

Title: Moduli of connexions on open varieties

Date: Aug 14, 2018 09:00 AM

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Abstract: This is a joint work with T. Pantev. In this talk, we will discuss moduli of \mathbb{P}^1 -bundles on smooth algebraic varieties, with possibly irregular singularities at infinity. For this, we use the notion of $\hat{\mathbb{A}}^1$ -formal boundary, previously studied by Ben Bassat-Temkin, Eilam Eyal and Hennion–Porta–Vezzosi, as well as the moduli of \mathbb{P}^1 -bundles at infinity. We prove that the fibers of the restriction map to infinity are representable. We also prove that this restriction map has a canonical Lagrangian structure in the sense of shifted symplectic geometry.



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Work with T. Parter
Module of connexions
on varieties (not necessarily compact)





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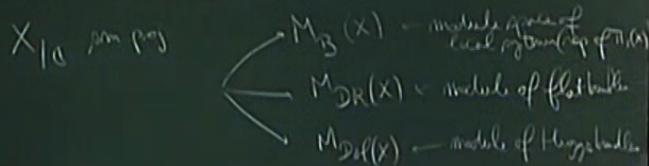
X non-compact

Mironaka: X behaves as if like a compact oriented manifold ∂X of $\dim \partial X = d-1$
 $X \sim (M, \partial M)$

Work with T Parker

Module of connections on varieties (not necessarily compact)

1) Reminders on the case of sm. proj var



want to make a step towards a generalization of Thm to the non-compact case

Q. Construct the module spaces
 In this talk, $M_{DR}(X) = ?$

Complement (3) (X sm. proj) $d = \dim X$
 $M_B(X)$, $M_{DR}(X)$ and $M_{Diff}(X)$ carry natural symplectic structures: are shifted by $(2, 2d)$

Thm (Non-abelian Hodge theory)

Hodge structures on $H^i(X, \mathbb{C})$

$$\left. \begin{array}{l} (1) M_B(X)^{sm} \simeq M_{DR}(X)^{sm} \text{ R.H.} \\ (2) M_{DR}(X) \xrightarrow{(\mathbb{C}^*)} M_{Diff}(X) \end{array} \right\}$$

And (1)+(2) are compatible with (3)
 (3) closely related to polarization on H^1



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X non-compact

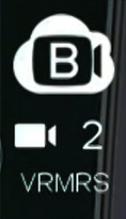
Mironaka: X behaves at as like a compact oriented manifold ∂X of $\dim 2d-1$
 $X \sim (M, \partial M)$

G reductive/ p ($G = \mathfrak{sl}_n$)

Thm (Caleque):

$$\begin{array}{ccc} \text{Map}(M, BG) & \xrightarrow{\text{res}} & \text{Map}(\partial M, BG) \\ \parallel & & \parallel \\ \text{denoted} & & \\ \text{Artin stack} & \ni & \text{Log}_G(M) \xrightarrow{\text{res}} \text{Log}_G(\partial M) \\ \text{Res} & & \text{carries a canonical Lagrangian structure} \\ & & \text{of degree } 2 \dim M \end{array}$$

Q. Construct the moduli spaces
 In this talk, $M_{DR}(X) = ?$
 (complement, (3) (X is non proj) $d = \dim X$)
 $M_B(X)$, $M_{DR}(X)$ and $M_{Dol}(X)$ carry natural symplectic structures, one shifted by $(2-2d)$



$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2 \dim M)$

Suppose X is a curve

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} & \bar{X} - X = 2p - 2pt \\ \text{Log}_G(X) & \longrightarrow & \Pi[G/G] \end{array}$$



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X non-compact

Mironaka. X behaves at as like a compact oriented manifold ∂X of $\dim \partial X = 2d-1$
 $X \sim (M, \mathcal{M})$

G reductive/ \mathcal{G} ($\mathcal{G} = \mathcal{G}_m$)

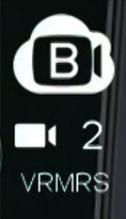
Thm (Caleque).

$$\text{Map}(M, BG) \xrightarrow{\text{res}} \text{Map}(M, B\mathcal{G})$$

$$\text{described Artin stack} \ni \text{Loc}_G(M) \xrightarrow{\text{res}} \text{Loc}_{\mathcal{G}}(M)$$

Res carries a canonical Lagrangian structure of degree $2-d \dim M$

Q. Construct the module spaces
 In this talk, $M_{DR}(X) = ?$
 (complement (3) (X is non proj) $d = \dim X$)
 $M_B(X)$, $M_{DR}(X)$ and $M_{Dol}(X)$ carry natural symplectic structures: one shifted by $(2-2d)$



$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2-d \dim M)$

Suppose X is a curve

$$\begin{array}{ccc} X \hookrightarrow \bar{X} & \bar{X} - X = 2p_1 - p_2 & \\ \text{Log}_G(X) \xrightarrow{\text{Res}} & \Pi[G/G] & \text{deg} \\ \text{supp} \nearrow & \uparrow & \\ \text{Loc}_G^{\text{st}}(X) & \longrightarrow & \Pi[\mathcal{O}_X/G] \end{array}$$



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X non-compact

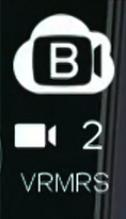
Mironaka. X behaves at as like a compact oriented manifold ∂X of $\dim 2d-1$
 $X \sim (M, \partial M)$

G reductive/ σ ($G = \text{Spin}$)

Thm (Calaque).

$$\begin{array}{ccc}
 \text{Map}(M, BG) & \xrightarrow{\text{res}} & \text{Map}(\partial M, BG) \\
 \parallel & & \parallel \\
 \text{denoted} & & \\
 \text{Artin stack} & \ni & \text{Log}_G(M) \xrightarrow{\text{res}} \text{Log}_G(\partial M) \\
 \text{Res} & & \text{carries a canonical Lagrangian structure} \\
 & & \text{of degree } 2 \cdot \dim M
 \end{array}$$

Q. Construct the moduli spaces
 In this talk, $M_{\text{DR}}(X) = ?$
 (complement, (3) (X is non proj) $d = \dim X$)
 $M_{\mathbb{B}}(X)$, $M_{\text{DR}}(X)$ and $M_{\text{DR}}(X)$ carry natural symplectic structures: one shifted by $(2-2d)$



$\Rightarrow \text{Map}(M, \text{SO})$ carries a canonical Poisson structure of deg $(2 \cdot \dim M)$

Suppose X is a curve

$$\begin{array}{ccc}
 X \hookrightarrow \bar{X} & \bar{X} - X = 2p_1 + \dots + 2p_l \\
 \text{Log}_G(X) & \xrightarrow{\text{Res}} & \Pi[G/G] \\
 \text{supple} \uparrow & & \uparrow \text{diag} \\
 \text{Log}_G^{\text{dlt}}(X) & \longrightarrow & \Pi[\text{Log}_G]
 \end{array}$$



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$$BG \xrightarrow{\text{res}} M_{\text{gp}}(\partial M, BG)$$

$$M \xrightarrow{\text{res}} \text{Loc}_G(\partial M)$$

Res carries a canonical lagrangian structure of degree $2 - \dim M$

② Case $M_{\mathbb{B}}(X)$ X non-compact

$$X \xrightarrow{\text{pm}/G} X$$

Mironaka: X behaves at ∞ like a compact oriented manifold ∂X of $\dim \partial X - 1$

$$X \sim (M, \partial M)$$

Q. Construct the moduli spaces
In this talk, $M_{\text{DR}}(X) = ?$

(complement, (3) (X non proj) $d = \dim X$)

$M_{\mathbb{B}}(X)$, $M_{\text{DR}}(X)$ and $M_{\text{DR}}(X)$ carry natural symplectic structures: one shifted by $(2-2d)$



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$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of degree $(2 - \dim M)$

Suppose X is a curve

$$\begin{array}{ccc}
 X \hookrightarrow \bar{X} & \bar{X} - X = 2p_1 - p_2 & \\
 \partial X = \coprod S^1 & \text{Loc}_G(X) \xrightarrow{\text{Res}} \pi[G/G] & \\
 \text{supp} \rightarrow & \text{Loc}_G^{\text{cl}}(X) \xrightarrow{\text{Res}} \pi[\text{Loc}_G/G] & \text{d.c.G}
 \end{array}$$



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$$BG \xrightarrow{\text{res}} \text{Map}(\mathcal{O}_M, BG)$$

$$\parallel$$

$$M \xrightarrow{\text{res}} \text{Loc}_G(\mathcal{O}_M)$$

Res carries a canonical Lagrangian structure of degree $2 \cdot \dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) \xrightarrow{\text{com dngly}} \text{SSets}$$

$$A \longmapsto \left. \begin{array}{l} D_{\text{res}} A \text{ dg mod} \\ \text{com } X \end{array} \right\}$$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2 \cdot \dim M)$

Suppose X is a curve

$$X \hookrightarrow \bar{X} \quad \bar{X} - X = \{p_1, \dots, p_n\}$$

$$DX = \Pi S' \xrightarrow{\text{Loc}_G(X) \text{ Res}} \Pi[G/G] \text{ d.c.G}$$

$$\text{supple} \rightarrow \text{Loc}_G^{\text{d.c.G}}(X) \rightarrow \Pi[\hat{O}_{X/G}]$$



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$$BG \xrightarrow{\text{res}} \text{Map}(\partial M, BG)$$

$$\parallel$$

$$M \xrightarrow{\text{res}} \text{Loc}_G(\partial M)$$

Res carries a canonical Lagrangian structure of degree $2 - \dim M$

⑤ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) : \text{Com dg alg} \longrightarrow \text{SSets}$$

$$A \longmapsto \left. \begin{array}{l} D_{\text{res}} A \text{ dg mod} \\ \text{Loc } X + \\ \text{loc free on } \mathcal{O}_X \end{array} \right\}$$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of degree $(2 - \dim M)$

Suppose X is a curve

$$X \hookrightarrow \bar{X} \quad \bar{X} - X = \{p_1, \dots, p_n\}$$

$$\text{Loc}_G(X) \xrightarrow{\text{Res}} \Pi[G/G] \quad \text{d.c.G.}$$

$$\text{supplite} \downarrow \text{Loc}_G^{\text{d.c.G.}}(X) \longrightarrow \Pi[\hat{\mathcal{O}}_G/G]$$



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$$\begin{array}{ccc}
 BG & \xrightarrow{\text{res}} & \text{Map}(\partial M, BG) \\
 & & \parallel \\
 M & \xrightarrow{\text{res}} & \text{Loc}_G(\partial M)
 \end{array}$$

Res carries a canonical Lagrangian structure of degree $2 - \dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) \text{, Com dglg} \xrightarrow{\quad} \text{Set mod}$$

$A \longmapsto \left\{ \begin{array}{l} \text{Set mod} \\ \text{in } \mathcal{O}_X \end{array} \right.$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2 - \dim M)$

Suppose X is a curve

$$\begin{array}{ccc}
 X \hookrightarrow \bar{X} & \bar{X} - X = \{p_1, \dots, p_n\} \\
 \text{d}X = \mathbb{H} S' & \text{Loc}_G(X) \xrightarrow{\text{Res}} \pi(G/G) & \text{d.c.G} \\
 \text{supplite} \nearrow & \text{Loc}_G^{\text{d}}(X) \longrightarrow \pi(\hat{\mathcal{O}}_{X/G})
 \end{array}$$

When X not compact this is pathological





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$$BG \xrightarrow{\text{res}} \text{Map}(\partial M, BG)$$

$$\parallel$$

$$M \xrightarrow{\text{res}} \text{Log}_G(\partial M)$$

Res carries a canonical Lagrangian structure of degree $2 \dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) : \text{Com dg alg} \longrightarrow \text{SSets}$$

$$A \longmapsto \left. \begin{array}{l} D_{\text{res}} A \text{ dg mod} \\ \text{Loc } X + \\ \text{loc free on } \mathcal{O}_X \end{array} \right\}$$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2 \dim M)$

Suppose X is a curve

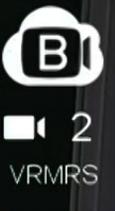
$$X \hookrightarrow \bar{X} \quad \bar{X} - X = \{p_1, \dots, p_n\}$$

$$DX = \coprod S^1 \xrightarrow{\text{Log}_G(X)} \text{res} \rightarrow \pi(G/G) \text{ d.c.G}$$

$$\text{symplicite} \rightarrow \text{Log}_G^{\text{d.c.G}}(X) \rightarrow \pi(\hat{\mathcal{O}}_X/G)$$

When X not compact this is pathological

(a) (V, ∇) is a G -pt. (i.e. a flat bundle on X)





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$$\begin{array}{ccc}
 BG & \xrightarrow{\text{res}} & \text{Map}(\partial M, BG) \\
 & & \parallel \\
 M & \xrightarrow{\text{res}} & \text{Loc}_G(\partial M)
 \end{array}$$

Res carries a canonical Lagrangian structure of degree $2 \dim M$

③ Case of M_{DR} :

exists a natural module functor

$$\text{Vect}^{\nabla}(X) : \text{Com dg alg} \longrightarrow \text{SSets}$$

$A \longmapsto \left. \begin{array}{l} \text{D, \mathbb{R}A dg mod} \\ \text{Loc } X + \\ \text{loc free} \end{array} \right\}$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of degree $(2 \dim M)$

Suppose X is a curve

$$\begin{array}{ccc}
 X \hookrightarrow \bar{X} & \bar{X} - X = \{p_1, \dots, p_n\} \\
 \text{Loc}_G(X) \xrightarrow{\text{Res}} \Pi[G/G] & \text{d.c.G} \\
 \text{supplite} \downarrow & \text{Loc}_G^{\text{d.c.G}}(X) \longrightarrow \Pi[\hat{O}_X/G]
 \end{array}$$

When X not compact this is pathological

(a) (V, ∇) is a G pt. (ie a flat bundle on X)

$$\prod_{(V, \nabla)} \text{Vect}^{\nabla}(X) \simeq H_1(X, \mathbb{Z}) \otimes \mathbb{Z}[1]$$





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$$BG \xrightarrow{\text{res}} \text{Map}(\mathcal{D}M, BG)$$

$$\parallel$$

$$M \xrightarrow{\text{res}} \text{Loc}_G(\mathcal{D}M)$$

Res carries a canonical Lagrangian structure of degree $2 - \dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) \xrightarrow{\text{can. dng.}} \text{Vect}^{\nabla}(X)$$

$$A \longmapsto \left. \begin{array}{l} \text{Drs} \\ \text{Loc} \\ \text{Loc} \end{array} \right\}$$

$\Rightarrow \text{Map}(M, BG)$ carries a canonical Poisson structure of deg $(2 - \dim M)$

Suppose X is a curve

$$X \hookrightarrow \bar{X} \quad \bar{X} - X = \{p_1, \dots, p_n\}$$

$$\text{Loc}_G(X) \xrightarrow{\text{Res}} \Pi[G/G] \xrightarrow{\text{d.c.G.}} \Pi[\hat{\mathcal{O}}_G/G]$$

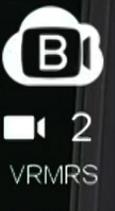
supplite \swarrow $\text{Loc}_G^{\text{d.c.G.}}(X)$

When X not compact this is pathological

(a) (V, ∇) is a G -pt. (ie a flat bundle on X)

$$\Pi_{(V, \nabla)} \text{Vect}^{\nabla}(X) \simeq H_1(X, \text{Vect}^{\nabla})[1]$$

finite dim.





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$$\begin{array}{ccc} \mathbb{B}G & \xrightarrow{\text{res}} & \text{Map}(\mathcal{O}_M, \mathbb{B}G) \\ & & \parallel \\ M & \xrightarrow{\text{res}} & \text{Loc}_G(\mathcal{O}_M) \end{array}$$

The universal commutative Hopf algebra structure
of $\text{res}^* \mathbb{B}G$.

③ Case of M_{DR}

\exists a natural module fibration

$$\begin{array}{ccc} \text{Vect}^{\mathbb{A}^1}(X) & \xrightarrow{\text{canonically}} & \text{Vect} \\ A \downarrow & \text{Descent datum} & \\ & \text{Loc } X + & \\ & \text{is pre-sheaf} & \end{array}$$

When X not compact this is pathological
 $\text{res}^*(\mathcal{O}, \mathcal{O})$ is a \mathbb{C} pt (ie a flat bundle)

$$\| \text{res}^*(\mathcal{O}) \|_{\text{top}} = \| \text{res}^*(\mathcal{O}, \mathcal{O}) \|_{\text{top}}$$

(iii) Then fields for \mathbb{A}^1 (finite time in general)



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$$\begin{array}{ccc}
 BG & \xrightarrow{res} & Map(\mathcal{O}_M, BG) \\
 & & \parallel \\
 M & \xrightarrow{res} & Loc_G(\mathcal{O}_M)
 \end{array}$$

Res carries a canonical Lagrangian structure of degree $2-dim M$

③ Case of M_{DR} :

\exists a natural module functor $Vect^{\nabla}(X)$: Com dgrly $A \rightarrow$ $\mathbb{S}Set$ dg mod \rightarrow $\mathbb{K}vec$ on Qst

When X not compact this is pathological

(a) (V, ∇) is a \in pt. (ie a flat bundle on X)

$$\prod_{(V, \nabla)} Vect^{\nabla}(X) \simeq H_{dR}^i(X, V \otimes V^*) [1]$$

① This fails for A-pt finite dim in general



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Ex: $\mathbb{C} \wedge^2$

$$\frac{d^2}{dt^2}$$



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$$BG \xrightarrow{res} \text{Map}(\partial M, BG)$$

$$\parallel$$

$$M \xrightarrow{res} \text{Loc}_G(\partial M)$$

Res carries a canonical Lagrangian structure of degree $2 - \dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\text{Vect}^{\nabla}(X) \xrightarrow{\text{alg}} \text{SSets}$$

\downarrow
 $\text{D}_{\text{res}} \text{ dg mod}$
 $\text{on } X +$
 $\text{tr free on } \mathcal{O}_X$

When X not compact this is pathological
 (a) (V, ∇) is a \in pt. (ie a flat bundle on X)

$$\prod_{(V, \nabla)} \text{Vect}^{\nabla}(X) \simeq H_{DR}^i(X, \text{Vect}^{\nabla}) [1]$$

① This fails for A-pt finite dim in general

Ex. $\mathbb{C} \wedge^2$

$$\mathbb{C} \frac{dt}{t} \oplus 0$$

$$\rightsquigarrow \pi \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}(i)$$

$\stackrel{?}{\cong} \text{PGh}(\mathbb{A}^1)$



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$$\begin{array}{ccc}
 BG & \xrightarrow{res} & Map(\mathcal{D}M, BG) \\
 & & \parallel \\
 M & \xrightarrow{res} & Loc_G(\mathcal{D}M)
 \end{array}$$

Res carries a canonical Lagrangian structure of degree $2-dim M$

③ Case of M_{DR} :

\exists a natural module functor

$$\begin{array}{ccc}
 Vect^{\nabla}(X) & \xrightarrow{\text{can dngly}} & \mathcal{S}Set \\
 A \longmapsto & \left. \begin{array}{l} D_{\text{can}} \text{ dng} \\ \text{can } X + \\ \text{loc free} \end{array} \right\} &
 \end{array}$$



When X not compact this is pathological
 (a) (V, ∇) is a \in pt. (ie a flat bundle on X)

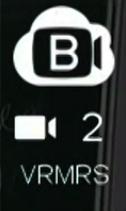
$$\prod_{(V, \nabla)} Vect^{\nabla}(X) \simeq H_{DR}^i(X, \mathbb{V}^{\nabla}) [1]$$

① This fails for A-pt finite dim in general

$$\underline{Ex} \quad \mathbb{Z} \subset \mathbb{A}^2$$

$$\mathbb{Z} \frac{dt}{t} \oplus \mathbb{Z} \xrightarrow{\sim} \mathbb{T} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}(i) \simeq \mathcal{D}Gh(\mathbb{A}^1)$$

$\Rightarrow Vect^{\nabla}(X)$ can not be representable by a derived Artin stack loc f type





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DR:

normal module functor

$$\text{Vect}^D(X) \xrightarrow{\text{can. dglg}} \text{SSets}$$

$$A \mapsto \left. \begin{array}{l} D_{\text{can}} A \text{ dgl. mod} \\ \text{on } X + \\ \text{loc. free on } \mathcal{O}_X \end{array} \right\}$$

G reductive/ \mathcal{O} ($G = \text{GL}$)

Thm (Caleque).

$$\text{Map}(M, BG) \xrightarrow{\text{res}} \text{Map}(\partial M, BG)$$

derived
Artin stack

$$\Downarrow$$

$$\text{Loc}_G(M) \xrightarrow{\text{res}} \text{Loc}_G(\partial M)$$

Res carries a canonical Lagrangian structure
of degree $2 \cdot \dim M$

When X not compact this is pathological

(a) (V, ∇) is a \mathbb{C} pt. (ie a flat bundle on X)

$$\prod_{(V, \nabla)} \text{Vect}^D(X) \simeq H_{\text{DR}}^1(X, \text{Vect}^D) [1]$$

(*) This fails for A-pt finite dim in general

Ex \mathbb{C}^2

$$\mathbb{C} \frac{dt}{t} \oplus \mathbb{C} \xrightarrow{\sim} \mathbb{T} \simeq \bigoplus_{\mathbb{Z}} \mathbb{C}(\alpha)$$

$$\text{DGL}(\mathbb{A}^1)$$

$\Rightarrow \text{Vect}^D(X)$ can not be representable by a derived Artin stack loc f type



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DR:

mal module functor

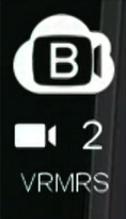
$\text{Vect}^{\mathcal{D}}(X)$ can only \rightarrow $\mathcal{S}\text{Sets}$
 $A \mapsto \left. \begin{array}{l} \text{D-mod} \\ \text{on } X + \\ \text{loc free on } \mathcal{D}\text{-st} \end{array} \right\}$

To solve this want to have
 a DR version of
 $\text{Loc}_G(X) \rightarrow \text{Loc}_G(\mathcal{D}X)$
 $\text{Vect}^{\mathcal{D}}(X) \rightarrow \text{Vect}^{\mathcal{D}}(?)$

$\underline{\text{Ex}} \quad \mathcal{D} \subset \mathcal{A}^2$
 $\mathcal{D} \frac{dt}{t} \oplus \mathcal{O} \quad \rightsquigarrow \quad \mathbb{T} \cong \bigoplus_{\mathbb{Z}} \mathcal{O}(n)$

$\mathcal{O}(\mathbb{C}h(\mathcal{A}^1))$

$\Rightarrow \text{Vect}^{\mathcal{D}}(X)$ can not be representable by a derived Artin stack loc f. type





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DR:

formal module functor

$$\text{Vect}^0(X) \xrightarrow{\text{formal mod}} \text{Sets}$$

$$A \mapsto \left. \begin{array}{l} \text{D.R. mod} \\ \text{on } X + \\ \text{loc free on } \mathcal{O}_X \end{array} \right\}$$

To solve this want to have
a DR version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\infty)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(\infty)$$

$$E \times \mathbb{A}^1$$

$$\mathbb{A}^1 \oplus \mathcal{O} \rightarrow \mathbb{A}^1 \oplus \mathcal{O}(\infty)$$

$$\varphi(\mathcal{O}_X(\infty))$$

$\Rightarrow \text{Vect}^0(X)$ can not be representable by
a derived Artin stack loc f. type



④ The formal boundary at ∞



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DR:

formal module functor

$\text{Vect}^0(X)$ comm. dgrly \rightarrow S-sets
 $A \mapsto \left. \begin{array}{l} \text{Drsn dg mod} \\ \text{on } X + \\ \text{loc free on } \mathcal{O}_X \end{array} \right\}$

To solve this want to have a DR version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\mathbb{P}^1)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(\mathbb{P}^1)$$



$$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

$$\mathbb{Z} \frac{dt}{t} \oplus \mathcal{O} \rightarrow \pi \cong \bigoplus_{\mathbb{Z}} \mathbb{C}(x)$$

$$\varphi(\text{Ch}(\mathbb{A}^1))$$

$\Rightarrow \text{Vect}^0(X)$ can not be representable by a derived Artin stack loc f. type



④ The formal boundary at ∞ :

- studied by:
- Temim Ben Basat, rigid geom
 - Hermion-Pata-Vezzosi: sheaf the diff
 - Lfiman: Cal. construction



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DR:

formal module functor

$\text{Vect}^0(X)$ can be thought of as $\mathcal{S}\text{Sets}$
 $A \mapsto \left. \begin{array}{l} D_{\text{res}} \text{ of mod} \\ \text{on } X + \\ \text{loc free on } D_{\text{res}} \end{array} \right\}$



Ex: \mathbb{A}^1

$$\mathbb{A}^1 \text{ is } \mathbb{A}^1 \oplus \mathcal{O} \rightarrow \mathbb{A}^1 \oplus \mathcal{O} \oplus \mathcal{O} \oplus \dots$$

$\Rightarrow \text{Vect}^0(X)$ can not be representable by a derived Artin stack loc f. type

To solve this want to have a DR version of

$$\text{Loc}_G(X) \rightarrow \text{Gr}(n, X)$$

$$\text{Vect}^0(X) \rightarrow \text{Gr}^0(n, X)$$

④ The formal boundary at ∞ :

- Studied by:
 - Temblin Ben Bassat, rigid geom
 - Haiman-Paula-Vezzosi: sheaf th def
 - Efimov: cat. construction

\Rightarrow We don't know what $?$ is but we still can define its cat of sheaves



BN-MATH-VC1

is. Want to have
version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\mathcal{O}_X)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(?)$$

③ Case of M_{DR} :

exists a natural module functor

$$\text{Vect}^0(X) : \text{Com dgalg} \rightarrow \text{SSets}$$

$$A \mapsto \begin{cases} D_{\text{com}} A \text{ dgmod} \\ \text{Com } X + \\ \text{loc free on } \mathcal{O}_X \end{cases}$$

④ The formal boundary at ∞ :

Studied by: - Tomer Ben-Bassat, rigid

- Hermann-Pata-Vezzosi: sheaf
th def

- Efimov Cal. construction

\Rightarrow We don't know what ? is but
We still can define its cat of presheaves



2

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$\Rightarrow \text{Map}(M, BG)$ carries
a canonical Poisson structure
of deg (2-dim M)

Suppose X is a curve

$$X \hookrightarrow \bar{X} \quad \bar{X} - X = 2p_1 + p_2$$

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_{\bar{X}} \quad \text{Loc}_G(X) \xrightarrow{\text{Res}} \Pi[G/G]$$

$$\text{supplite} \uparrow \text{Loc}_G^{\text{supplite}}(X) \xrightarrow{\text{supplite}} \Pi[\mathcal{O}_X/G]$$



BN-MATH-VC1

is want to have
realization of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\hat{\partial}X)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(?)$$

We can define $\text{Vect}^0(\hat{\partial}X)$

④ The formal boundary at ∞ :

Studied by: - Temblam Ben Bassat, rigid

- Hermann-Pata-Vezzosi: sheaf
th. diff

- Efimov Cal. construction

\Rightarrow We don't know what ? is but
We still can define its Cal. properties



2

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is. Want to have
 version of

$$\mathrm{Loc}_G(X) \rightarrow \mathrm{Loc}_G(\hat{\partial}X)$$

$$\mathrm{Vect}^0(X) \rightarrow \mathrm{Vect}^0(?)$$

We can define $\mathrm{Vect}^0(\hat{\partial}X)$

$$X \hookrightarrow \bar{X} \twoheadrightarrow D$$

④ The formal boundary at ∞

Studied by:

- Temblam Ben Bassat, rigid
- Hermann-Pala-Vezzosi: sheaf
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is. Want to have
 version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\partial X)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(?)$$

BN-MATH-VC1

We can define $\text{Vect}^0(\hat{\partial}X)$

$$X \hookrightarrow \bar{X} \twoheadrightarrow D$$

$$\hat{\partial}X = \hat{D} - D''$$

④ The formal boundary at ∞ :

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- Hermann-Pala-Vezzosi: sheaf
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 We still can define its Cal. properties

B
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is want to have
 version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\partial X)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(?)$$

BN-MATH-VC1

We can define $\text{Vect}^0(\hat{\partial}X)$

$$X \hookrightarrow \bar{X} \hookrightarrow D$$

$$\hat{\partial}X = \hat{D} - D''$$

\mathcal{D}_3 sheaf on D

④ The formal boundary at ∞

Studied by:

- Temblam Ben Bassat, rigid
- Hermann-Pata-Vezzosi: sheaf
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\Rightarrow We don't know what $?$ is but
 We still can define its cat of presheaves





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We want to have
a version of

$$\text{Loc}_G(X) \rightarrow \text{Loc}_G(\partial X)$$

$$\text{Vect}^0(X) \rightarrow \text{Vect}^0(?)$$

We can define $\text{Vect}^0(\hat{\partial}X)$

$$X \hookrightarrow \bar{X} \rightarrow D$$

$$\hat{\partial}X = \hat{D} - D'$$

$$\mathcal{D}_3 \text{ sheaf on } D \quad \text{Vect}^0(\partial X) = \mathcal{D}_3 \text{H}^0 \text{-mod} \\ + \text{ordibm}$$

where $t = D$

④ The formal boundary at ∞ :

studied by: - Tembim Ben Bassat, rigid

- Hermann-Pata-Vezzosi: sheaf
th. def

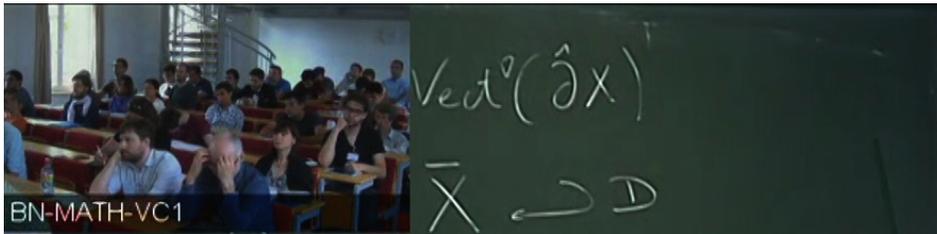
- Efimov Cal. construction

\Rightarrow We don't know what $?$ is but
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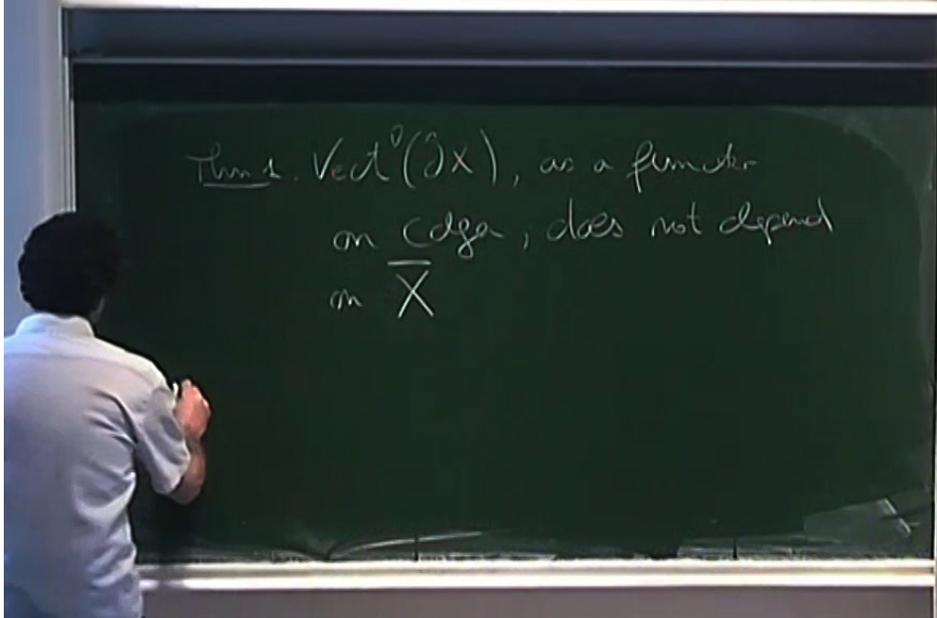
$$\text{Vect}^0(\hat{\partial}X)$$

$$\bar{X} \hookrightarrow D$$

$$\hat{\partial}X = \hat{D} - D'$$

$$\mathcal{D}_3 \text{ Sheaf on } D \quad \text{Vect}^0(\partial X) = \mathcal{D}_3 \text{H}^0\text{-mod} \\ + \text{condition}$$

$$\text{where } t = D$$



Thm 1. $\text{Vect}^0(\partial X)$, as a functor
on Edge , does not depend
on \bar{X}

④ The formal boundary at ∞ :

Studied by: - Temblam Ben Bassat, rigid

- Hermann-Pala-Vezzosi: sheaf
th. def

- Efimov Cal. construction

\Rightarrow We don't know what ? is but
We still can define its cat. properties



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$$\text{Vect}^0(\hat{\partial}X)$$

$$\bar{X} \hookrightarrow D$$

$$\hat{\partial}X = \hat{D} - D'$$

Sheaf on D $\text{Vect}^0(\hat{\partial}X) = \mathcal{D}_D \text{H}^0\text{-mod}$
+ condition
 $\text{Vect}^0(\hat{\partial}X) = D$

Thm 1. $\text{Vect}^0(\hat{\partial}X)$, as a functor
on Cdg , does not depend
on \bar{X}

Proof. Reduction to Efimov's

④ The formal boundary at ∞ :

Studied by: - Temblam Ben Bassat, rigid

- Hermann-Pata-Vezzosi: sheaf
th dif

- Efimov: Cal. construction

\Rightarrow We don't know what $?_n$ is but
We still can define its Cal. presheaves



2

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BN-MATH-VC1

$$\text{Vect}^0(\hat{\partial}X)$$

$$\bar{X} \hookrightarrow D$$

$$\hat{\partial}X = \hat{D} - D'$$

\mathcal{D}_3 Sheaf on D $\text{Vect}^0(\hat{\partial}X) = \mathcal{D}_3 \text{Hom} \text{-mod}$
 $+ \text{ordilin}$
 $\text{where}(t) = D$

Thm 1. $\text{Vect}^0(\hat{\partial}X)$, as a functor
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 - Efimov Cal. construction

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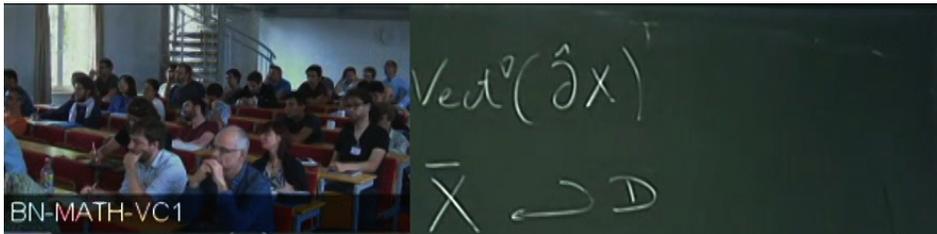


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$\hat{\partial}X = \text{Spf } \mathbb{C}((t))$





BN-MATH-VC1

$$\text{Vect}^0(\hat{\partial}X)$$

$$\bar{X} \hookrightarrow D$$

$$\hat{\partial}X = \hat{D} - D'$$

\mathcal{D}_3 Sheaf on D $\text{Vect}^0(\partial X) = \mathcal{D}_3 \text{H}^0\text{-mod}$
 + condition
 $\text{Wher}(t) = D$



④ The formal boundary at ∞ :

- Studied by:
- Temblam Ben Bassat, rig
 - Hermann-Pala-Vezzosi: sheaf th dif
 - Efimov Cal. construction

\Rightarrow We don't know what $? is but$

We still can define its cat of presheaves

Thm 1. $\text{Vect}^0(\partial X)$, as a functor
 on Cdg , does not depend
 on \bar{X}

Proof. Reduction to Efimov's

Ex. X over $D = \mathbb{R}^n$
 $\hat{\partial}X = \text{Spc } \mathbb{C}((t))$
 $\text{Vect}^0(\partial X)$





BN-MATH-VC1

$$\text{Vect}^0(\hat{\partial}X)$$

$$\bar{X} \hookrightarrow \mathbb{D}$$

$$\hat{\partial}X = \hat{D} - D'$$

Sheaf on \mathbb{D} $\text{Vect}^0(\partial X) = \mathcal{D}_S \mathbb{H}^3 \text{ mod } + \text{condition}$
 $\text{Wher}(t) = \mathbb{D}$

Thm 1. $\text{Vect}^0(\partial X)$, as a functor on Cdg , does not depend on \bar{X}

Proof. Reduction to Efimov's

④ The formal boundary at ∞ :

- Studied by:
- Temim Ben Bassat, rig
 - Hermann-Pala-Vezosi, Sheaf th
 - Efimov Cal. construction

\Rightarrow We don't know what $? is but We still can define its Cal. properties$



$$\underline{\Delta} \hookrightarrow \mathbb{D} \text{ via } \mathbb{D} \text{ etc}$$

$$\hat{\partial}X = \text{Spq } \mathcal{C}(\mathbb{D})$$

$\text{Vect}^0(\partial X)$ — the functor studied by S Ramshu

$$A \longmapsto \left. \begin{array}{l} \text{All } A\text{-modules } M \\ \downarrow \partial X \\ M \longrightarrow M \end{array} \right\}$$



BN-MATH-VC1

\mathcal{O}_X , as a sheaf
idea, does not depend
on \bar{X}

Proof: Reduction to Efmov's

⑤ Results

Thm 2: (1) There is a restriction
res: $\text{Vect}^v(X) \rightarrow \text{Vect}^v(\mathcal{O}_X)$
(2) Res has a canonical
Lagrangian structure of
deg $2 - 2\dim X$

$\bar{\mathcal{O}}_X = \text{Spec } \mathbb{C}(t)$
 $\text{Vect}^v(\bar{\mathcal{O}}_X)$ — the functor studied
by S Rambo

$$A \longmapsto \left\{ \begin{array}{l} A(t)\text{-modules } M \\ M \xrightarrow{\mathcal{O}_X} M \end{array} \right\}$$



2

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BN-MATH-VC1

$\mathcal{V}(X)$, as a functor
 does not depend
 on \bar{X}

Proof: Reduction to Efmov's

⑤ Results

Thm 2: (1) There is a restriction
 $\text{res}: \text{Vect}^v(X) \rightarrow \text{Vect}^v(\mathcal{V}X)$
 (2) Res has a canonical
 Lagrangian structure of
 $\text{deg } 2 - 2\dim X$

$\mathcal{V}X = \text{Spec } \mathbb{C}(t)$
 $\text{Vect}^v(\mathcal{V}X)$ — the functor studied
 by S Rambois

$$A \longmapsto \left. \begin{array}{l} \text{A}(t)\text{-modules } M \\ \downarrow \text{res} \\ M \end{array} \right\}$$



2

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Thm 3. $\forall (V, \mathcal{V}) \in \text{Vect}^v(\mathcal{V}X)$
 then the fiber of res at (V, \mathcal{V})
 is representable by a derived
 Artin stack of finite type.





BN-MATH-VC1

\mathcal{O}_X), as a sheaf of \mathcal{O}_X -
modules, does not depend
on \bar{X}

Proof: Reduction to Efmov's

⑤ Results

Thm 2: (1) There is a restriction
 $\text{res}: \text{Vect}^d(X) \rightarrow \text{Vect}^d(\mathcal{O}_X)$
(2) Res has a canonical
Lagrangian structure of
 $\text{deg } 2 - 2\dim X$

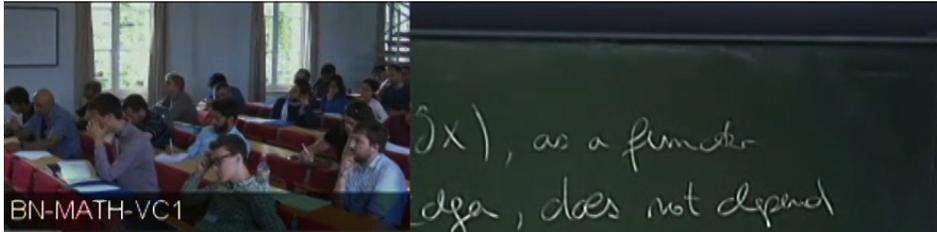
Thm 3: $\forall (V, \mathcal{D}) \in \text{Vect}^d(\mathcal{O}_X)$
then the fiber of res at (V, \mathcal{D})
is representable by a deformed
Artin stack of finite type

Idea Proof: (3) Need Artin's rep.
+ Mochizuki's extension
 (V, \mathcal{D})



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BN-MATH-VC1

\mathcal{O}_X , as a sheaf of \mathcal{O}_X -
algebras, does not depend
on \bar{X}

Proof: Reduction to Efmov's

⑤ Results

Thm 2: (1) There is a restriction
 $res: Vect^d(X) \rightarrow Vect^d$
(2) res has a canonical
Lagrangian structure
 $deg \mathbb{Z} - 2dim X$

Thm 3, $\forall (V, \mathcal{D}) \in Vect^d(\mathbb{A}^n)$
then the fiber of res at (V, \mathcal{D})
is representable by a derived
Artin stack of finite type

Idea Proof. (3) Need Artin's rep.
+ Mochizuki's extension
 (V, \mathcal{D}) , then $\exists \bar{X}$ st V
extends



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BN-MATH-VC1

There is a restriction
 $\text{res}: \text{Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$
 (2) Res has a canonical
 Lagrangian structure of
 $\text{deg } 2\text{-dim } X$

For the closed 2-form on $\text{Vect}(\tilde{X})$
 We use a general thm.
 \mathcal{C} is a derived to rigid α -cat

Thm 3. $\forall (V, \mathcal{D}) \in \text{Vect}^{\vee}(\tilde{X})$
 then the fiber of res at (V, \mathcal{D})
 is representable by a derived
 Artin stack of finite type.

sketch Proof. (3) Need Artin's rep.
 + Mochizuki's extension
 (V, \mathcal{D}) , then $\exists \bar{X}$ st V
 extends



2

VRMRS



BN-MATH-VC1

there is a restriction

$$\text{res}: \text{Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$$

(2) Res has a canonical
Lagrangian structure of
deg 2 - $2\dim X$

For the closed 2-form on $\text{Vect}(\tilde{X})$
We use a general thm.

$$\mathbb{C} \text{ is a derived to rigid } \alpha \rightarrow \text{cat} \\ + \mathbb{C}(A, B) \xrightarrow{\int} \mathbb{C}[-d]$$

Thm 3. $\forall (V, \mathcal{D}) \in \text{Vect}^{\vee}(\tilde{X})$

then the fiber of res at (V, \mathcal{D})
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VRMRS



BN-MATH-VC1

here is a restriction

$$\text{res}: \text{Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$$

(2) Res has a canonical Lagrangian structure of $\deg 2 - 2\dim X$

For the closed 2-form on $\text{Vect}(\tilde{X})$
We use a general thm.

$$\mathcal{C} \text{ is a derived to rigid } \mathfrak{g}\text{-cat} \\ + \mathcal{C}(A, B) \xrightarrow{\quad} \mathbb{C}[d]$$

the derived stack underlying \mathcal{C}
carries a canonical closed 2-form of $\deg 2 - d$

Thm 3. $\forall (V, \mathcal{D}) \in \text{Vect}^{\vee}(\tilde{X})$

then the fiber of res at (V, \mathcal{D})
is representable by a derived
Artin stack of finite type.

sketch Proof. (3) Need Artin's rep.
+ Mochizuki's extension
 (V, \mathcal{D}) , then $\exists \bar{X}$ st V
extends



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BN-MATH-VC1

here is a restriction

$$\text{res}: \text{Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$$

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For the closed 2-form on $\text{Vect}(\tilde{X})$
We use a general thm.

$$\mathcal{C} \text{ is a derived to rigid } \mathfrak{g}\text{-cat}$$
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+ Mochizuki's extension
 (V, \mathcal{D}) , then $\exists \bar{X}$ st V
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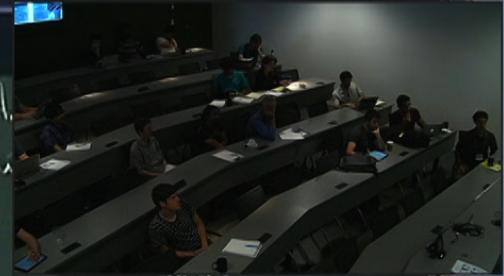
here is a restriction
 $\text{res}: \text{Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$
 (2) Res has a canonical
 Lagrangian structure of
 deg $2 - 2\dim X$

For the closed 2-form on $\text{Vect}^{\vee}(X)$
 We use a general thm.
 \mathcal{C} is a derived to rigid \mathcal{A} - \mathcal{B} -cat
 $+ \mathcal{C}(\mathcal{A}, \mathcal{B}) \xrightarrow{\int} \mathbb{C}[-d]$
 \Rightarrow the derived stack underlying \mathcal{C}
 carries a canonical closed 2-form of deg $2-d$

Thm 3. $\forall (V, \mathcal{D}) \in \text{Vect}^{\vee}(\tilde{X})$
 then the fiber of res at (V, \mathcal{D})
 is representable by a derived
 Artin stack of finite type.



class Proof. (3) Need \mathcal{A}
 $+ \text{Mod}$
 (V, \mathcal{D})
 $X \subset \tilde{X} = D_1 \cup D_2$





BN-MATH-VC1

here is a restriction

$$\text{res: Vect}^{\vee}(X) \rightarrow \text{Vect}^{\vee}(\tilde{X})$$

(2) Res has a canonical
Lagrangian structure of
deg 2-2dim X

For the closed 2-form on $\text{Vect}(\tilde{X})$
We use a general thm.

$$\mathcal{C} \text{ is a derived to rigid } \mathcal{A} \text{--cat}$$

$$+ \mathcal{C}(A, B) \xrightarrow{\int} \mathbb{C}[-d]$$

\Rightarrow the derived stack underlying \mathcal{C}
carries a canonical closed 2-form of deg 2-d

Thm 3. $\forall (V, \mathcal{D}) \in \text{Vect}^{\vee}(\tilde{X})$
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2
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sketch Proof. (3) Need Artin's rep.
+ Mochizuki's extension
 (V, \mathcal{D}) , then $\exists \bar{X}$ st V
 $X \hookrightarrow \bar{X} = \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ extends