Title: Single-shot interpretations of von Neumann entropy

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Abstract: <p>In quanum physics, the von Neumann entropy usually arises in i.i.d settings, while single-shot settings are commonly characterized by smoothed min- or max-entropies. In this talk, I will discuss new results that give single-shot interpretations to the von Neumann entropy under appropriate conditions. I first present new results that give a single-shot interpretation to the Area Law of entanglement entropy in many-body physics in terms of compression of quantum information on the boundary of a region of space. Then I show that the von Neumann entropy governs single-shot transitions whenever one has access to arbitrary auxiliary systems, which have to remain invariant in a state-transition ("catalysts"), as well as a decohering environment. Getting rid of the decohering environment yields the "catalytic entropy conjecture", for which I present some supporting arguments.</p>

<p>If time permits, I also discuss some applications of these result to thermodynamics and speculate about consequences for quantum information theory and holography.</p>

Single-shot interpretations of von Neumann entropy

Henrik Wilming, ETH Zurich
Joint work with Paul Boes, Jens Eisert, Rodrigo Gallego, Markus Müller

Perimeter Institute, Waterloo, August 29th 2018

General wisdom in quantum information theory

Asymptotic *iid* setting:

$$\rho^{\otimes n} \longrightarrow \sigma^{\otimes m}$$

Operational tasks controlled by **von Neumann entropy** (or relatives like relative entropy, mutual information etc.)

Single-shot setting:

$$\rho \longrightarrow \sigma$$

(these may be high-dimensional and consist of many correlated subsystems)

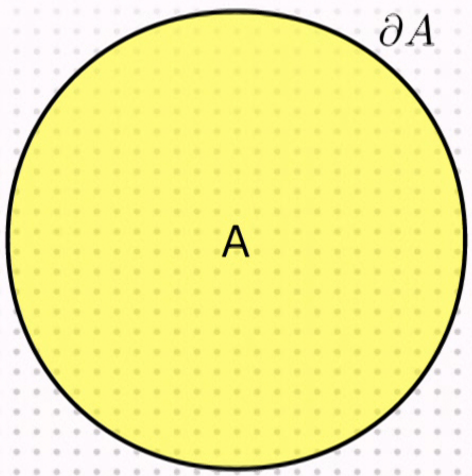
Tasks controlled by **(smooth) Rényi entropies** (or their relatives)

Is the *iid* limit really necessary or can we give single-shot interpretations to standard entropic quantities?

Motivation: Area Laws in many-body physics

Area Law conjecture:
Consider a pure groundstate of a **gapped, local**, lattice Hamiltonian. There exists a constant k>0 such that the **entanglement entropy** of any region A is bounded in terms of the **surface area** of A:

$$S(A) \leq k|\partial A|.$$



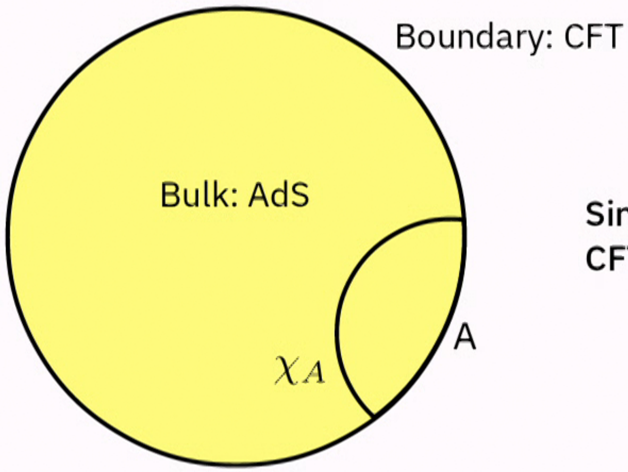
What does the Area Law operationally mean for a single system?

Motivation: AdS/CFT

Geometric quantities in the bulk can be expressed as standard entropic quantities on the boundary. Example:

$$S(A) = \frac{\text{Area}(\chi_A)}{4G_N} + \ldots$$

Ryu, Takayanagi (2006)



Single-shot interpretation on the CFT side?

Talk today

1. Single-shot interpretation of Area Law in many-body physics

   Joint work with Jens Eisert, soon on arXiv.

2. Single-shot interpretation of von Neumann entropy using "catalysts": The catalytic entropy conjecture

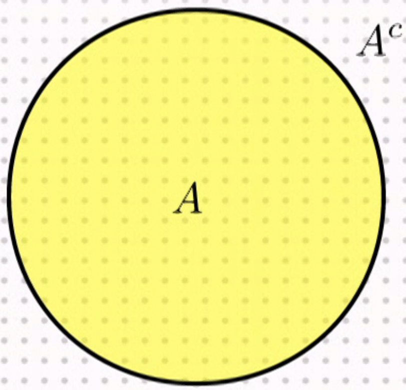
   Joint work with Paul Boes, Jens Eisert, Rodrigo Gallego, Markus Müller
   ArXiv:1807.08773

A few results, a conjecture and more open problems.

Area Law: Conventions

Consider **pure states** on any **regular lattice** of finite dimensional Hilbert-spaces

$$\mathcal{H} = \bigotimes_{x \in \Lambda} \mathcal{H}_x, \qquad \dim(\mathcal{H}_x) = d, \qquad \log(d) := 1$$



Surface area of A: $|\partial A|$

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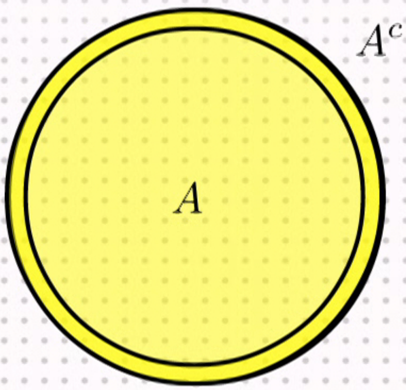

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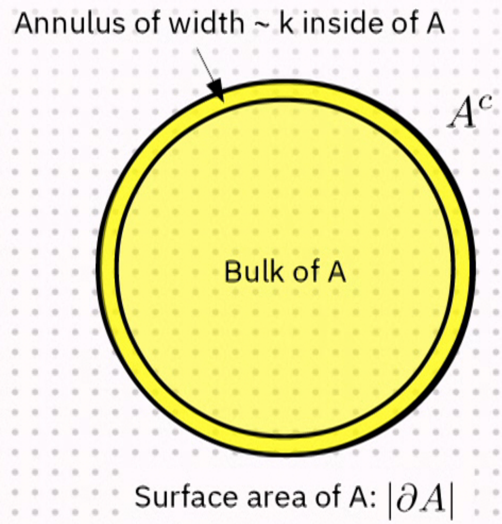

For large, smooth regions A:

The annulus of width k contains roughly $k|\partial A|$ sites and has Hilbert-space dimension $\sim d^{k|\partial A|}$.

Area Law

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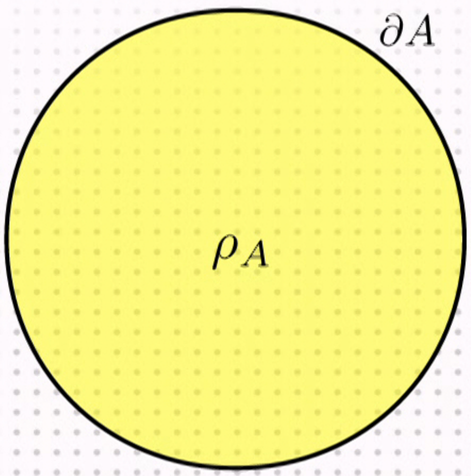

Important: The statement is non-trivial only for large, smooth regions (scaling behaviour).

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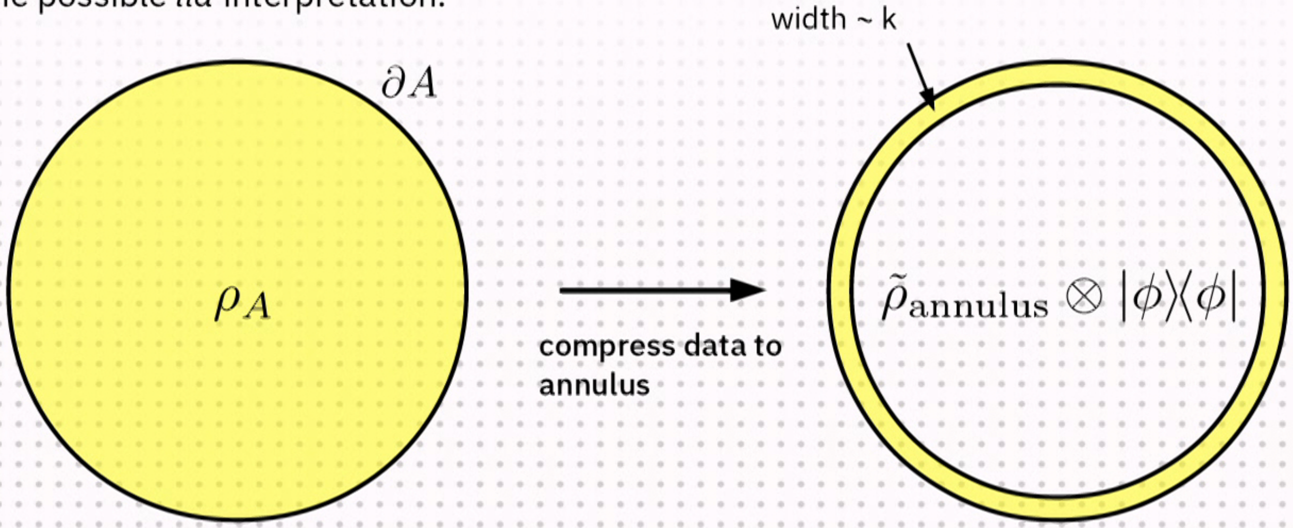
One possible *iid* interpretation:

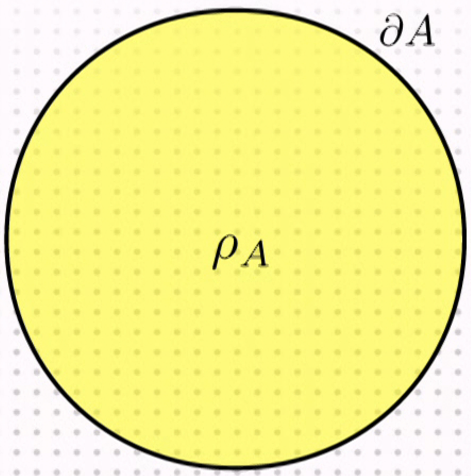


Area Law: Main result

Theorem (Holographic compression, informal)
Consider a pure state $|\Psi\rangle$ on a lattice fulfilling an area law. Then for any $\varepsilon > 0$ and any region *A* there exists a unitary supported in *A* such that

$$U_A|\Psi\rangle \approx_\epsilon |\chi\rangle_{A^c \cup \mathrm{annulus}} \otimes |\phi\rangle_{\mathrm{bulk}(A)},$$

where the annulus has width $\sim k/\varepsilon$ and the error is measured in fidelity.



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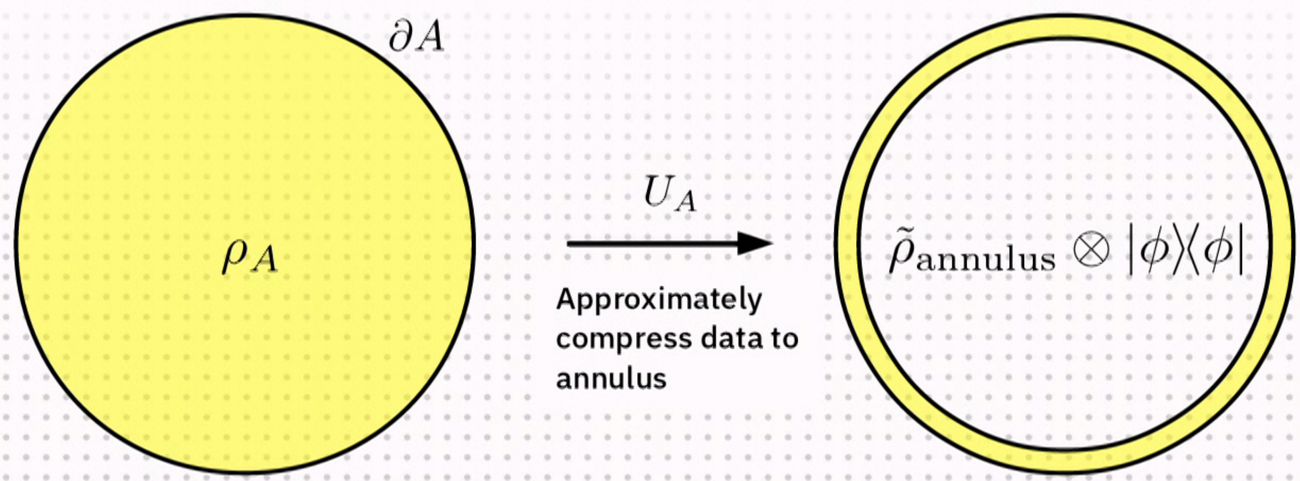
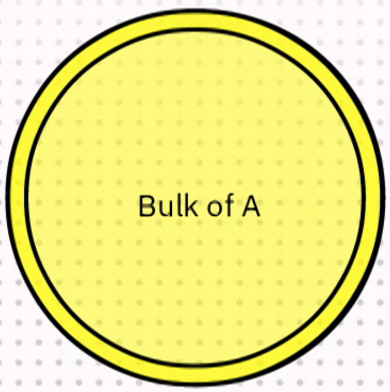


Area Law: A simple fact

Schmidt-decomposition:

$$|\psi\rangle = \sum_{j=1}^{r} \sqrt{p_j} |j\rangle_A \otimes |j\rangle_{A^c}, \qquad p_1 \geq p_2 \geq p_3 \geq \cdots, \quad P_M := \sum_{j=1}^{M} |j\rangle_A \langle j|$$



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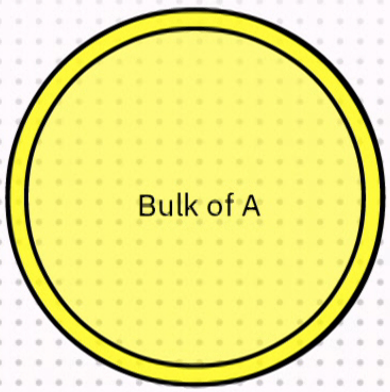

If Schmidt-rank equal to Hilbert-space dimension of annulus, there exists a unitary on A such that:

$$U_A |j\rangle_A = |\tilde{j}\rangle_{\text{annulus}} \otimes |\phi\rangle, \quad j = 1, \ldots, r$$

$$U_A P_r U_A^\dagger = \tilde{P}_r \otimes |\phi\rangle\langle\phi|$$

$$U_A \rho_A U_A^\dagger = \tilde{\rho}_{\text{annulus}} \otimes |\phi\rangle\langle\phi|$$

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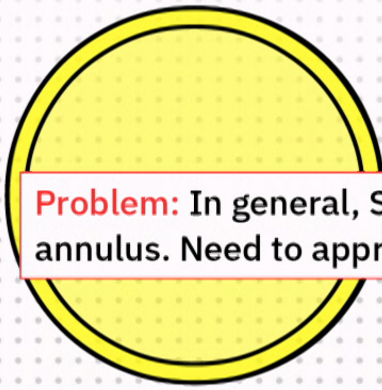

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Problem: In general, Schmidt-rank is maximal and state does not “fit” on annulus. Need to approximate the state by one with small Schmidt-rank.

$$U_A \rho_A U_A^\dagger = \tilde{\rho}_{\text{annulus}} \otimes |\phi\rangle\langle\phi|$$

Area Law: The basic problem to solve

Need to approximate the state by one with small Schmidt-rank, but the only information we have is a bound on the entropy.

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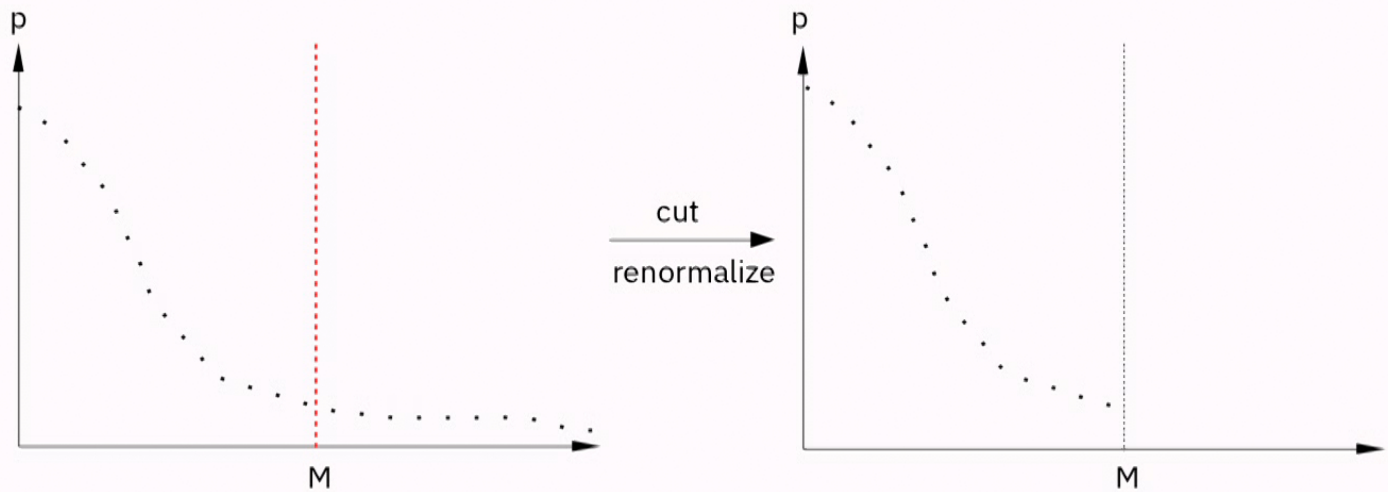

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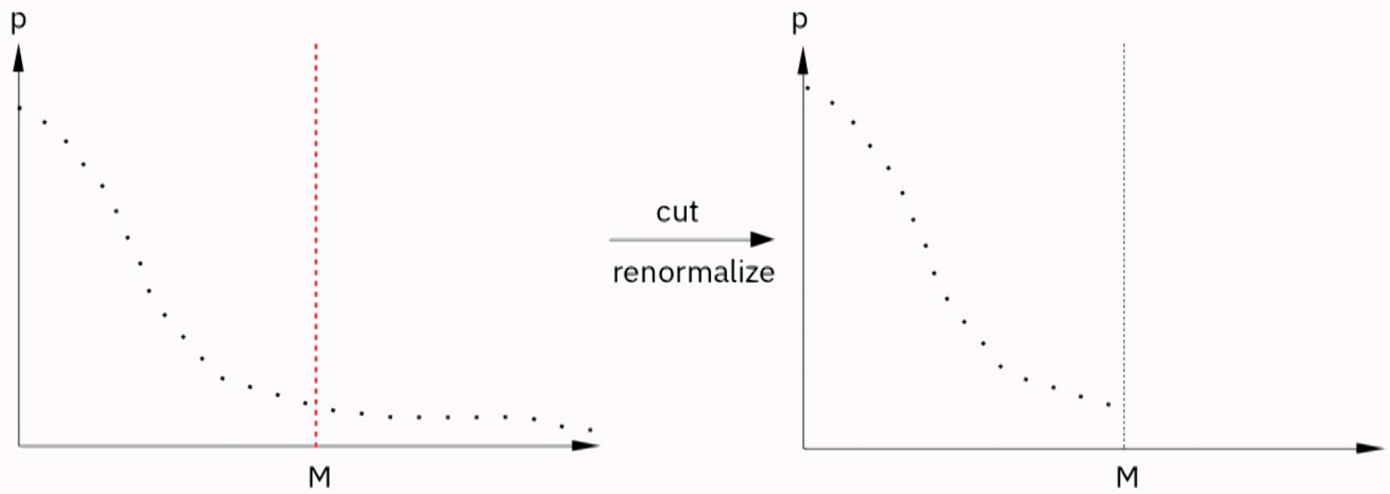

If we cut a probability distribution after the M's largest entry, how big is the error?

Area Law: The basic problem to solve

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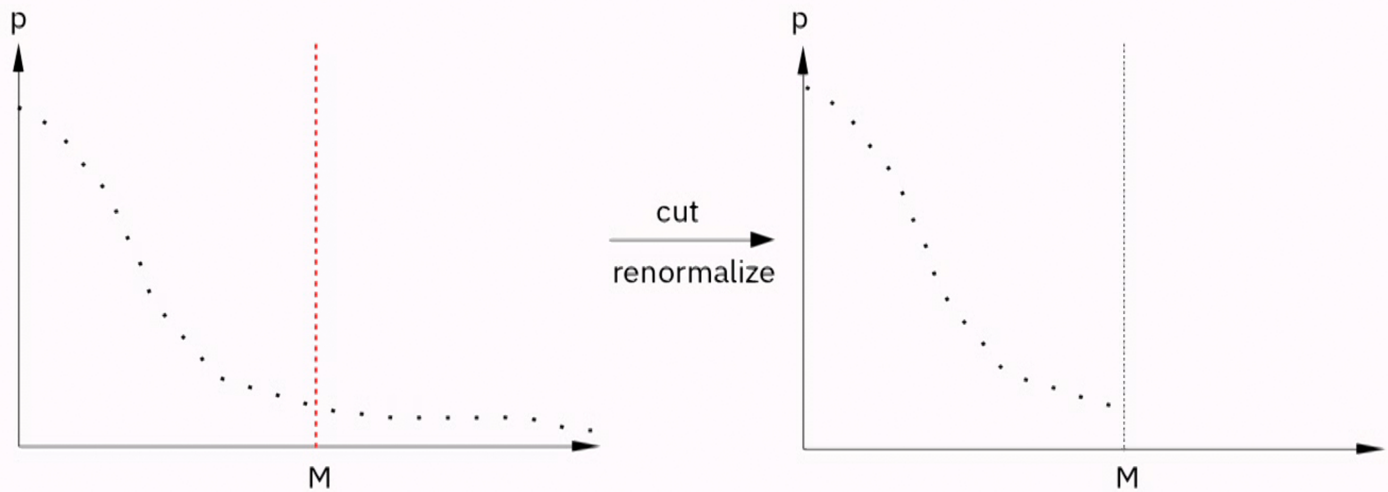

If we cut a probability distribution after the M's largest entry, how big is the error?

Intuition: if distribution has small entropy, most weight is carried by few entries.

Area Law: A simple Lemma

Lemma (Approximating distributions, simple version)
Let $\mathbf{p}$ be any probability distribution, ordered non-increasingly. Then

$$\sum_{j=1}^{M} p_j \geq 1 - \frac{S(\mathbf{p})}{\log(M)}$$

- Due to monotonicity of Rényi entropies, it also holds for all Rényi entropies with **α** < 1.
- Does **not** hold for Rényi entropies with **α** > 1 (counter-example).
- For smooth max-entropy it implies $\varepsilon S_0^{\varepsilon} \leq S$

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Corollary (Single-shot compression)
Let $|\psi\rangle$ be any bi-partite pure state on AA^c. Consider the normalized state

$$|\psi_M\rangle := \frac{P_M \otimes \mathbf{1}}{\sqrt{\sum_{j=1}^{M} p_j}} |\psi\rangle \propto \sum_{j=1}^{M} \sqrt{p_j} |j\rangle_A \otimes |j\rangle_{A^c}$$

Then:

$$|\langle\psi|\psi_M\rangle|^2 = \sum_{j=1}^{M} p_j \geq 1 - \frac{S(A)}{\log(M)}$$

Area Law: Proof of theorem

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4. From Corollary we learn:

$$|\langle\psi_M|U_A^\dagger U_A|\psi\rangle|^2 = |\langle\psi_M|\psi\rangle|^2$$
$$\geq 1 - \frac{S(A)}{\log(M)} = 1 - \varepsilon$$

■

Area Law

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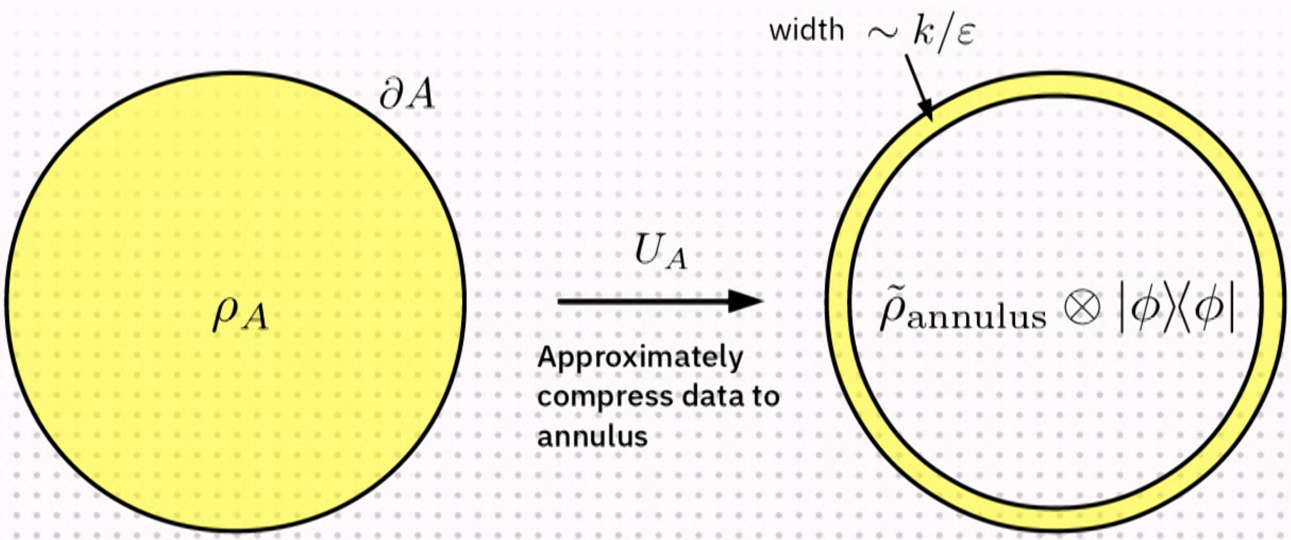
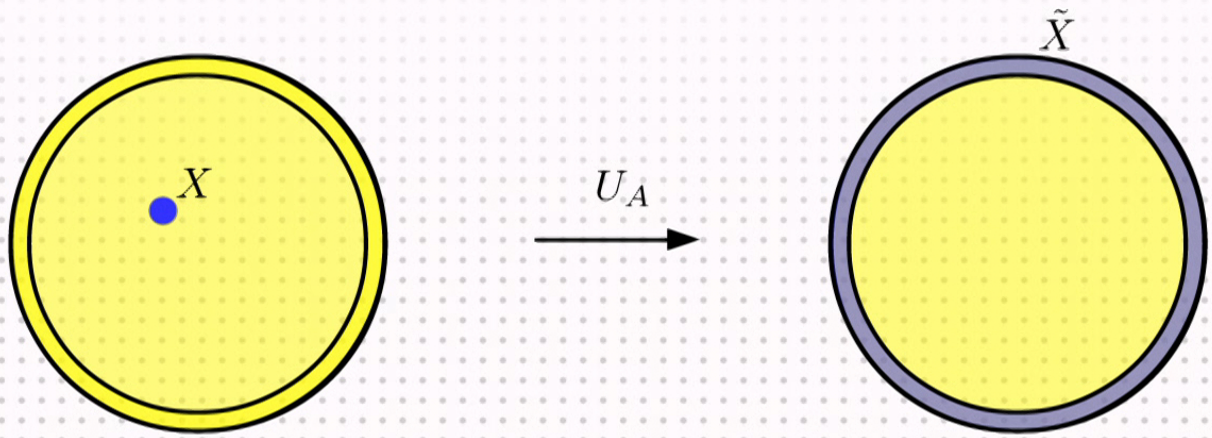


Area Law: Operator correspondence

All operators supported completely within P_M are mapped exactly to boundary:

$$X = P_M X P_M \longrightarrow U_A X \otimes \mathbf{1}_{A^c} U_A = \tilde{X} \otimes |\phi\rangle\langle\phi| \otimes \mathbf{1}_{A^c}$$

$$\mathrm{Tr}(\tilde{X}\tilde{\rho}_{\mathrm{annulus}}) \approx \langle\psi| X \otimes \mathbf{1}_{A^c} |\psi\rangle$$



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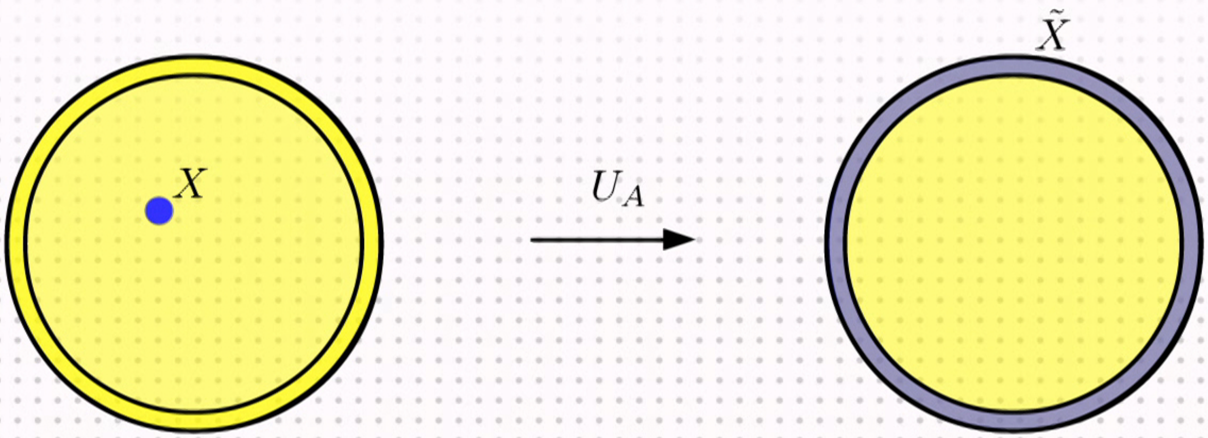
Open question: Do local operators (approximately) fulfill this condition for groundstates of gapped, local Hamiltonians?



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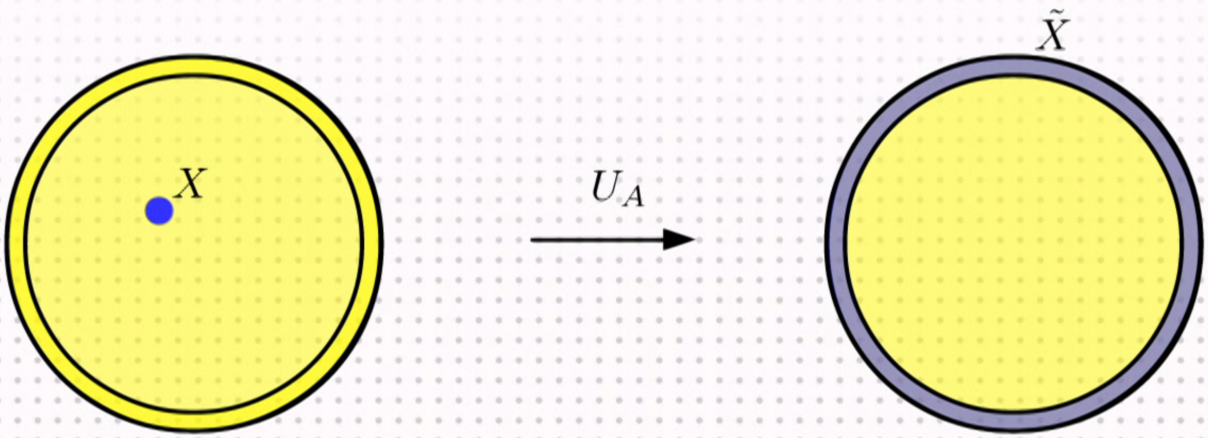
Open question: How much can the mapping preserve locality and what's the complexity of the unitary?



Interpretation as “rotated” quantum markov chains

Quantum markov chain:
Tripartite state on *ABC* s.t. there exsists a channel $\mathcal{R}_B^{AB}$ on *B* such that:

$$\rho_{ABC} = \mathcal{R}_B^{AB}(\rho_{BC})$$

State is quantum markov chain iff conditional mutual information vanishes:

$$I(A : C|B) = 0$$

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"Rotated quantum markov chain":
Tripartite state on *ABC* for which there exsists a unitary on *AB* such that $U_{AB}\rho_{ABC}U_{AB}^{\dagger}$ is a quantum markov chain.

Our result: States fulfilling area law are approximate rotated quantum markov chains.

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Our result: States fulfilling area law are approximate rotated quantum markov chains.

Open problem: Is there a simple criterion that determines whether a state is a rotated quantum markov chain?

Area Law: Summary

- Area Law in terms of von Neumann entropy implies approximate holographic compression.
  (Not possible to show from Rényi entropies with **α** > 1)

- For any **ε**>0, groundstate defines sub-algebra of operators that are mapped to annulus of width ~1/**ε**.
  Open question: Are local operators in this algebra for physical models?

- **Open question:** How much can U_A preserve locality (gauge freedom)?

- **Open question:** Can we say anything about the complexity of U_A?

- Results follow from simple inequality:

Lemma (Approximating distributions, simple version)
Let $\mathbf{p}$ be any probability distribution, ordered non-increasingly. Then

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2. Catalytic entropy conjecture

Joint work with Paul Boes, Jens Eisert, Rodrigo Gallego, Markus Müller
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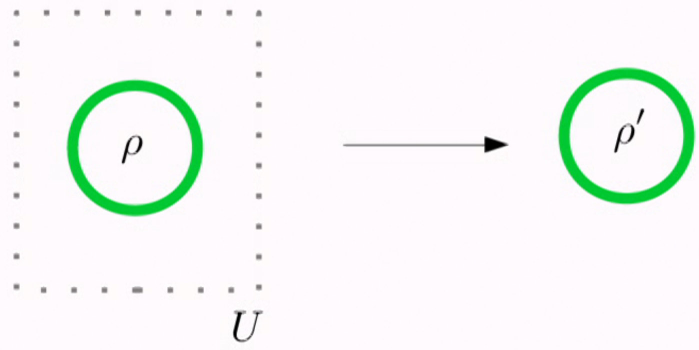
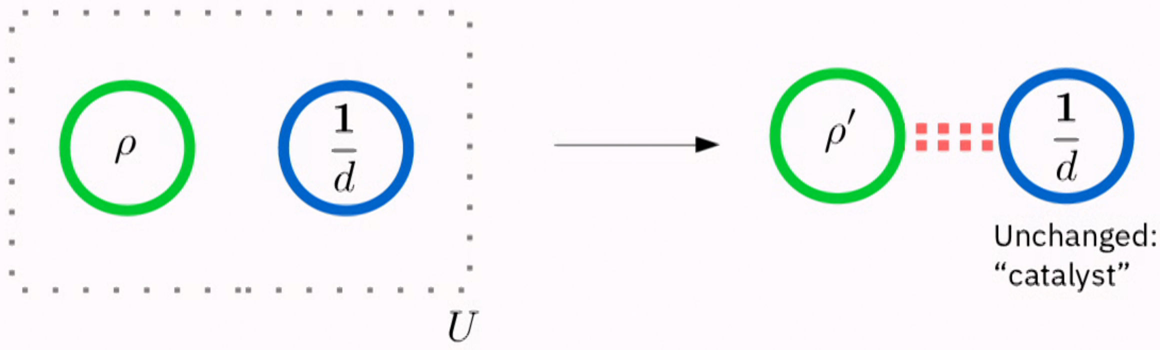
A series of questions



Q: Which states can be reached?

A series of questions



d: larger or equal to dimension of ρ

Q: Which states can be reached by choosing different unitaries?

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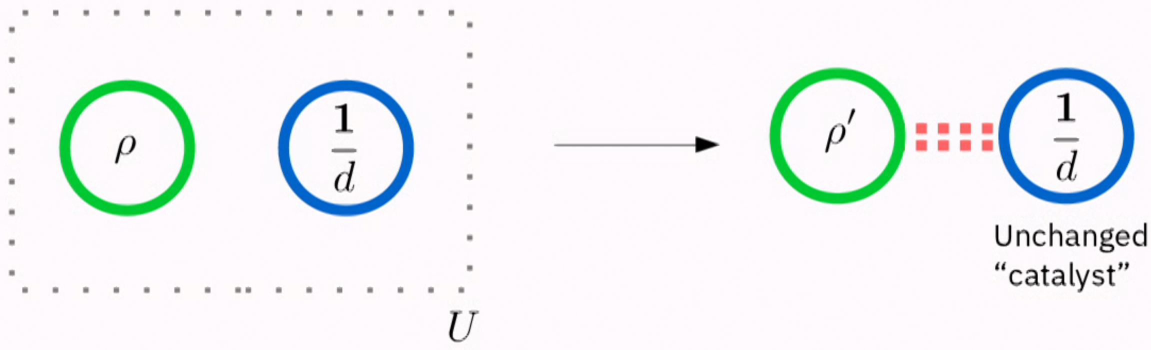

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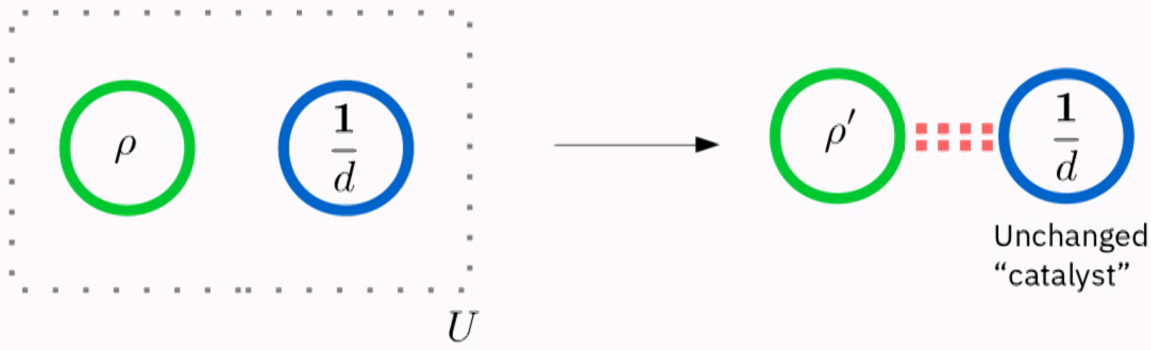
Q: Which states can be reached by choosing different unitaries?

A: All states that are **majorized** by ρ *.

* Suffices that d is larger than $\sqrt{\dim(\rho)}$. See P. Boes, H.W., R. Gallego, J. Eisert, arXiv:1804.03027

A series of questions



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→ **Rényi-entropies can only increase!**

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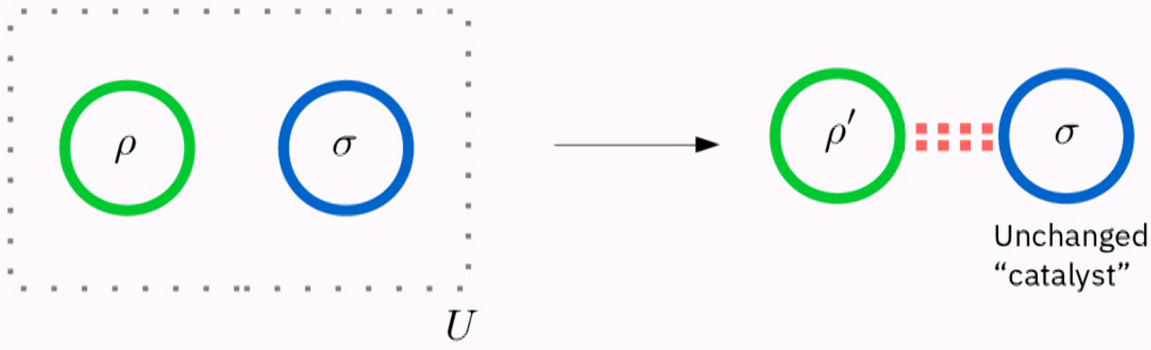
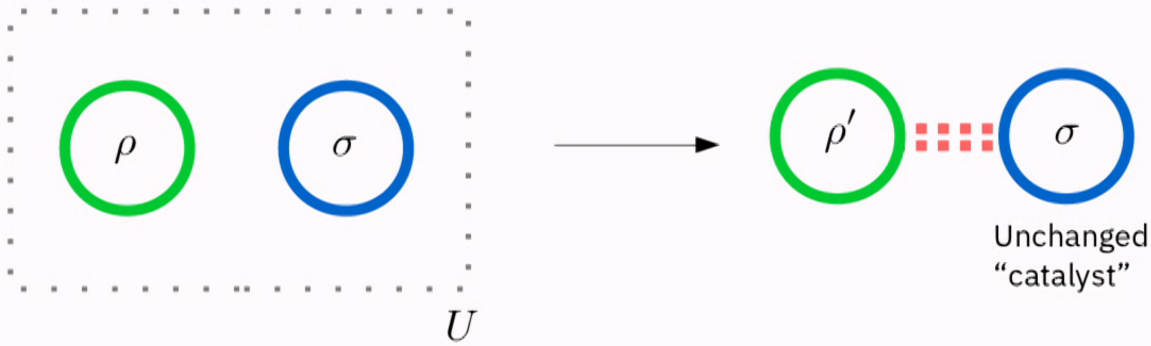
Catalytic transitions



Q: Which states can be reached if we can choose catalyst and unitary?

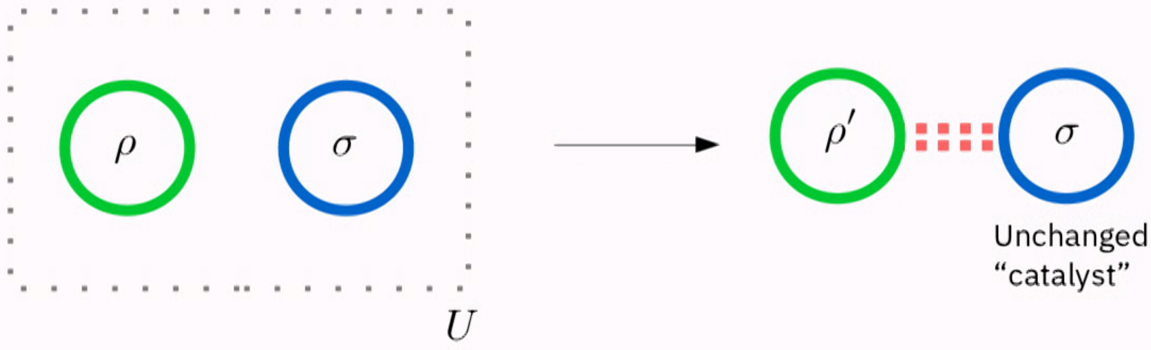
Catalytic transitions: Classical example



q_2p_3	q_1p_3	p_3
q_2p_2	q_1p_2	p_2
q_2p_1	q_1p_1	p_1
q_2	q_1	

Step 1: Write bipartite distribution as table.

Catalytic transitions: Classical example



q_2p_3	q_1p_3	p_3
q_2p_2	q_1p_2	p_2
q_2p_1	q_1p_1	p_1
q_2	q_1	

Example:
$p=(1/2,1/2,0)$
→

0	0	0
$q_2/2$	$q_1/2$	1/2
$q_2/2$	$q_1/2$	1/2
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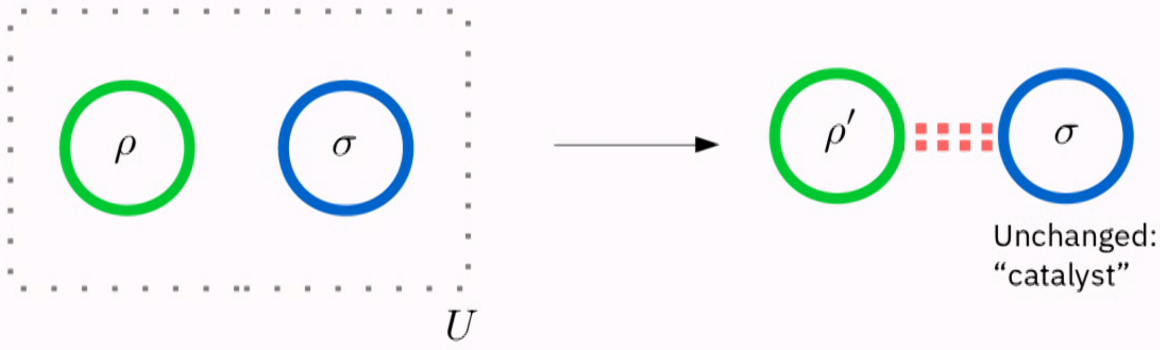

0	0	0
$q_2/2$	$q_1/2$	1/2
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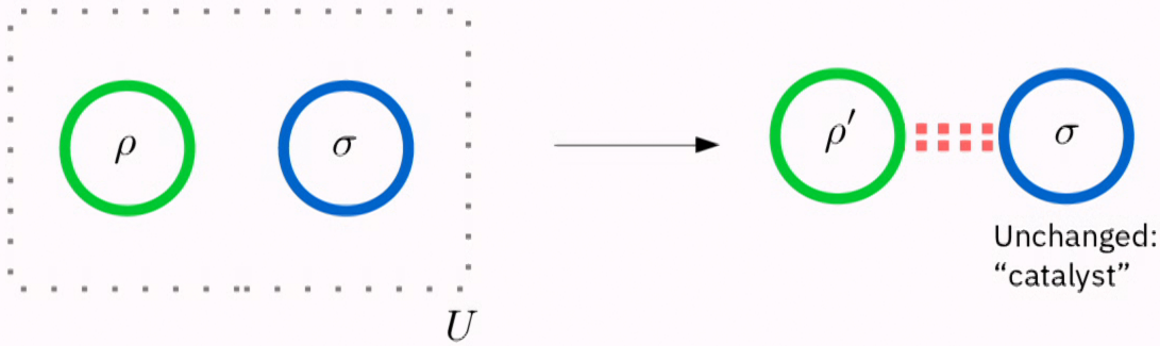
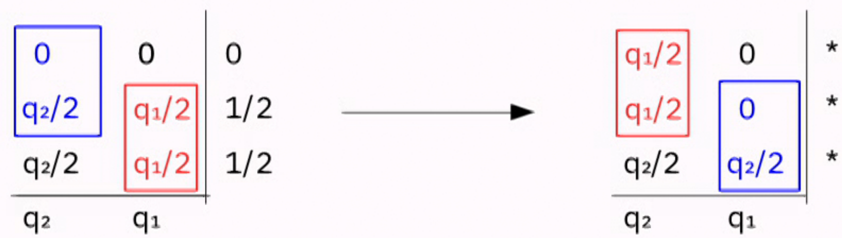
Step 2: Choose a permutation on the joint-distribution.

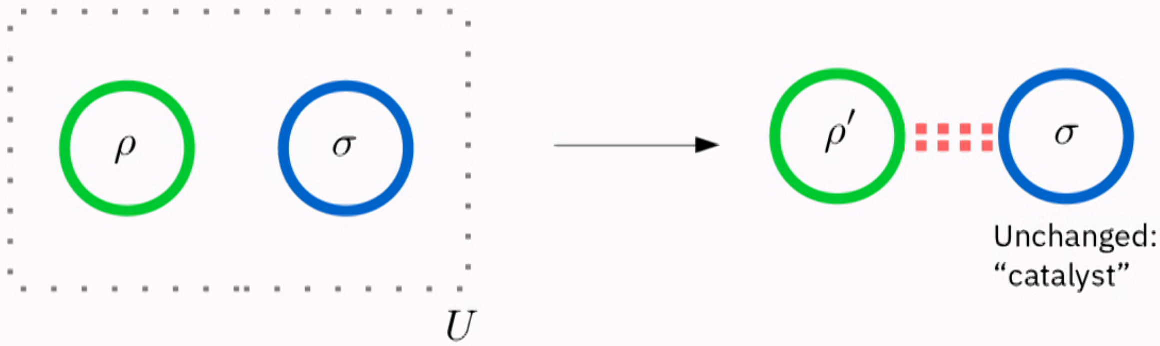
Catalytic transitions: Classical example





Step 3: Solve equations to ensure "catalyticity".

Catalytic transitions: Classical example



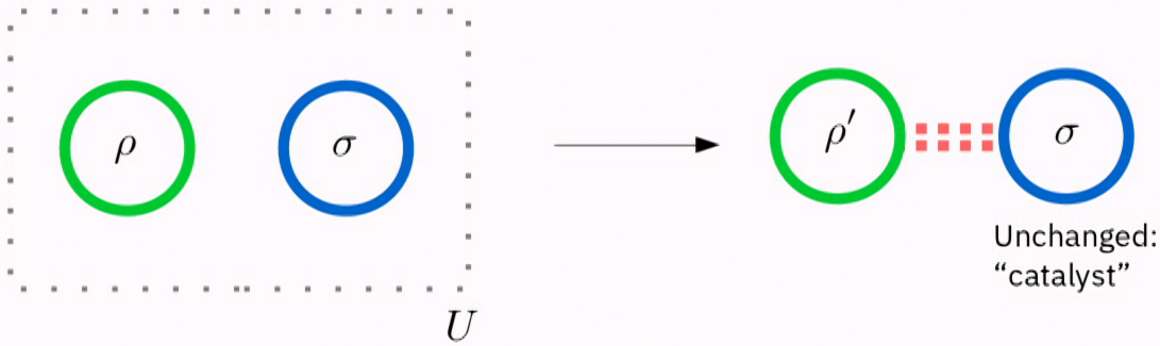
0	0	0
2/6	1/6	1/2
2/6	1/6	1/2
2/3	1/3	

→

1/6	0	1/6
1/6	0	1/6
2/6	2/6	2/3
2/3	1/3	

Step 3: Solve equations to ensure “catalyticity”.

Catalytic transitions: Classical example



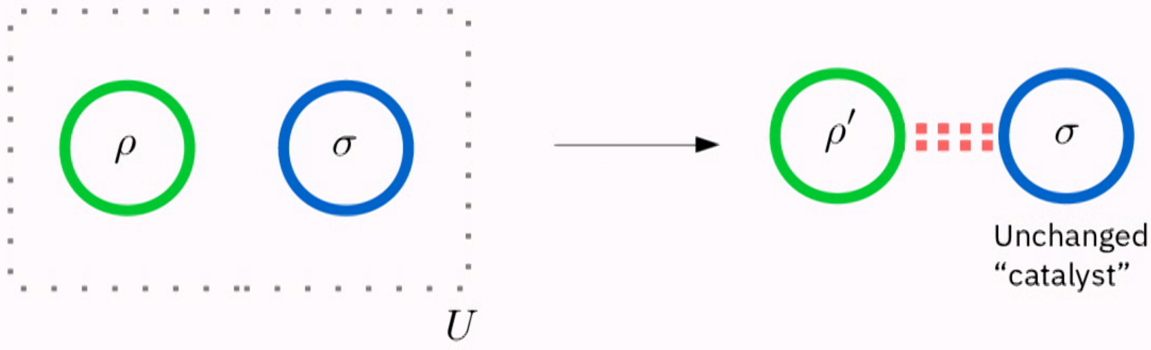
0	0	0
2/6	1/6	1/2
2/6	1/6	1/2
2/3	1/3	

→

1/6	0	1/6
1/6	0	1/6
2/6	2/6	2/3
2/3	1/3	

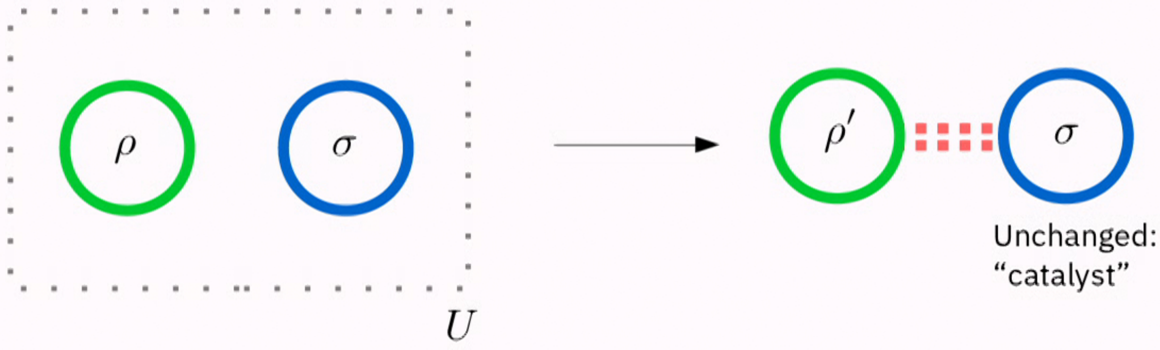
Largest probability increased from 1/2 to 2/3.
Thus **min-entropy decreased.**

Catalytic transitions



Q: Which states can be reached if we can choose catalyst and unitary?

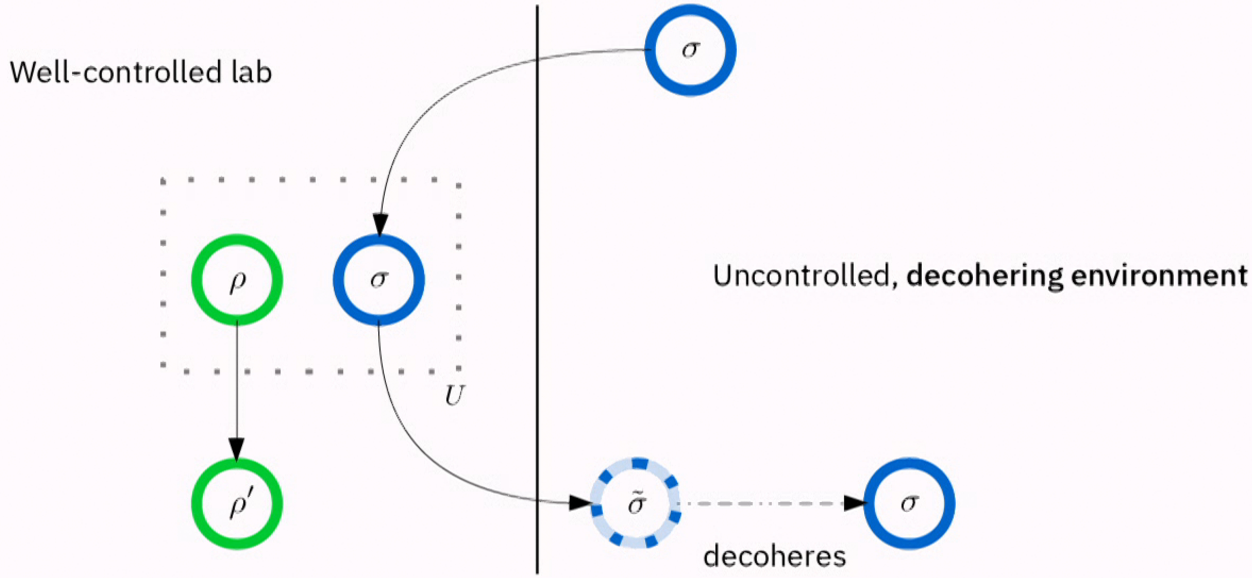
Catalytic transitions



Q: Which states can be reached if we can choose catalyst and unitary?

A: **??? (but we have a conjecture)**

A more general setting



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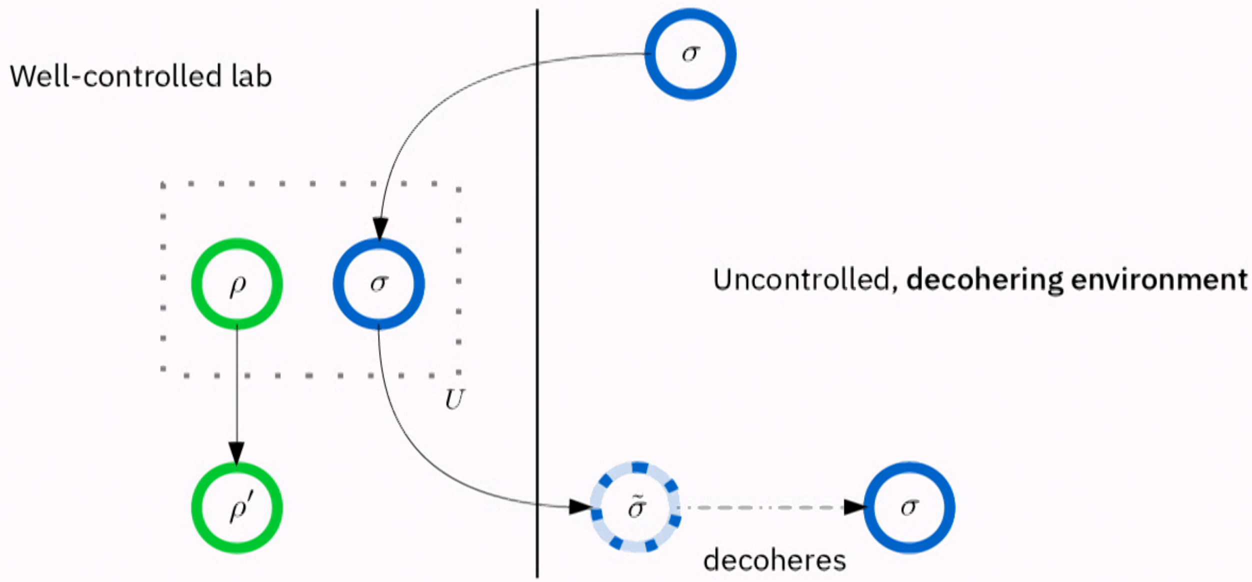

$\rho \xrightarrow{\text{dec.}} \rho'$ $:\Leftrightarrow$ Can find corresponding unitary and catalyst

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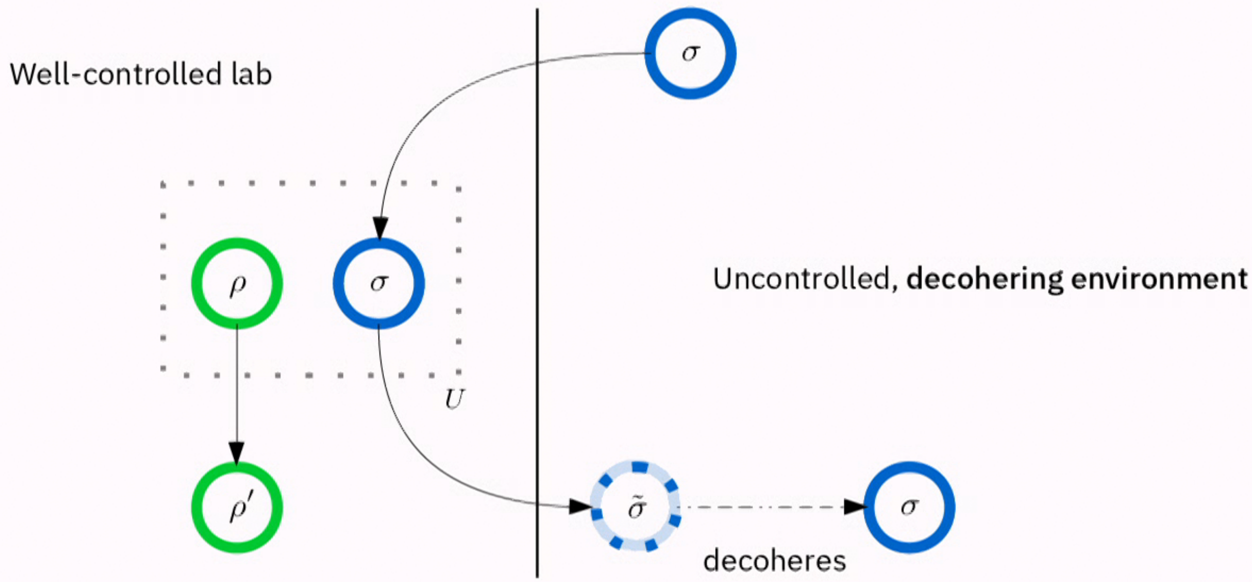


$$\rho \xrightarrow{\text{dec.}} \rho' \quad :\Leftrightarrow$$

Exists unitary and catalyst such that:

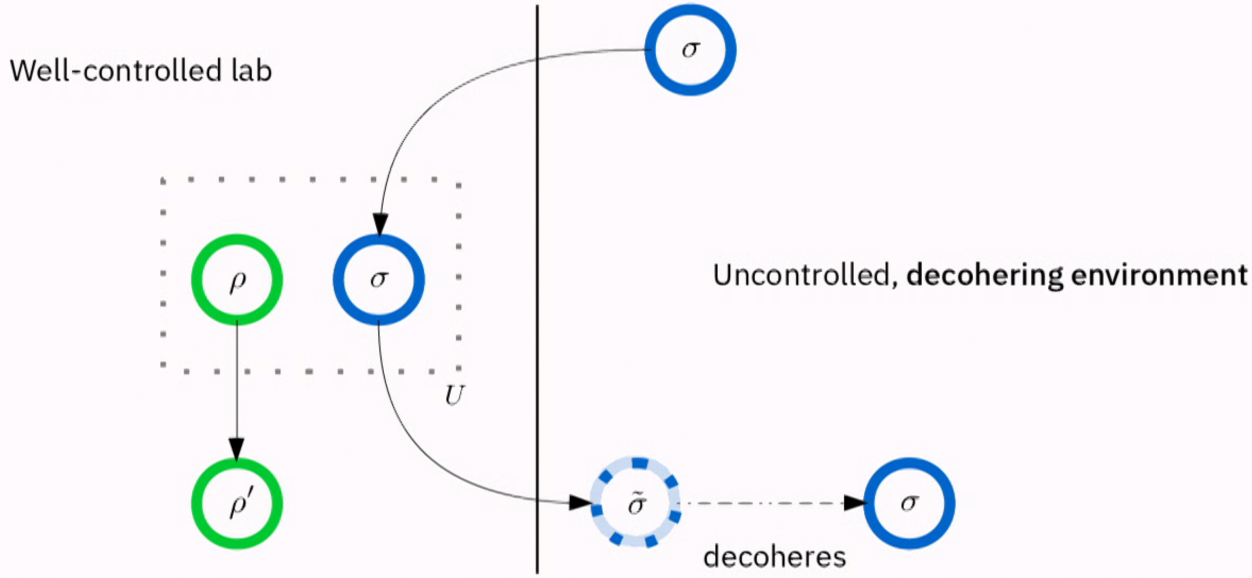
$$\mathrm{Tr}_1\left(U\rho\otimes\sigma U^\dagger\right) = \rho'$$

$$\mathcal{D}\left[\mathrm{Tr}_2\left(U\rho\otimes\sigma U^\dagger\right)\right] = \sigma$$

$\mathcal{D}$: Decoherence in fixed basis

Q: Which states can be reached?

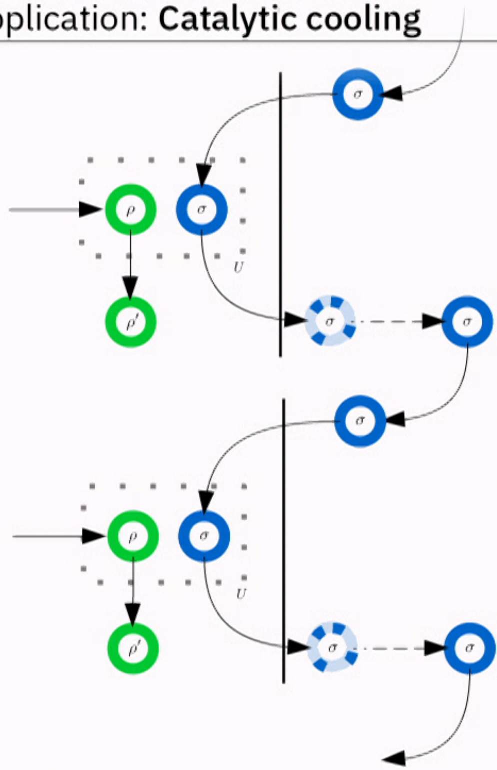
A characterization of von Neumann entropy without iid limit



Theorem
Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then:

$$\rho \xrightarrow{\text{dec.}} \rho' \quad \Leftrightarrow \quad S(\rho') > S(\rho) \ \text{ and } \ \mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$$

Application: **Catalytic cooling**



Can use a **single** catalyst to transform arbitrary many copies. Each undergoes transition

$$\rho \longrightarrow \rho'$$

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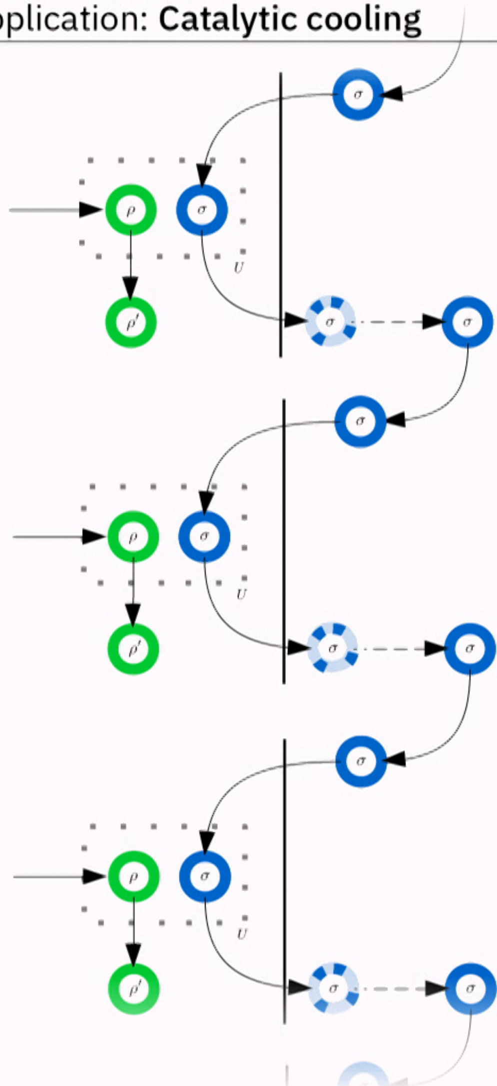

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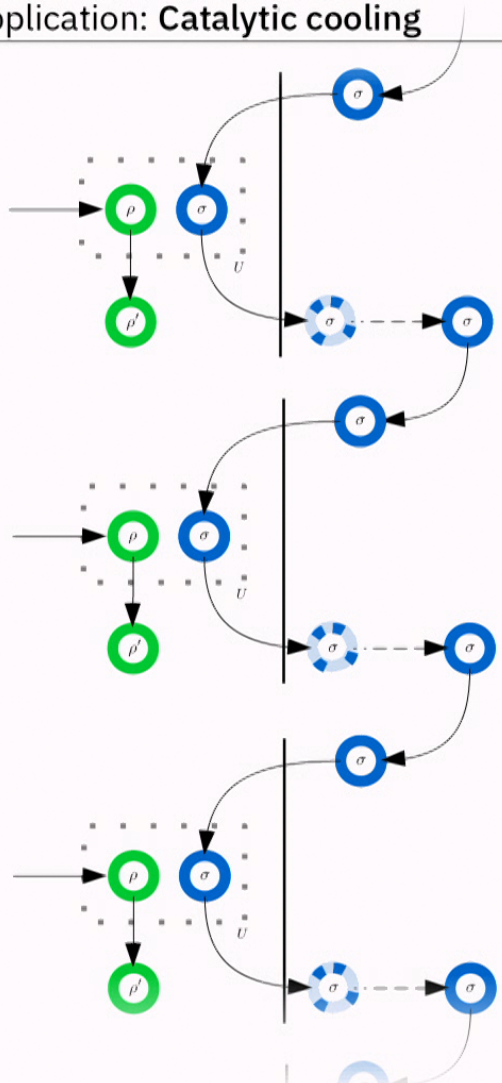

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Example:

$$\rho = \chi \otimes \chi, \quad S(\chi) < \frac{1}{2}\log(2)$$

$$\rho' = \frac{\mathbf{1}}{2} \otimes |0\rangle\langle 0|_{\varepsilon}$$

Full-rank state arbitrarily close to |0>

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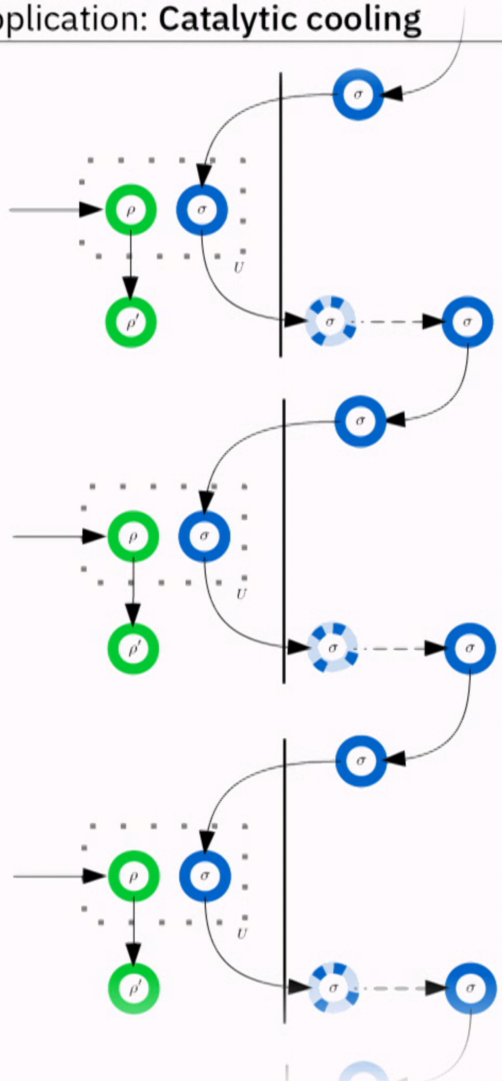

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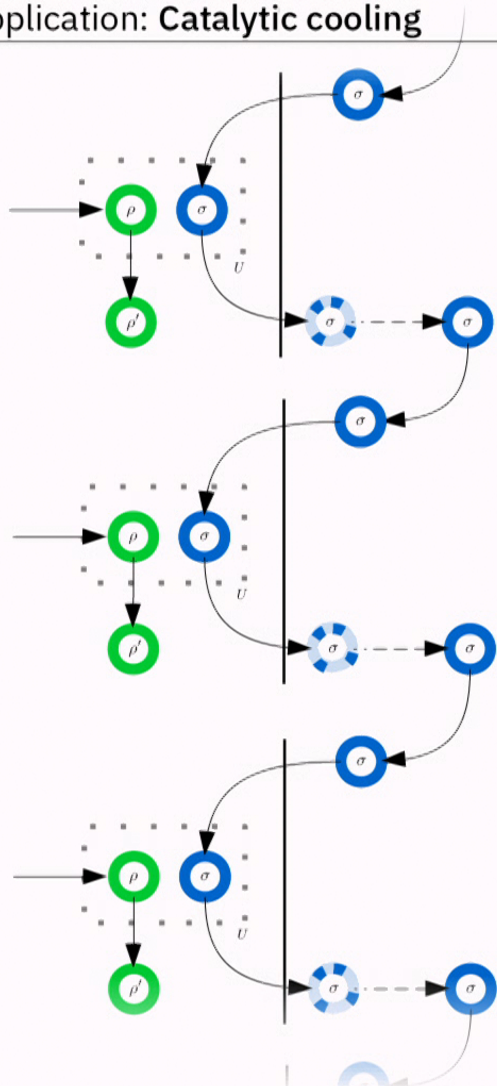

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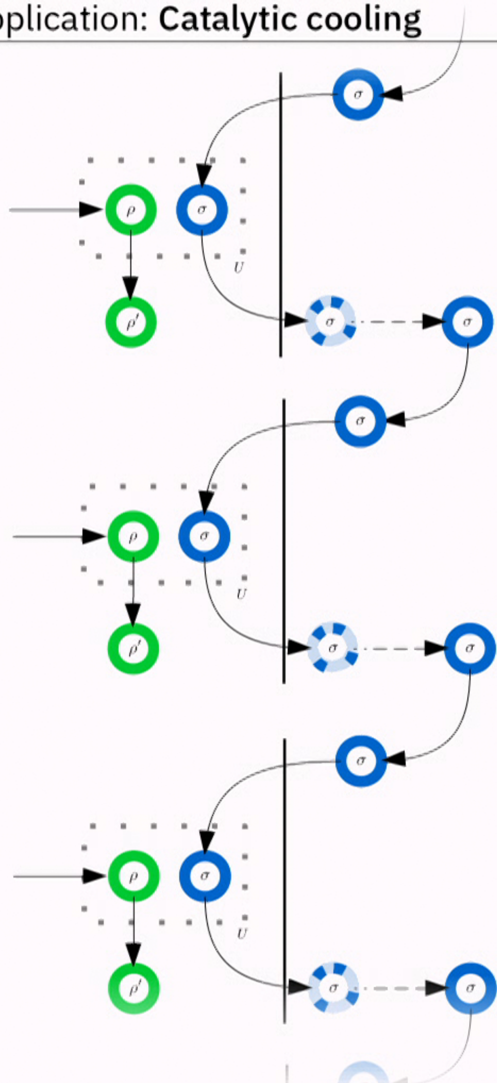

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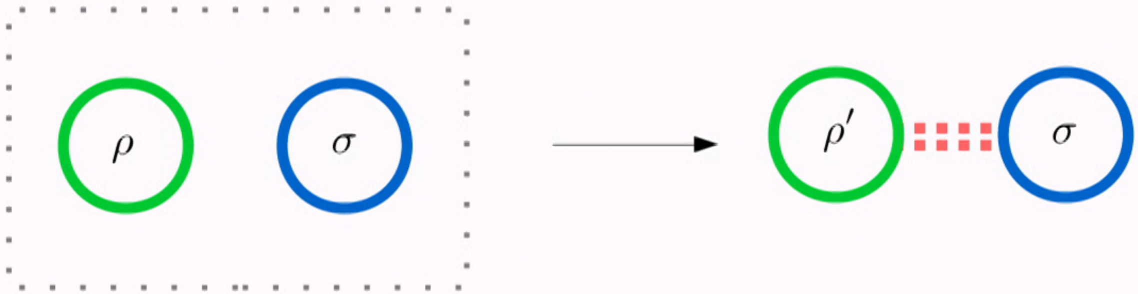
Like algorithmic cooling without iid limit.

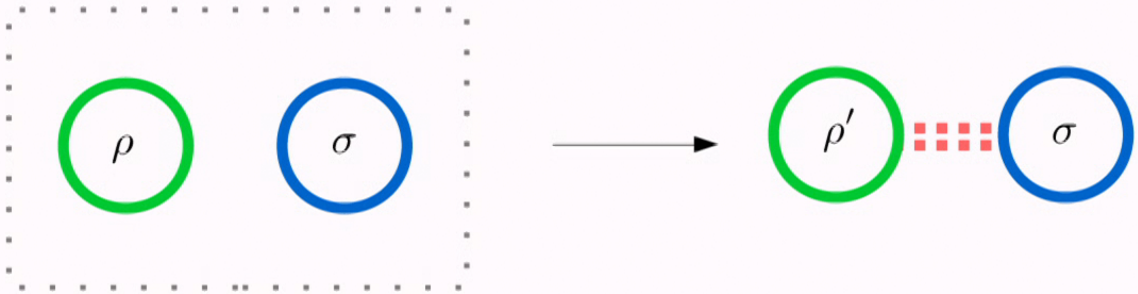
However, **correlations** are created:

$$\rho^{\otimes n} \longrightarrow \rho'_{1,\ldots,n} \neq \rho'^{\otimes n}$$

Back to catalytic transitions



Back to catalytic transitions



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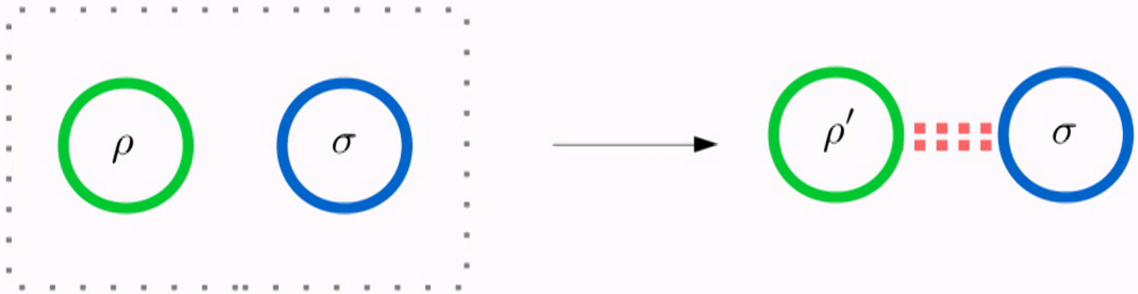
Catalytic entropy conjecture

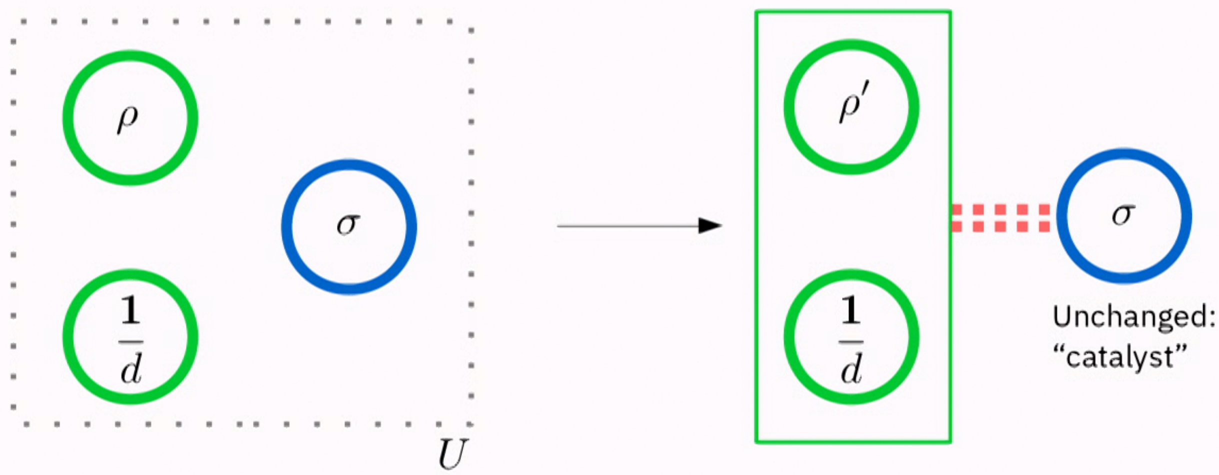


Catalytic entropy conjecture

Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then:

$$\rho \longrightarrow \rho' \quad \Leftrightarrow \quad S(\rho') > S(\rho) \quad \text{and} \quad \mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$$

Weak solution to catalytic entropy conjecture



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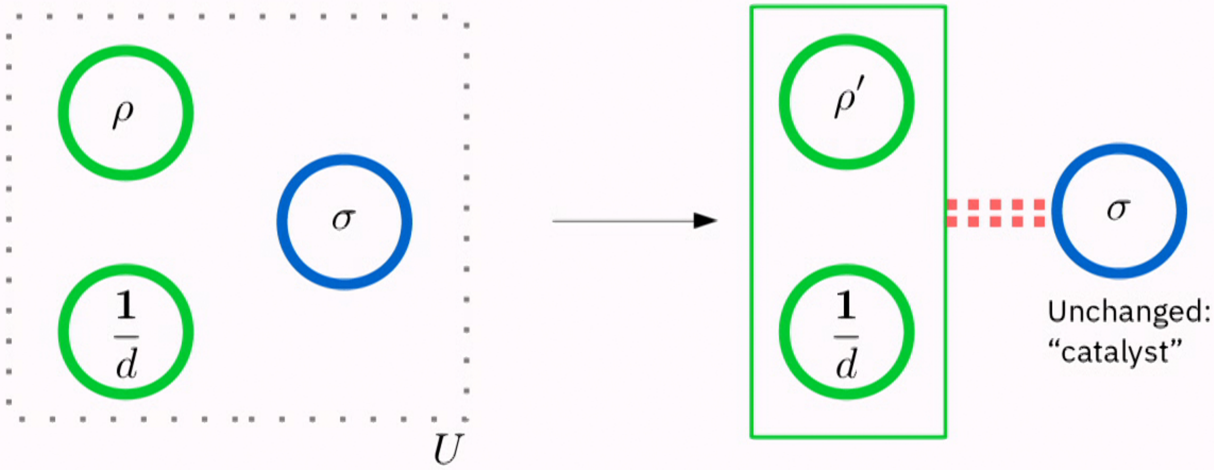

Lemma (Weak solution to conjecture)
Let $\mathrm{spec}(\rho) \neq \mathrm{spec}(\rho')$. Then the following are equivalent:

i) $S(\rho') > S(\rho)$ and $\mathrm{rank}(\rho') \geq \mathrm{rank}(\rho)$

ii) There exists some finite dimension *d* such that

$$\rho \otimes \frac{\mathbf{1}}{d} \longrightarrow \rho' \otimes \frac{\mathbf{1}}{d}$$

Implication of weak solution

Monotone: Any function *f* on the set of density matrices such that

$$\rho \longrightarrow \rho' \quad \Rightarrow \quad f(\rho) < f(\rho')$$

Call *f* **additive** if $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$.

Implication of weak solution

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Call f **additive** if $f(\rho_1 \otimes \rho_2) = f(\rho_1) + f(\rho_2)$.

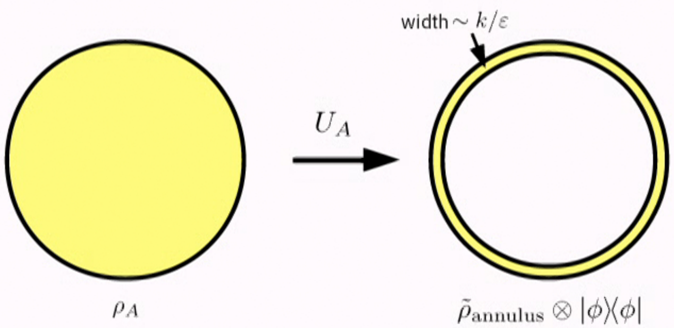
Proposition (Quasi-unique monotone)
Let f be a monotone on catalytic transitions. Then exactly one of the following statements is true.

i) $S(\rho') > S(\rho) \quad \Leftrightarrow \quad f(\rho') > f(\rho)$

ii) The function f is **non-additive** or **discontinuous**.

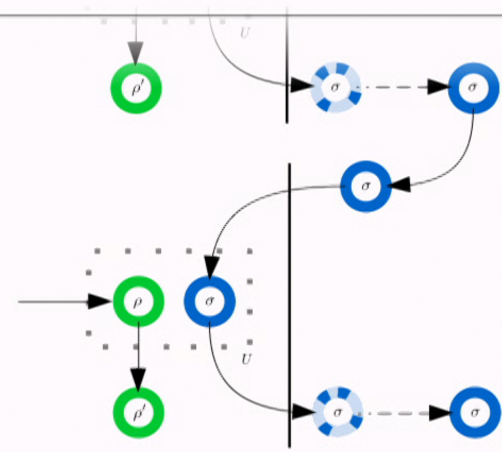
In particular, this rules out all Rényi entropies as monotones.

Thank you for your attention!



Lemma

$$\sum_{j=1}^{M} p_j \geq 1 - \frac{S(\mathbf{p})}{\log(M)}$$



Catalytic entropy conjecture

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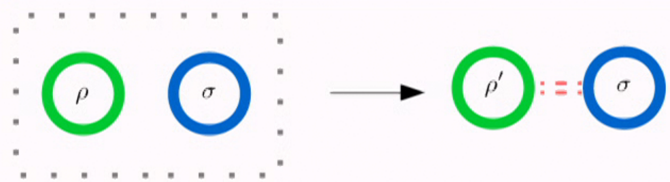

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With Jens Eisert, soon on arXiv.

ArXiv:1807.08773
With Paul Boes, Jens Eisert, Rodrigo Gallego, Markus Müller