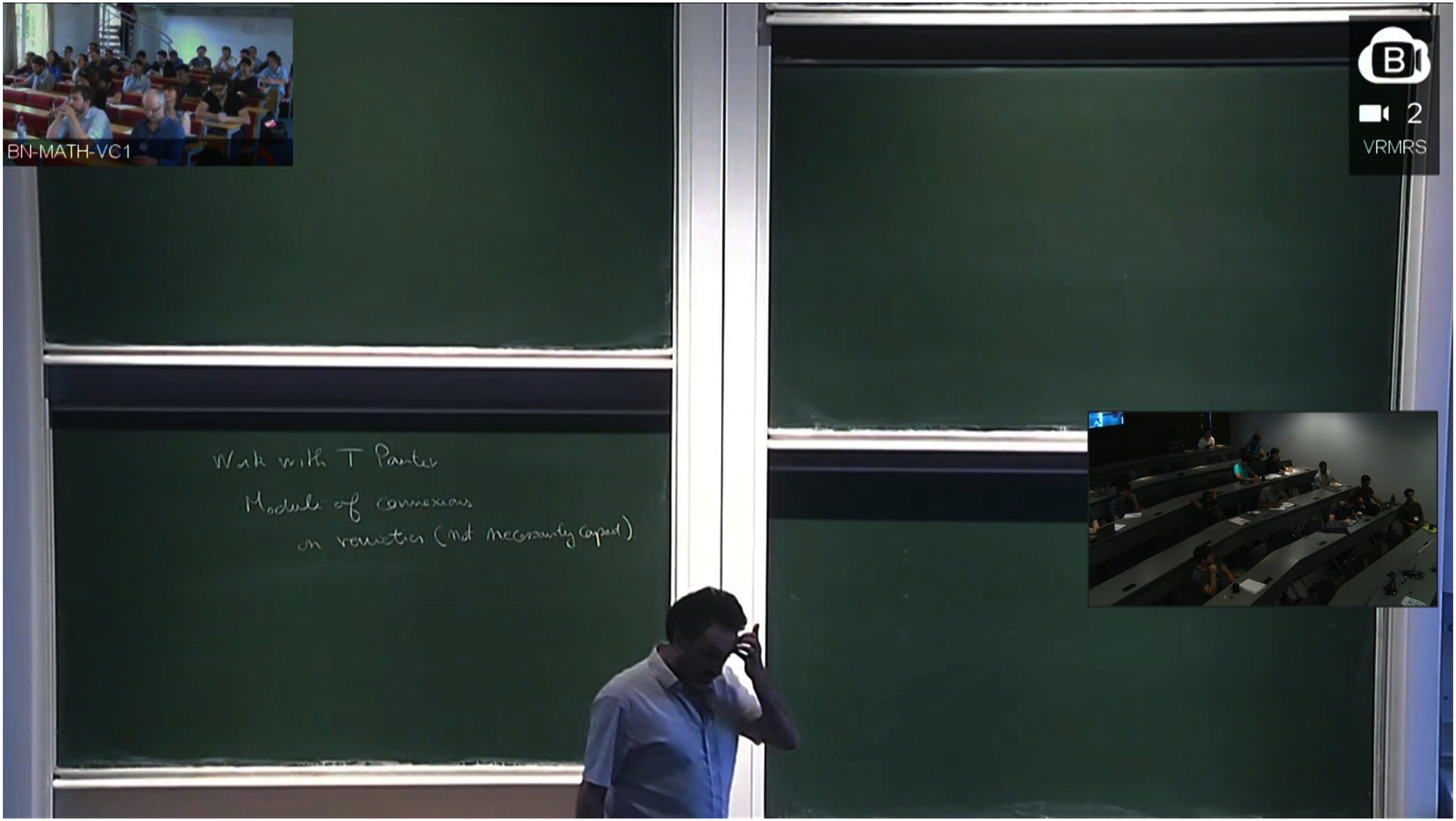
Title: Moduli of connexions on open varieties

Date: Aug 14, 2018 09:00 AM

URL: http://pirsa.org/18080080

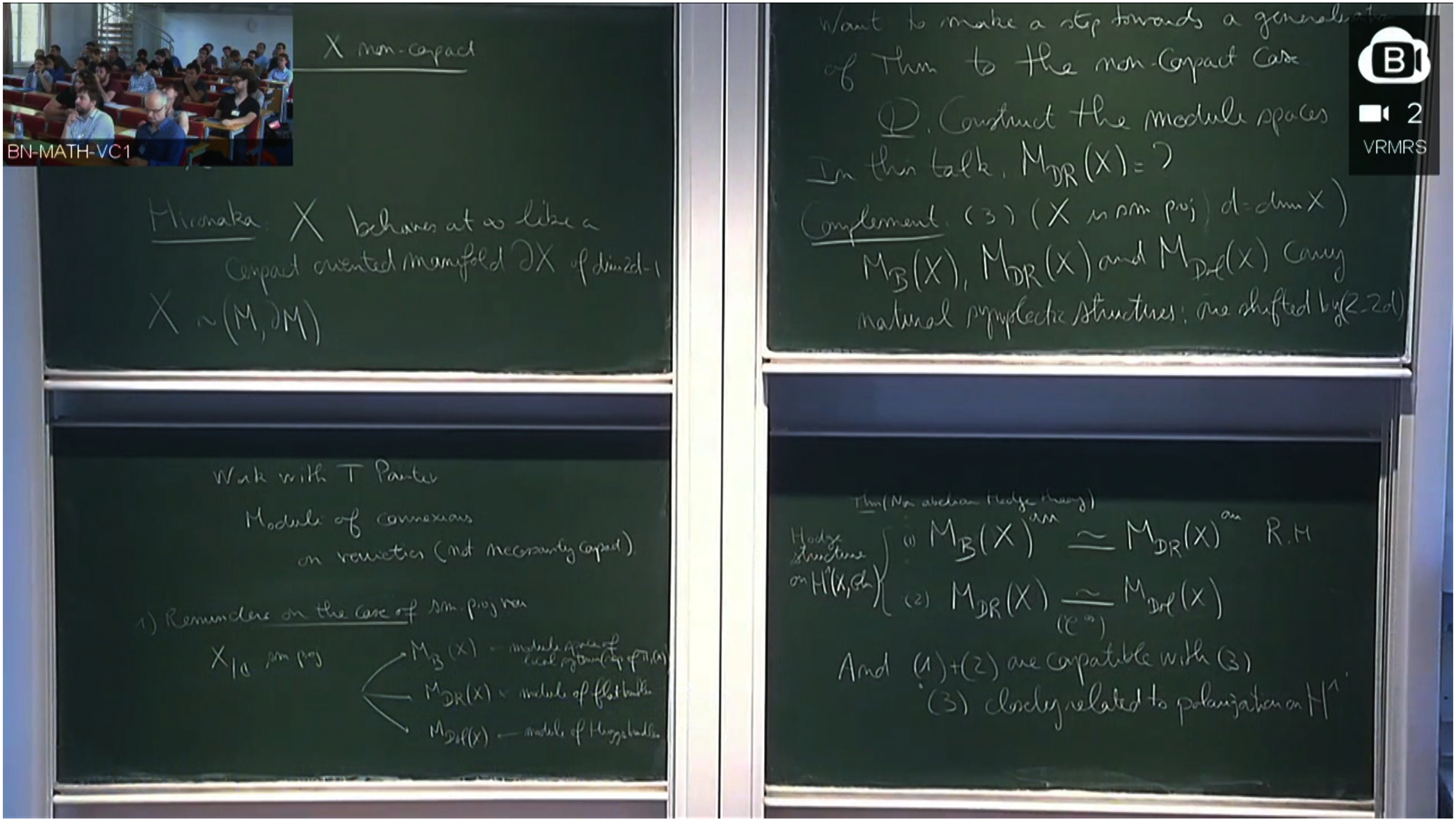
Abstract: This is a joint work with T. Pantev. In this talk, we will discuss moduli of ï¬‚at bundles on smooth algebraic varieties, with possibly irregular singularities at inï¬•nity. For this, we use the notion of â€œformal boundaryâ€•, previously studied by Ben Bassat-Temkin, Eï¬•mov and Hennionâ€“ Portaâ€“Vezzosi, as well as the moduli of ï¬‚at bundles at inï¬•nity. We prove that the ï¬•bers of the restriction map to inï¬•nity are representable. We also prove that this restriction map has a canonical Lagrangian structure in the sense of shifted symplectic geometry.



Work with T Pantev

Moduli of connexions

on varieties (not necessarily compact)



Work with T Pantev

Moduli of connexions
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1) Reminders on the case of sm. proj var

$$\begin{tikzcd} & M_B(X) & \text{moduli space of local systems (rep of } \pi_1(X)) \arrow[l] \\ X_{/\mathbb{C}} \text{ sm proj} \arrow[ur] \arrow[r] \arrow[dr] & M_{DR}(X) & \text{moduli of flat bundles} \arrow[l] \\ & M_{Dol}(X) & \text{moduli of Higgs bundles} \arrow[l] \end{tikzcd}$$

Thm (Non abelian Hodge theory)

Hodge structure on $H^1(X, \mathbb{C})$
$$\left\{ \begin{array}{ll} (1) & M_B(X)^{an} \simeq M_{DR}(X)^{an} \quad \text{R.H} \\ (2) & M_{DR}(X) \underset{(C^\infty)}{\simeq} M_{Dol}(X) \end{array} \right.$$

And (1)+(2) are compatible with (3)
(3) closely related to polarization on H^1

Want to make a step towards a generali
of Thm to the non-compact case

Q. Construct the moduli spaces

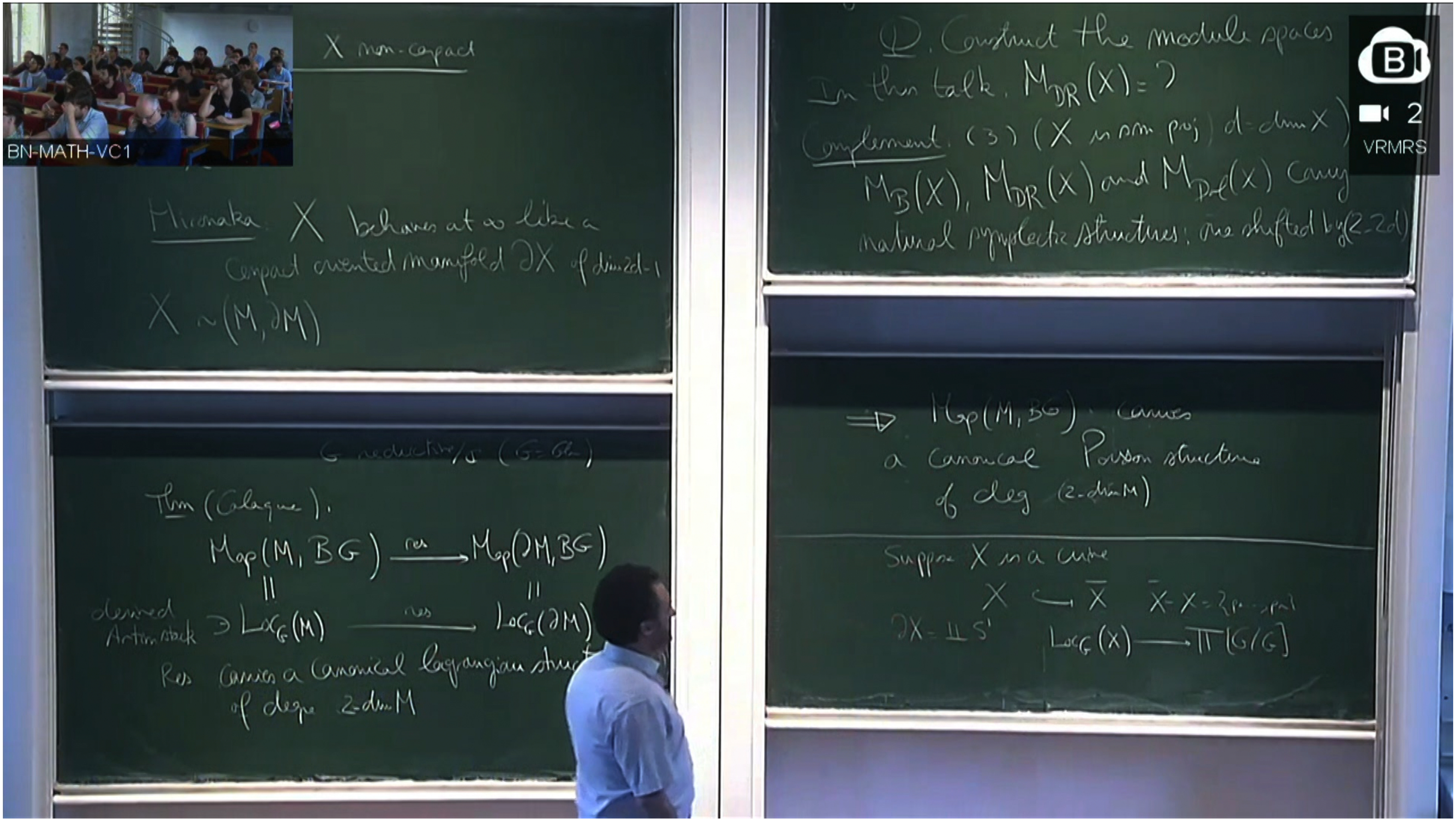
In this talk, $M_{DR}(X) = ?$

Complement. (3) (X is sm. proj, $d = \dim X$)
$M_B(X)$, $M_{DR}(X)$ and $M_{Dol}(X)$ carry natural symplectic structures; are shifted by $(2-2d)$

X non-compact

Hironaka: X behaves at ∞ like a compact oriented manifold ∂X of dim $2d-1$

$X \sim (M, \partial M)$



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G reductive/$\mathbb{C}$ ($G = GL_n$)

Thm (Calaque):

$$\begin{tikzcd} & Map(M, BG) \arrow[r, "res"] \arrow[d, equal] & Map(\partial M, BG) \arrow[d, equal] \\ \text{derived Artin stack} \ni & Loc_G(M) \arrow[r, "res"] & Loc_G(\partial M) \end{tikzcd}$$

Res carries a canonical lagrangian struct
of degree $2 - \dim M$

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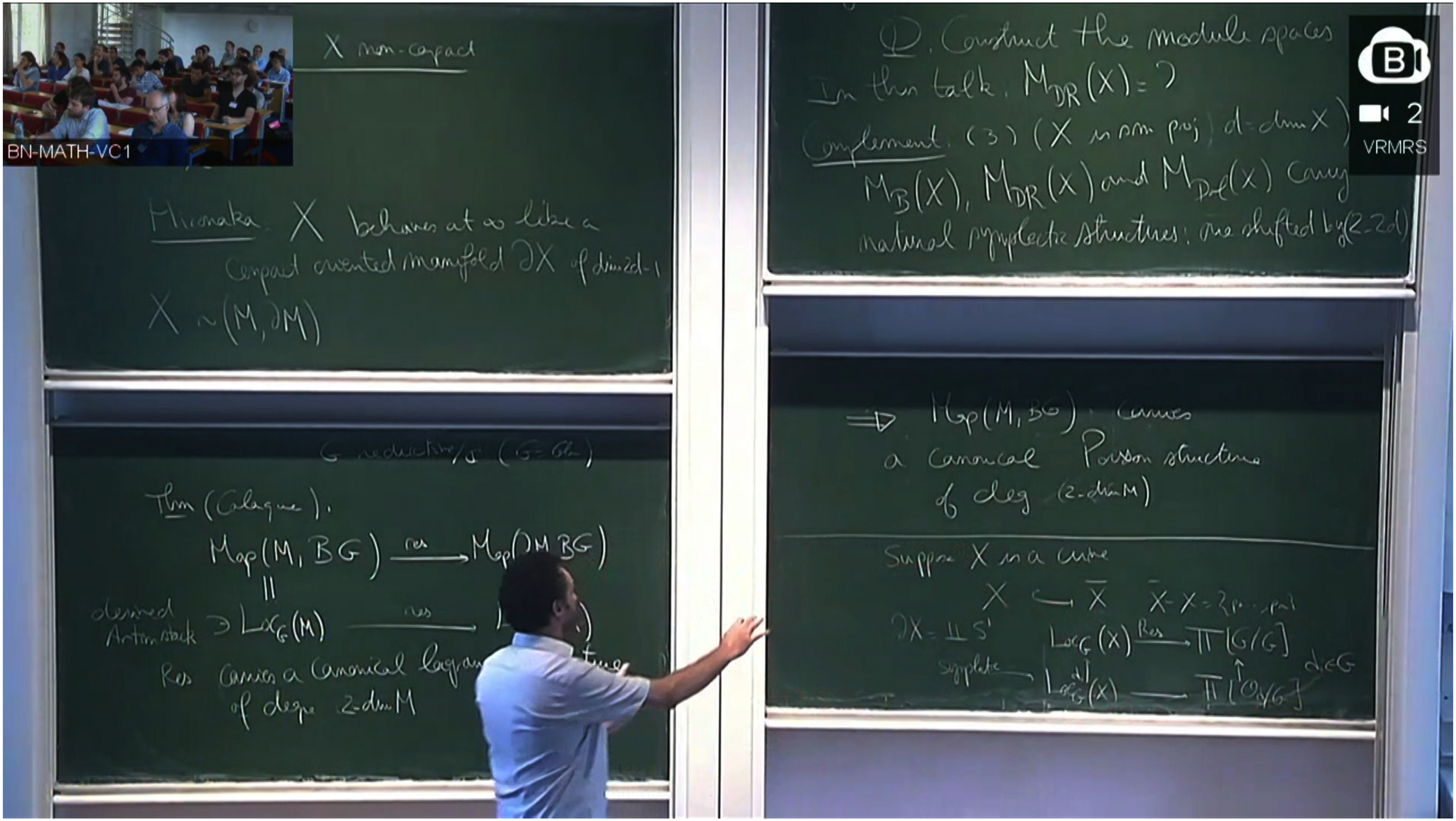
Complement: (3) (X is non proj, $d = \dim X$)
$M_B(X)$, $M_{DR}(X)$ and $M_{Dol}(X)$ carry natural symplectic structures: one shifted by $(2-2d)$

$\Longrightarrow$ $Map(M, BG)$ carries a canonical Poisson structure of deg $(2 - \dim M)$

Suppose X is a curve

$X \hookrightarrow \bar{X}$ $\quad \bar{X} - X = \{p_1, \dots, p_n\}$

$\partial X = \coprod S^1$ $\quad Loc_G(X) \longrightarrow \prod [G/G]$



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derived Artin stack ∋

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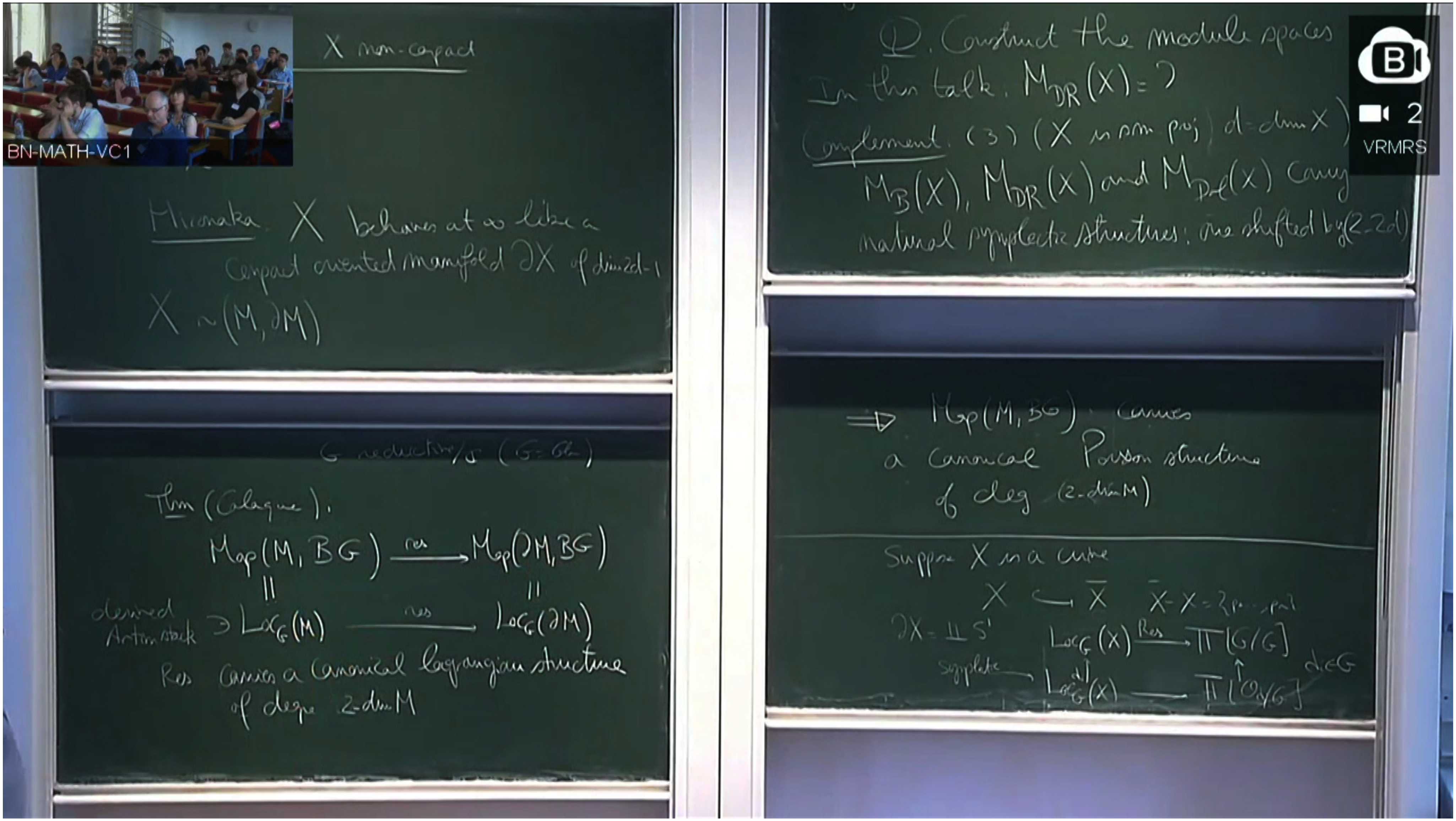
$$\begin{tikzcd} \mathrm{Loc}_G(X) \arrow[r, "Res"] & \prod [G/G] \\ \mathrm{Loc}_G^{d.}(X) \arrow[u] \arrow[r] & \prod [O_i/G] \arrow[u] \end{tikzcd}$$

symplectic →

$O_i \in G$



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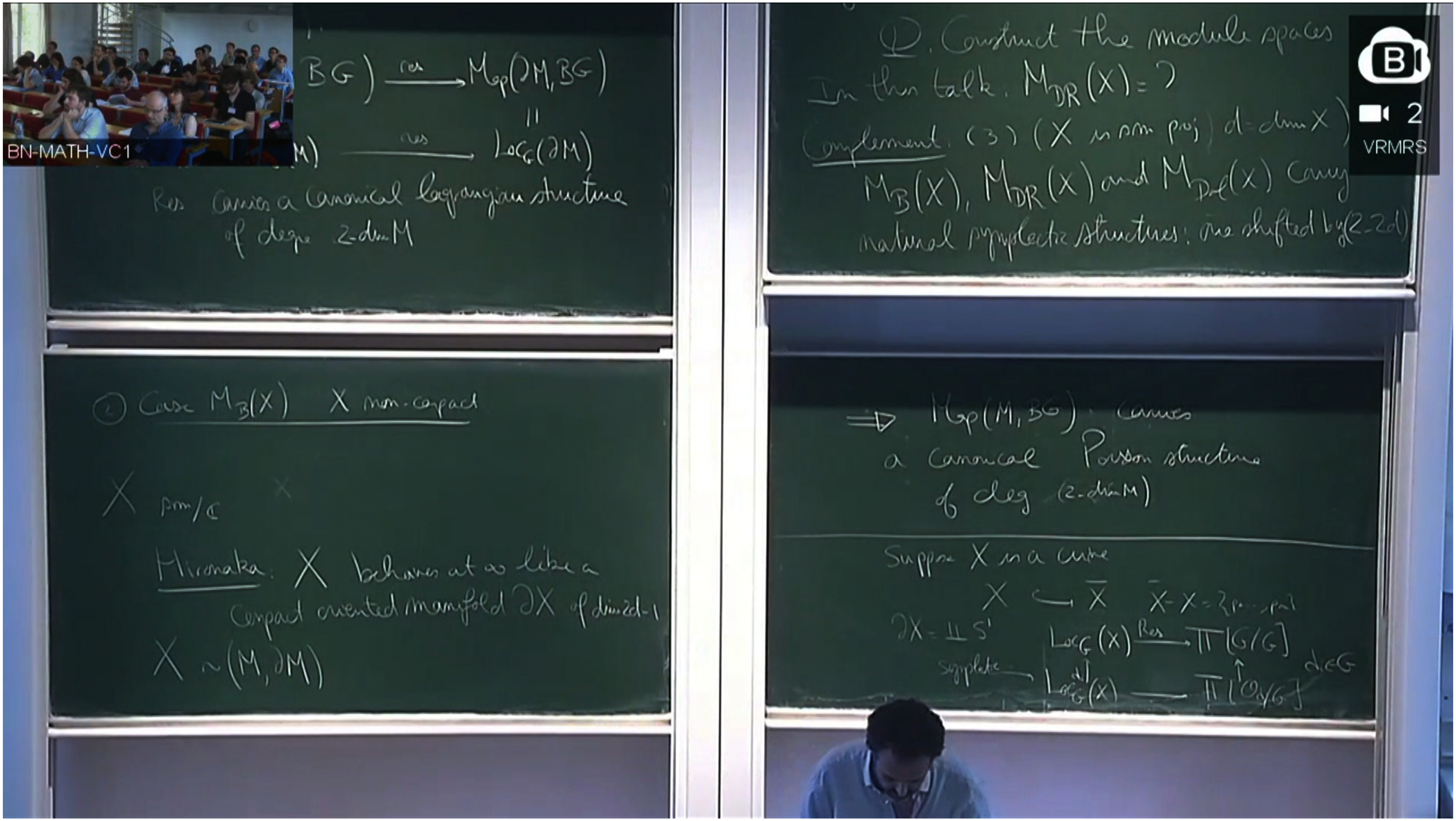
$$\begin{tikzcd} & \mathrm{Loc}_G(X) \arrow[r, "\mathrm{Res}"] & \prod [G/G] \\ \text{symplectic} \arrow[r] & \mathrm{Loc}_G^{c}(X) \arrow[u] \arrow[r] & \prod [\mathcal{O}_{c_i}/G] \arrow[u] \end{tikzcd}$$

$c_i \in G$



$$\begin{tikzcd} BG) \arrow[r, "res"] & \mathrm{Map}(\partial M, BG) \arrow[d, equal] \\ M) \arrow[r, "res"] & \mathrm{Loc}_G(\partial M) \end{tikzcd}$$

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② Case $M_B(X)$ X non-compact

X sm/$\mathbb{C}$

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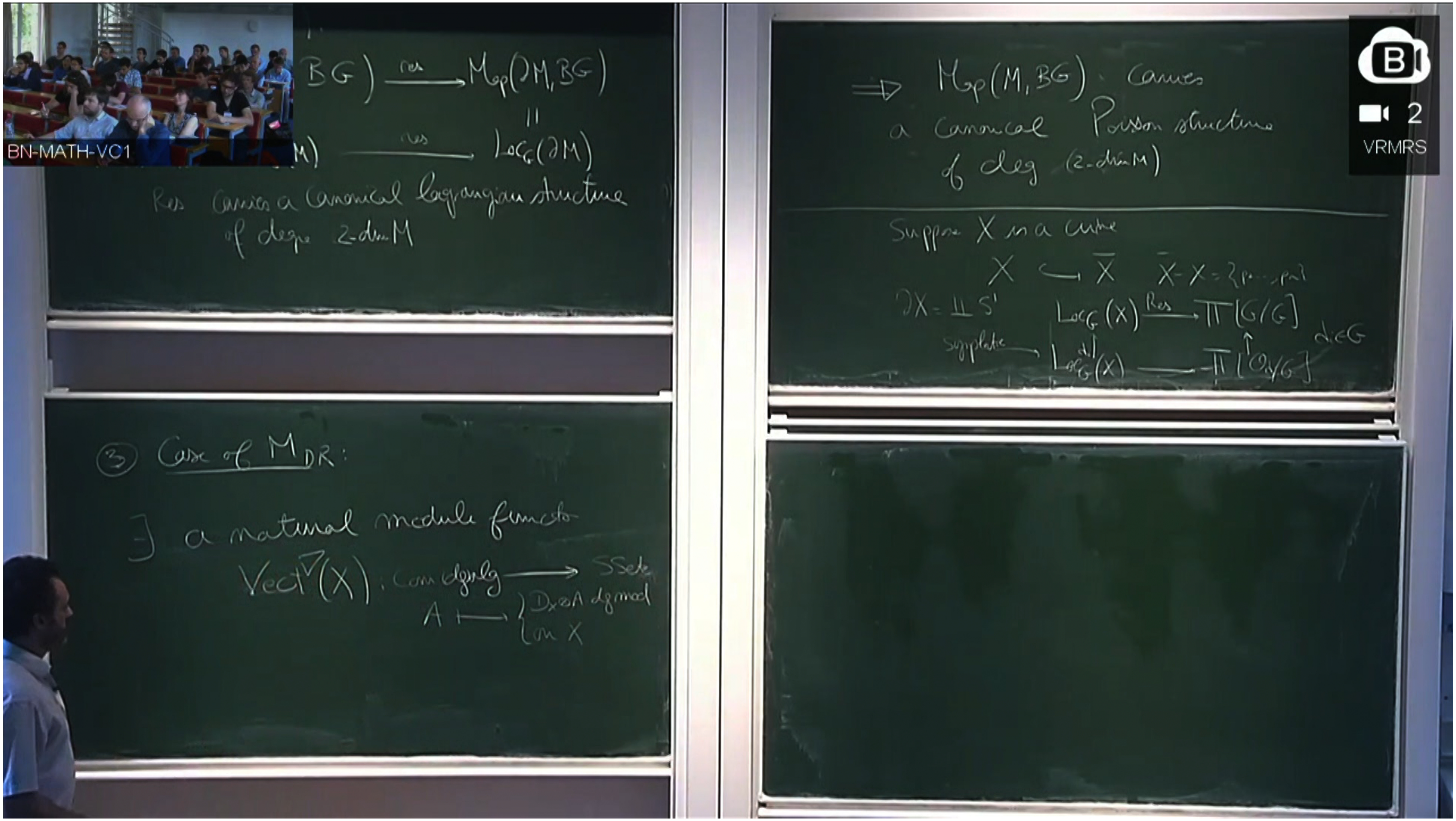
$$\begin{tikzcd} & \mathrm{Loc}_G(X) \arrow[r, "Res"] & \prod [G/G] \\ \text{symplectic} \arrow[r] & \mathrm{Loc}_G^{\alpha}(X) \arrow[u] \arrow[r] & \prod [\mathcal{O}_{\alpha}/G] \arrow[u] \end{tikzcd}$$

$\alpha_i \in G$



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$$\begin{tikzcd} & \mathrm{Loc}_G(X) \arrow[r, "Res"] & \prod [G/G] \\ \arrow[r, "symplectic"] & \mathrm{Loc}_G^{d.l}(X) \arrow[u] \arrow[r] & \prod [O_i/G] \arrow[u] \end{tikzcd}$$

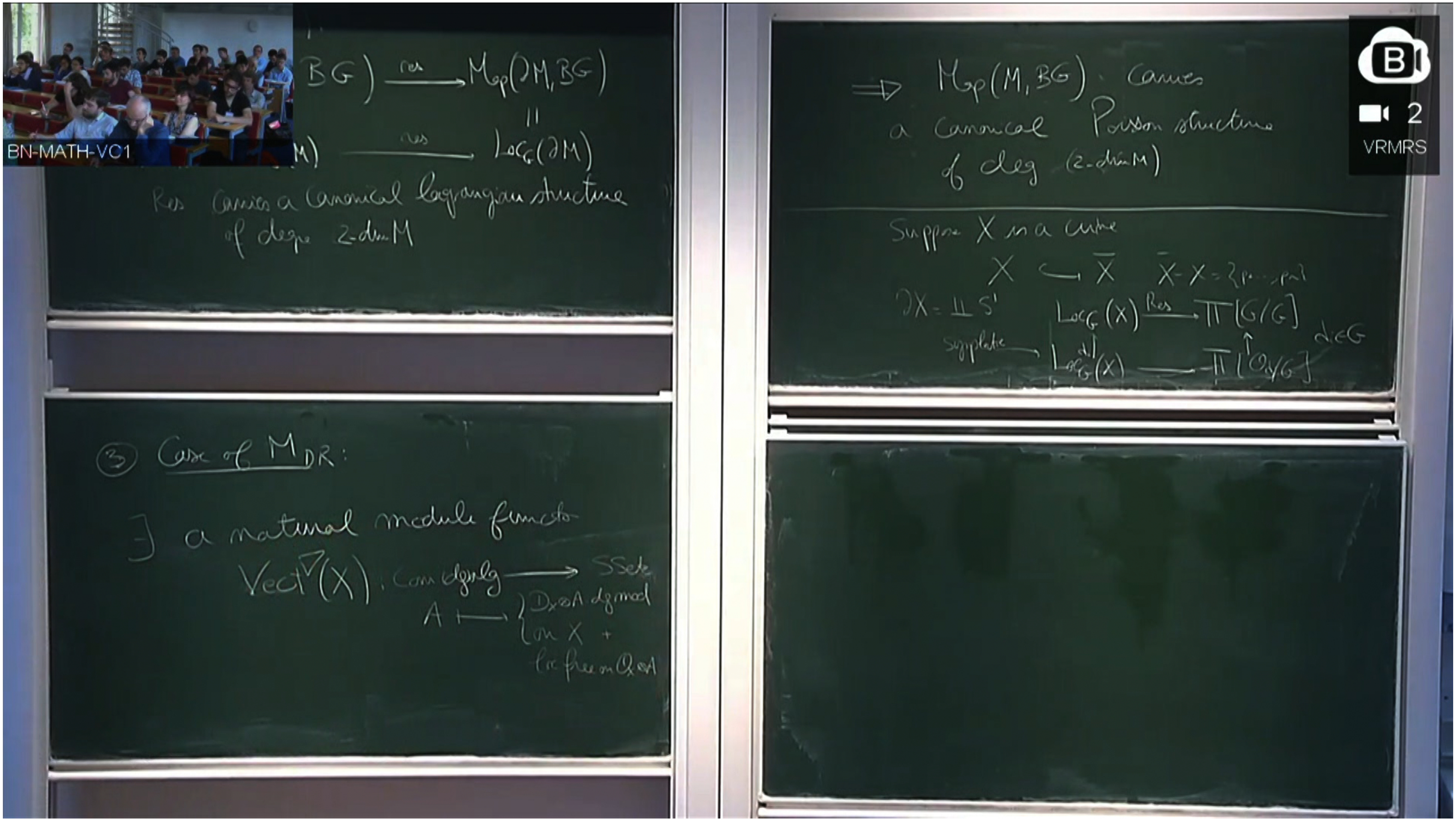
$O_i \in G$

③ <u>Case of M_{DR}</u>:

$\exists$ a natural module functor

$$\mathrm{Vect}^{\nabla}(X) : \text{comm dg alg} \longrightarrow \text{SSets}$$

$$A \longmapsto \left\{ \begin{array}{l} D_X \otimes A \text{ dg mod} \\ \text{on } X \end{array} \right.$$



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$O_i \subset G$

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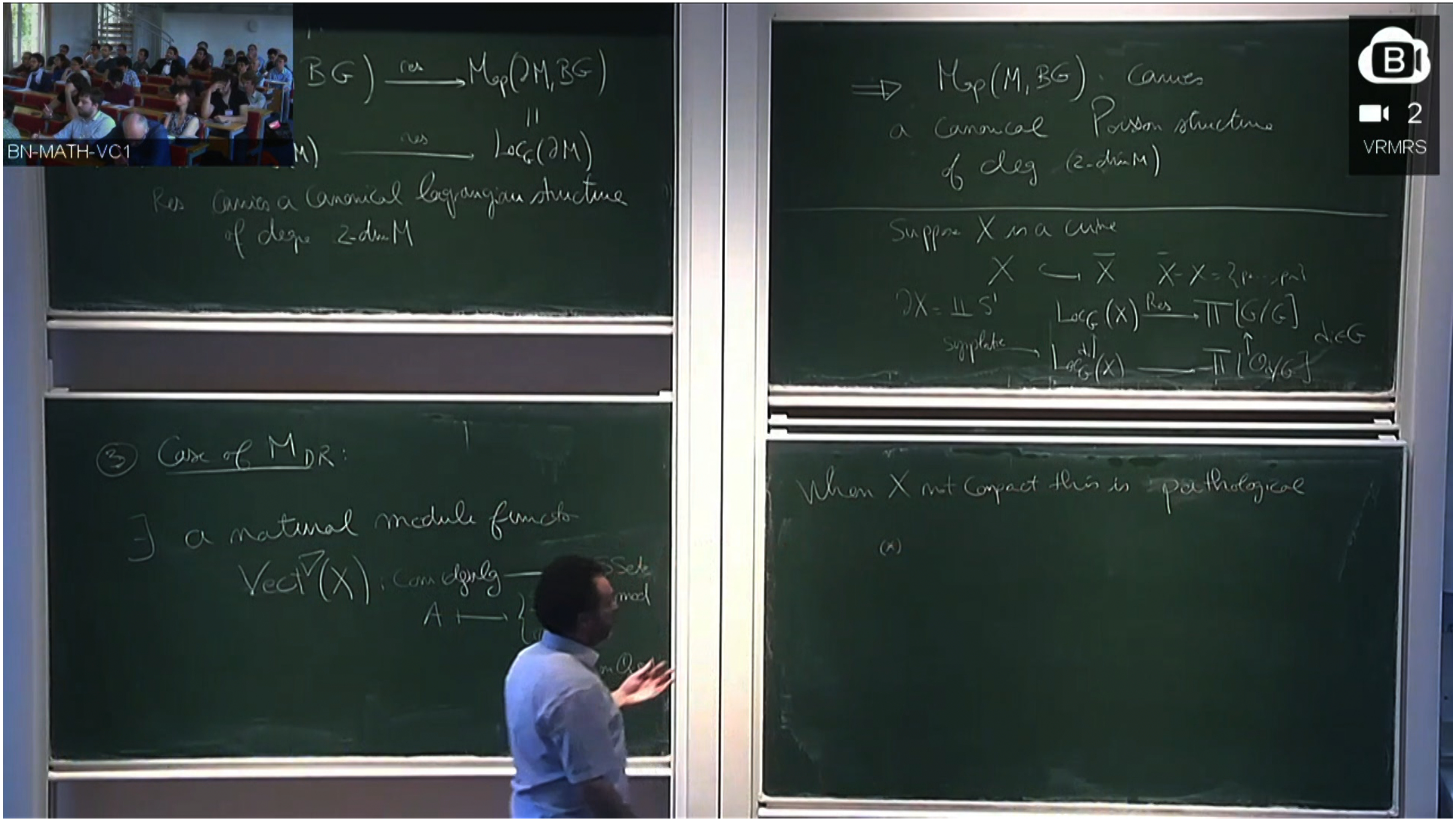


$$\begin{tikzcd} BG) \arrow[r, "res"] & \mathrm{Map}(\partial M, BG) \arrow[d, phantom, "\parallel"] \\ M) \arrow[r, "res"] & \mathrm{Loc}_G(\partial M) \end{tikzcd}$$

Res carries a canonical lagrangian structure of degree $2-\dim M$

③ Case of M_{DR}:

$\exists$ a natural moduli functor

$\mathrm{Vect}^{\nabla}(X)$: Com dgalg ⟶ SSets

$A \longmapsto \{$ mod

m$\mathcal{O}$

$\Longrightarrow$ $\mathrm{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2-\dim M)$

Suppose X is a curve

$X \hookrightarrow \overline{X}$ $\quad \overline{X}-X=\{p_1 \dots p_n\}$

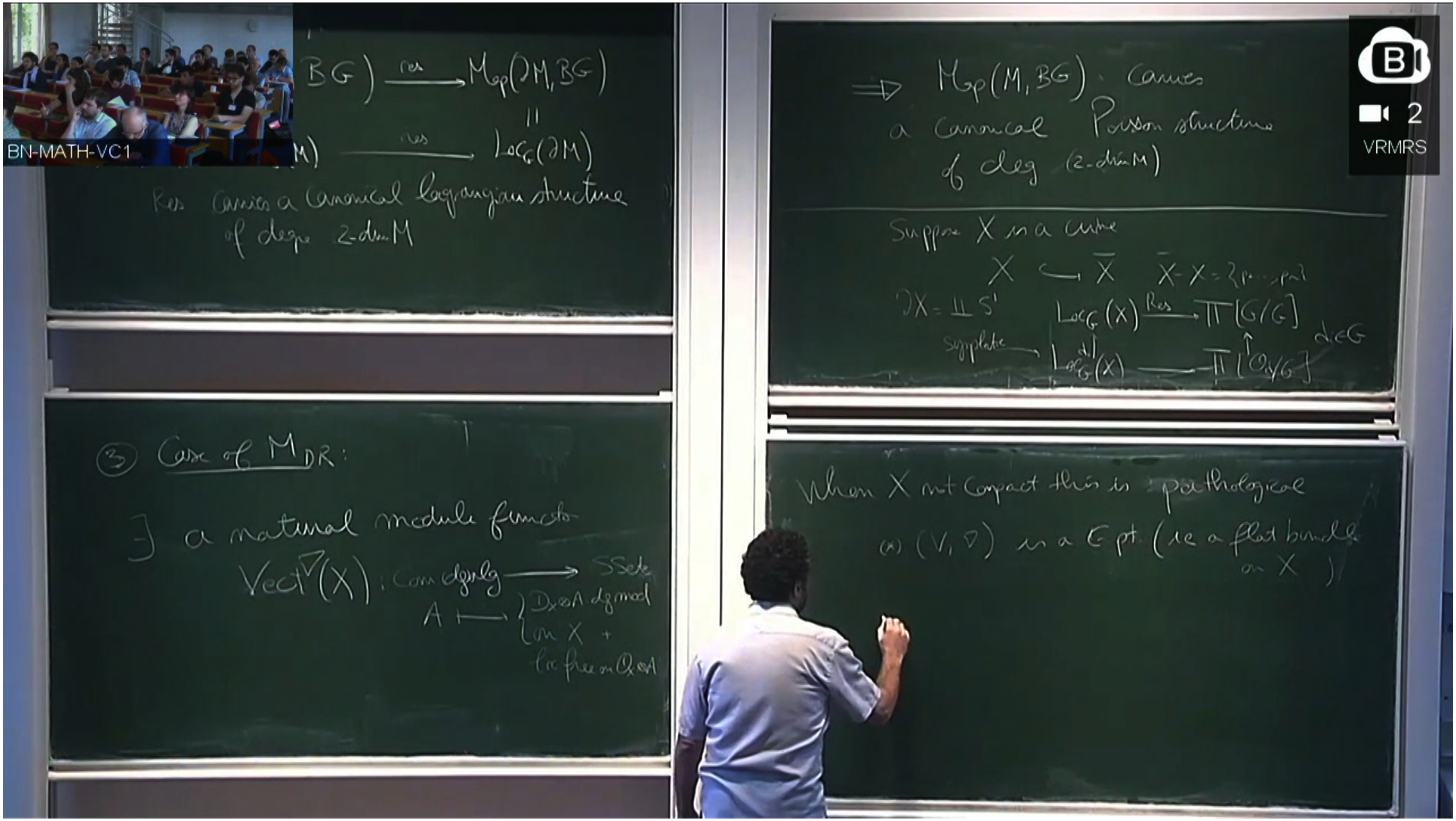
$\partial X = \amalg S^1$

$$\begin{tikzcd} & \mathrm{Loc}_G(X) \arrow[r, "Res"] & \prod [G/G] \\ \text{symplectic} \arrow[r] & \mathrm{Loc}_G^{d.l}(X) \arrow[u, phantom, "\cup"] \arrow[r] & \prod [\mathcal{O}_{c_i}/G] \arrow[u] \end{tikzcd}$$

$c_i \in G$

When X not compact this is pathological

(*)



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$\Longrightarrow$ $\mathrm{Map}(M, BG)$ carries a canonical Poisson structure of deg (2-dim M)

Suppose X is a curve

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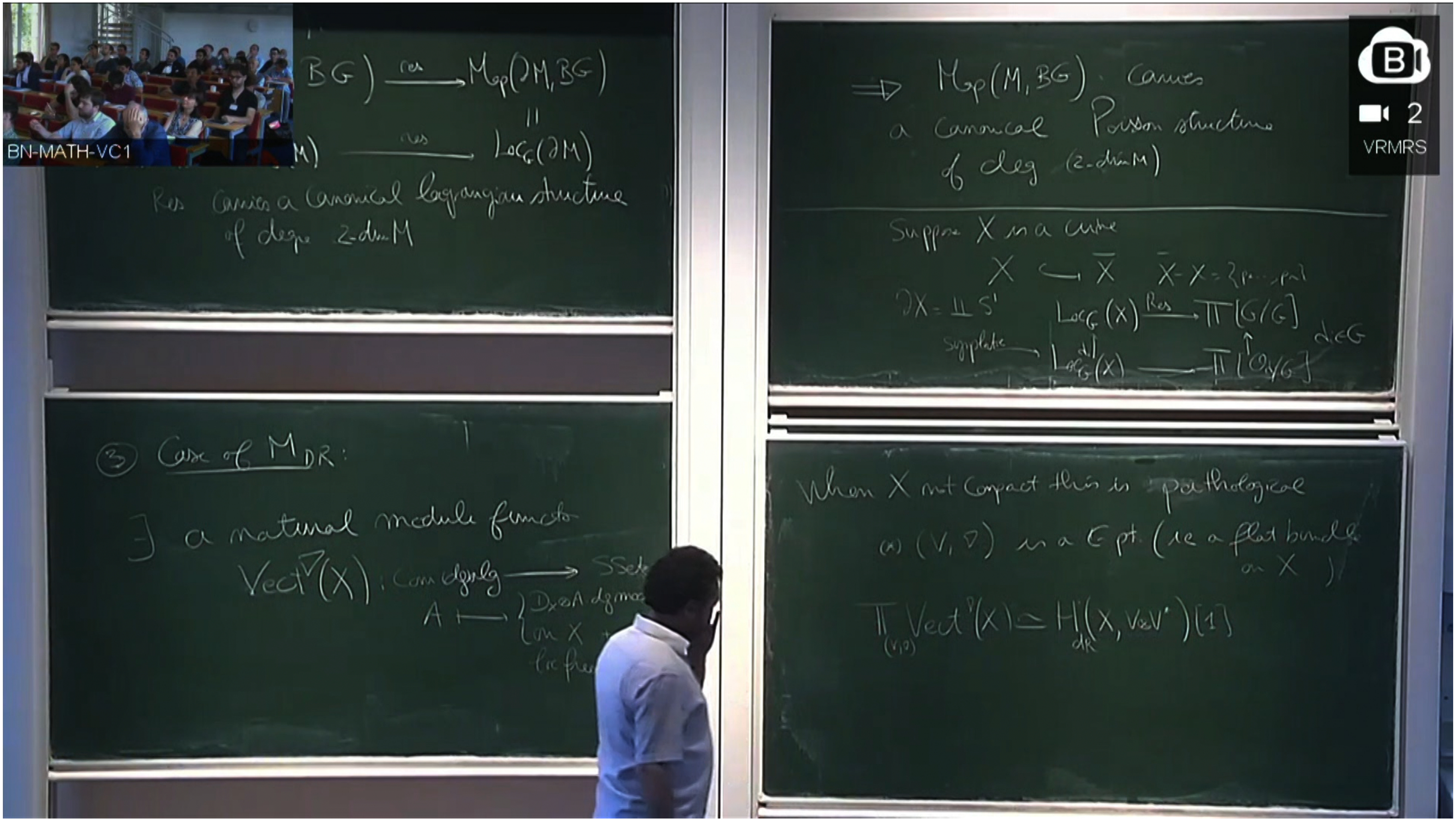
(*) (V, ∇) is a $\mathbb{C}$ pt (ie a flat bundle on X)



$$\begin{tikzcd} \ldots BG) \arrow[r, "\text{res}"] & \mathrm{Map}(\partial M, BG) \arrow[d, equal] \\ \ldots M) \arrow[r, "\text{res}"] & \mathrm{Loc}_G(\partial M) \end{tikzcd}$$

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$\Rightarrow$ Map(M, BG) carries a canonical Poisson structure of deg (2-dim M)

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$\mathcal{O}_i \subset G$

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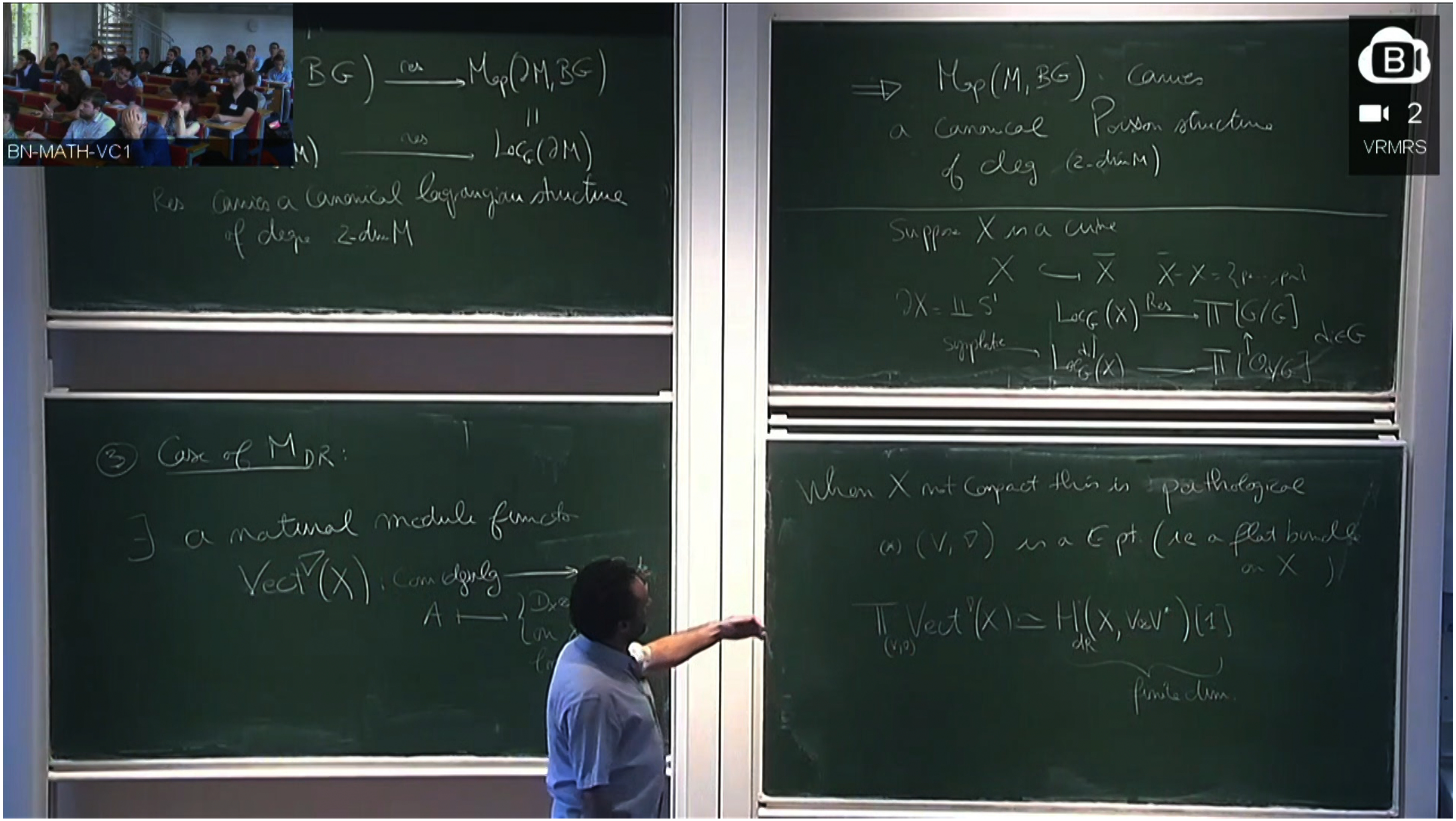
$$\mathrm{Vect}^{\nabla}(X) : \mathrm{Comm\ dgalg} \longrightarrow \mathrm{SSets}$$

$$A \longmapsto \left\{ \begin{array}{l} D_X \otimes A \text{ dg mod} \\ \text{on } X + \\ \text{loc free} \end{array} \right.$$

When X not compact this is pathological

(x) (V, ∇) is a $\mathbb{C}$ pt. (ie a flat bundle on X)

$$\mathbb{T}_{(V,\nabla)} \mathrm{Vect}^{\nabla}(X) \simeq H^{\bullet}_{dR}(X, V \otimes V^*)[1]$$



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Suppose X is a curve

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symplectic $\longrightarrow$

$\mathcal{O}_i \subset G$

③ <u>Case of M_{DR}</u>:

$\exists$ a natural module functor

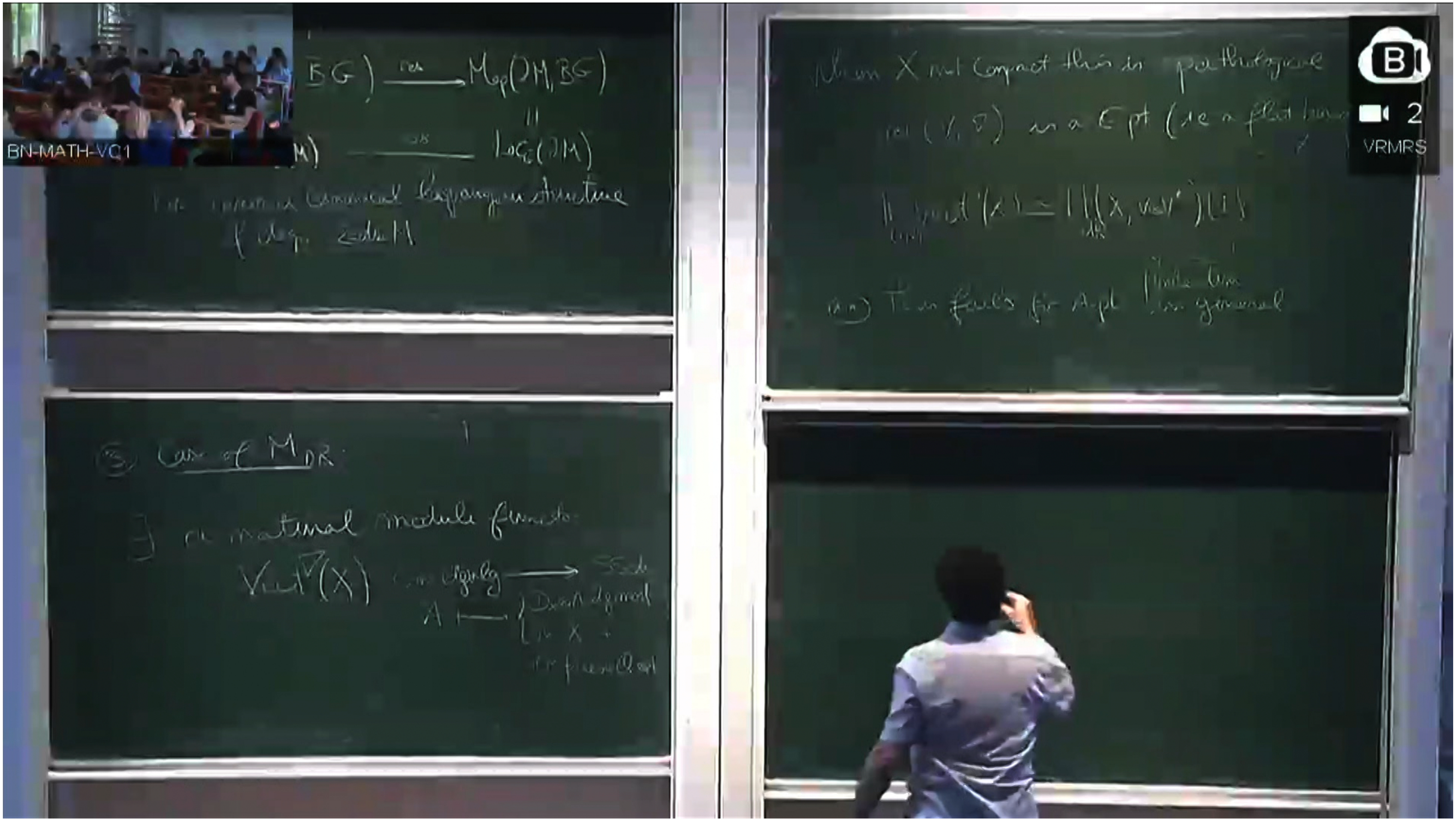
$\mathrm{Vect}^{\nabla}(X) : \mathrm{Com\ dg\ alg} \longrightarrow$

$A \longmapsto \{ D_X \otimes$

When X not compact this is pathological

(x) (V, ∇) is a $\mathbb{C}$-pt. (ie a flat bundle on X)

$$T_{(V,\nabla)} \mathrm{Vect}^{\nabla}(X) \simeq \underbrace{H^{\bullet}_{dR}(X, V \otimes V^*)[1]}_{\text{finite dim.}}$$



$BG) \xrightarrow{res} Map(\partial M, BG)$

$\|$

$M) \xrightarrow{\simeq} Loc_G(\partial M)$

has canonical Lagrangian structure

(deg. 2−dim M

When X not compact this is pathological

ex (Y, ∇) is a ℂ pt (i.e a flat bun

$H^{*}_{(\ldots)}(X) \simeq H_{dR}(X, \ldots)[1]$

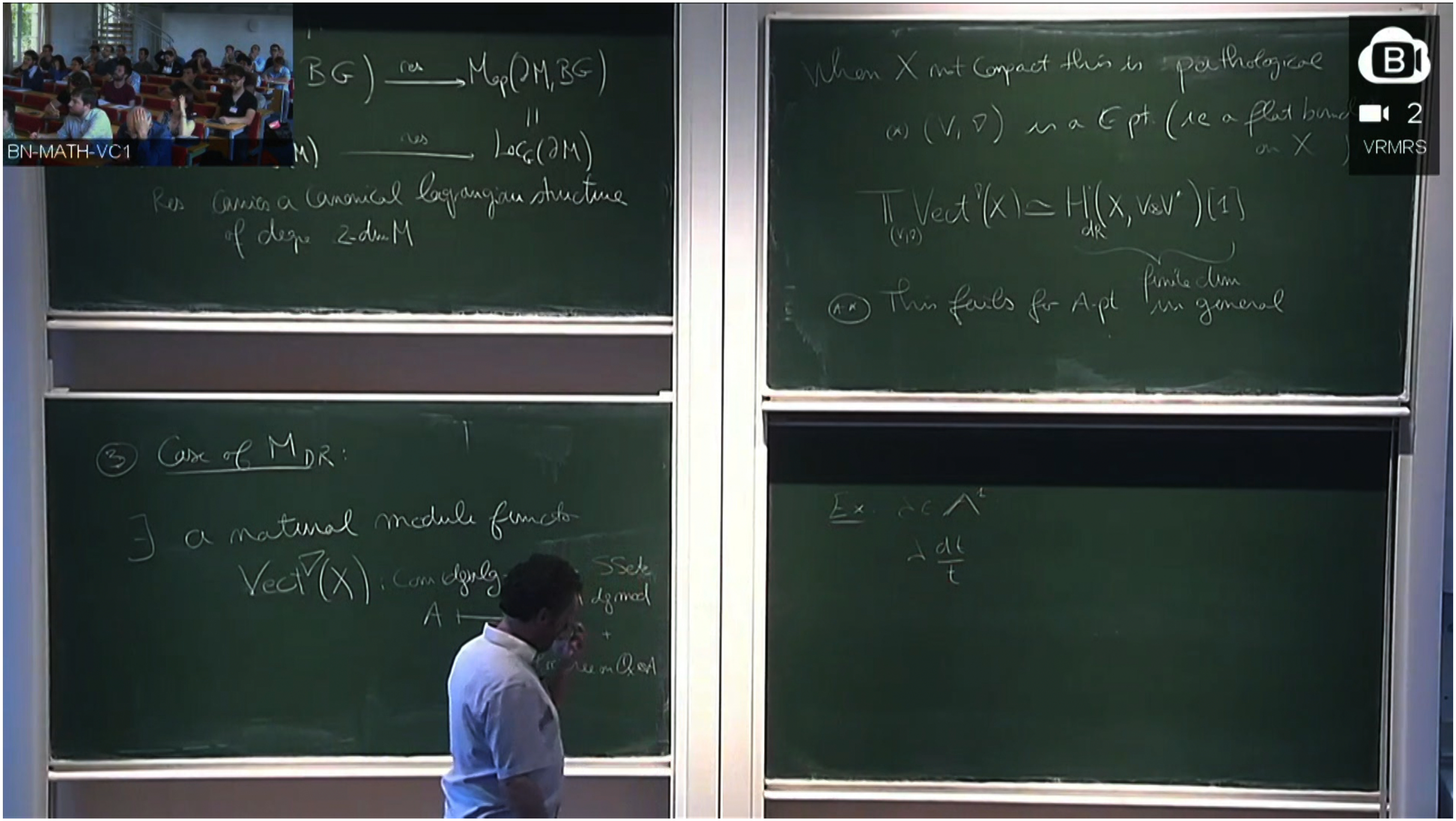
(**) This fails for A-pts | finite dim / in general

③ Case of M_{DR}:

∃ a natural module functor

$Vect^{\nabla}(X)$ cdga $\longrightarrow$ SSets

$A \longmapsto \{$ Derived deformal / in X + … flat conn



$$\begin{tikzcd} \ldots BG) \arrow[r, "res"] & Map(\partial M, BG) \arrow[d, phantom, "||"] \\ \ldots M) \arrow[r, "res"] & Lag_G(\partial M) \end{tikzcd}$$

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$Vect^\nabla(X)$: Com dgalg → SSets

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When X not compact this is pathological

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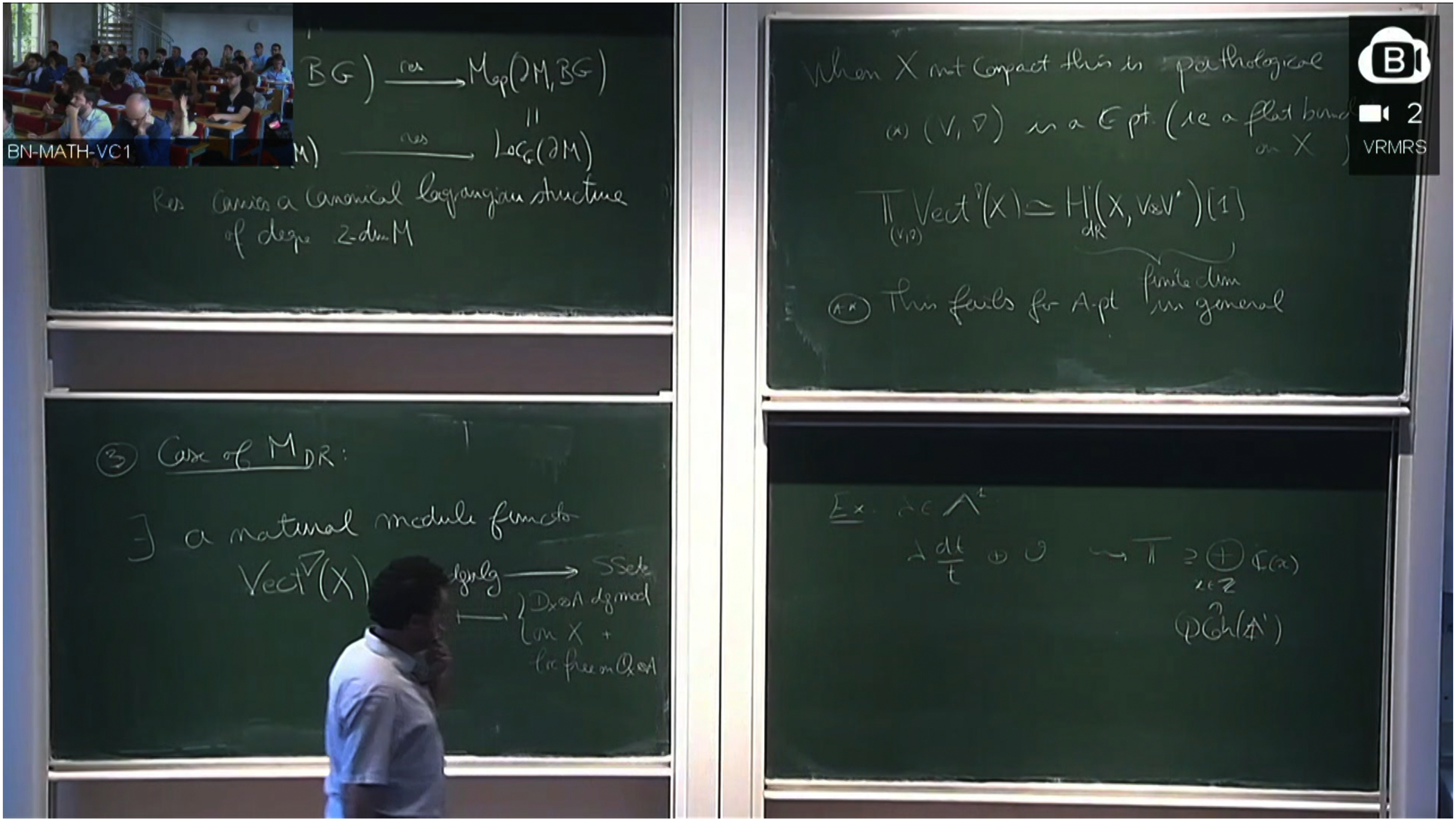
$$\mathbb{T}_{(V,\nabla)} Vect^\nabla(X) \simeq \underbrace{H^\cdot_{dR}(X, V\otimes V^*)}_{\text{finite dim}}[1]$$

(**) This fails for A-pt in general

Ex. $\lambda \in \mathbb{A}^1$

$\lambda \frac{dt}{t}$



$$\ldots BG) \xrightarrow{res} M_{gp}(\partial M, BG)$$

$$\|$$

$$\ldots M) \xrightarrow{res} Loc_G(\partial M)$$

Res carries a canonical lagrangian structure of degree 2-dim M

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When X not compact this is pathological

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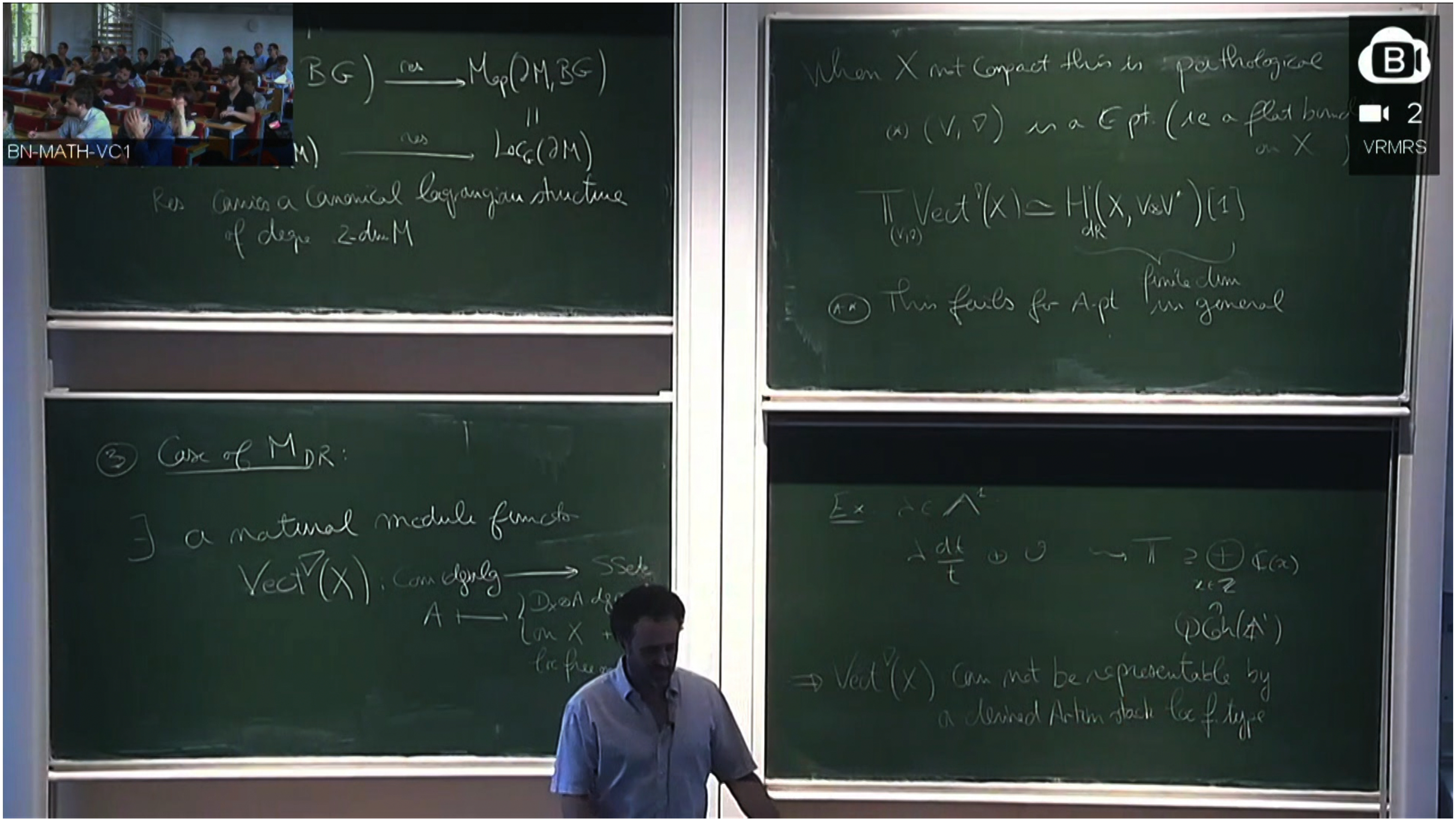
(A-*) This fails for A-pt in general

Ex. $\lambda \in \mathbb{A}^1$

$$\lambda \frac{dt}{t} \oplus \mathcal{O} \rightsquigarrow T \supseteq \bigoplus_{x \in \mathbb{Z}} \mathbb{C}(x)$$

$$\cap$$

$$\bigoplus \mathcal{O}h(\mathbb{A}^1)$$



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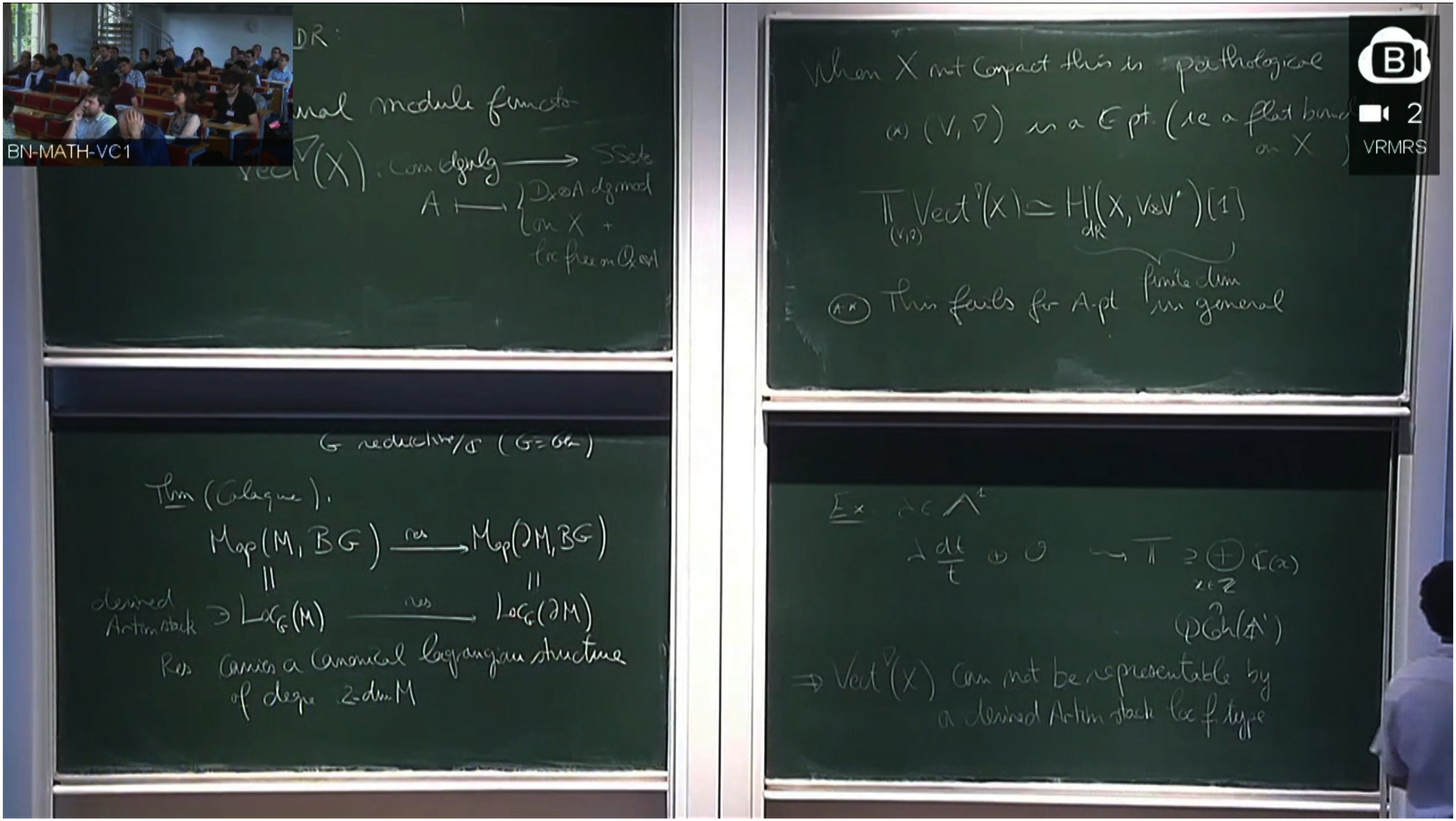
$\cap$

$\mathbb{D}Coh(\mathbb{A}^1)$

$\Rightarrow Vect^{\nabla}(X)$ can not be representable by a derived Artin stack loc f. type



DR:

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$Vect^{\nabla}(X)$: Com dgalg $\longrightarrow$ SSets

$A \longmapsto \left\{ \begin{array}{l} D_X \otimes A \text{ dg mod} \\ \text{on } X + \\ \text{loc free on } O_X \otimes A \end{array} \right.$

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$$QCoh(\mathbb{A}^1)$$

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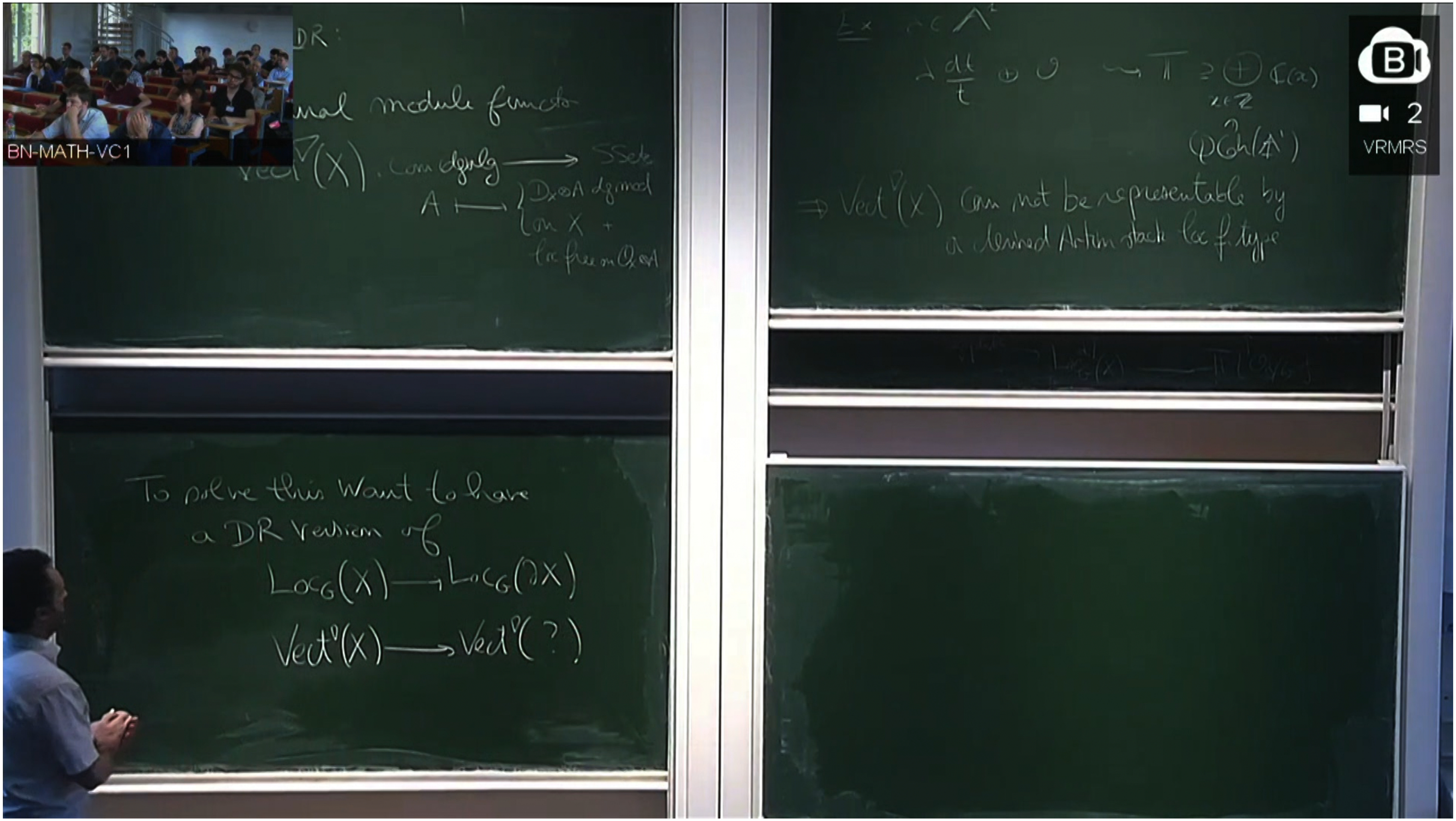


DR:

nal module functor

$\mathrm{Vect}^{\nabla}(X) : \mathrm{cdga} \longrightarrow \mathrm{SSets}$

$A \longmapsto \left\{ \begin{array}{l} D_X \otimes A \text{ dg mod} \\ \text{on } X \; + \\ \text{loc free on } \mathcal{O}_X \otimes A \end{array} \right.$

Ex $\subset \mathbb{A}^1$

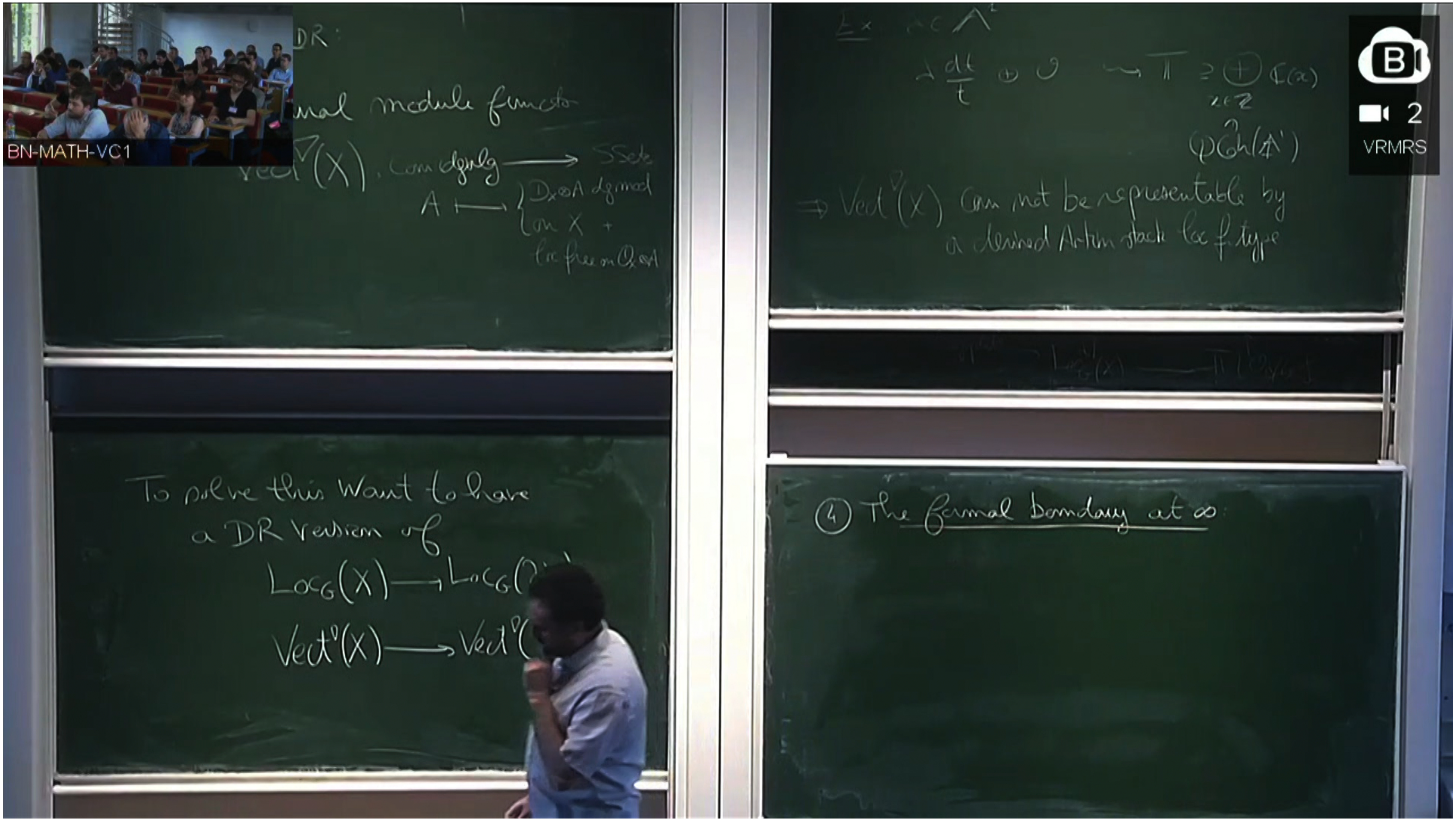
$\lambda \frac{dt}{t} \oplus \mathcal{O} \rightsquigarrow \mathbb{T} \simeq \bigoplus_{x \in \mathbb{Z}} \mathbb{C}(x) \overset{?}{\subset} \mathcal{D}Coh(\mathbb{A}^1)$

$\Rightarrow \mathrm{Vect}^{\nabla}(X)$ can not be representable by a derived Artin stack loc f. type

To solve this want to have a DR version of

$\mathrm{Loc}_G(X) \longrightarrow \mathrm{Loc}_G(\partial X)$

$\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\,?\,)$



DR:

mal module functor

$\mathrm{Vect}^{\nabla}(X) : \mathrm{Com\ dgalg} \longrightarrow \mathrm{SSets}$

$A \longmapsto \{ D_X \otimes A$ dg mod on X + loc free on $\mathcal{O}_X \otimes A$

Ex $x \in \mathbb{A}^1$

$\lambda \frac{dt}{t} + \partial \rightsquigarrow \Pi \cong \bigoplus_{x \in \mathbb{Z}} \mathbb{C}(x)$

$\cap$

$\mathrm{QCoh}(\mathbb{A}^1)$

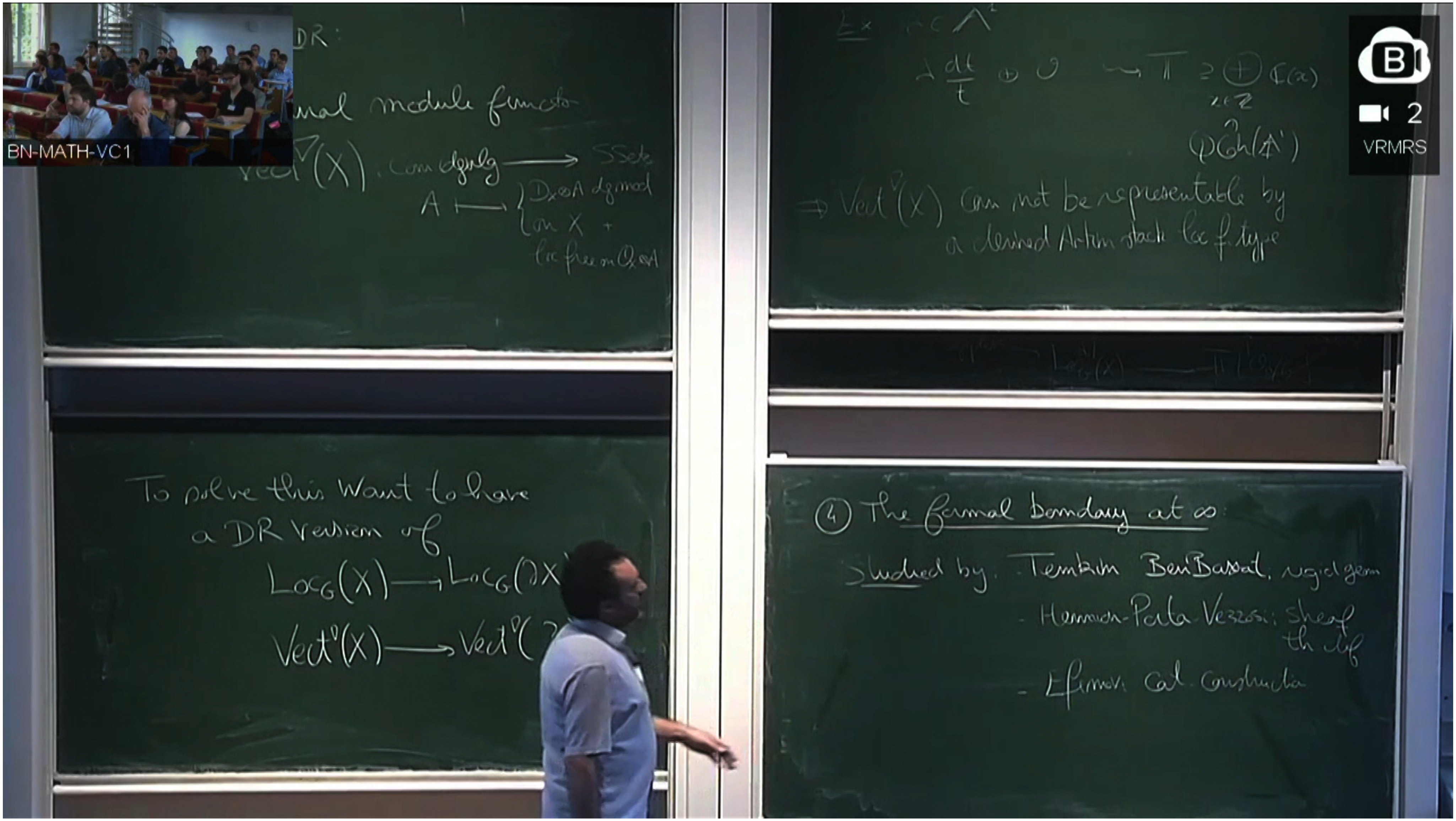
$\Rightarrow \mathrm{Vect}^{\nabla}(X)$ can not be representable by a derived Artin stack loc f. type

To solve this want to have a DR version of

$\mathrm{Loc}_G(X) \longrightarrow \mathrm{Loc}_G($

$\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}($

④ The formal boundary at ∞



DR:

nal module functo

$\mathrm{Vect}^{\nabla}(X)$: com dgalg $\longrightarrow$ SSets

$A \longmapsto \left\{ \begin{array}{l} D_X \otimes A \text{ dg mod} \\ \text{on } X \; + \\ \text{loc free on } \mathcal{O}_X \otimes A \end{array} \right.$

To solve this Want to have a DR version of

$$\mathrm{Loc}_G(X) \longrightarrow \mathrm{Loc}_G(\partial X$$

$$\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}($$

Ex $X \subset \mathbb{A}^1$

$$\lambda \frac{dt}{t} \oplus \partial \rightsquigarrow \Pi \supseteq \bigoplus_{x \in \mathbb{Z}} \mathbb{C}(x)$$

$$\overset{?}{\cap}$$

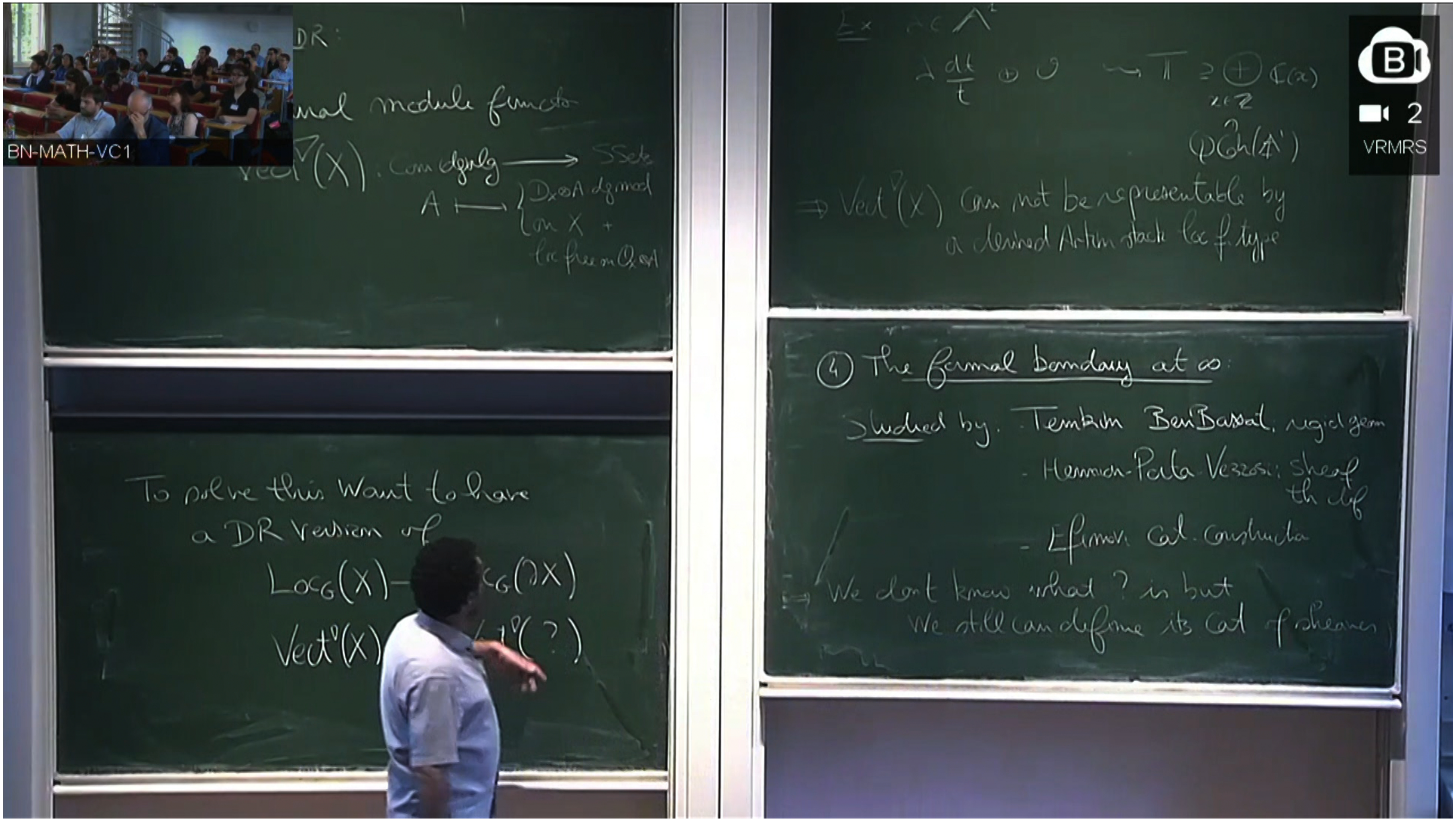
$$\varphi \, \mathrm{Coh}(\mathbb{A}^1)$$

$\Rightarrow$ $\mathrm{Vect}^{\nabla}(X)$ can not be representable by a derived Artin stack loc f. type

④ The formal boundary at ∞:

Studied by:

- Temkin Ben Bassat, rigid geom
- Hennion-Porta-Vezzosi: sheaf th dif
- Efimov: cat. construction



DR:

mal module functo

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To solve this want to have
a DR version of

$\mathrm{Loc}_G(X) -$ … $_G(\partial X)$

$\mathrm{Vect}^{\nabla}(X)$ … $^{\nabla}(\ ?\ )$

Ex $x \in \mathbb{A}^1$

$\lambda \frac{dt}{t} \oplus \mathcal{O} \rightsquigarrow \Pi \supseteq \bigoplus_{x \in \mathbb{Z}} \mathbb{C}(x)$

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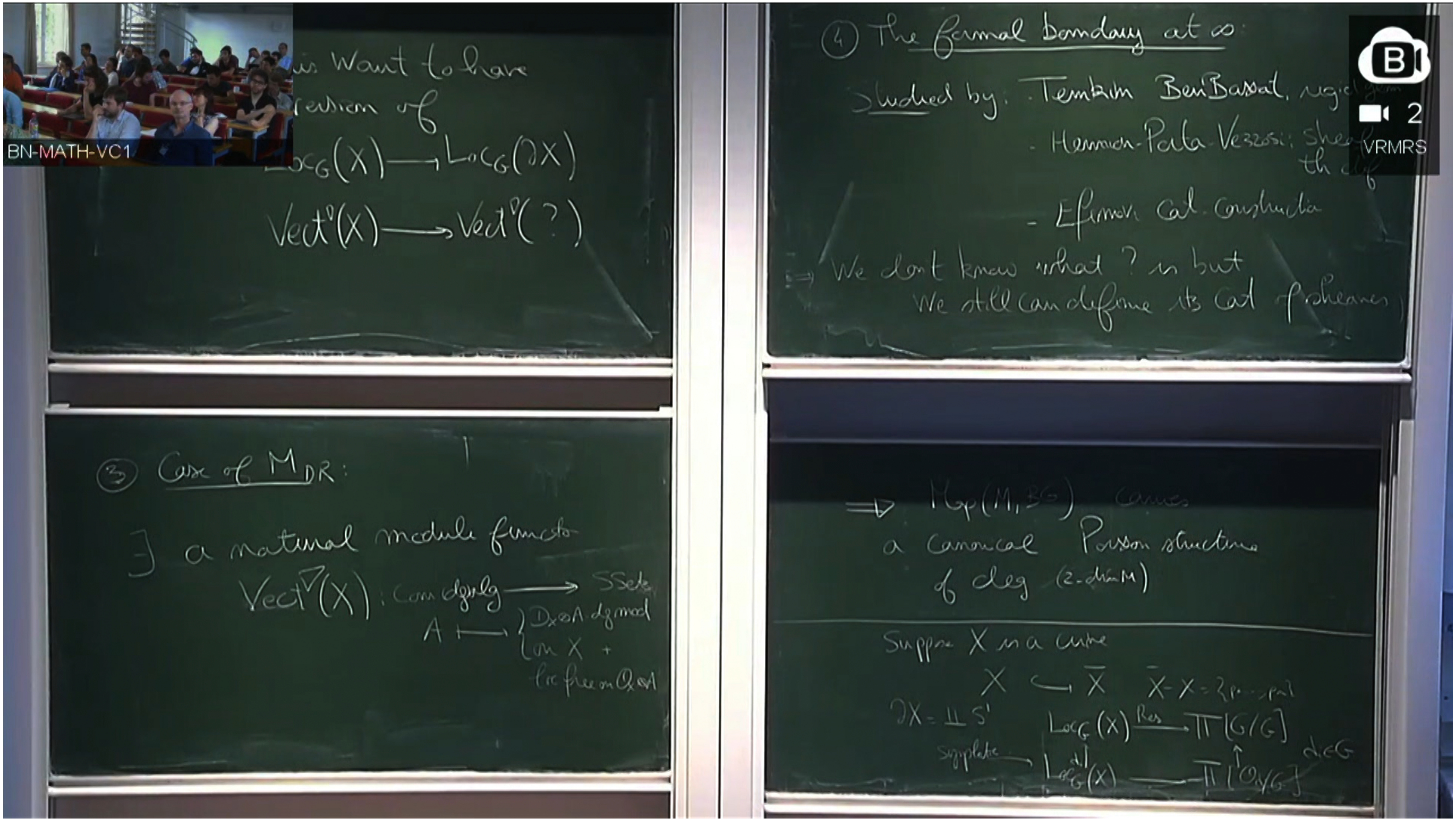
⇒ We don't know what ? is but
We still can define its cat of sheaves



is Want to have
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$$\mathrm{Loc}_G(X) \longrightarrow \mathrm{Loc}_G(\partial X)$$

$$\mathrm{Vect}^\nabla(X) \longrightarrow \mathrm{Vect}^\nabla(\,?\,)$$

③ Case of M_{DR}:

∃ a natural module functor

$$\mathrm{Vect}^\nabla(X) : \text{Com dgalg} \longrightarrow \text{SSets}$$

$$A \longmapsto \left\{\begin{array}{l} D_X\otimes A\text{-dgmod} \\ \text{on } X \ + \\ \text{loc free on } \mathcal{O}_X\otimes A \end{array}\right.$$

④ The formal boundary at ∞:

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· Efimov cat. construction

⟹ We don't know what ? is but
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⟹ $\mathrm{Map}(M, BG)$ carries
a canonical Poisson structure
of deg $(2-\dim M)$

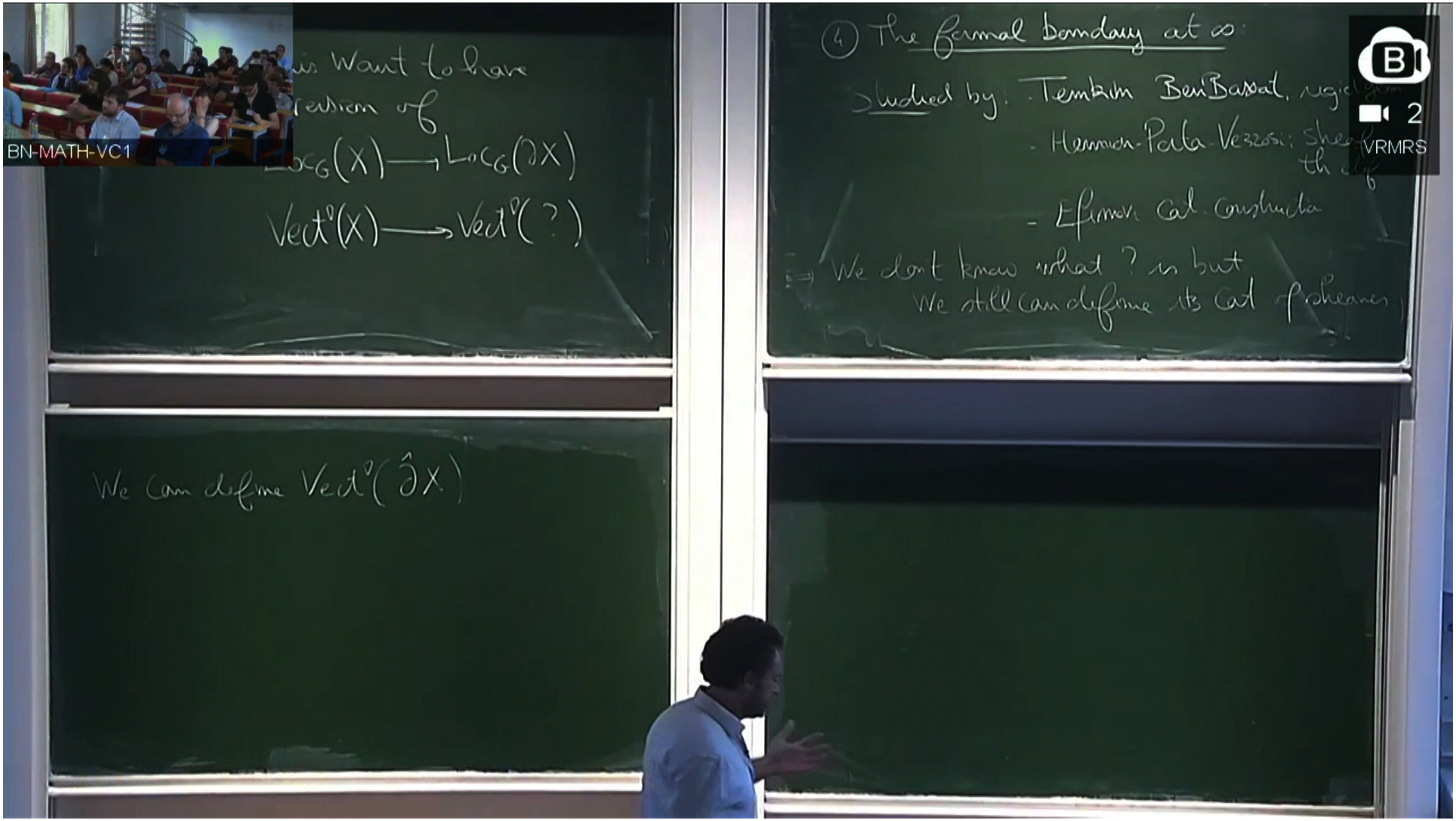
Suppose X is a curve

$$X \hookrightarrow \bar{X} \qquad \bar{X} - X = \{p_1, \dots, p_n\}$$

$\partial X = \amalg S^1$

$$\begin{tikzcd} & \mathrm{Loc}_G(X) \arrow[r, "\mathrm{Res}"] & \prod [G/G] \\ \text{symplectic} \arrow[r] & \mathrm{Loc}_G(X) \arrow[u, hook] \arrow[r] & \prod [\mathcal{O}_i/G] \arrow[u] \end{tikzcd}$$

$\mathcal{O}_i \subset G$



is Want to have
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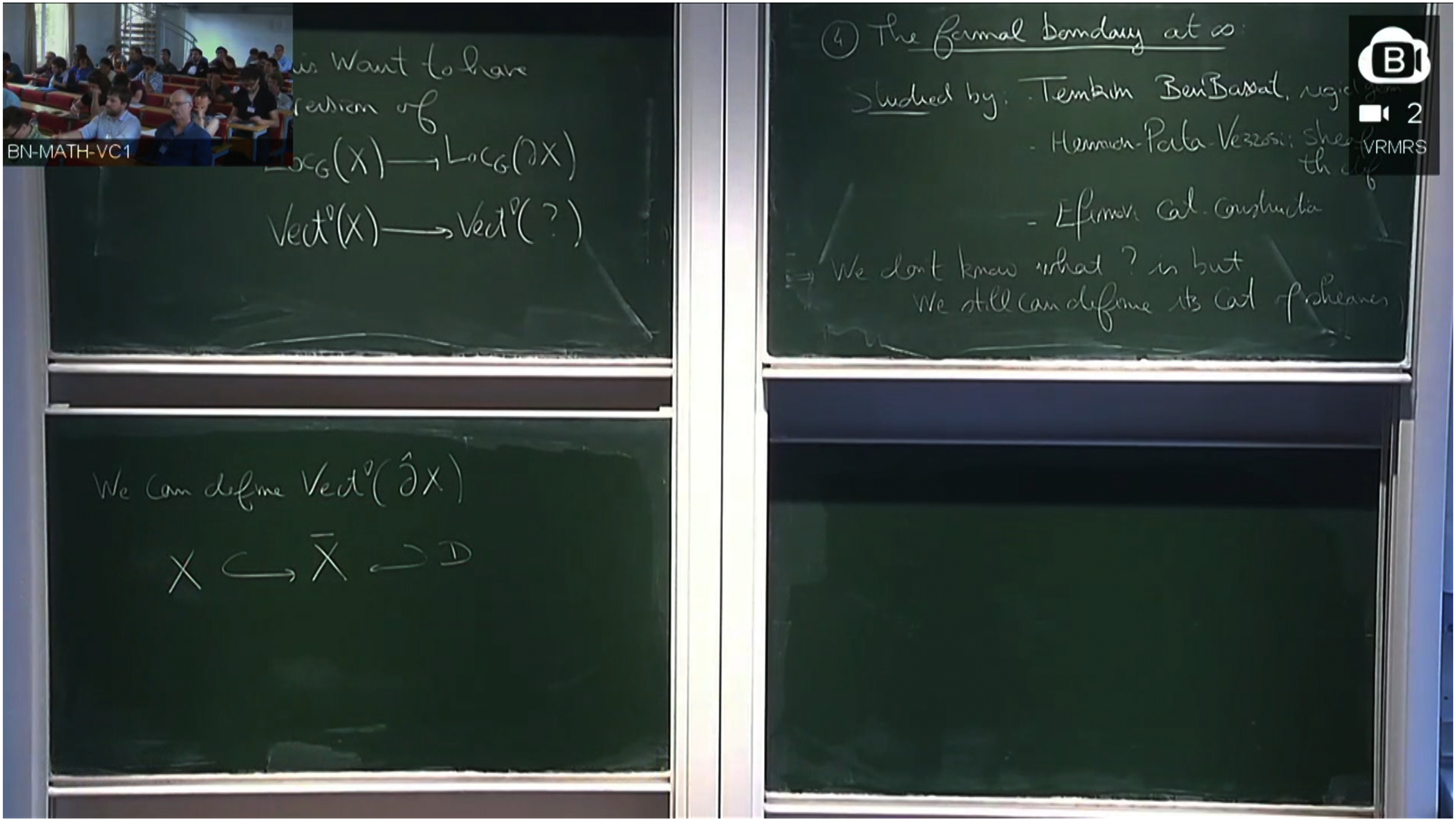
We can define $\mathrm{Vect}^0(\hat{\partial} X)$



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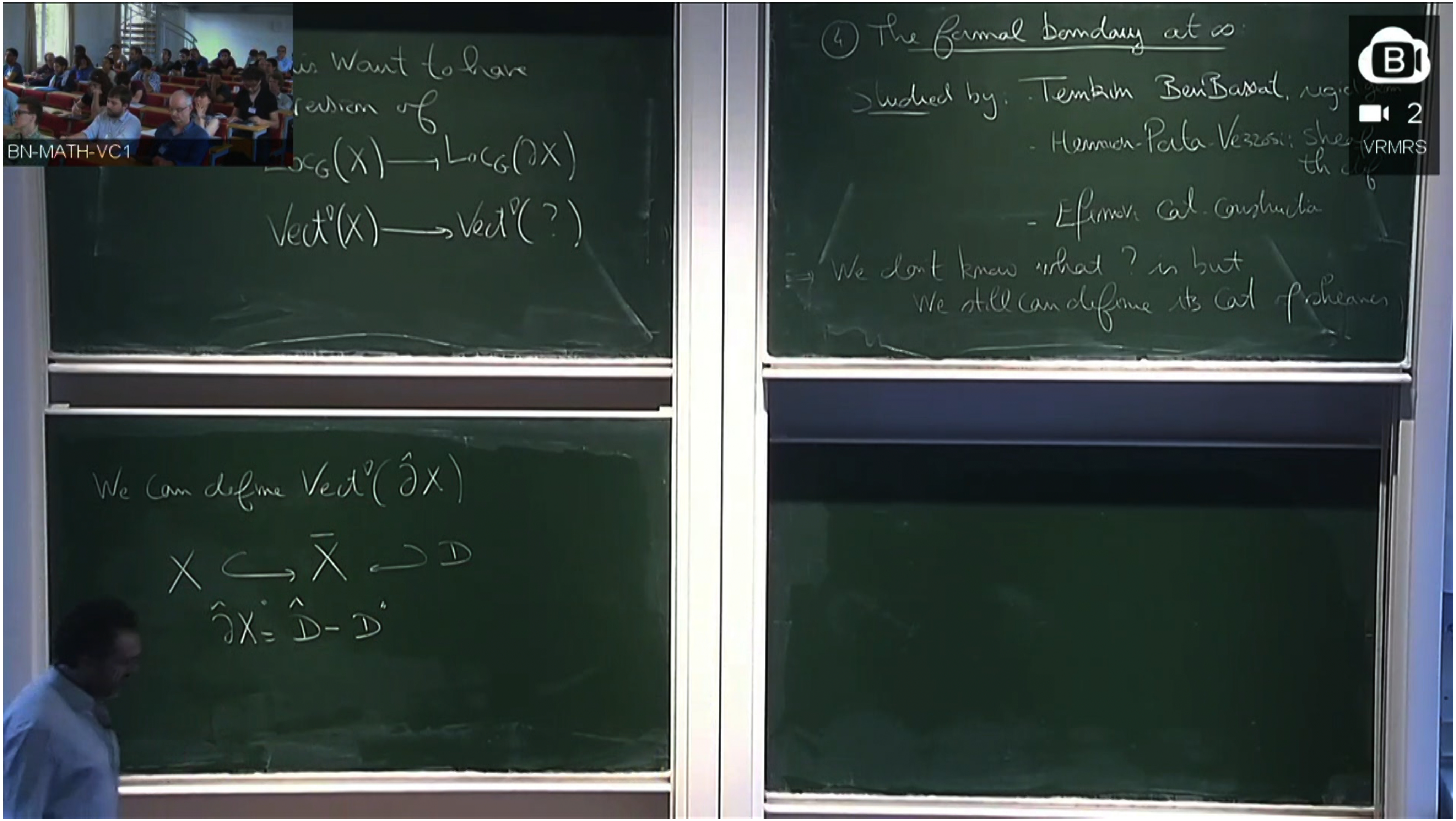

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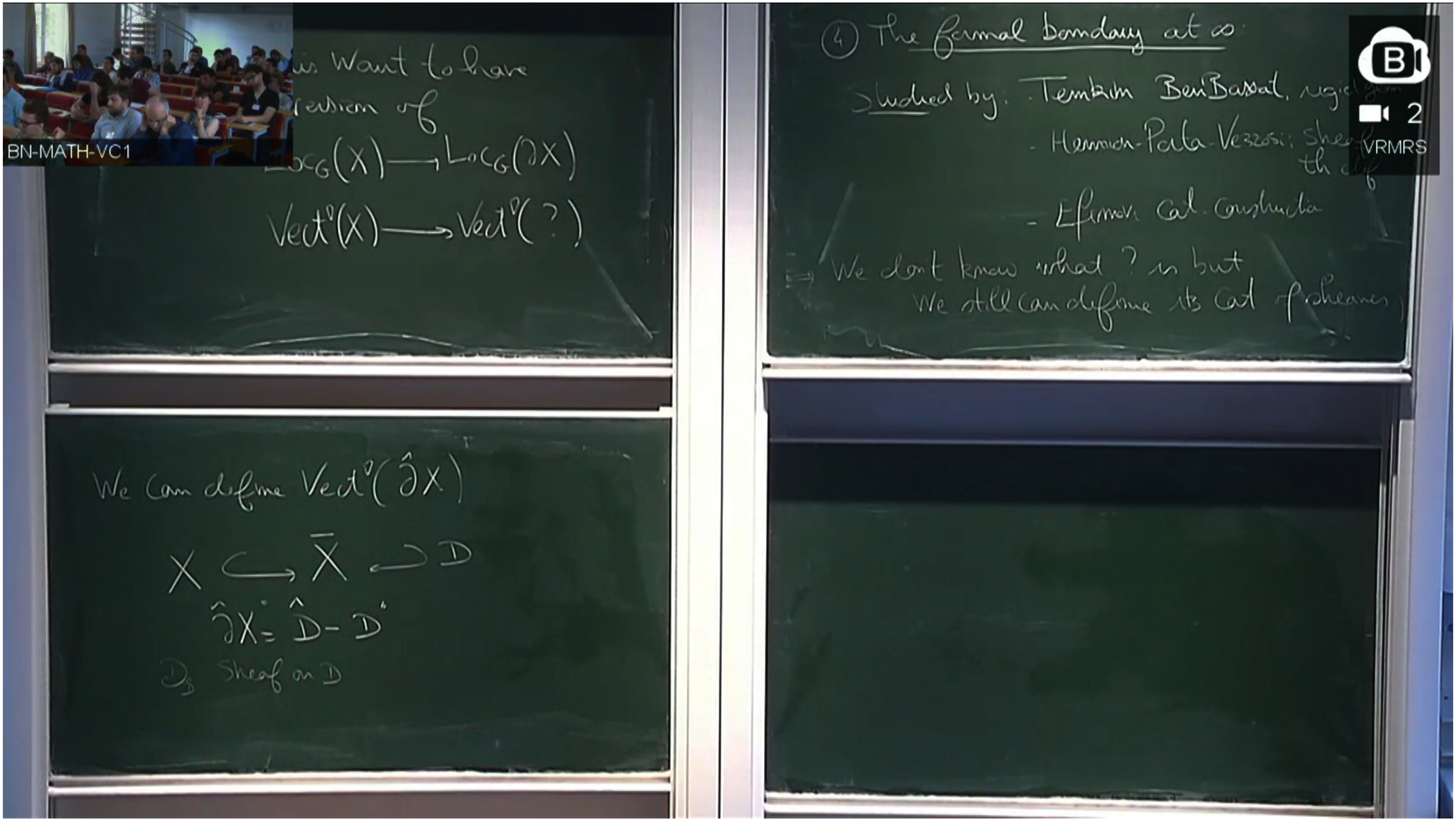

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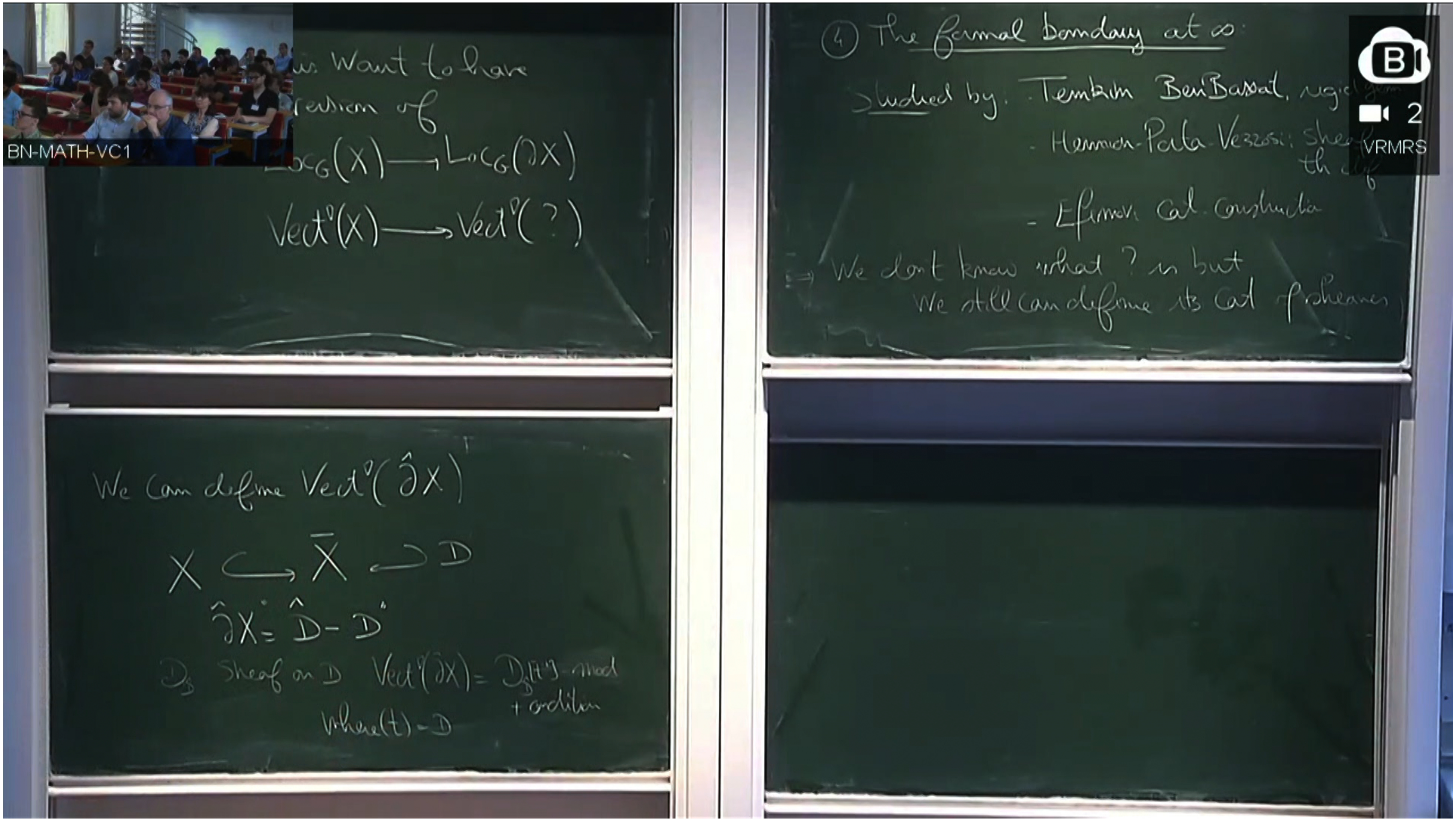
We can define $\mathrm{Vect}^0(\hat{\partial} X)$

$$X \hookrightarrow \bar{X} \hookleftarrow D$$

$$\hat{\partial} X := \hat{D} - D$$

$\mathcal{O}_D$ Sheaf on D



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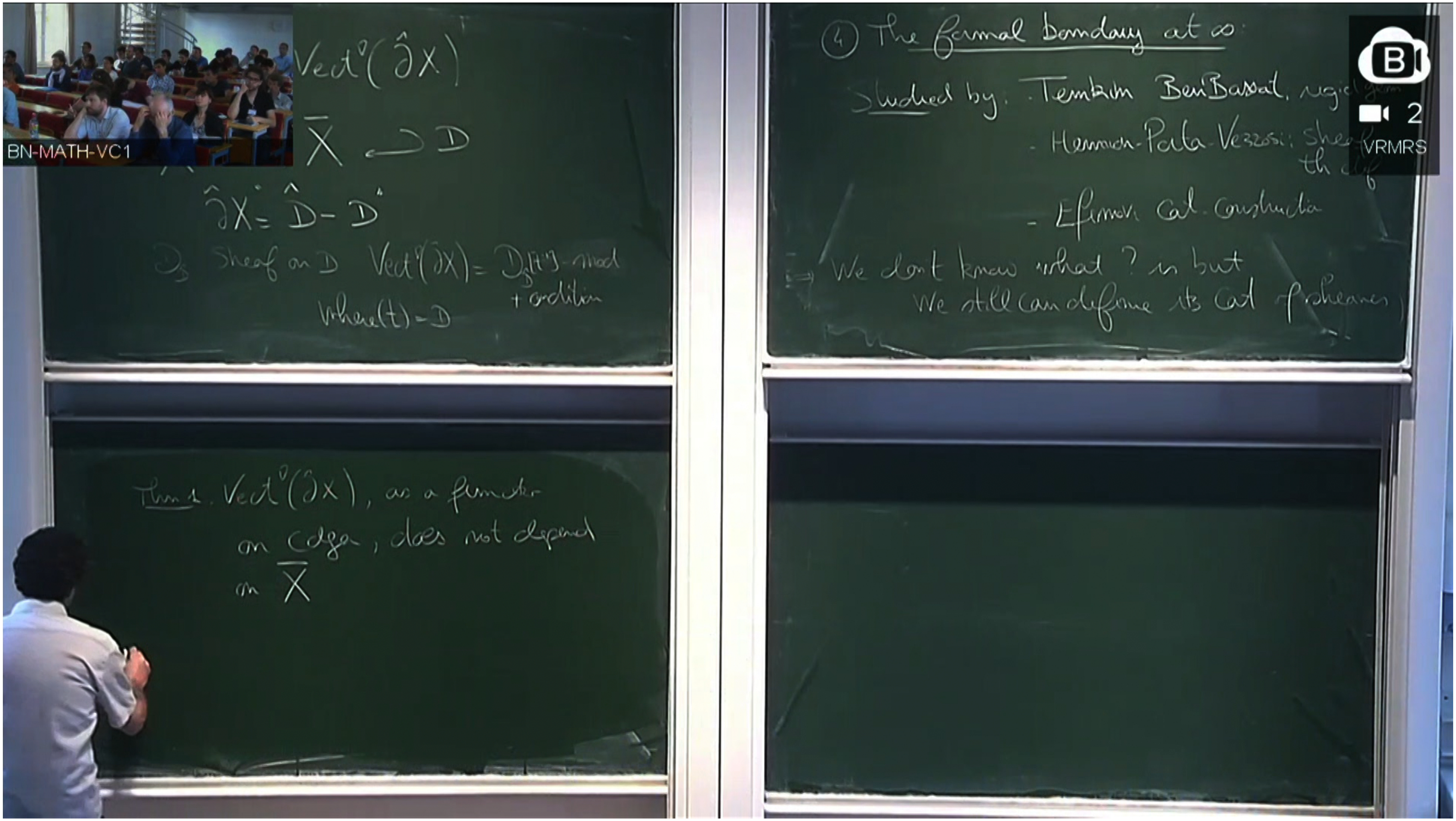
$\mathcal{D}_D$ sheaf on D $\mathrm{Vect}^0(\partial X) = \mathcal{D}_D[t^{-1}]$-mod + condition

where $(t) = D$



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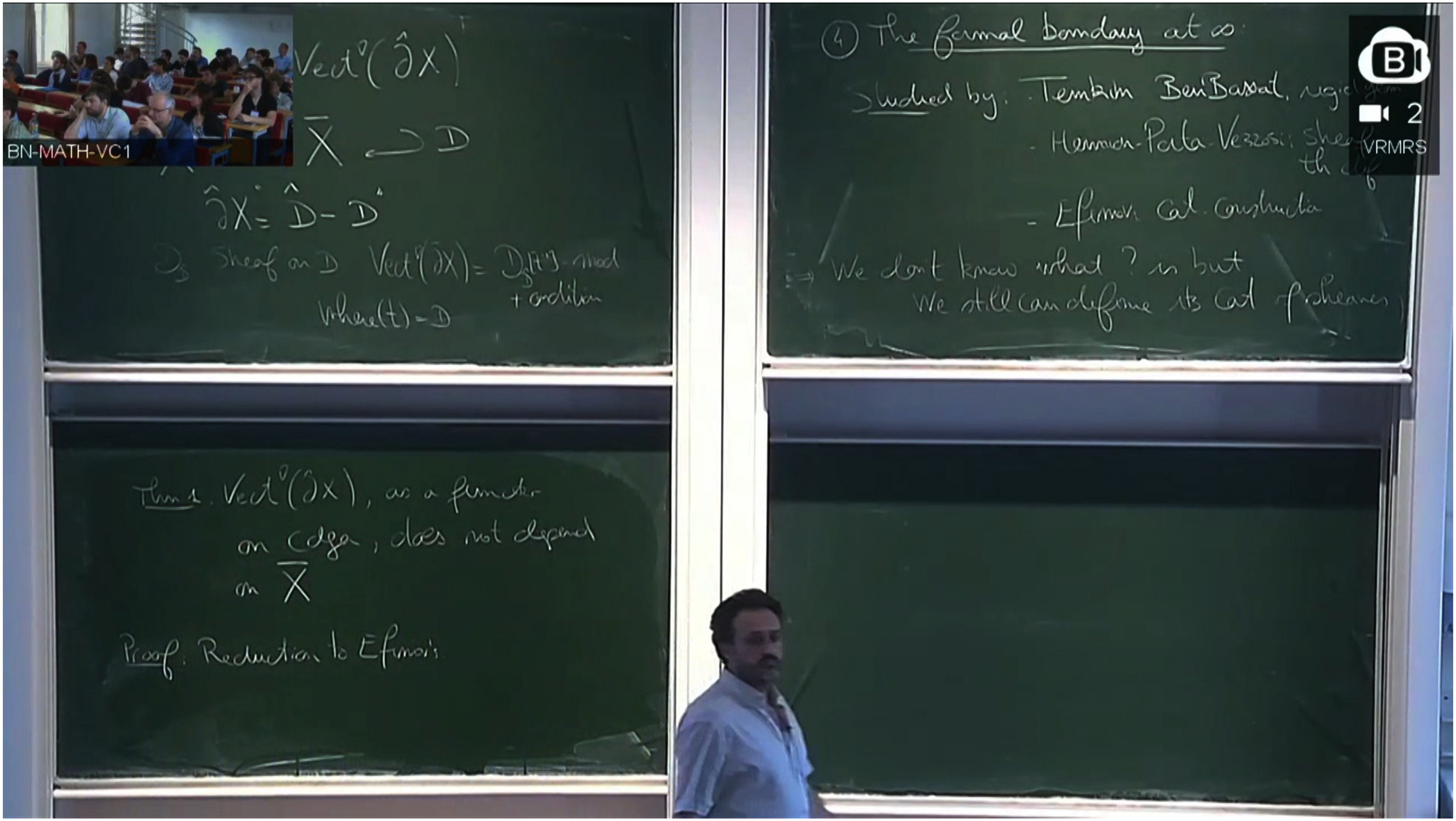
Thm 1. $\mathrm{Vect}^0(\partial X)$, as a functor
on Cdga, does not depend
on $\bar{X}$



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Thm 1. $\mathrm{Vect}^0(\partial X)$, as a functor on Cdga, does not depend on $\bar{X}$

Proof: Reduction to Efimov's

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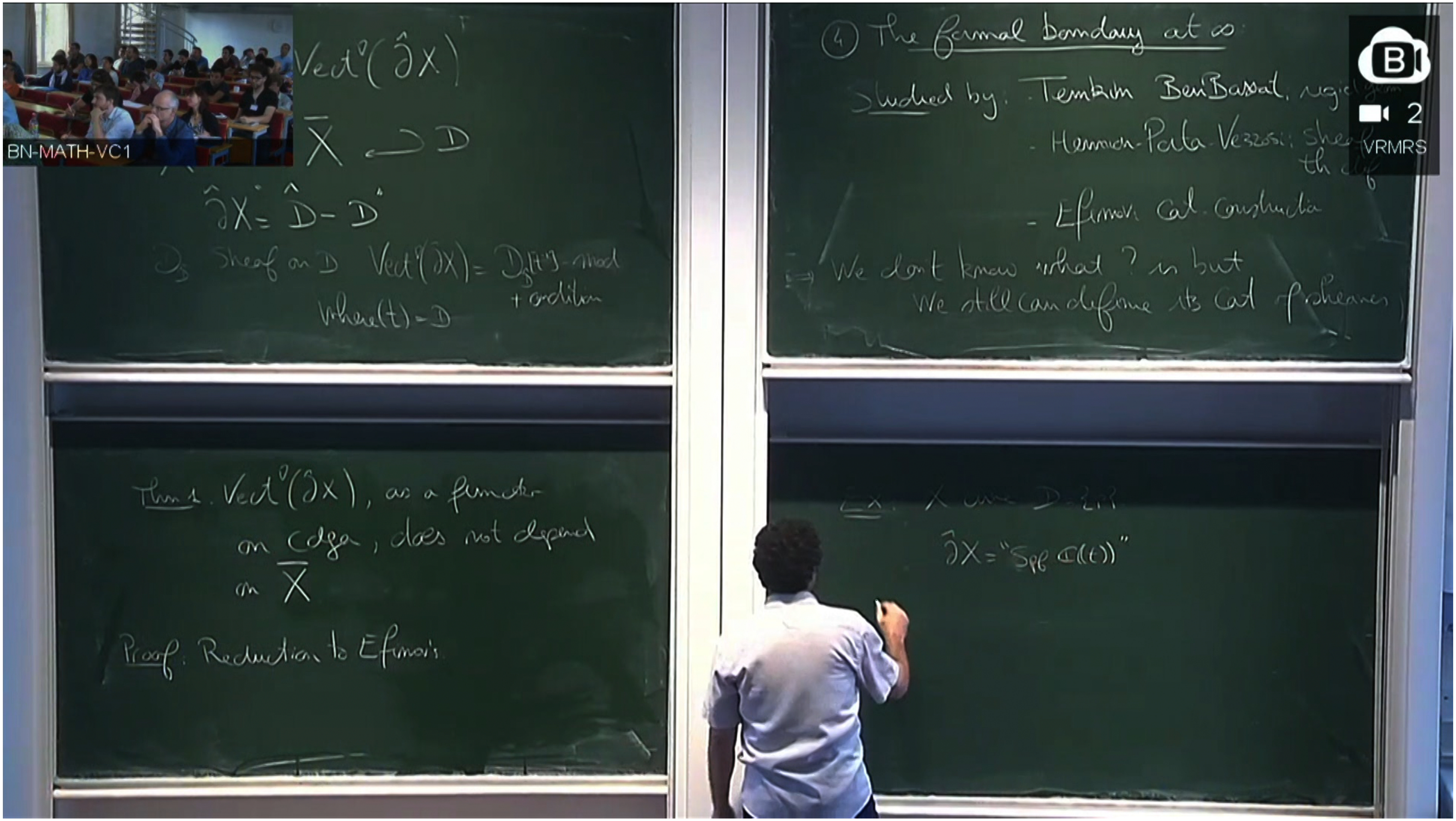

$\mathrm{Vect}^0(\hat{\partial} X)$

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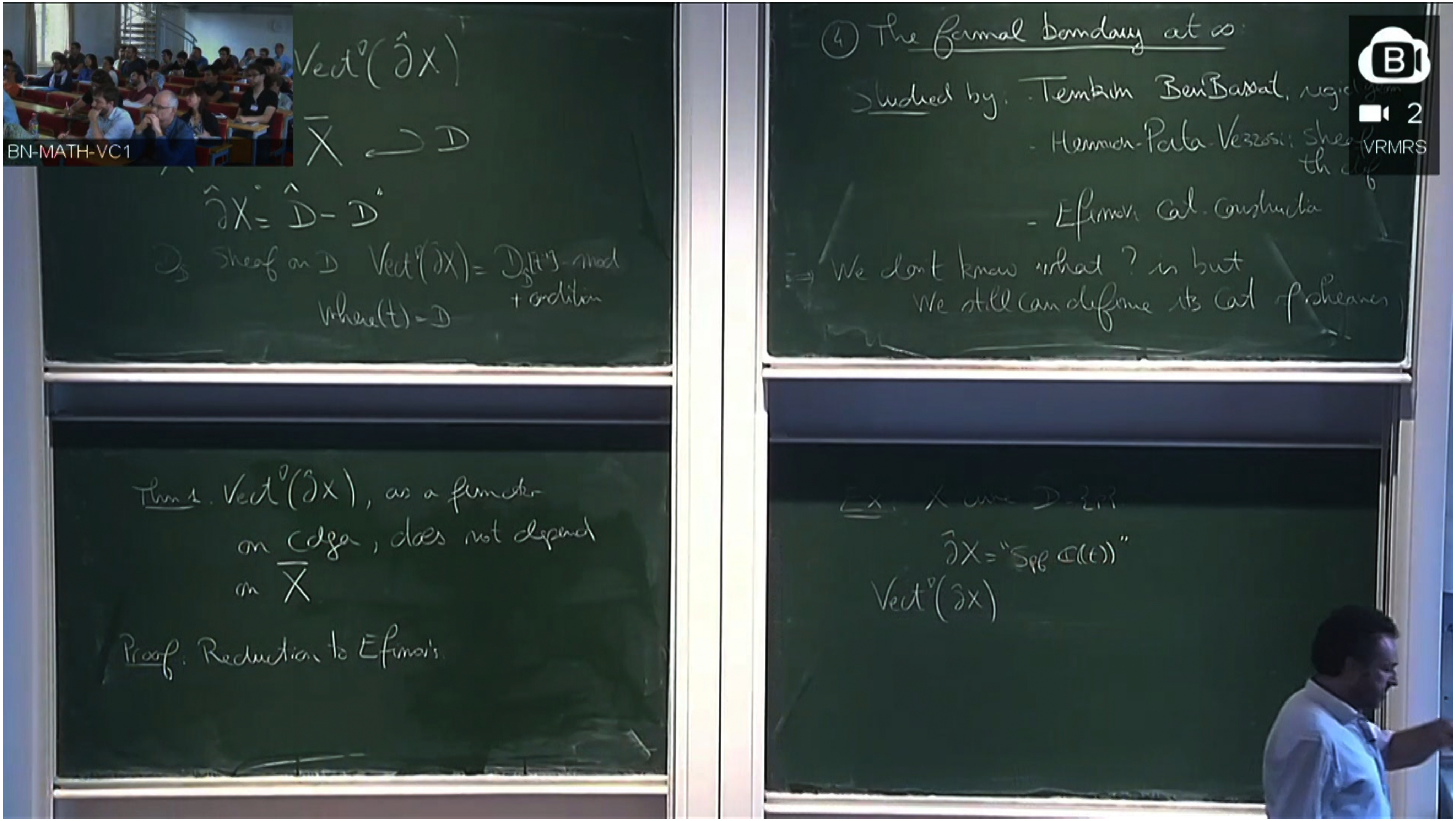
Ex. X curve D = {p}

$\hat{\partial} X =$ "Spf $\mathbb{C}((t))$"



$\mathrm{Vect}^0(\hat{\partial} X)$

$\bar{X} \hookleftarrow D$

$\hat{\partial} X^{\circ} = \hat{D} - D^{\circ}$

$\mathcal{D}_D$ Sheaf on D $\mathrm{Vect}^0(\partial X) = \mathcal{D}_D[t^{-1}]$-mod

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Proof: Reduction to Efimov's

Ex. X curve D = {p}

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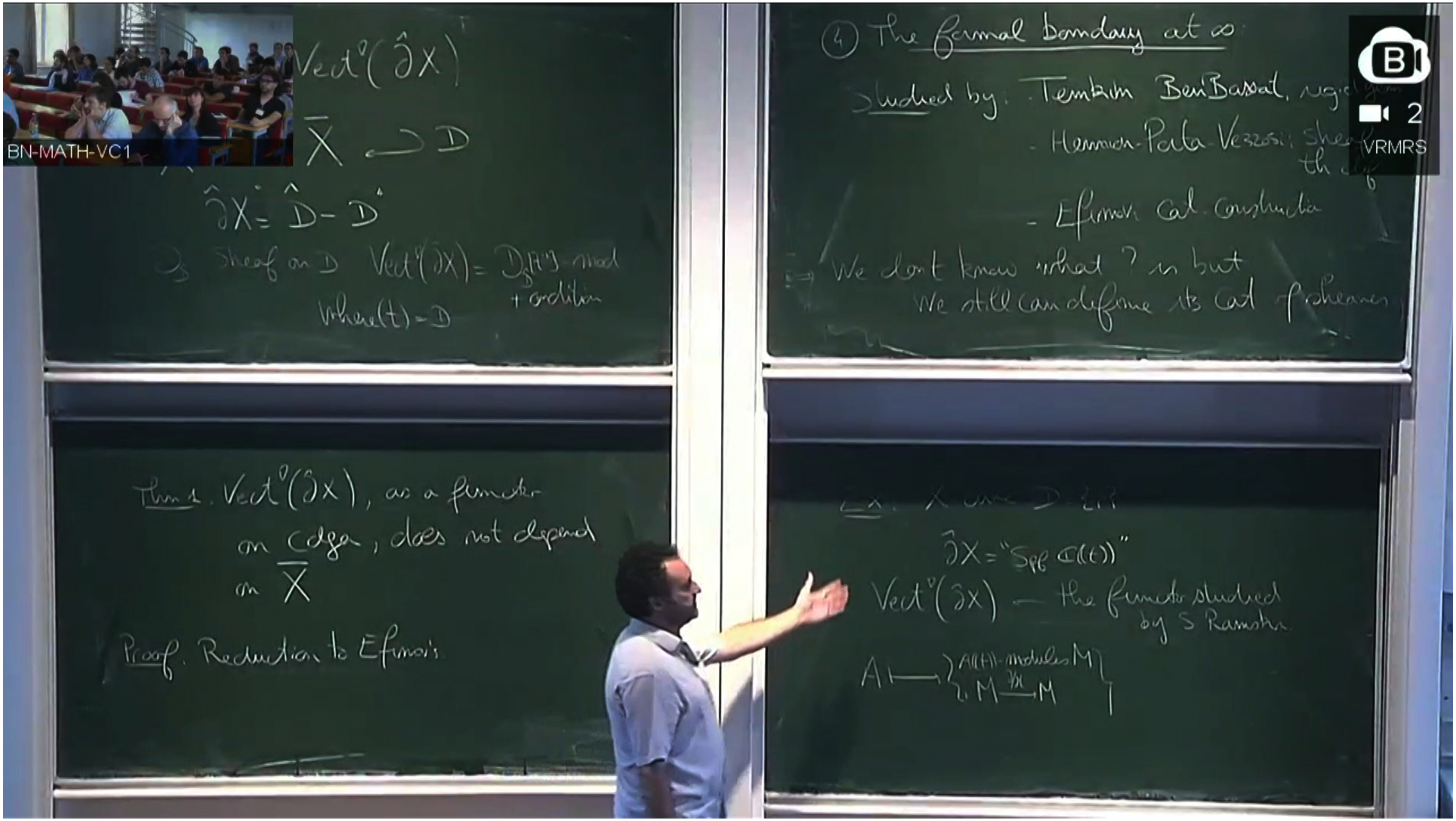

$\mathrm{Vect}^0(\hat{\partial} X)$

$\bar{X} \hookleftarrow D$

$\hat{\partial} X = \hat{D} - \overset{\circ}{D}$

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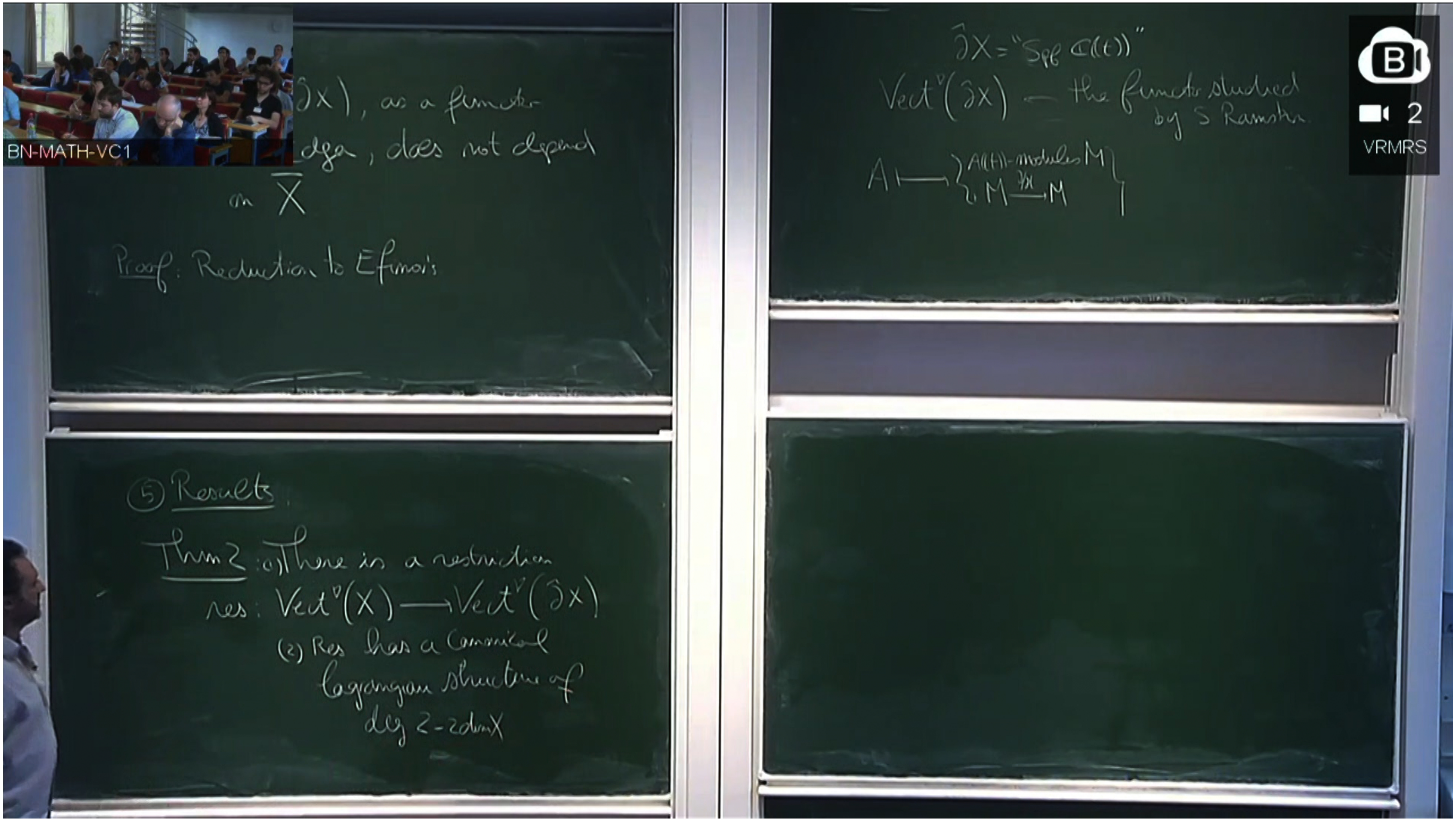
Ex. $X = D - \{0\}$

$\hat{\partial} X =$ "Spf $\mathbb{C}((t))$"

$\mathrm{Vect}^0(\partial X)$ — the functor studied by S Raskin

$A \longmapsto \left\{ \begin{array}{l} A((t))\text{-modules } M \\ M \xrightarrow{\partial_t} M \end{array} \right\}$



∂X), as a functor
dga, does not depend
on $\overline{X}$

<u>Proof</u>: Reduction to Efimov's

(5) <u>Results</u>:

<u>Thm 2</u>: (1) There is a restriction
res: $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\hat{\partial} X)$
(2) Res has a canonical
lagrangian structure of
deg $2 - 2\dim X$

$\hat{\partial} X =$ "$\mathrm{Spf}\, \mathbb{C}((t))$"

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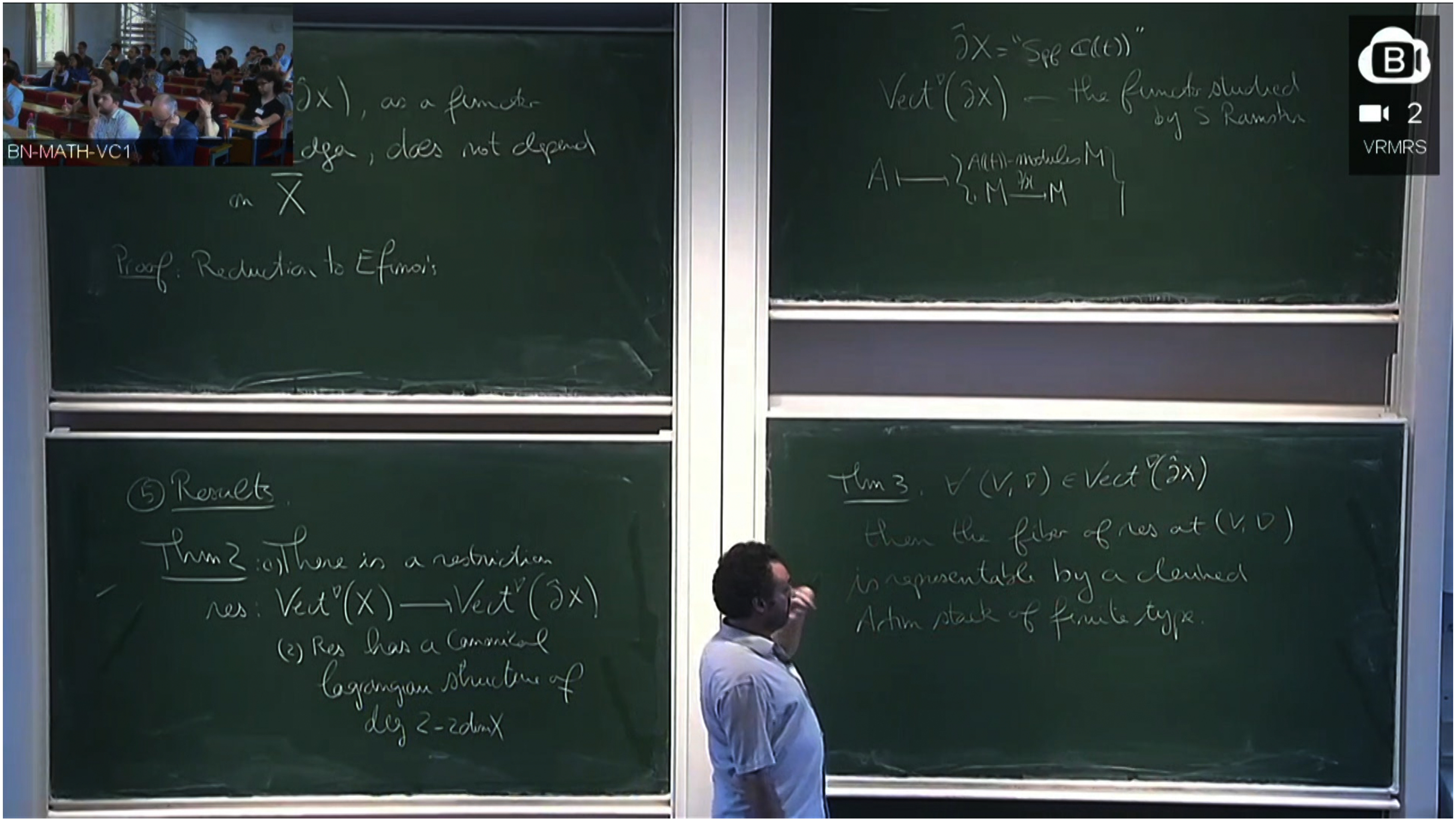

$\partial X)$, as a functor

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<u>Proof</u>: Reduction to Efimov's

⑤ <u>Results</u>.

<u>Thm 2</u>: (1) There is a restriction

res: $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$

(2) Res has a Canonical Lagrangian structure of deg $2-2\dim X$

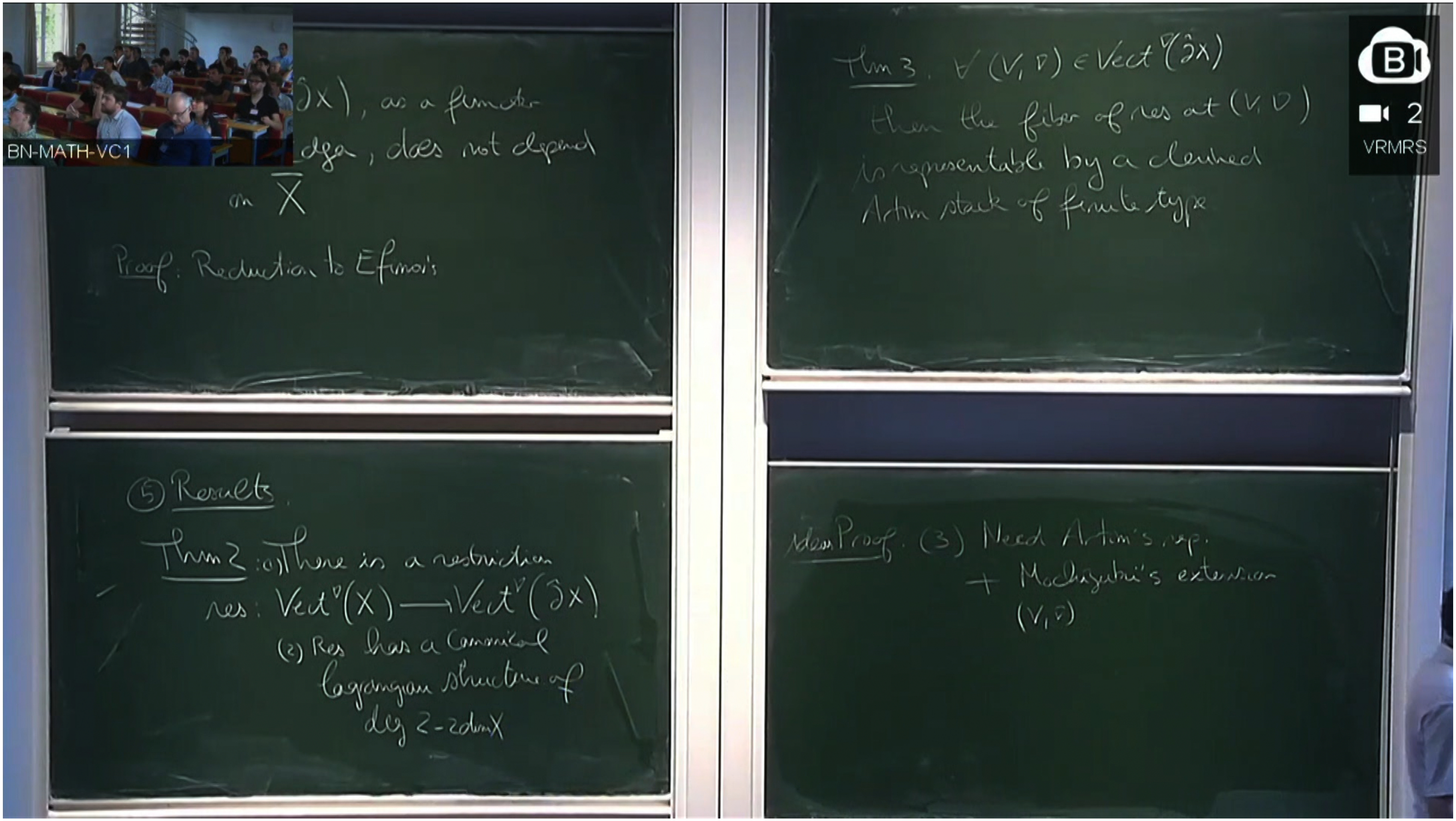
$\hat{\partial} X =$ "$\mathrm{Spf}\ \mathbb{C}((t))$"

$\mathrm{Vect}^{\nabla}(\hat{\partial} X)$ — the functor studied by S Raskin

$$A \longmapsto \left\{ \begin{matrix} A((t))\text{-modules } M \\ \nabla: M \xrightarrow{\partial_t} M \end{matrix} \right\}$$

<u>Thm 3</u>. $\forall (V, \nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$

then the fiber of res at (V, ∇) is representable by a derived Artin stack of finite type.



$\partial X)$, as a functor

dga, does not depend

on $\overline{X}$

Proof: Reduction to Efimov's

⑤ Results.

Thm 2: (1) There is a restriction

$\mathrm{res}: \mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$

(2) Res has a canonical lagrangian structure of deg $2-2\dim X$

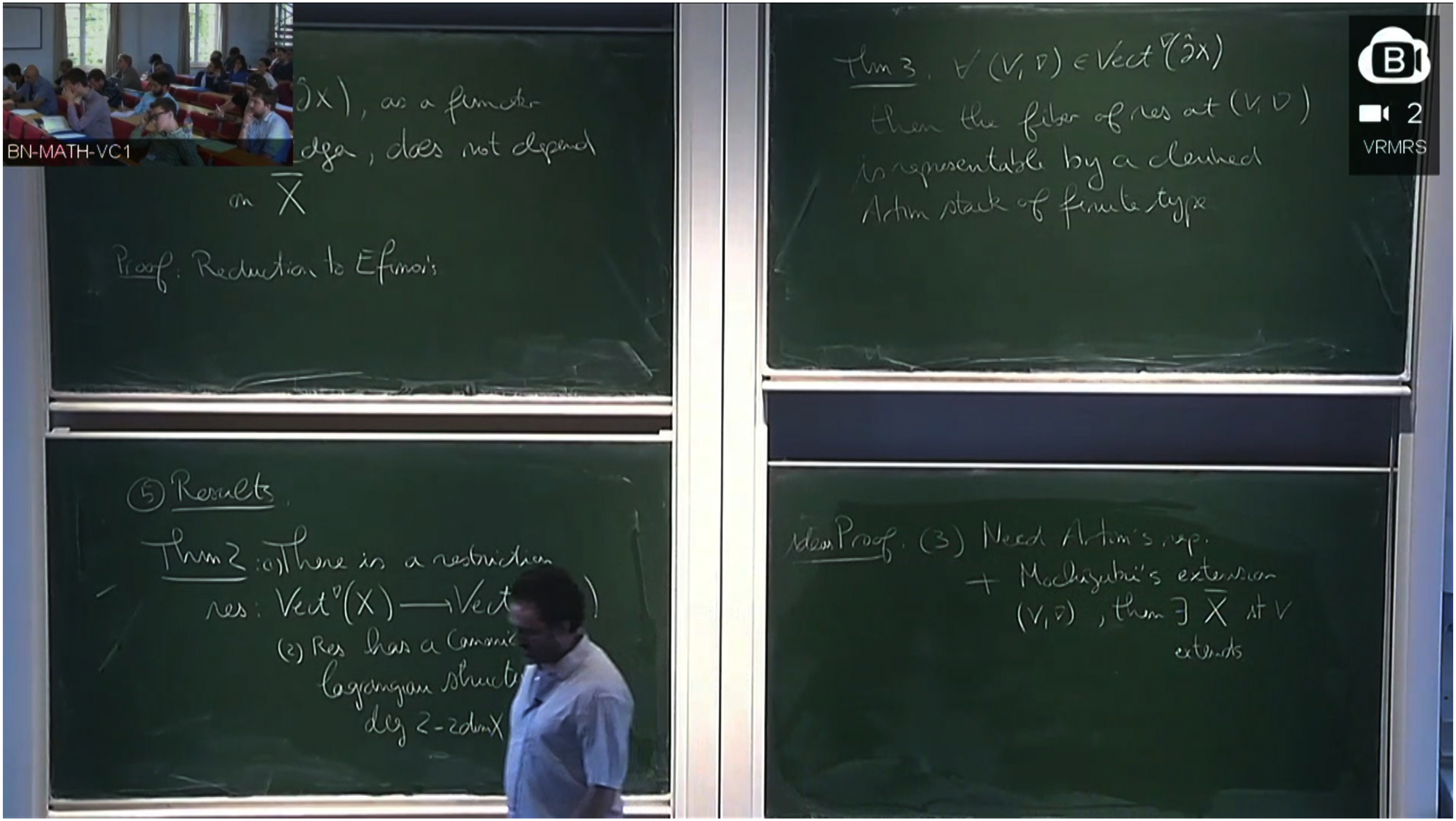
Thm 3. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$

then the fiber of res at (V,∇) is representable by a derived Artin stack of finite type

Idea Proof. (3) Need Artin's rep.

+ Mochizuki's extension

(V,∇)



∂X), as a functor

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Proof: Reduction to Efimov's

⑤ Results.

Thm 2: (1) There is a restriction

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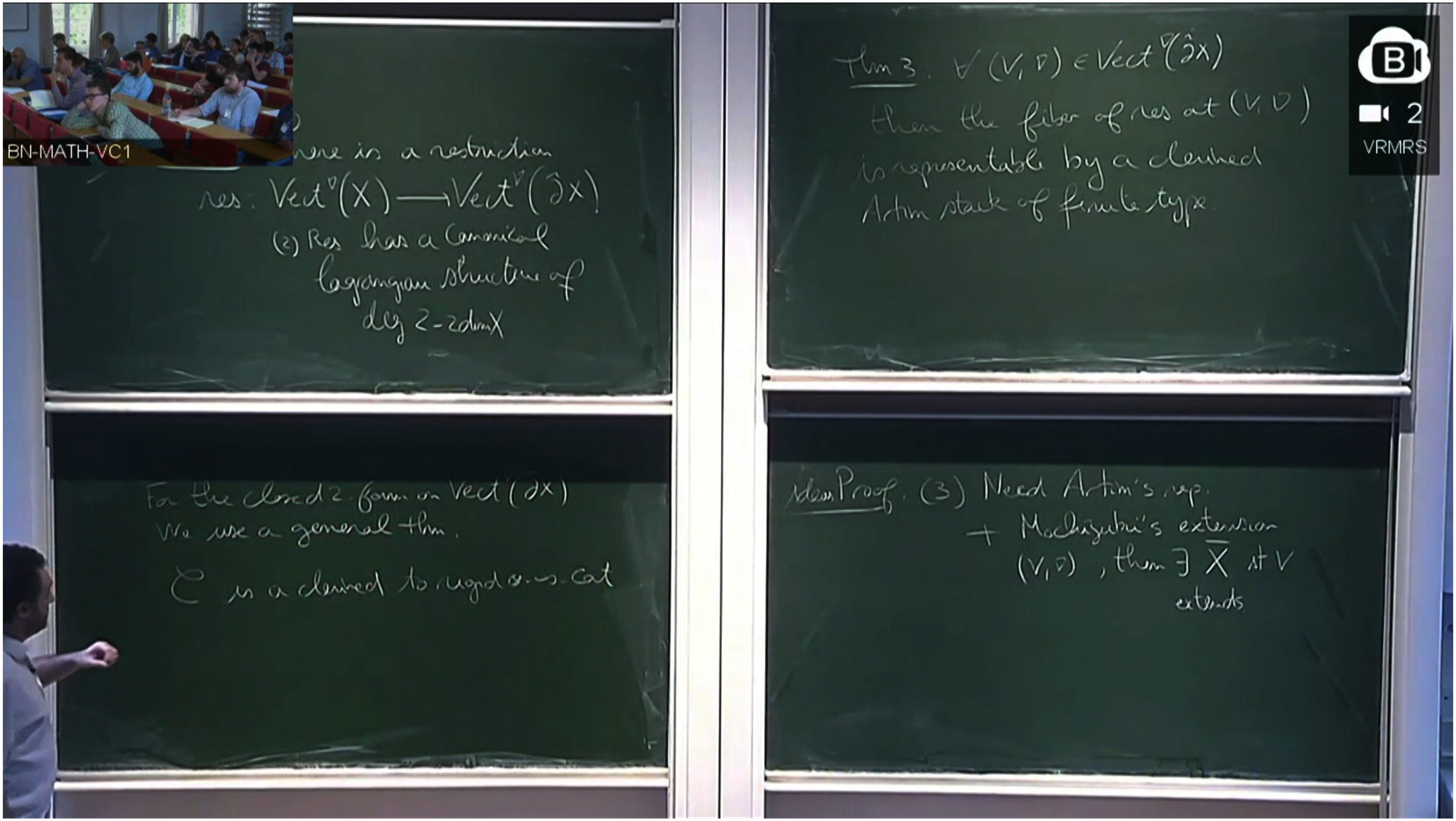
(V,∇), then $\exists\ \overline{X}$ st V

extends



)
…ere is a restriction

res: $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\hat{\partial} X)$

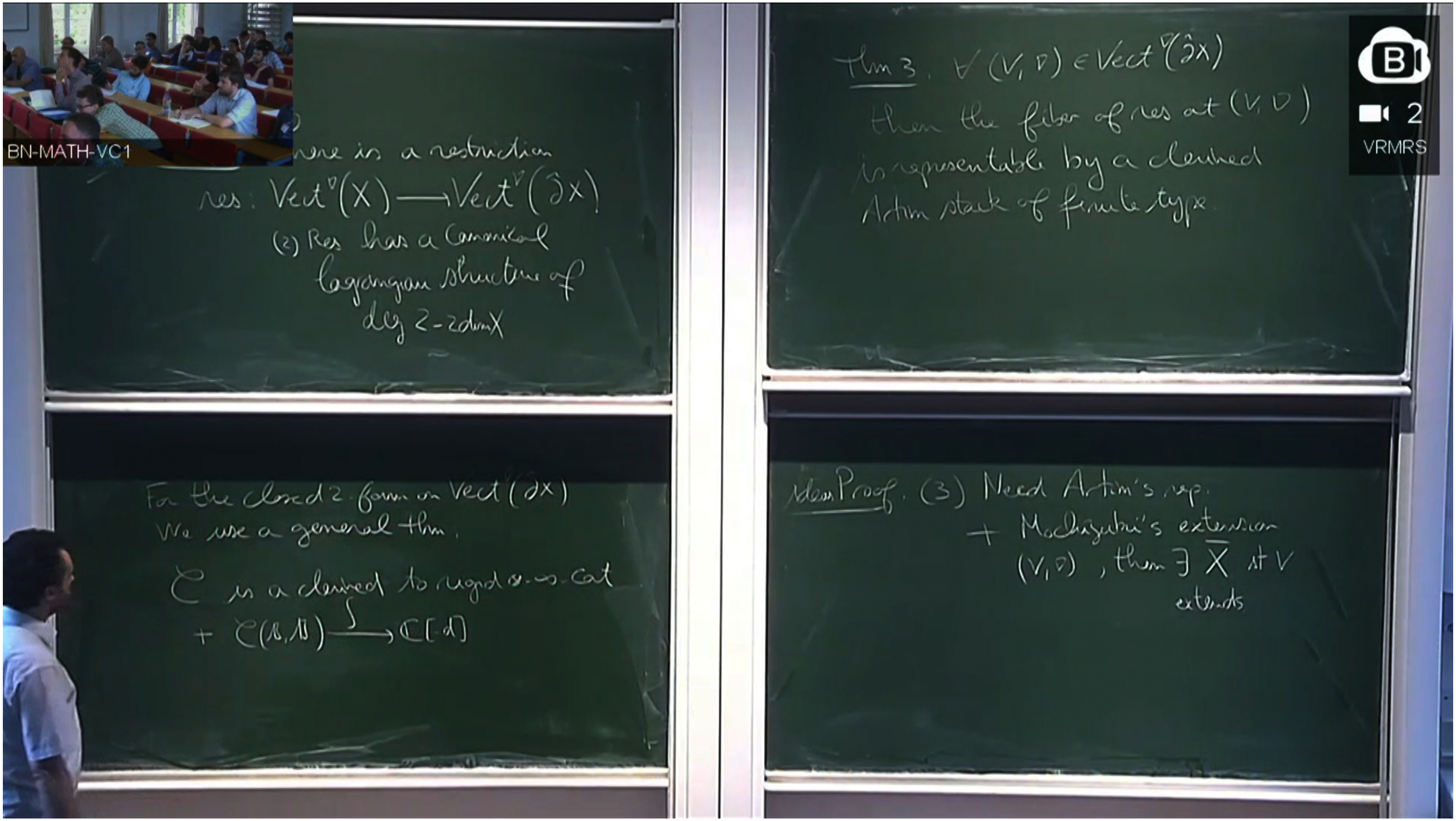
(2) Res has a canonical lagrangian structure of deg $2-2\dim X$

For the closed 2-form on $\mathrm{Vect}^{\nabla}(\partial X)$ we use a general thm.

$\mathcal{C}$ is a derived lo. rigid ⊗-∞-cat

<u>Thm 3</u>. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$ then the fiber of res at (V,∇) is representable by a derived Artin stack of finite type.

<u>Idea Proof</u>. (3) Need Artin's rep.
+ Mochizuki's extension
(V,∇), then $\exists\ \overline{X}$ st V extends



ere is a restriction

res : $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$

(2) Res has a canonical Lagrangian structure of deg $2 - 2\dim X$

For the closed 2-form on $\mathrm{Vect}(\partial X)$ we use a general thm.

$\mathcal{C}$ is a derived to rigid ⊗-∞-cat

$+ \ \mathcal{C}(1,1) \xrightarrow{\int} \mathbb{C}[-d]$

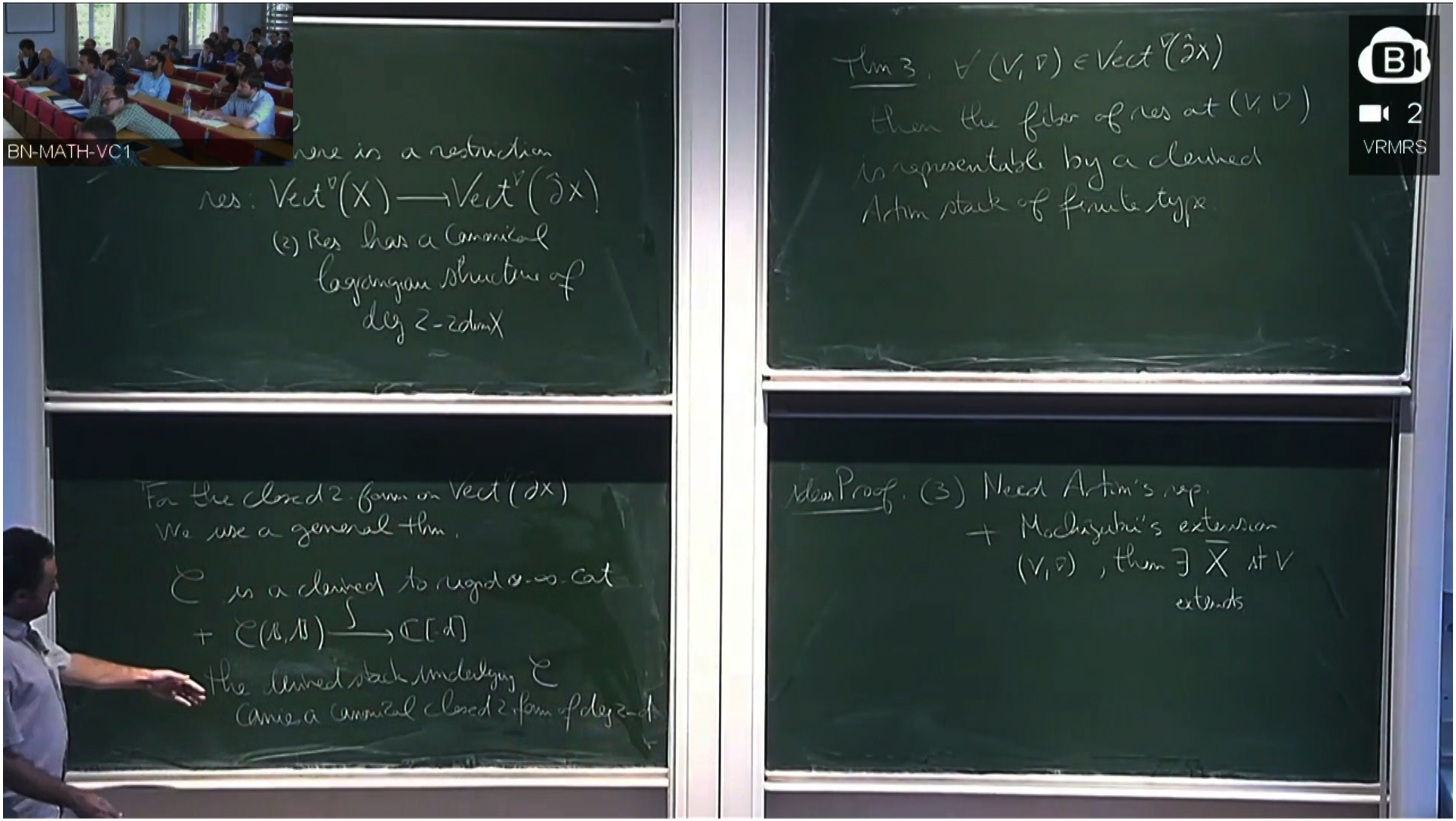
__Thm 3.__ $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$

then the fiber of res at (V,∇) is representable by a derived Artin stack of finite type.

__Idea Proof.__ (3) Need Artin's rep.

$+$ Mochizuki's extension

(V,∇), then $\exists\ \overline{X}$ st V extends



ere is a restriction

res: $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$

(2) Res has a canonical Lagrangian structure of deg $2-2\dim X$

For the closed 2-form on $\mathrm{Vect}(\partial X)$

We use a general thm.

$\mathcal{C}$ is a derived to rigid $\otimes$-∞-cat

+ $\mathcal{C}(1,1) \xrightarrow{\int} \mathbb{C}[-d]$

the derived stack underlying $\mathcal{C}$ carries a canonical closed 2-form of deg $2-d$

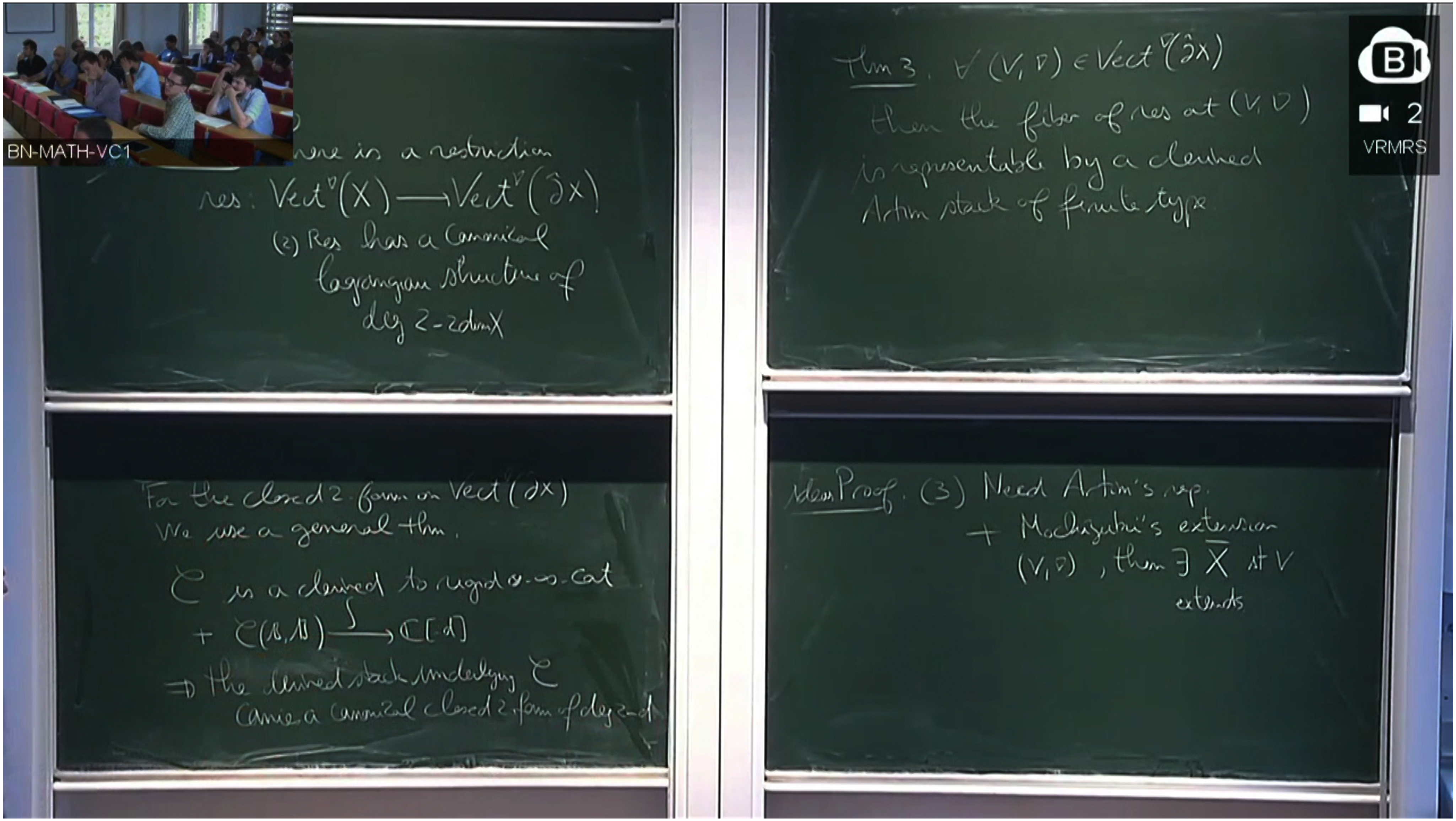
Thm 3. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\partial X)$

then the fiber of res at (V,∇) is representable by a derived Artin stack of finite type

Idea Proof. (3) Need Artin's rep.

+ Mochizuki's extension

(V,∇), then $\exists\ \overline{X}$ st V extends



)

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For the closed 2-form on $\mathrm{Vect}(\partial X)$
We use a general thm.

$\mathcal{C}$ is a derived to rigid ⊗-∞-cat

+ $\mathcal{C}(\mathbb{1},\mathbb{1}) \xrightarrow{\int} \mathbb{C}[-d]$

$\Rightarrow$ the derived stack underlying $\mathcal{C}$
carries a canonical closed 2-form of deg $2-d$

Thm 3. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\partial X)$
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Idea Proof. (3) Need Artin's rep.
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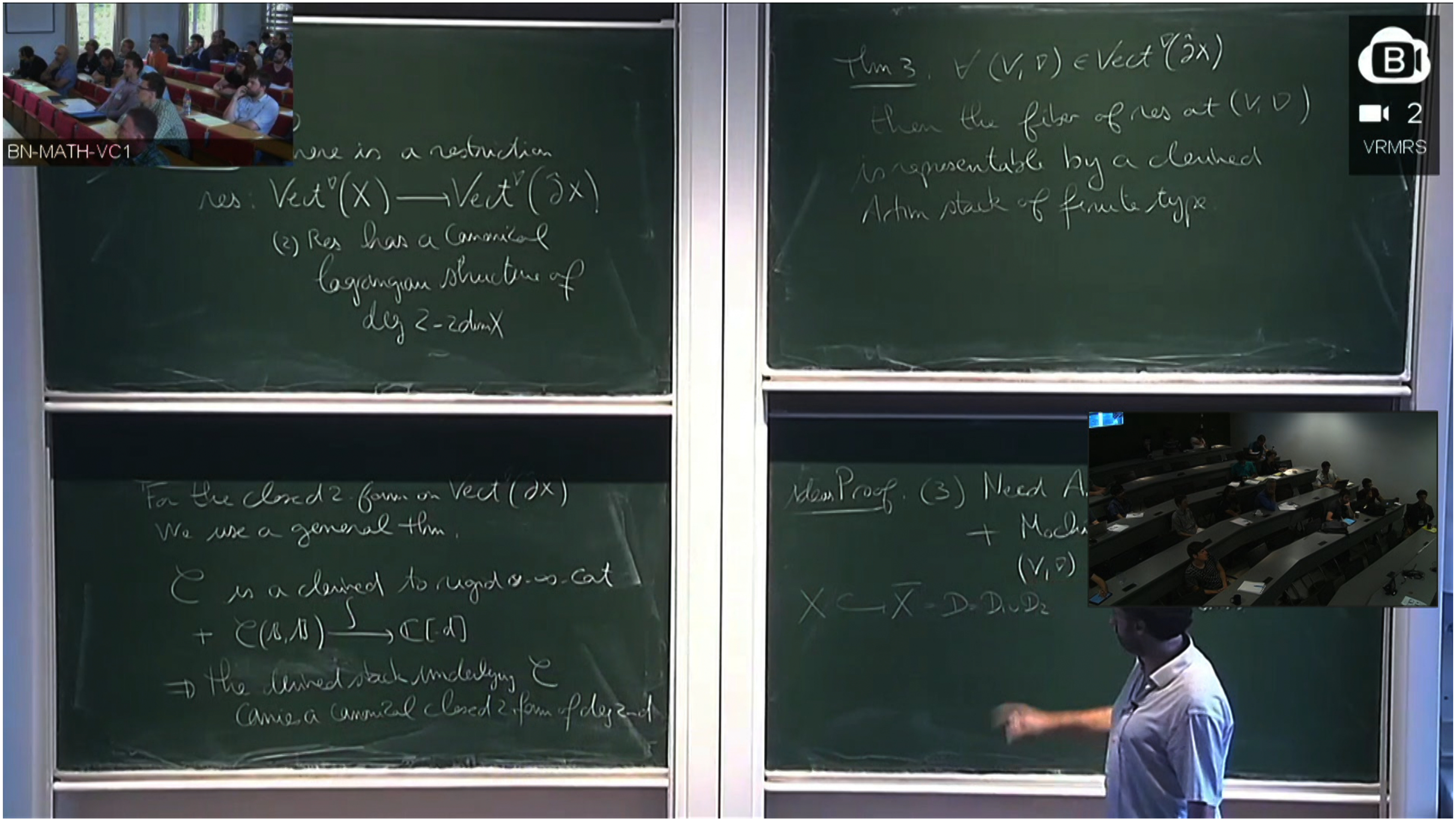
(V,∇), then $\exists\ \overline{X}$ st V
extends



here is a restriction

res: $\mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$

(2) Res has a Canonical Lagrangian structure of deg $2-2\dim X$

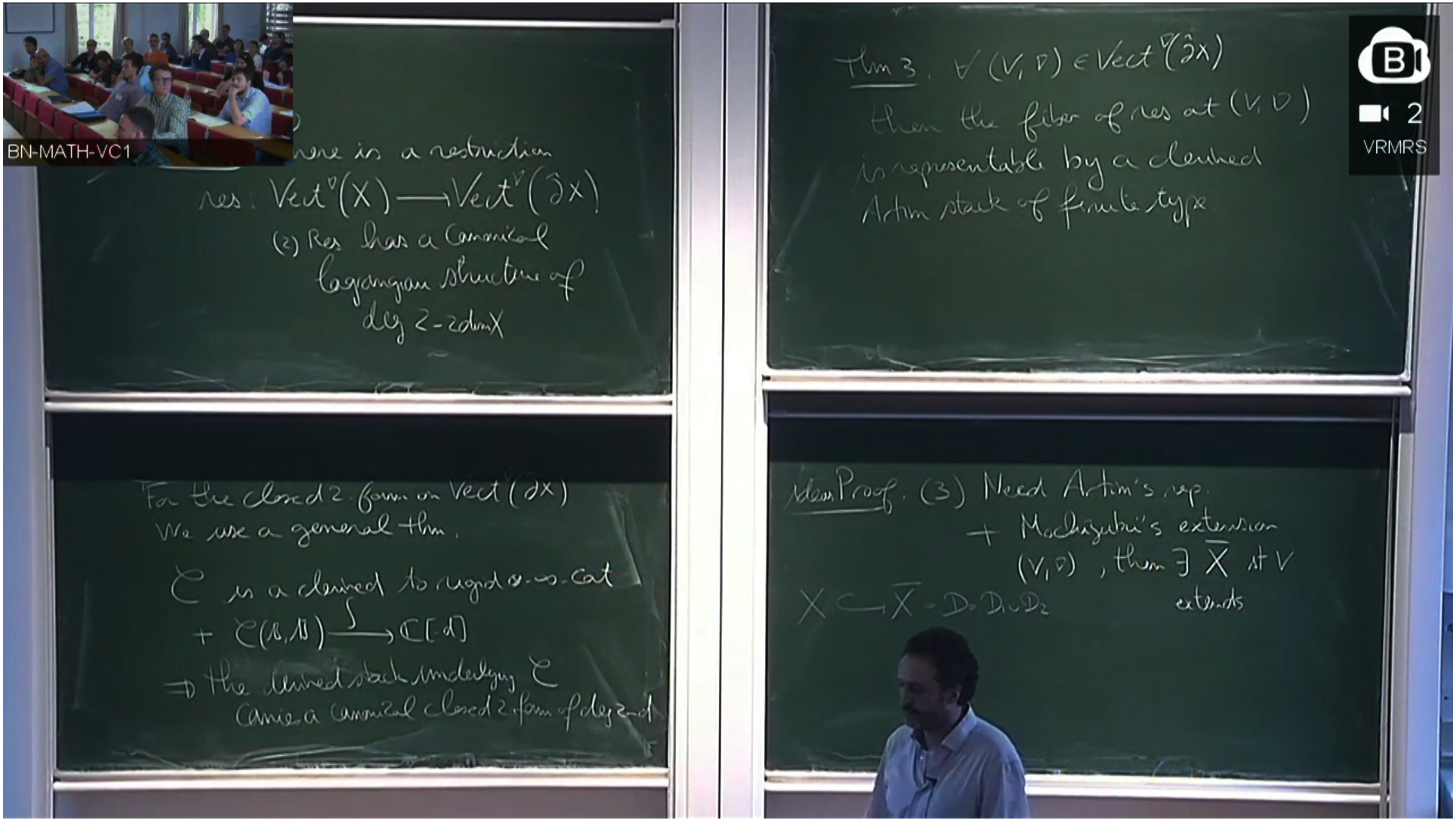
For the closed 2-form on $\mathrm{Vect}(\partial X)$
We use a general thm.

$\mathcal{C}$ is a derived to rigid $\otimes$-∞-cat
+ $\mathcal{C}(1,1) \xrightarrow{\int} \mathbb{C}[-d]$
$\Rightarrow$ the derived stack underlying $\mathcal{C}$
carries a canonical closed 2-form of deg $2-d$

<u>Thm 3</u>. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$
then the fiber of res at (V,∇)
is representable by a derived
Artin stack of finite type.

<u>Idea Proof</u>. (3) Need A
+ Mochi
(V,∇)

$X \hookrightarrow \overline{X} - D = D_1 \cup D_2$



ere is a restriction

$$\mathrm{res}: \mathrm{Vect}^{\nabla}(X) \longrightarrow \mathrm{Vect}^{\nabla}(\partial X)$$

(2) Res has a canonical lagrangian structure of deg $2-2\dim X$

For the closed 2-form on $\mathrm{Vect}^{\nabla}(\partial X)$ we use a general thm.

$\mathcal{C}$ is a derived to rigid $\otimes$-∞-cat

$$+ \ \mathcal{C}(\mathbb{1},\mathbb{1}) \xrightarrow{\int} \mathbb{C}[-d]$$

$\Rightarrow$ the derived stack underlying $\mathcal{C}$ carries a canonical closed 2-form of deg $2-d$

Thm 3. $\forall (V,\nabla) \in \mathrm{Vect}^{\nabla}(\hat{\partial} X)$ then the fiber of res at (V,∇) is representable by a derived Artin stack of finite type

Idea Proof. (3) Need Artin's rep. + Mochizuki's extension

(V,∇), then $\exists\ \overline{X}$ st V extends

$$X \hookrightarrow \overline{X} - D = D_1 \cup D_2$$