

VOLUME ELEMENT
$g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$
$\det g_{\mu\nu} < 0$
$= \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$
$= \det \frac{\partial x}{\partial x'} \det \frac{\partial x}{\partial x'} g$
$= \left|\frac{\partial x}{\partial x'}\right|^2 g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$

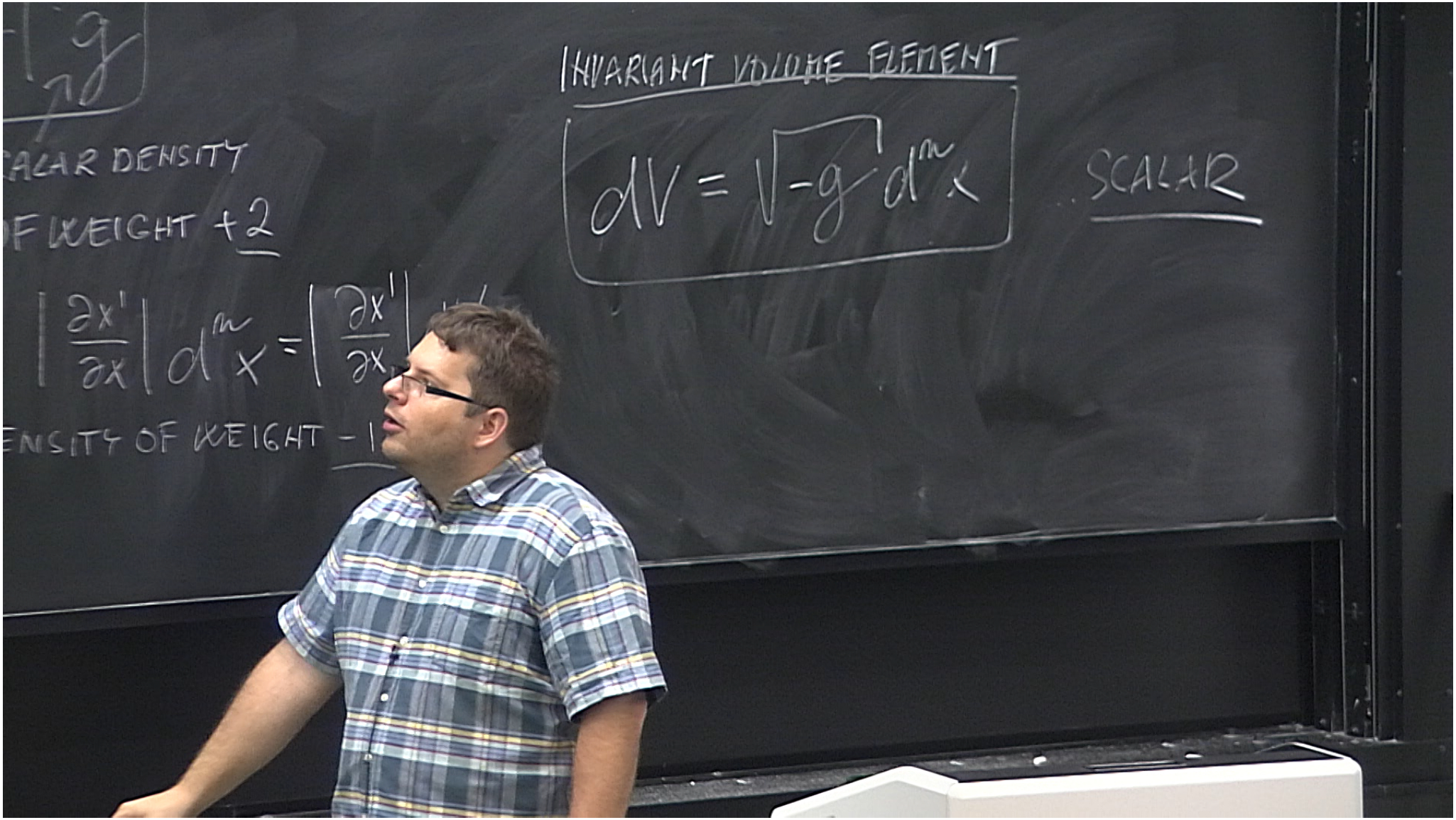

VOLUME ELEMENT
$\det g_{\mu\nu} < 0$
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$g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$
SCALAR DENSITY
OF WEIGHT +2


VOLUME ELEMENT
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SCALAR DENSITY
OF WEIGHT +2
$dV' = d^n x'$


VOLUME ELEMENT
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SCALAR DENSITY
OF WEIGHT +2
$dV' = d^n x' = \left|\frac{\partial x'}{\partial x}\right| d^n x$


LUME ELEMENT
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$g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$
SCALAR DENSITY
OF WEIGHT +2
$dV' = d^n x' = \left|\frac{\partial x'}{\partial x}\right| d^n x = \left|\frac{\partial x'}{\partial x}\right| dV$
SCALAR DENSITY OF WEIGHT -1


g
SCALAR DENSITY
OF WEIGHT +2
$\left|\frac{\partial x'}{\partial x}\right| d^n x = \left|\frac{\partial x'}{\partial x}\right| dV$
DENSITY OF WEIGHT -1
INVARIANT VOLUME ELEMENT
$dV = \sqrt{-g}\, d^n x$


INVARIANT VOLUME ELEMENT
$dV = \sqrt{-g}\, d^n x$
SCALAR
$\left|\frac{\partial x'}{\partial x}\right| d^n x = \left|\frac{\partial x'}{\partial x}\right|$

$$= \left|\frac{\partial x}{\partial x'}\right|^2 g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$$

• LEVI-CIVITA TENSOR:

$$[\mu\nu\rho\sigma] \leftarrow \begin{cases} +1 & [0,1,2,3] \text{ \& EVEN PERM.} \\ -1 & \text{ODD PERM.} \\ 0 & \text{OTHERWISE.} \end{cases}$$

$$= \left|\frac{\partial x}{\partial x'}\right|^2 g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$$

LEVI-CIVITA TENSOR:

$$[\mu\nu\rho\sigma] \leftarrow \begin{cases} +1 \\ -1 \\ 0 \end{cases}$$

$[0,1,2,3]$ & EVEN PERM.

ODD PERMS

OTHERWISE.

$$\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,[\mu\nu\rho\sigma]$$

$$= \left|\frac{\partial x}{\partial x'}\right|^2 g = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$$

• LEVI-CIVITA TENSOR:

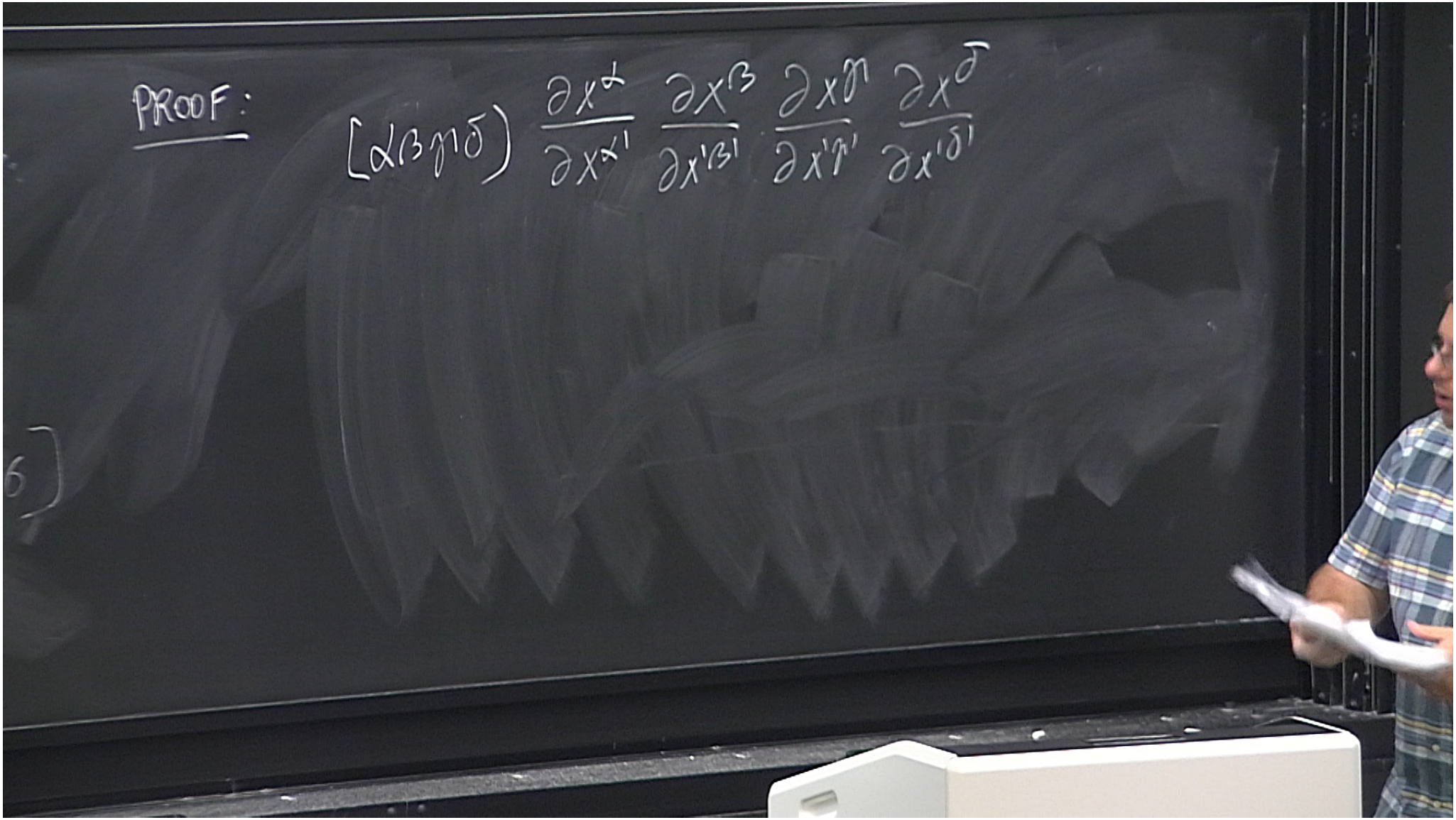
$$[\mu\nu\rho\sigma] \leftarrow \begin{matrix} +1 \\ -1 \\ 0 \end{matrix}$$

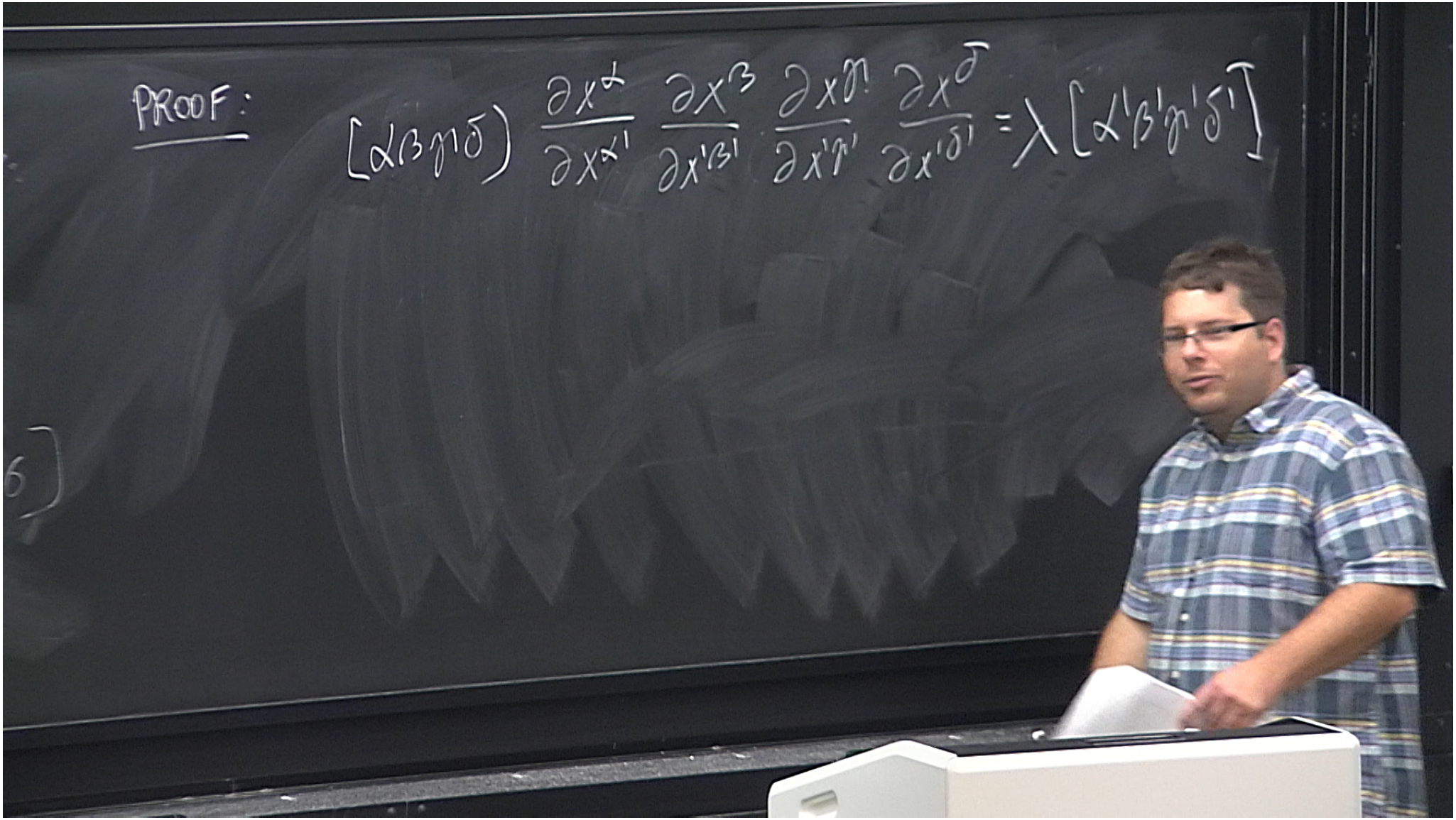
$[0,1,2,3]$ & EVEN PERM.

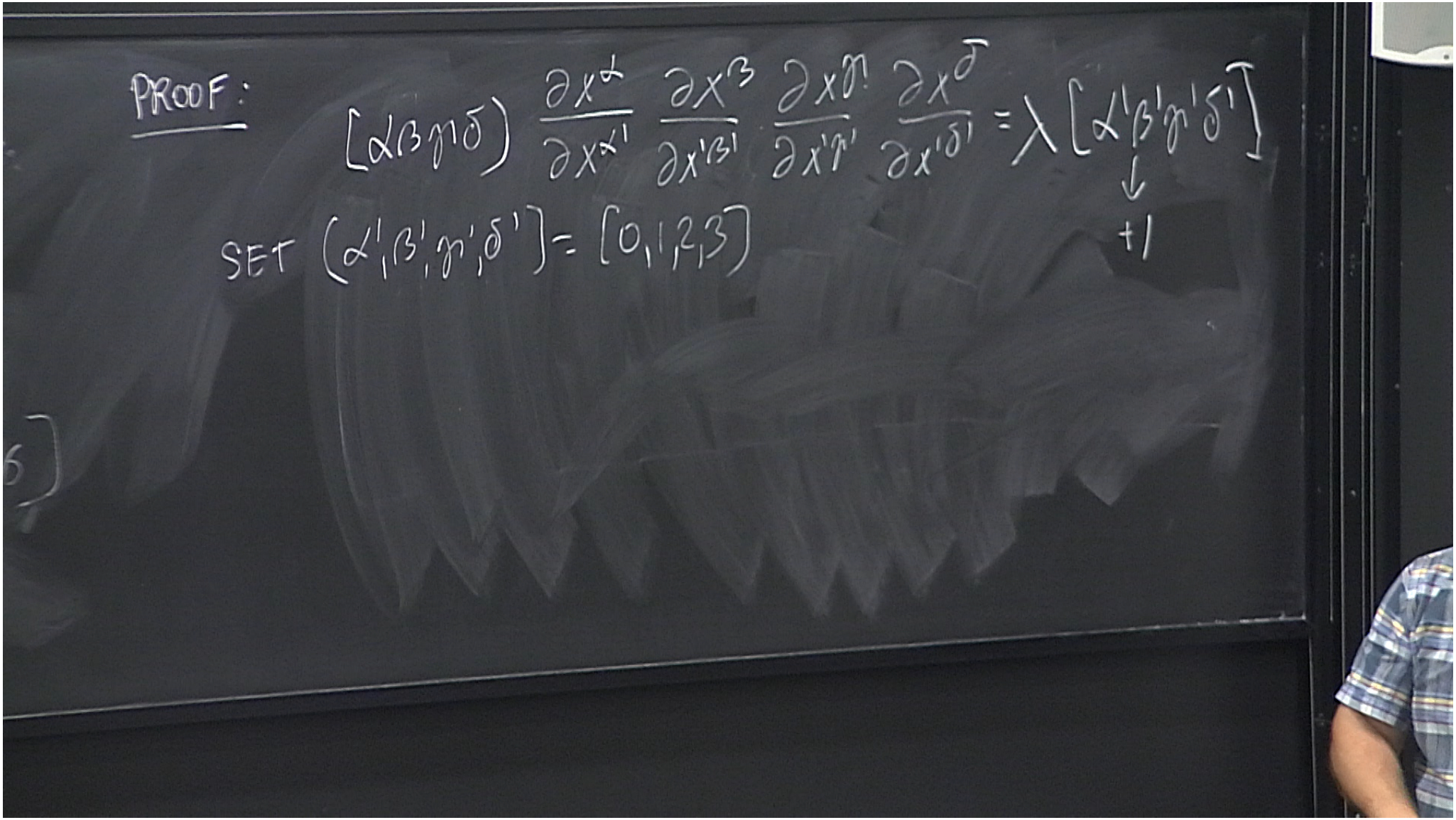
ODD PERM

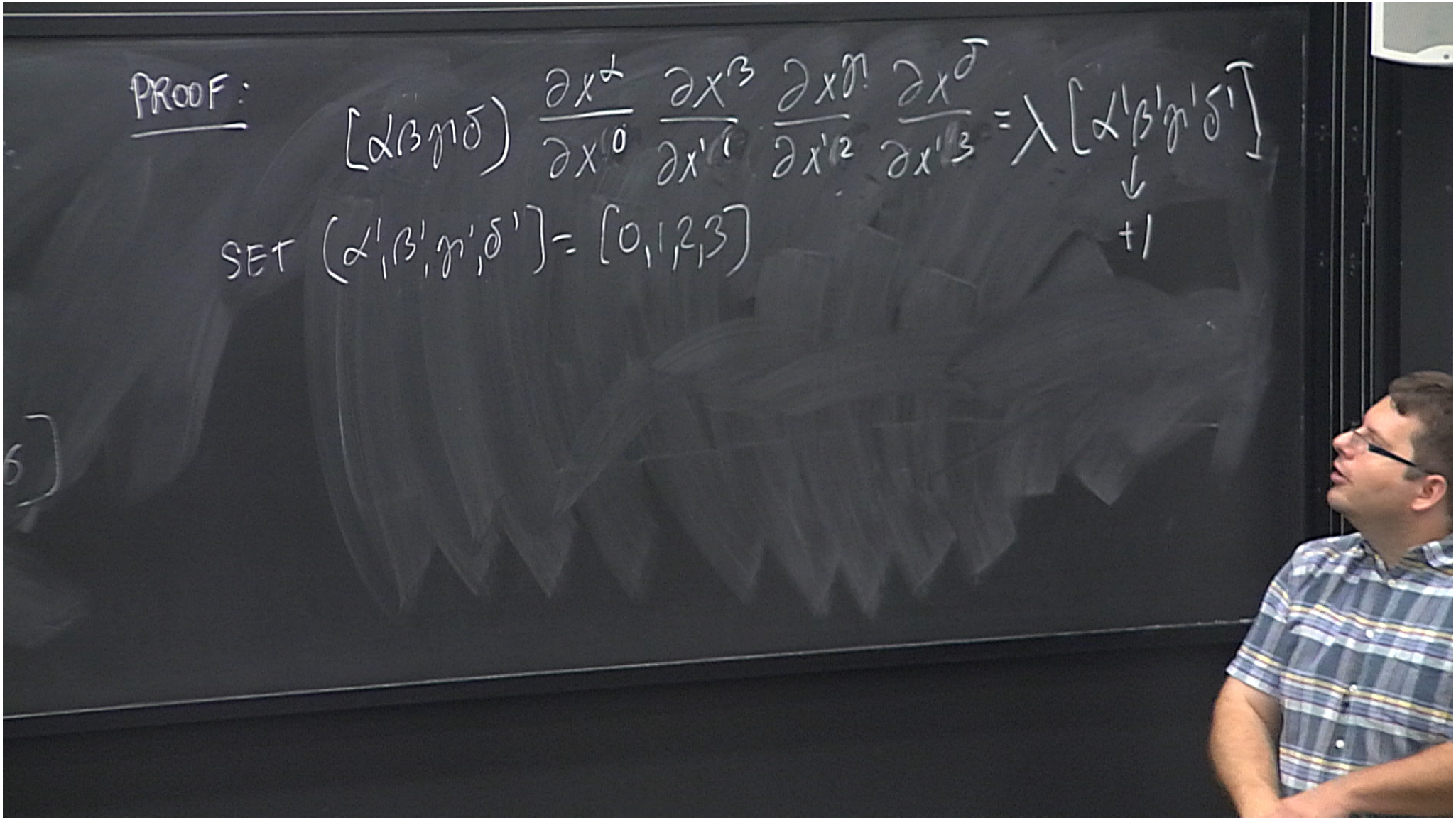
OTHERWISE

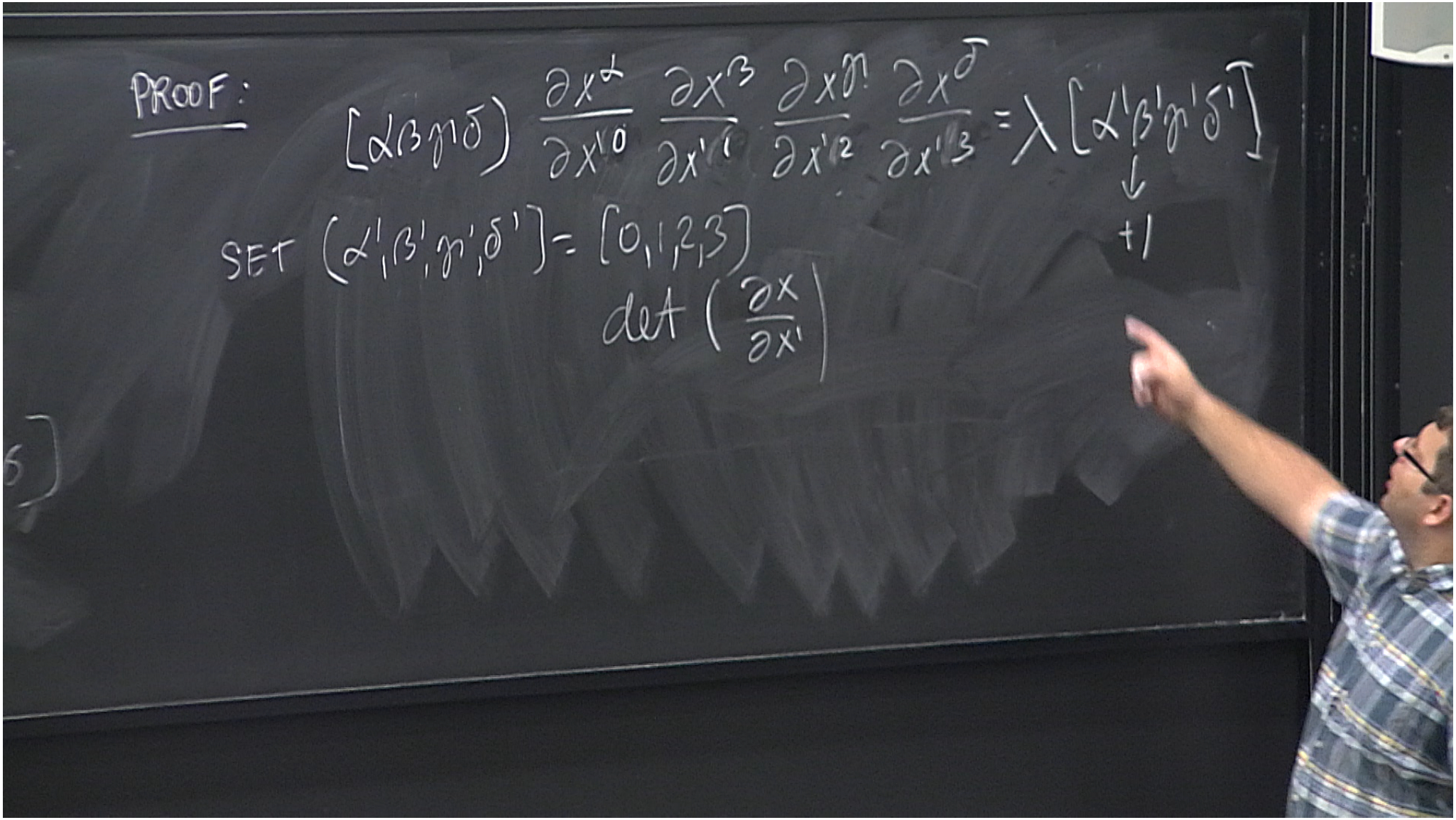
$$\underline{\varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\,[\mu\nu\rho\sigma]} \qquad \varepsilon^{\mu\nu\rho\sigma} = -\frac{1}{\sqrt{-g}}[\mu\nu\rho\sigma]$$

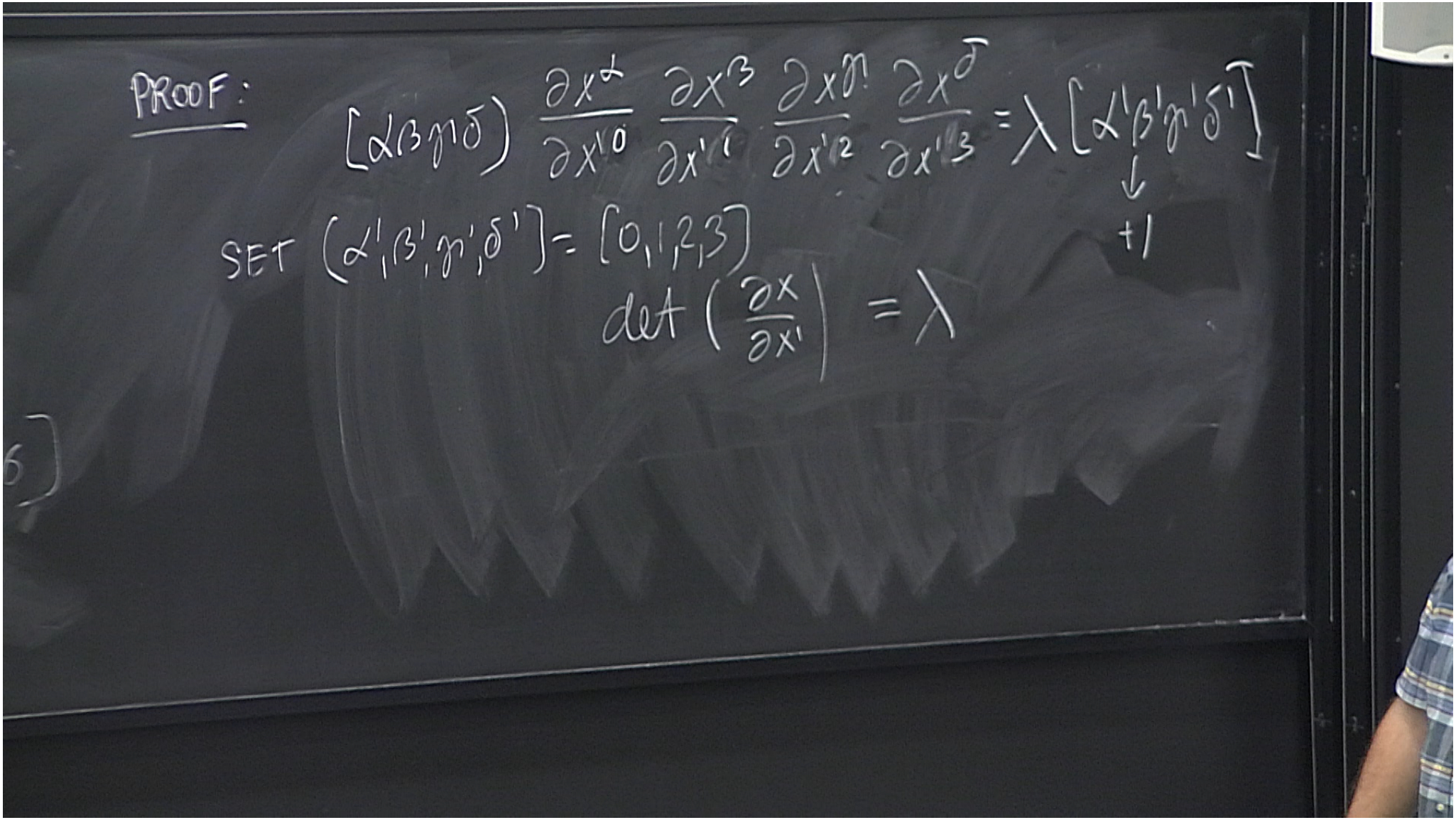

PROOF:
$[\alpha\beta\gamma\delta) \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} \frac{\partial x^\gamma}{\partial x'^{\gamma'}} \frac{\partial x^\delta}{\partial x'^{\delta'}}$
6]

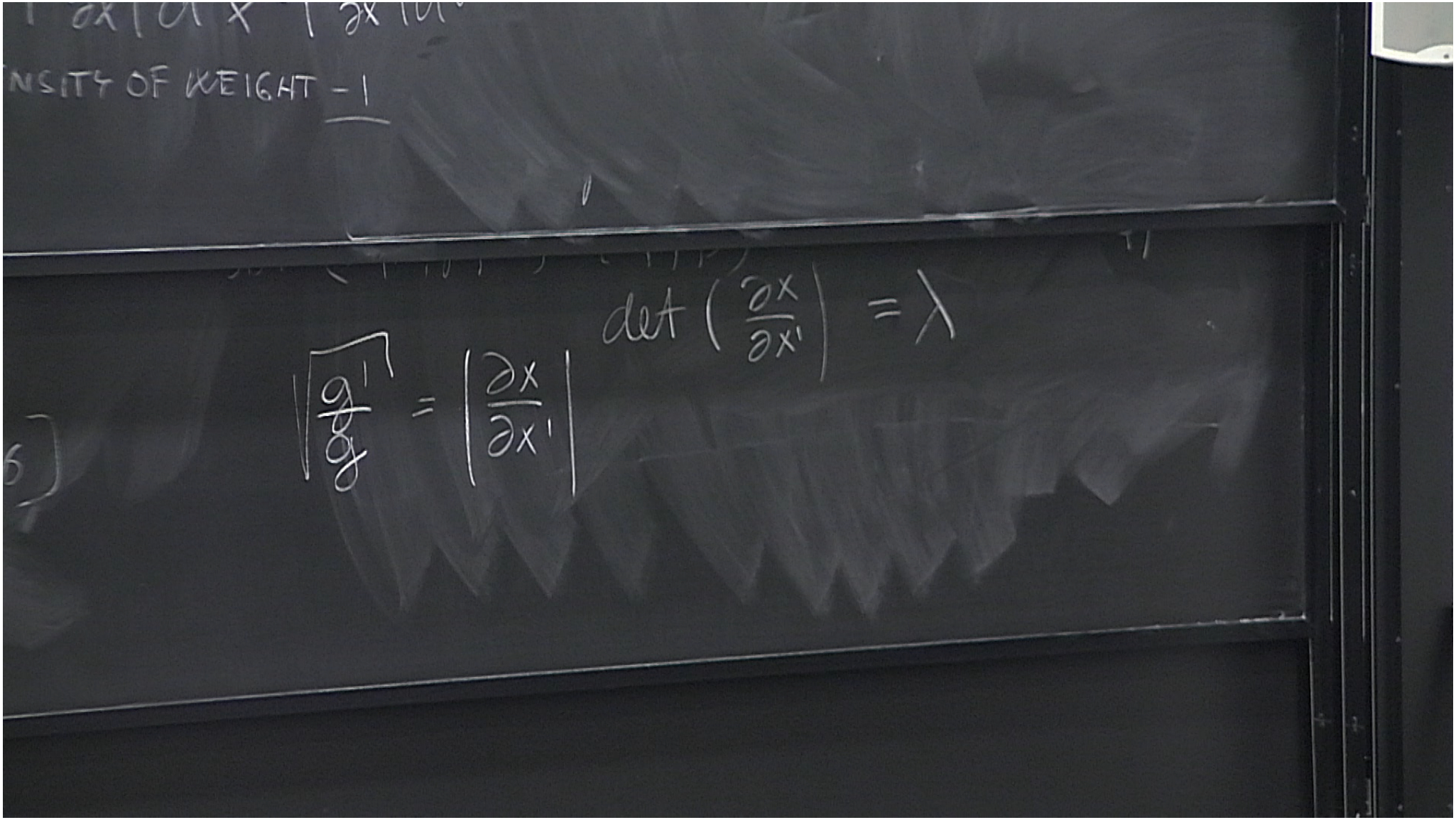

PROOF:
$[\alpha\beta\gamma\sigma)\ \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\frac{\partial x^{\gamma}}{\partial x^{\gamma'}}\frac{\partial x^{\sigma}}{\partial x^{\sigma'}} = \lambda\,[\alpha'\beta'\gamma'\sigma']$
6]

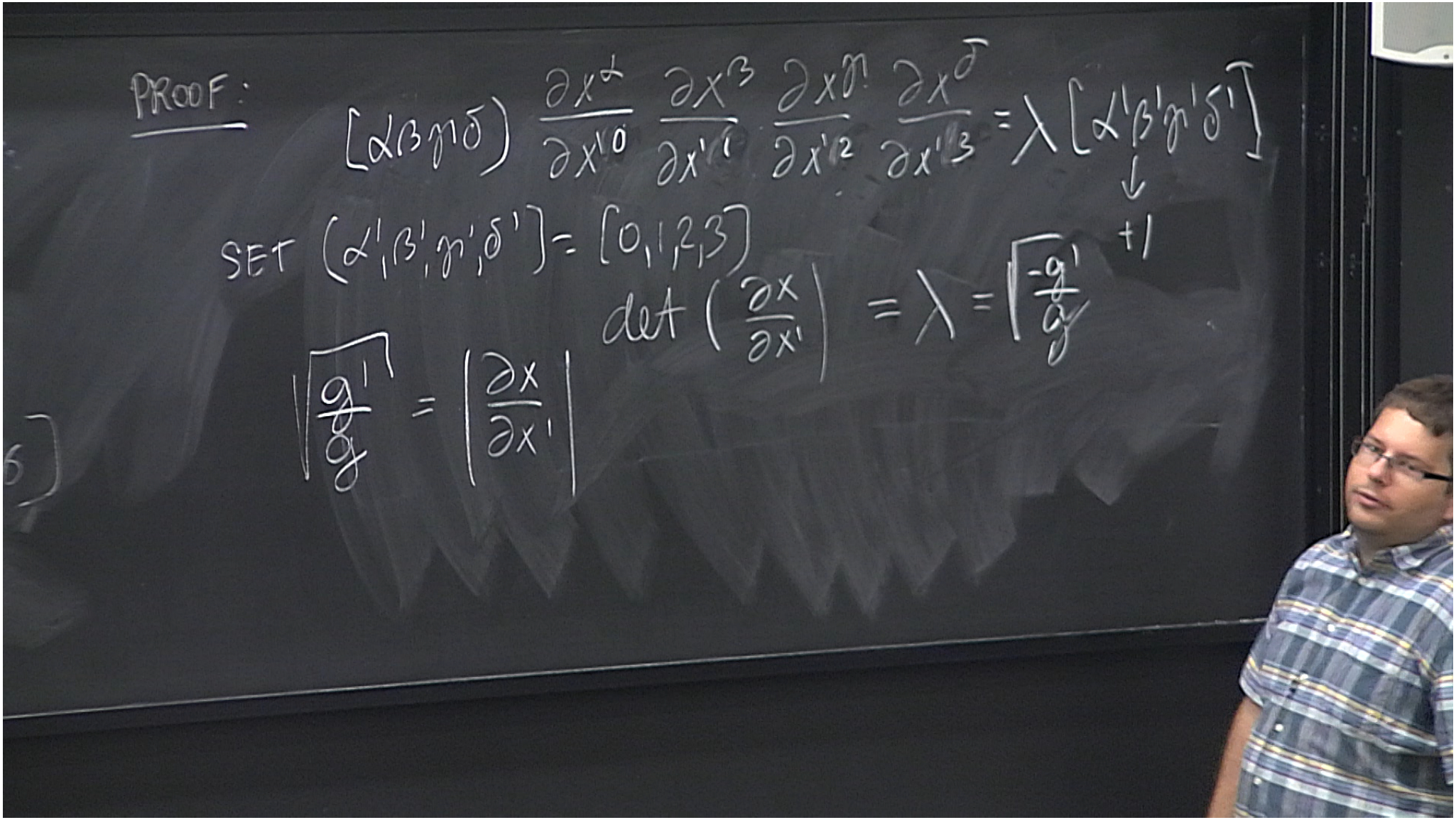

PROOF:
$[\alpha\beta\gamma\delta]\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\frac{\partial x^\delta}{\partial x^{\delta'}} = \lambda[\alpha'\beta'\gamma'\delta']$
SET $[\alpha',\beta',\gamma',\delta'] = [0,1,2,3]$
$+1$

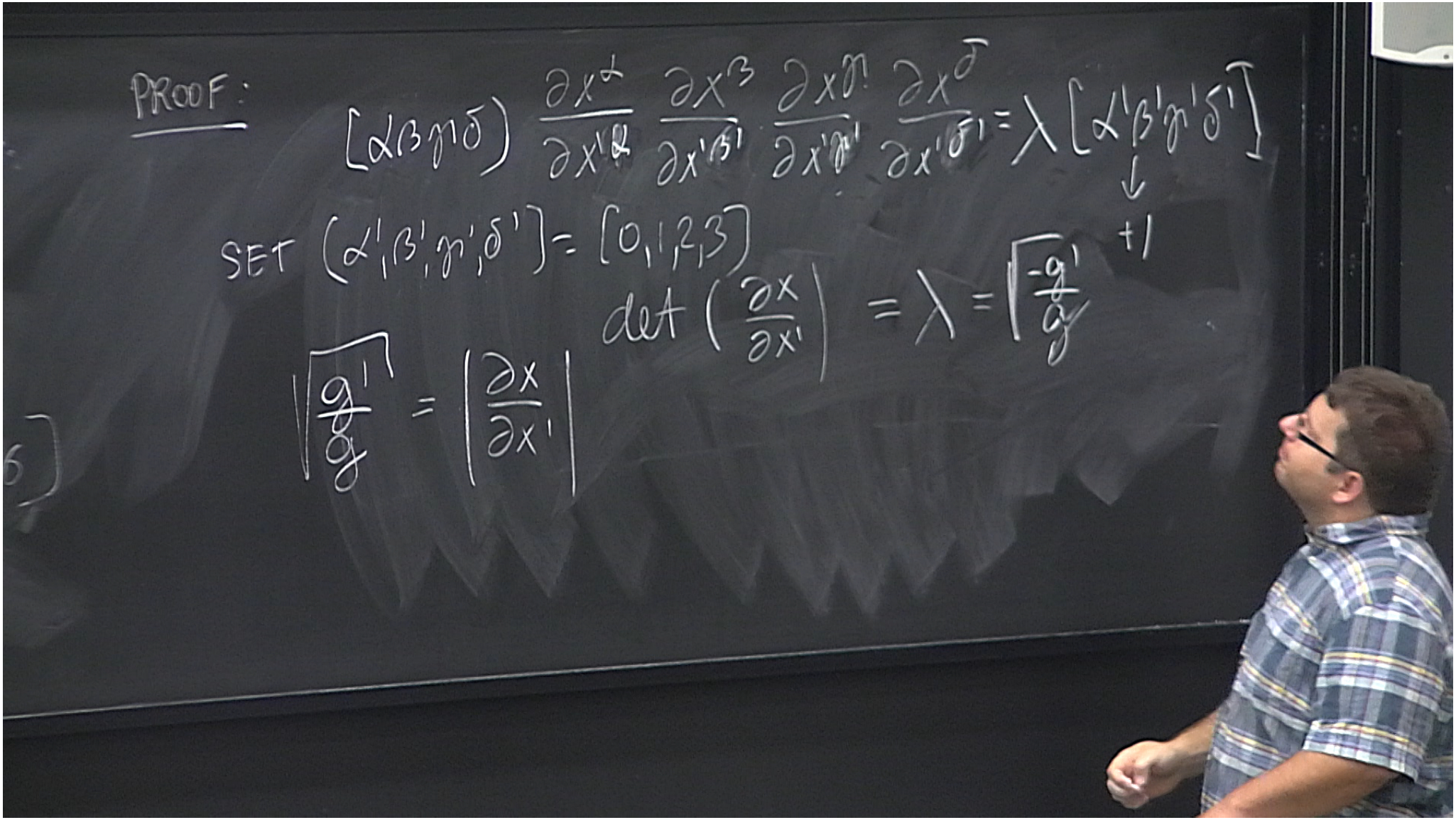

PROOF:
$[\alpha\beta\gamma\delta)\ \frac{\partial x^\alpha}{\partial x'^0}\frac{\partial x^\beta}{\partial x'^1}\frac{\partial x^\gamma}{\partial x'^2}\frac{\partial x^\delta}{\partial x'^3} = \lambda\,[\alpha'\beta'\gamma'\delta']$
SET $(\alpha',\beta',\gamma',\delta'] = [0,1,2,3]$
$+1$

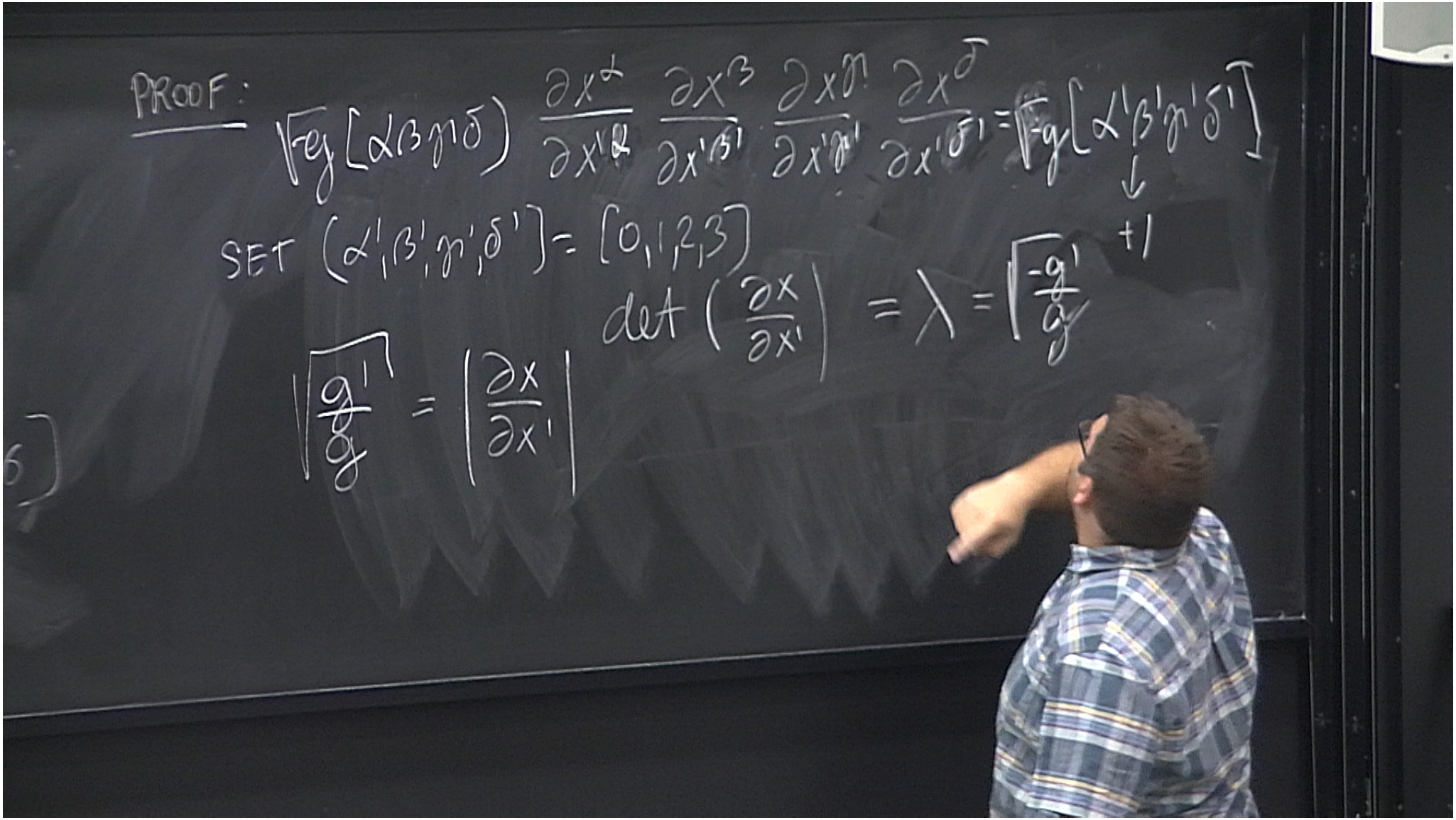

PROOF:
$[\alpha\beta\gamma\delta)\frac{\partial x^\alpha}{\partial x'^0}\frac{\partial x^\beta}{\partial x'^1}\frac{\partial x^\gamma}{\partial x'^2}\frac{\partial x^\delta}{\partial x'^3}=\lambda[\alpha'\beta'\gamma'\delta']$
SET $(\alpha',\beta',\gamma',\delta']=[0,1,2,3]$
$\det\left(\frac{\partial x}{\partial x'}\right)$
+1

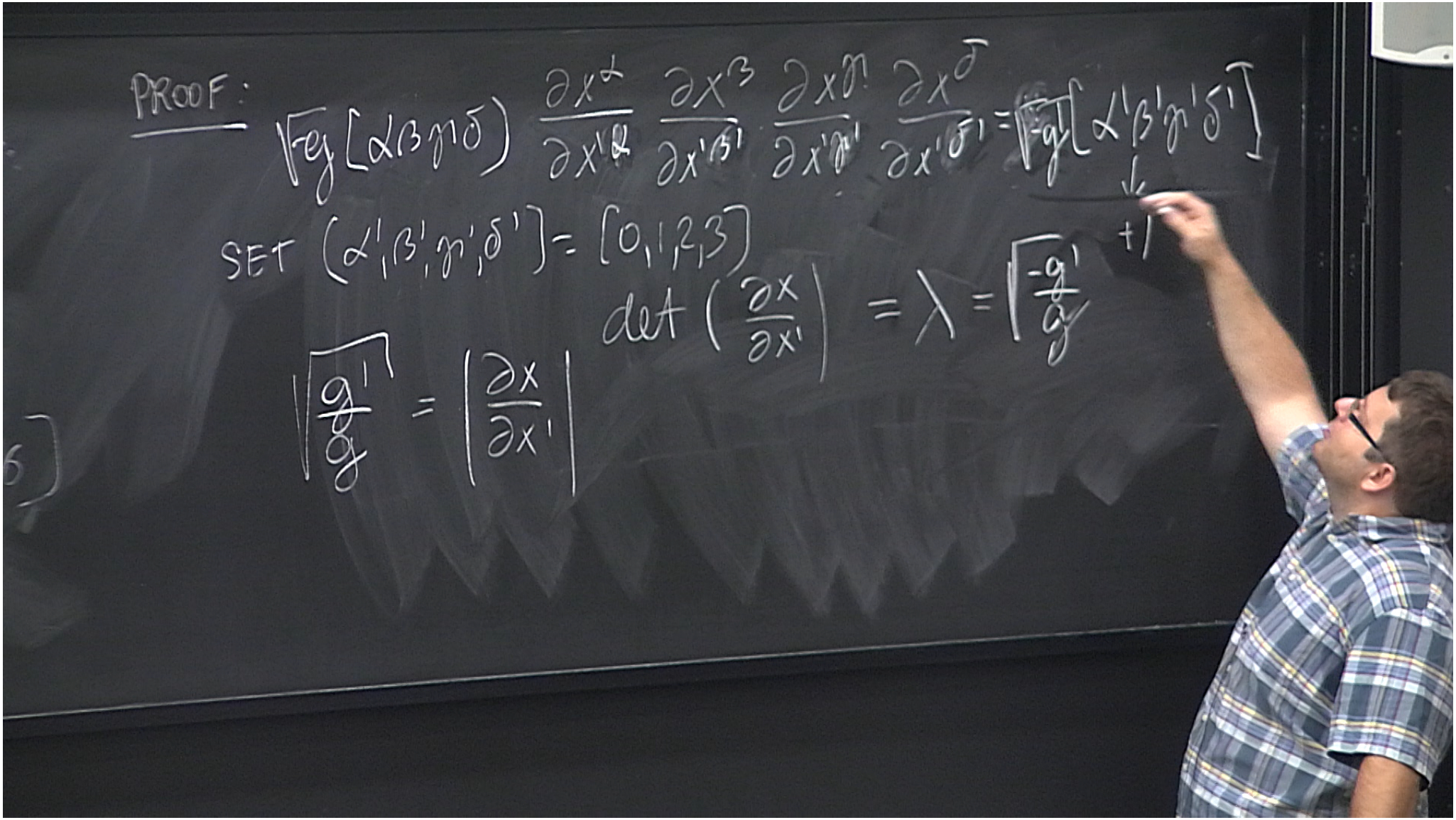

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$+1$
SET $[\alpha',\beta',\gamma',\delta'] = [0,1,2,3]$
$\det\left(\frac{\partial x}{\partial x'}\right) = \lambda$

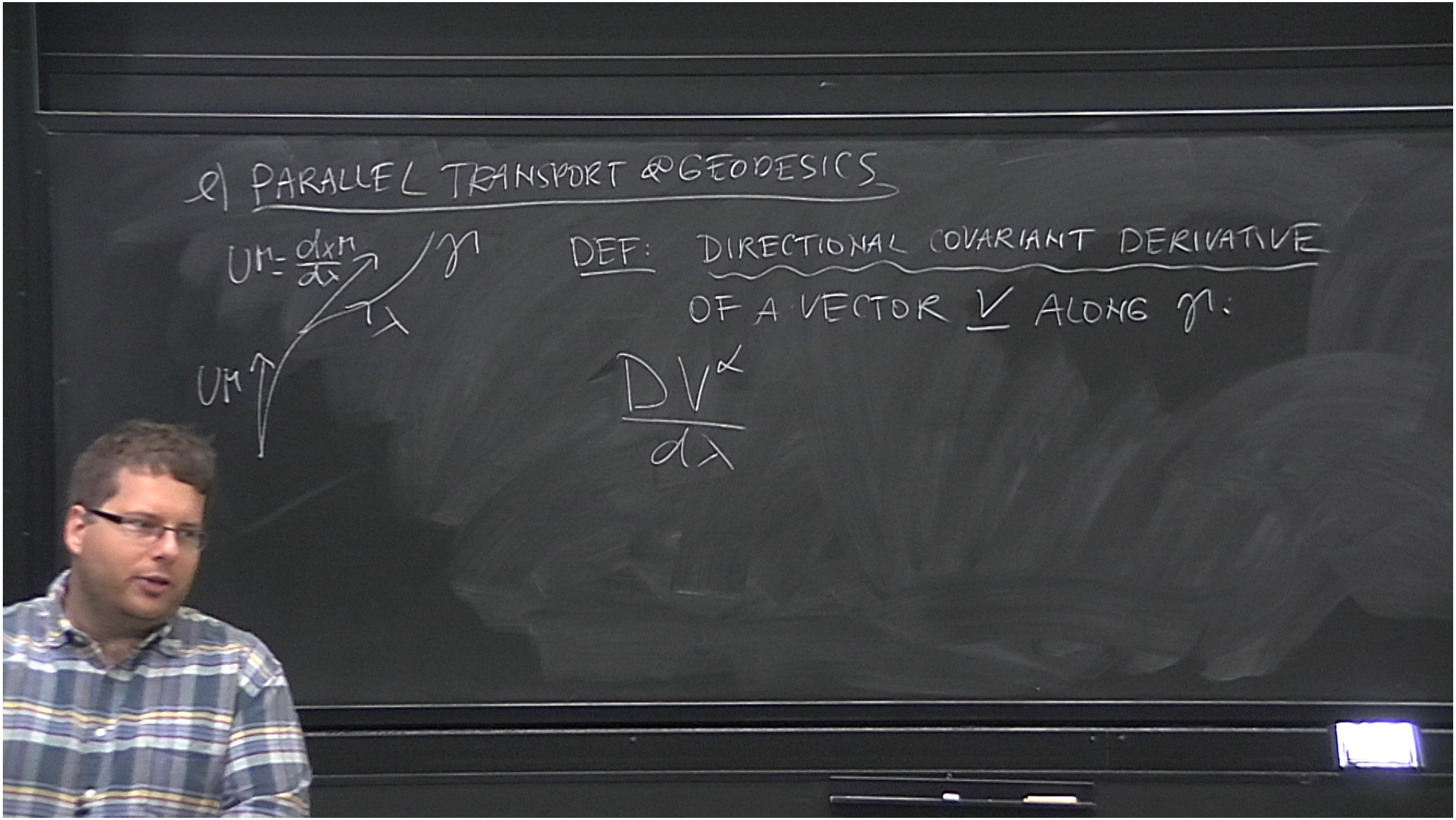

NSITY OF WEIGHT -1
$\sqrt{\frac{g'}{g}} = \left|\frac{\partial x}{\partial x'}\right|$
$\det\left(\frac{\partial x}{\partial x'}\right) = \lambda$

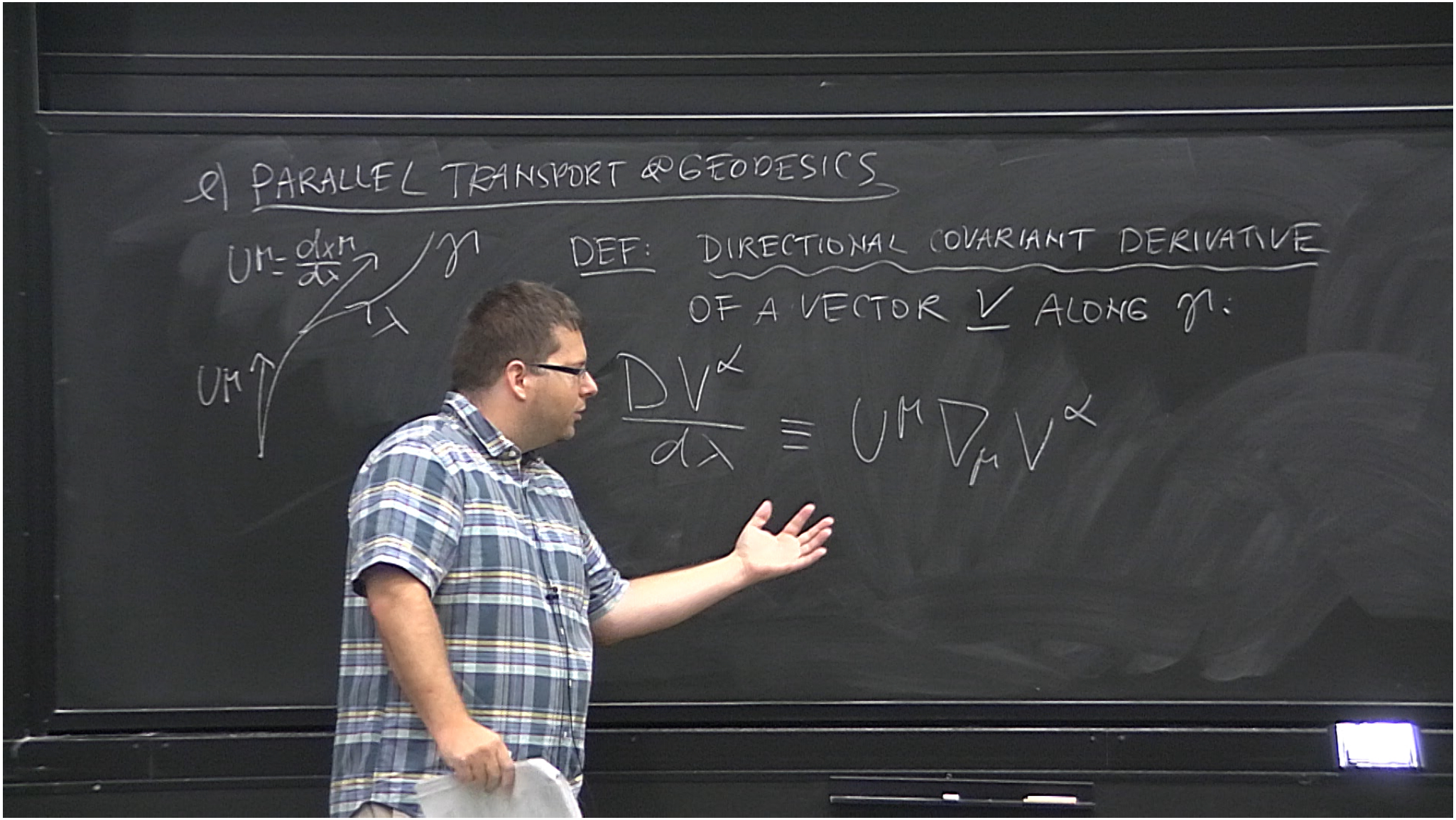

PROOF:
$[\alpha\beta\gamma\delta)\ \frac{\partial x^\alpha}{\partial x'^0}\frac{\partial x^\beta}{\partial x'^1}\frac{\partial x^\gamma}{\partial x'^2}\frac{\partial x^\delta}{\partial x'^3} = \lambda\,[\alpha'\beta'\gamma'\delta']$
SET $(\alpha',\beta',\gamma',\delta'] = [0,1,2,3)$
$+1$
$\det\left(\frac{\partial x}{\partial x'}\right) = \lambda = \sqrt{\frac{-g'}{g}}$
$\sqrt{\frac{g'}{g}} = \left|\frac{\partial x}{\partial x'}\right|$

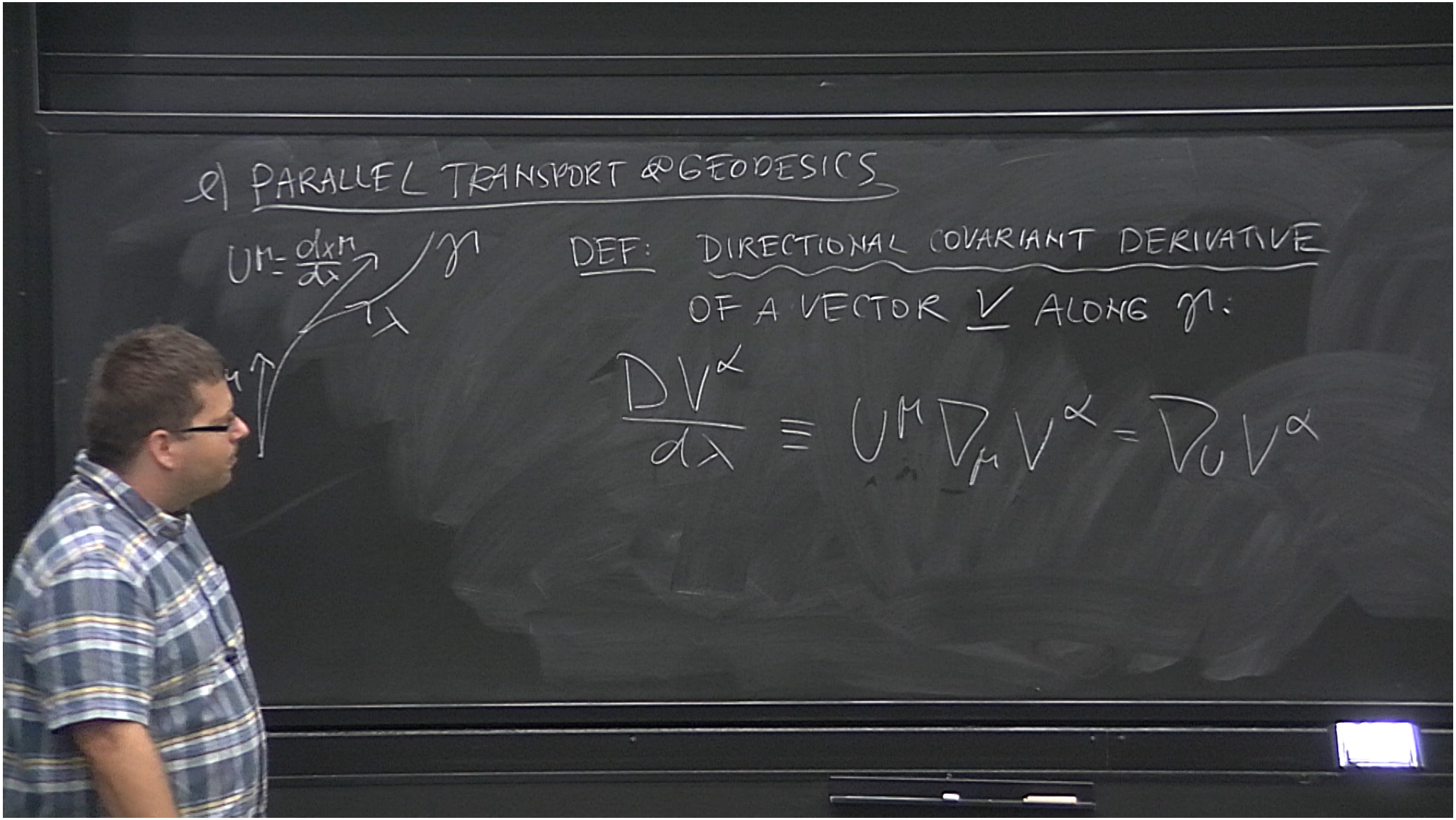

PROOF:
$[\alpha\beta\gamma\delta]\frac{\partial x^\alpha}{\partial x'^\alpha}\frac{\partial x^\beta}{\partial x'^\beta}\frac{\partial x^\gamma}{\partial x'^\gamma}\frac{\partial x^\delta}{\partial x'^\delta}=\lambda[\alpha'\beta'\gamma'\delta']$
SET $(\alpha',\beta',\gamma',\delta']=[0,1,2,3]$
$\det\left(\frac{\partial x}{\partial x'}\right)=\lambda=\sqrt{\frac{-g'}{g}}$
$+1$
$\sqrt{\frac{g'}{g}}=\left|\frac{\partial x}{\partial x'}\right|$

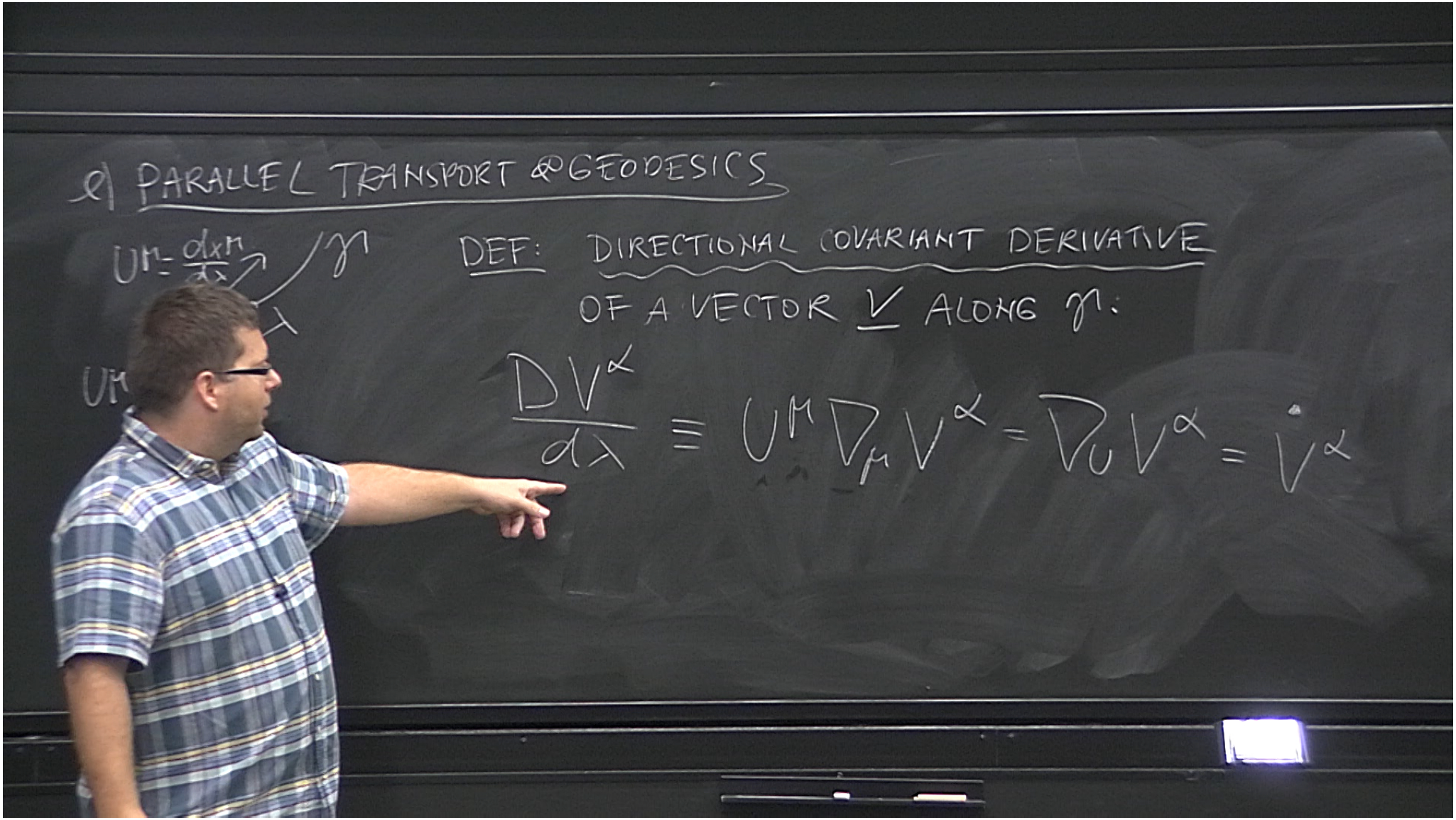

PROOF:
$\sqrt{-g}\,\varepsilon[\alpha\beta\gamma\delta]\,\frac{\partial x^\alpha}{\partial x'^{\alpha'}}\frac{\partial x^\beta}{\partial x'^{\beta'}}\frac{\partial x^\gamma}{\partial x'^{\gamma'}}\frac{\partial x^\delta}{\partial x'^{\delta'}} = \sqrt{-g}\,\varepsilon[\alpha'\beta'\gamma'\delta']$
$\varepsilon[\alpha'\beta'\gamma'\delta'] = [0,1,2,3]$
$+1$
$\det\left(\frac{\partial x}{\partial x'}\right) = \lambda = \sqrt{\frac{-g'}{g}}$
$\sqrt{\frac{g'}{g}} = \left|\frac{\partial x}{\partial x'}\right|$


PROOF:
$\sqrt{-g}\,[\alpha\beta\gamma\delta]\,\frac{\partial x^\alpha}{\partial x'^\alpha}\frac{\partial x^\beta}{\partial x'^\beta}\frac{\partial x^\gamma}{\partial x'^\gamma}\frac{\partial x^\delta}{\partial x'^\delta} = \sqrt{-g}\,[\alpha'\beta'\gamma'\delta']$
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$\sqrt{\frac{g'}{g}} = \left|\frac{\partial x}{\partial x'}\right|$
+1

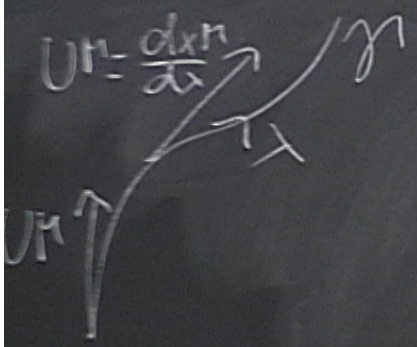

8| PARALLEL TRANSPORT & GEODESICS
$U^\mu = \frac{dx^\mu}{d\lambda}$
γ
T_λ
U^μ
DEF: DIRECTIONAL COVARIANT DERIVATIVE
OF A VECTOR V ALONG γ:
$\frac{DV^\alpha}{d\lambda}$


8) PARALLEL TRANSPORT & GEODESICS
$U^\mu = \frac{dx^\mu}{d\lambda}$
γ
γ_λ
U^μ
DEF: DIRECTIONAL COVARIANT DERIVATIVE
OF A VECTOR V ALONG γ:
$\frac{DV^\alpha}{d\lambda} \equiv U^\mu D_\mu V^\alpha$


8| PARALLEL TRANSPORT & GEODESICS
$U^\mu = \frac{dx^\mu}{d\lambda}$
γ
DEF: DIRECTIONAL COVARIANT DERIVATIVE
OF A VECTOR V ALONG γ:
$\frac{DV^\alpha}{d\lambda} \equiv U^\mu \nabla_\mu V^\alpha = \nabla_U V^\alpha$


e) PARALLEL TRANSPORT & GEODESICS
$U^\mu = \frac{dx^\mu}{d\lambda}$
γ
DEF: DIRECTIONAL COVARIANT DERIVATIVE
OF A VECTOR V ALONG γ:
$\frac{DV^\alpha}{d\lambda} \equiv U^\mu \nabla_\mu V^\alpha = \nabla_U V^\alpha = \dot{V}^\alpha$

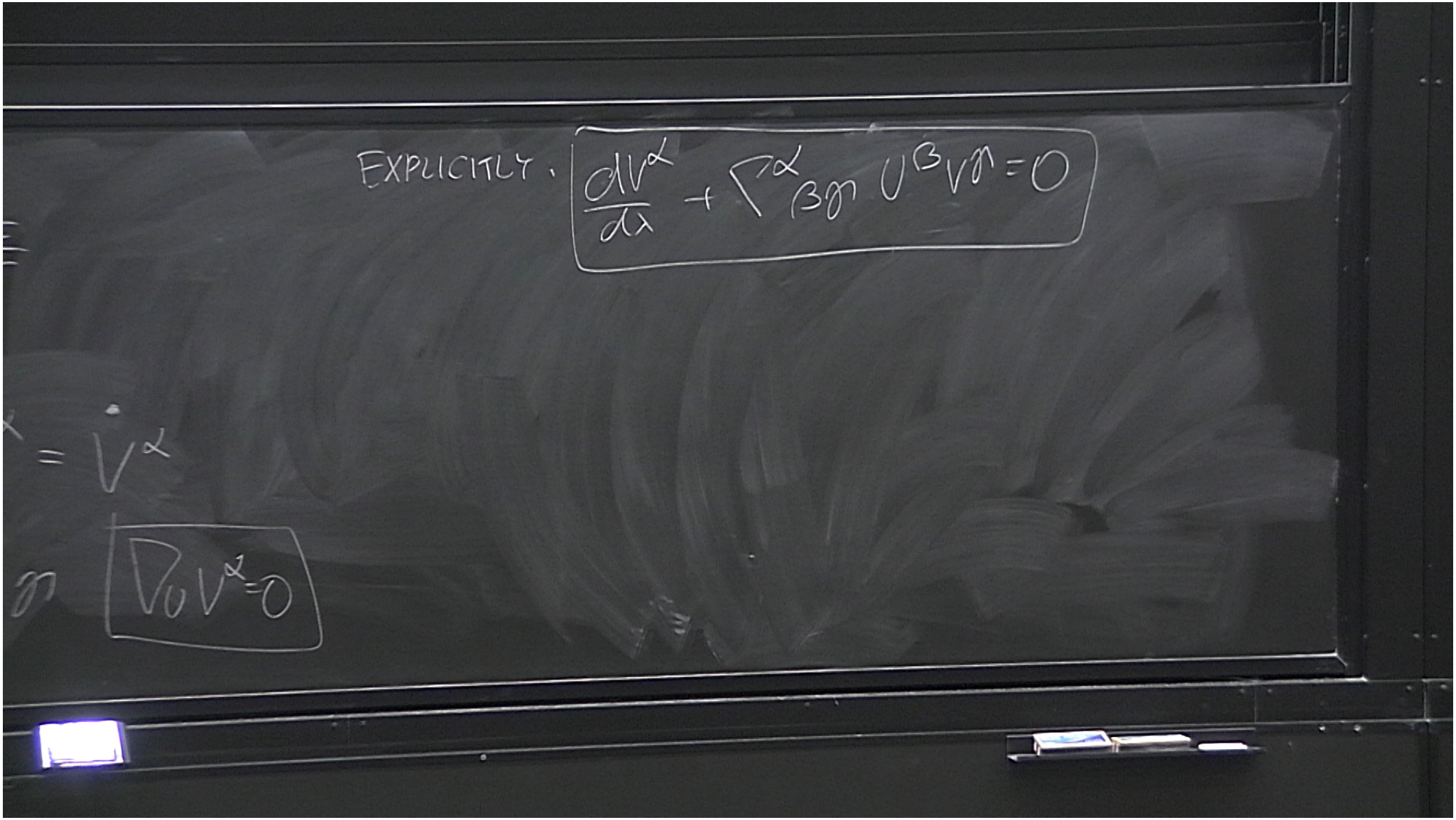
PARALLEL TRANSPORT & GEODESICS

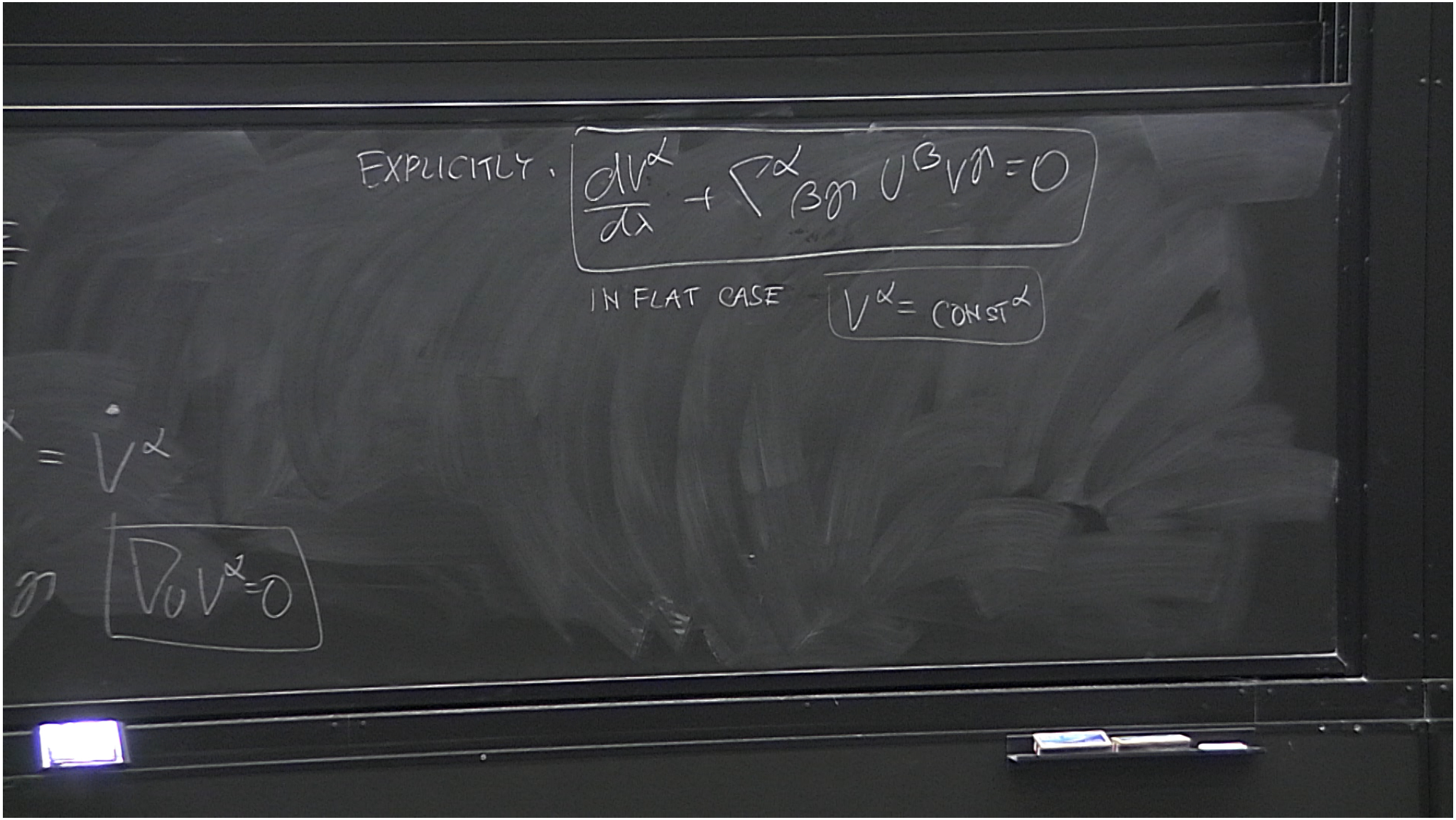


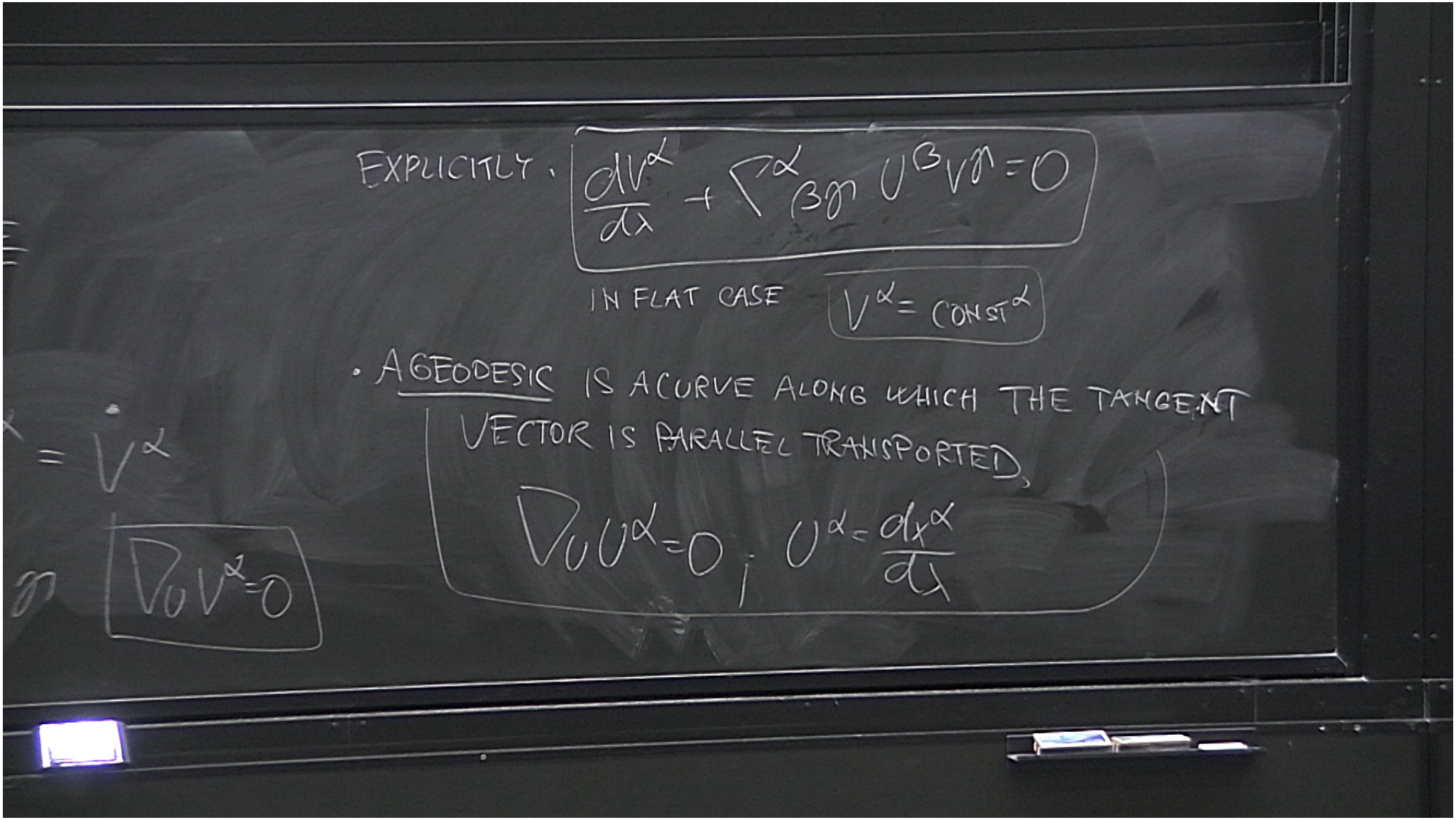
DEF: DIRECTIONAL COVARIANT DERIVATIVE OF A VECTOR V ALONG γ:

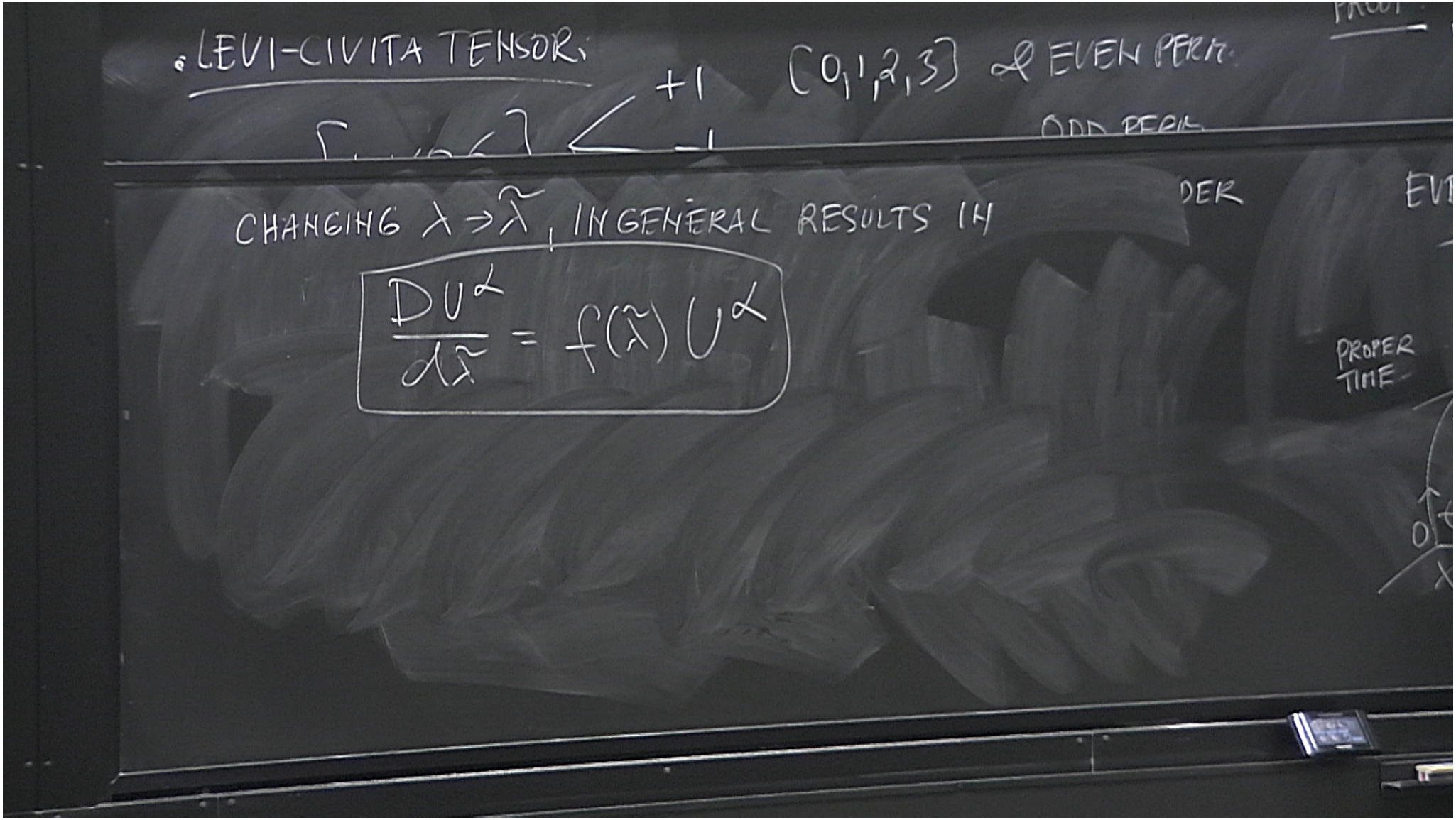
$$\frac{DV^\alpha}{d\lambda} \equiv U^\mu \nabla_\mu V^\alpha = \nabla_U V^\alpha = \dot{V}^\alpha$$

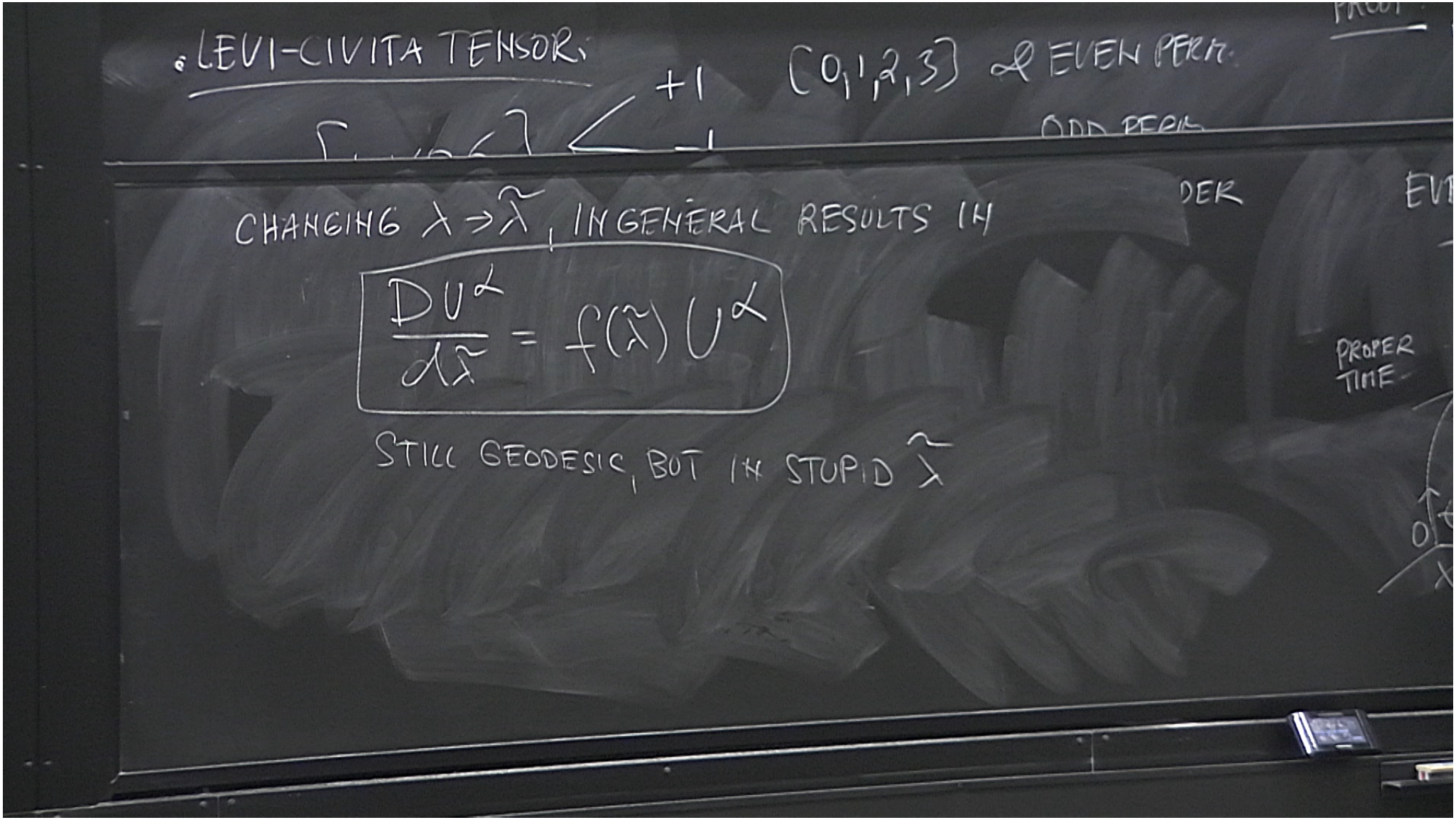
- VECTOR IS PARALLEL TRANSPORTED ALONG γ $\boxed{\nabla_U V^\alpha = 0}$

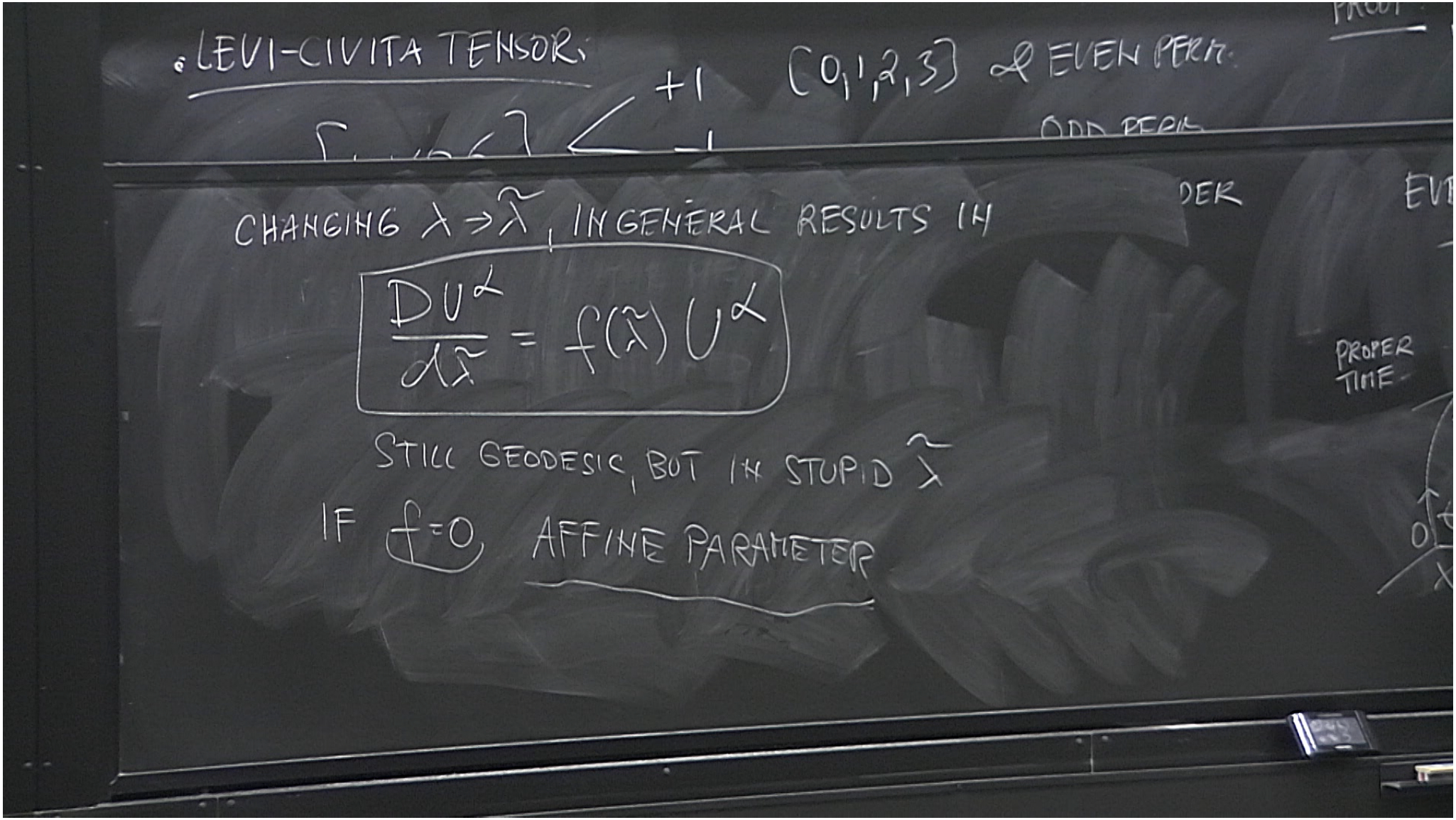

EXPLICITLY: $\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} U^\beta V^\gamma = 0$
$= \dot{V}^\alpha$
$\nabla_U V^\alpha = 0$


EXPLICITLY .
$\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} U^\beta V^\gamma = 0$
IN FLAT CASE
$V^\alpha = \text{CONST}^\alpha$
$= \dot{V}^\alpha$
$\nabla_U V^\alpha = 0$


EXPLICITLY.
$\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma} U^\beta V^\gamma = 0$
IN FLAT CASE
$V^\alpha = \text{CONST}^\alpha$
• A GEODESIC IS A CURVE ALONG WHICH THE TANGENT VECTOR IS PARALLEL TRANSPORTED.
$\nabla_U U^\alpha = 0 \; ; \; U^\alpha = \frac{dx^\alpha}{d\lambda}$
$= \dot{V}^\alpha$
$\nabla_U V^\alpha = 0$


• LEVI-CIVITA TENSOR:
+1 (0,1,2,3) & EVEN PERM.
CHANGING $\lambda \to \tilde{\lambda}$, IN GENERAL RESULTS IN
$\frac{DU^\alpha}{d\tilde{\tau}} = f(\tilde{\lambda}) U^\alpha$
PROPER TIME


• LEVI-CIVITA TENSOR:
+1 [0,1,2,3] & EVEN PERM.
-1
ODD PERM.
CHANGING $\lambda \to \tilde{\lambda}$, IN GENERAL RESULTS IN
$\frac{DU^\alpha}{d\tilde{\lambda}} = f(\tilde{\lambda}) U^\alpha$
STILL GEODESIC, BUT IN STUPID $\tilde{\lambda}$
PROPER TIME


• LEVI-CIVITA TENSOR:
+1
[0,1,2,3] & EVEN PERM.
ODD PERM.
PROOF
CHANGING $\lambda \Rightarrow \tilde{\lambda}$, IN GENERAL RESULTS IN
$\frac{DU^\alpha}{d\tilde{\lambda}} = f(\tilde{\lambda}) U^\alpha$
STILL GEODESIC, BUT IN STUPID $\tilde{\lambda}$
IF $f=0$ AFFINE PARAMETER
PROPER TIME