

Title: A natural refinement of the Euler characteristic

Date: Aug 17, 2018 11:00 AM

URL: <http://pirsa.org/18080079>

Abstract: The Euler characteristic of a compact complex manifold M is a classical cohomological invariant. Depending on the viewpoint, it is most natural to interpret it as an index of an elliptic differential operator on M , or as a supersymmetric index in superconformal field theories on M . Refining the Euler characteristic but keeping with both index theoretic interpretations, one arrives at the notion of complex elliptic genera. We argue that superconformal field theory motivates further refinements of these elliptic genera which result in a choice of several new invariants, all of which have lost their interpretation in terms of index theory. However, at least if M is a K3 surface, then superconformal field theory and higher algebra select the same new invariant as a natural refinement of the complex elliptic genus.



A natural refinement of the Euler characteristic

arXiv:1705.07104

- PLAN
1. Refining the Euler characteristic
 2. More algebra
 3. Interpretation in conformal field theory

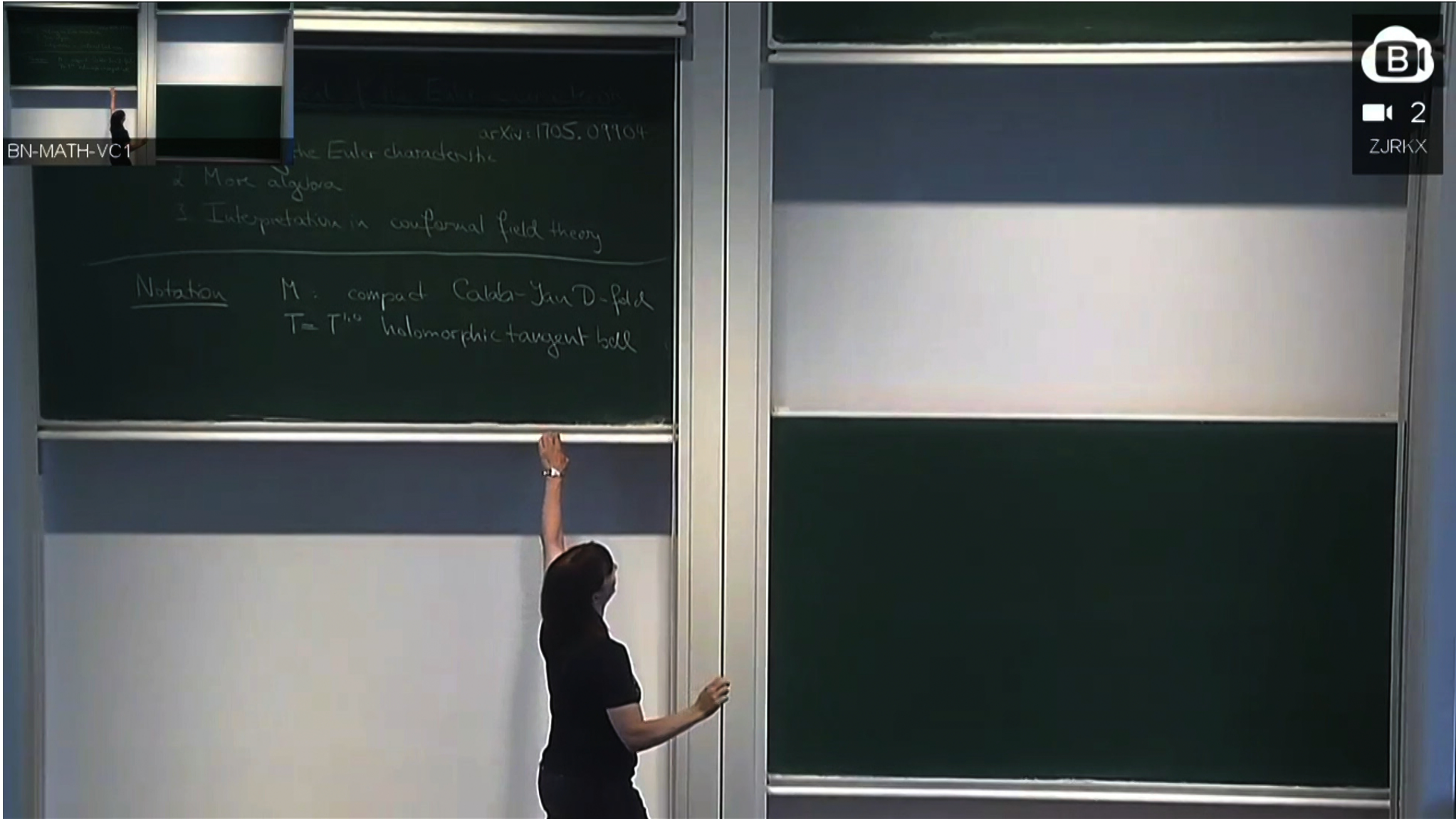
A natural refinement of the Euler characteristic

arXiv:1705.09904

- PLAN:
1. Refining the Euler characteristic
 2. More algebra
 3. Interpretation in conformal field theory

Notation: M : compact Calabi-Yau D -fold





Refining the Euler characteristic

arXiv:1705.09904

Refining the Euler characteristic

2. More algebra

3. Interpretation in conformal field theory

Notation M = compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{kj}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

Refining the Euler characteristic

arXiv:1705.07704

1. Refining the Euler characteristic
2. More algebra
3. Interpretation in conformal field theory

Notation M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{k,j}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

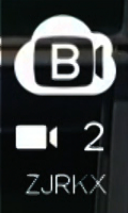
Hirzebruch χ_y -genus

$$\chi_y(M) = \chi(\Lambda_y T^*)$$



... of the Euler characteristic
 arXiv:1705.09904
 ... the Euler characteristic
 2. More algebra
 3. Interpretation in conformal field theory

Notation M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle



1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{k,j}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \text{ with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

Hirzebruch χ_y -genus:

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

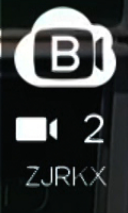


Refining the Euler characteristic

arXiv:1705.09904

- 1. Refining the Euler characteristic
- 2. More algebra
- 3. Interpretation in conformal field theory

Notation M : compact Calabi-Yau D -fold
 $T = T^{hol}$ holomorphic tangent bundle



1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \text{ with } y = -1, E = T$$

$$S_y E = \bigoplus_k y^k S^k E$$

Hirzebruch χ_y -genus.

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

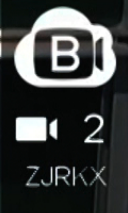
Def [Hirzebruch, Witten &]

$$\mathbb{E}_{y_1, y_2} = (-y_1)^{\text{pt}} \Lambda_{y_1} T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y_1^n} T^* \otimes \Lambda_{y_2^n} T \otimes S_{y_1} T^* \otimes S_{y_2} T)$$

... of the Euler characteristic
 arXiv:1705.09904

1. Computing the Euler characteristic
 2. More algebra
 3. Interpretation in conformal field theory

Notation M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle



Hirzebruch χ -genus

$$\chi(M) = \sum_k (-1)^k h^k(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \text{ with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

Hirzebruch χ_y -genus

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch, Witten 83]

$$IE_{g_1, y} = (-y)^{2g_1} \Lambda_y T^* \otimes \hat{\bigotimes}_{n=1}^{2g_1} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T \otimes S_{y^{2n}} T^* \otimes S_{y^{-2n}} T)$$

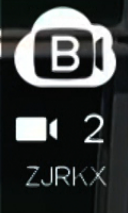
COMPLEX ELLIPTIC GENUS of M

$$\chi(IE_{g_1, y})$$

... of the Euler characteristic
 arXiv:1705.09904

1. Computing the Euler characteristic
 2. More algebra
 3. Interpretation in conformal field theory

Notation M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle



Hirzebruch χ_y -genus:
 $\chi_y(M) = \sum_k (-1)^k \chi^k(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$
 $= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$
 $\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$

Hirzebruch χ_y -genus:
 $\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$

Def [Hirzebruch, Witten & ...]
 $E_{g,y} = (-y)^{2g} \Lambda_y T^* \otimes \bigotimes_{n=1}^g (\Lambda_{y^n} T \otimes \Lambda_{y^{-n}} T \otimes S_{y^n} T^* \otimes S_{y^{-n}} T^*)$
 COMPLEX ELLIPTIC GENUS of M
 $E(M, T^2) = \chi(E_{g,y})$ ($(t, z) = (y, y)$
 $q = e^{2\pi i t}, y = e^{2\pi i z}$)

the Euler characteristic
algebra
ation in conformal field theory

Notation · M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

Properties

· χ is a regularized $U(1)$ -equivariant index of a Dirac operator on Cosp space

$\chi(M) = \chi(\Lambda_y T^*)$ Atiyah-Singer χ

Def [Hirzebruch, Witten &]

$$\mathbb{E}_{q,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\dim} (\Lambda_{y^n} T^* \otimes \Lambda_{y^n} T \otimes S_y T^* \otimes S_y T)$$

COMPLEX ELLIPTIC GENUS of M

$$\chi(M, \tau, \epsilon) = \chi(\mathbb{E}_{q,y}) \quad (\tau, \epsilon \in \mathbb{C} \times \mathbb{C}^*)$$

$q = e^{2\pi i \tau}, y = e^{2\pi i \epsilon}$

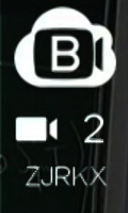
1. Refining $\chi(M)$

$$\chi(M) = \sum_{i,k} (-1)^{i+k} h^{i,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$



arXiv:1705.09904

the Euler characteristic
 algebra
 in conformal field theory

Notation
 M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

Properties

χ is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

use $c(T) = \prod_{j=1}^D (1 + x_j)$

$$\chi(M; \tau, \bar{\tau}) = \sum_M \prod_{j=1}^D \frac{x_j}{1 - e^{-x_j}}$$

↑
Td

Atiyah-Singer $\chi(M)$

Def [Hirzebruch, Witten &]

$$\mathbb{E}_{q, y} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^n} T \otimes S_y T^* \otimes S_y T)$$

COMPLEX ELLIPTIC GENUS of M

$$\chi(M; \tau, \bar{\tau}) = \chi(\mathbb{E}_{q, y}) \quad (\tau, \bar{\tau} \in \mathbb{H} \times \mathbb{H})$$

$q = e^{2\pi i \tau}, y = e^{2\pi i \bar{\tau}}$



1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

arXiv:1705.09904
 the Euler characteristic
 algebra
 in conformal field theory

BN-MATH-VC1

Notation
 M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

Properties

χ is a regularized $U(1)$ -equivariant index of a Dirac operator on Cosp space

use $c(T) = \prod_{j=1}^D (1 + x_j)$

$$\chi(M; \tau, \zeta) = \sum_M \prod_{j=1}^D \frac{x_j}{1 - e^{-x_j}} (1 - y e^{-x_j})$$

\uparrow \uparrow
 Td $\text{ch}(\Lambda_{-y} T^*)$

Atiyah-Singer χ
 Def [Hirzebruch & Witten & ...]
 $\chi_{g,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T \otimes S_n T^*)$
 COMPLEX ELLIPTIC GENUS of M
 $\chi(M; \tau, \zeta) = \chi(\chi_{g,y})$ $(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^*$
 $q = e^{2\pi i \tau}, y = e^{2\pi i \zeta}$



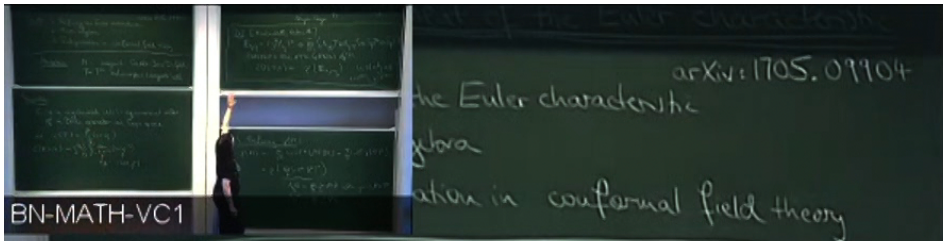
1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \text{ with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$



BN-MATH-VC1

Notation M : compact Calabi-Yau D-fold
 $T = T^{1,0}$ holomorphic tangent bundle

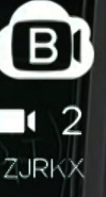
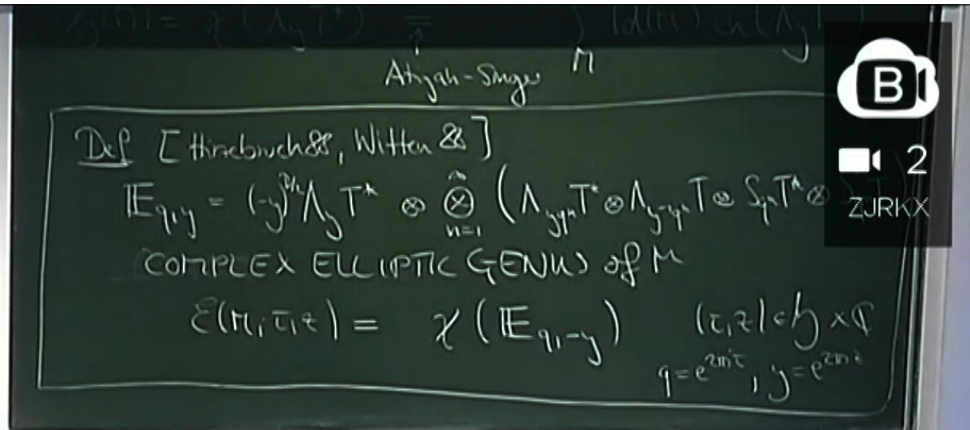
Properties:

- \mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on loop space

use $c(T) = \prod_{j=1}^D (1 + x_j)$

$$\mathcal{E}(M; \tau, \zeta) = y^{-D/2} \sum_{\mu} \prod_{j=1}^D \frac{x_j}{1 - e^{-x_j}} (1 - y e^{-x_j})$$

\uparrow \uparrow
 Td $ch(\Lambda_{-y} T^*)$



1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$
 $S_y E = \bigoplus_k y^k S^k E$

arXiv:1705.09904
 the Euler characteristic
 algebra
 in conformal field theory

BN-MATH-VC1

Notation
 M : compact Calabi-Yau D -fold
 $T = T^{1,0}$ holomorphic tangent bundle

Properties

χ is a regularized $U(1)$ -equivariant index of a Dirac operator on Cosp space

use $c(T) = \prod_{j=1}^D (1 + x_j)$

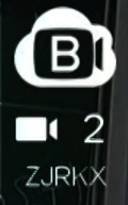
$$\chi(M; \tau_1, \tau_2) = \int_M \prod_{j=1}^D \frac{x_j}{1 - e^{-x_j}} (1 - y e^{-x_j}) \prod_{s=1}^D \frac{(1 - y q^{2s-1})(1 - y^{-1} q^{2s})}{(1 - q^{2s} e^{-x_j})(1 - q^{2s} e^{x_j})}$$

\uparrow Td \uparrow $\text{ch}(\Lambda_y T^*)$



$\chi_y(M) = \chi(\Lambda_y T^*) \overline{\int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)}$
 Atiyah-Singer

Def [Hirzebruch, Witten & ...]
 $\chi_y(M) = (-y)^{\dim M} \chi(T^*) \otimes \bigotimes_{n=1}^D (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T \otimes S_n T^* \otimes S_n T)$
 COMPLEX ELLIPTIC GENUS of M
 $\chi(\tau_1, \tau_2) = \chi(\chi_{y_1, y_2})$ $(\tau_1, \tau_2 = h \times \eta)$
 $q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$



1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \text{ with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

arXiv:1705.09904
 the Euler characteristic
 algebra
 in conformal field theory

Notation: M - compact Calabi-Yau 3D RLT

Properties:

E is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

use $c(T) = \prod_{j=1}^D (1+x_j)$

$$E(\pi; \tau, z) = y^{-D/2} \sum_{\pi} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

$$= \sum_{\pi} \prod_{j=1}^D x_j \frac{\text{Td}(\text{ch}(\Lambda_{-y} T^*))}{\mathcal{D}_1(\tau_1, z - x_j)}$$

$$\chi(\pi) = \chi(\Lambda_y T^*) \overline{\text{Atiyah-Singer}} \int_M \text{Td}(\pi) \text{ch}(\Lambda_y)$$

Def [Hirzebruch, Witten &]

$$E_{q,y} = (-y)^{D/2} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y q^n} T^* \otimes \Lambda_{y^{-1} q^n} T^* \otimes S_q T^* \otimes S_q T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$E(\pi; \tau, z) = \chi(E_{q,y}) \quad (\tau, z) \in \mathfrak{h} \times \mathfrak{g}$$

$$q = e^{2\pi\tau}, y = e^{2\pi iz}$$

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

$$\chi(M) = \chi(\Lambda_y T^*) \overline{\chi} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

$$\prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

$$= \int_M \prod_{j=1}^D \frac{\text{Td}(\Lambda_{y^{-1}} T^*) \text{ch}(\Lambda_y T^*)}{\theta_1(\tau_1, z - x_j)} \Rightarrow \left\{ \begin{array}{l} \text{Jacobi form} \\ \text{modular of wt 0} \\ \text{and index } D/2 \end{array} \right.$$

$$\chi_y(M) = \chi(\Lambda_y T^*) \overline{\chi} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch, Witten &]

$$\mathbb{E}_{q,y} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_n T^* \otimes S_n T^*)$$
 COMPLEX ELLIPTIC GENUS of M

$$\mathbb{E}(M, \tau, z) = \chi(\mathbb{E}_{q,y}) \quad (|\tau, z| = h \times \theta)$$

$$q = e^{2\pi i \tau}, y = e^{2\pi i z}$$

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

$\chi(M) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y)$

operator on loop space

$$\prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y^n q^{2n}) (1-y^n q^{2n})}{(1-q^n e^{-x_j}) (1-q^n e^{x_j})}$$

\uparrow $\text{Td}(\Lambda_y T^*)$

$$= \int_M \prod_{j=1}^D x_j \frac{\mathcal{D}_1(\tau_1, z - x_j)}{\mathcal{D}_1(\tau_1 - x_j)} \Rightarrow \left[\begin{array}{l} \text{Jacobi form} \\ \text{modular of wt 0} \\ \text{and index } D/2 \end{array} \right]$$

BN-MATH-VC1

$\chi(M) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y)$

Def [Thurston, Witten &]

$$\mathbb{E}_{g,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_n T^* \otimes S_n)$$

COMPLEX ELLIPTIC GENUS OF M

$$\mathcal{E}(M; \tau, z) = \chi(\mathbb{E}_{g,y}) \quad (\tau, z) \in \mathcal{H} \times \mathbb{C}$$

$q = e^{2\pi i \tau}, y = e^{2\pi i z}$



2

ZJRKX

it's a genus

refine $\mathcal{E}(M; \tau, z)$

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$

$S_y E = \bigoplus_k y^k S^k E$



BN-MATH-VC1

Heat kernel - equivalent index
operator on loop space

$$\prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

\uparrow \uparrow \uparrow
 Td $\text{ch}(\Lambda_{-y} T^*)$

$$= \int_{\mathbb{D}} \prod_{j=1}^D x_j \frac{\theta_1(\tau, z - x_j)}{\theta_1(\tau, -x_j)} = \left[\text{Jacobi form multiplier of wt 0 and index } D/2 \right]$$

it's a genus

refine $\chi(\pi) \xleftarrow{z=0} \mathcal{E}(\pi, \tau, z) \xleftarrow{r=0} \mathcal{E}(\pi, \tau, z, r)$

$$\chi_g(\pi) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_{\pi} \text{Td}(\pi) \text{ch}(\Lambda_y T^*)$$

Def [Thurston, Witten & ...]

$$\mathbb{E}_{q,y} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_n T^* \otimes \dots)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(\pi, \tau, z) = \chi(\mathbb{E}_{q,y}) \quad (\tau, z) \in \mathfrak{h} \times \mathfrak{g}$$

$q = e^{2\pi i \tau}, y = e^{2\pi i z}$

1. Refining $\chi(M)$

$$\chi(\pi) = \sum_{j,k} (-1)^{j+k} h^{j,k}(\pi) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi \left(\bigoplus_k (-1)^k \Lambda^k T^* \right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^*$$

$$S_y E = \bigoplus_k y^k S^k E$$

BN-MATH-VC1

Real unit-equivalent index
operator on loop space

$$\prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^{2n-1}) (1-y^{-1} q^{2n})}{(1-q^n e^{-x_j}) (1-q^n e^{x_j})}$$

\uparrow \uparrow \uparrow

Td $ch(\Lambda_{-y} T^*)$

$$= \int_M \prod_{j=1}^D x_j \frac{\mathcal{D}_1(\tau, z-x_j)}{\mathcal{D}_1(\tau, -x_j)} = \left[\begin{array}{l} \text{Jacobi form} \\ \text{modular of wt 0} \\ \text{and index } D/2 \end{array} \right]$$

$\chi(\mathbb{P}^1) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M Td(M) ch(\Lambda_y T^*)$

Def [Thurston, Witten & ...]

$\mathbb{E}_{q,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T^* \otimes S_n T^* \otimes S_n T^*)$

COMPLEX ELLIPTIC GENUS of M

$\mathcal{E}(M, \tau, z) = \chi(\mathbb{E}_{q,y}) \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}$
 $q = e^{2\pi i \tau}, y = e^{2\pi i z}$

it's a genus

refine $\chi(M) \xleftarrow{z=0} \mathcal{E}(M, \tau, z) \xleftarrow{r=0} \mathcal{E}(M, \tau, z, r)$

Def [Kahn/Thripathy 16]

Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$

$S_y E = \bigoplus_k y^k S^k E$

BN-MATH-VC1

operator on loop space

$$\prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^{2n-1}) (1-y^{-1} q^{2n})}{(1-q^{2n} e^{-x_j}) (1-q^{2n} e^{x_j})}$$

\uparrow $\text{Td}(\Lambda_{y,T^*})$ \uparrow $\text{ch}(\Lambda_{y,T^*})$

$= \int \prod_{j=1}^D x_j \frac{\mathcal{D}(\tau, z-x_j)}{\mathcal{A}(\tau-x_j)} = \text{Jacobi form multiplier of wt 0}$

$\chi_g(M) = \chi(\Lambda_y T^*) \overline{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$

Def [Thurston, Witten & ...]

$\mathbb{E}_{q,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T^* \otimes S_{q^n} T^*)$

COMPLEX ELLIPTIC GENUS of M

$\mathcal{E}(M, \tau, z) = \chi(\mathbb{E}_{q,y}) \quad (\tau, z) \in \mathfrak{h} \times \mathfrak{g}$
 $q = e^{2\pi\tau}, y = e^{2\pi iz}$

it's a genus

refine $\chi(M) \xleftarrow{z=0} \mathcal{E}(M, \tau, z) \xleftarrow{r=0} \mathcal{E}(M, \tau, z, r) \xrightarrow{\text{HECT}}$

Def [Kahn / Tripathy 16]

$\mathbb{E}_{q,y} = y^{-\dim} \bigoplus_{\mathcal{E}_{im}} q^{\mathcal{E}_{im}} \mathcal{T}_{\mathcal{E}_{im}}$

$y^{-\dim} \sum_j (-1)^j \sum_{\mathcal{E}_{im}} q^{\mathcal{E}_{im}} \dim H^j(M, \mathcal{T}_{\mathcal{E}_{im}})$

1. Refining $\chi(M)$

$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$

$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$

$\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$

$S_y E = \bigoplus_k y^k S^k E$

Notation M : compact Calabi-Yau D-fold

$T = T^{n,0}$ holomorphic tangent bundle

BN-MATH-VC1

red unit - equivalent index operator on loop space

$$\prod_{j=1}^g \frac{x_j}{1-e^{-x_j}} \prod_{n=1}^{\infty} \frac{(1-yq^{2n-1})(1-y^{-1}q^{2n})}{(1-q^{2n}e^{-x_j})(1-q^{2n}e^{x_j})}$$

↑ $\text{Td}(\Lambda_{-y} T^*)$ | Jacobi form

it's a genus

refine $\chi(\pi) \xleftarrow{z=0} \mathcal{E}(\pi, \tau, z) \xleftarrow{r=0} \mathcal{E}(\pi, \tau, z, r) \stackrel{\text{HEG}}{\leftarrow}$

Def [Kachm/Tripathy 16]

$$E_{g,y} = y^{-2g} \bigoplus_{c,m} q^c (-y)^m \mathcal{T}_{c,m}, \quad r \in \mathbb{C}, \quad u = e^{2\pi i r}$$

$$\mathcal{E}^{\text{HEG}}(\pi, \tau, z) = (u)^{Dz} \sum_j (-u)^j \sum_{c,m} q^c (-y)^m \dim H^j(\pi, \mathcal{T}_{c,m})$$

Notation M - compact Gaudin-Jacobi field
 $T = T^{1,0}$ holomorphic tangent bundle

$\chi_g(\pi) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_{\pi} \text{Td}(\pi) \text{ch}(\Lambda_y T^*)$

Def [Thurston, Witten & ...]

$E_{g,y} = (-y)^{2g} \Lambda_y T^* \otimes \bigotimes_{n=1}^g (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T^* \otimes S_n T^* \otimes \dots)$

COMPLEX ELLIPTIC GENUS OF M

$\mathcal{E}(\pi, \tau, z) = \chi(E_{g,y}) \quad (z, \tau) \in \mathcal{H} \times \mathcal{H}$
 $q = e^{2\pi i \tau}, y = e^{2\pi i z}$

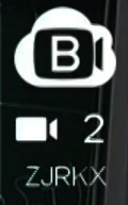
Blank chalkboard

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^*)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^*\right)$$

$\Lambda_y E = \bigoplus_k y^k \Lambda^k E$ with $y = -1, E = T^*$
 $S_y E = \bigoplus_k y^k S^k E$



ZJKX



BN-MATH-VC1

$$\chi(M) = \sum_{r=0}^{\dim M} (-1)^r \dim H^r(M, \mathbb{R})$$

$$\chi(M, \tau, \epsilon) = \sum_{r=0}^{\dim M} (-1)^r \dim H^r(M, \tau, \epsilon)$$

This [Kadomtshypathy 16]
 If M is a 4-manifold or a K3 surface then $\chi(M, \tau, \epsilon)$ is an invariant

$$\chi(M) = \chi(\Lambda_y T^* M) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{td}(M) \text{ch}(\Lambda_y T^* M)$$

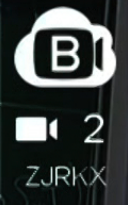
Def [Thurston, Witten & ...]

$$\mathbb{E}_{g,y} = (-y)^{\dim M} \Lambda_y T^* M \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* M \otimes \Lambda_{y^{-n}} T^* M \otimes S_n T^* M)$$

COMPLEX ELLIPTIC GENUS of M

$$\chi(M, \tau, \epsilon) = \chi(\mathbb{E}_{q,y}) \quad (\tau, \epsilon) = (h, \nu)$$

$$q = e^{2\pi\tau}, y = e^{2\pi\epsilon}$$



ZJRKX

Vertical refinement of the Euler characteristic

- PLAN
1. Refining the Euler char
 2. More algebra
 3. Interpretation in conf

Notation: M : compact ... n -manifold
 $T = T^{n,0}$ holomorphic tangent bundle

1. Refining $\chi(M)$

$$\chi(M) = \sum_{j,k} (-1)^{j+k} h^{j,k}(M) = \sum_k (-1)^k \chi(\Lambda^k T^* M)$$

$$= \chi\left(\bigoplus_k (-1)^k \Lambda^k T^* M\right)$$

$$\Lambda_y E = \bigoplus_k y^k \Lambda^k E \quad \text{with } y = -1, E = T^* M$$

$$S_y E = \bigoplus_k y^k S^k E$$



BN-MATH-VC1

$$\begin{aligned}
 & \{6\} \\
 & q^e(-y)^m \mathcal{T}_{em}, \quad r \in \mathbb{C} \\
 & u = e^{2\pi i r} \\
 & \sum_{em} q^e(-y)^m \dim H^j(\mathbb{P}^1, \mathcal{T}_{em})
 \end{aligned}$$

Thm [Kadomtshipathy 16]
 If M is a spinors or a K3 surface then E^{HEG} is an invariant

Properties

E_r is a regularized $U(1)$ -equivariant index of a Dirac operator on loop space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$E(\mathbb{P}^1; \tau, z) = y^{-D/2} \int_{\mathbb{P}^1} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

$$= \int_{\mathbb{P}^1} \prod_{j=1}^D x_j \frac{\text{Tr } \text{ch}(\Lambda_{-j} T^*)}{\mathcal{D}_1(\tau_1 - x_j)} \Rightarrow \left[\begin{array}{l} \text{Focki form} \\ \text{modular of wt 0} \\ \text{and index } D/2 \end{array} \right]$$

$$\begin{aligned}
 & E_{g,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T \otimes S_n T^* \otimes S_n T) \\
 & \text{COMPLEX ELLIPTIC GENUS of } M \\
 & E(\mathbb{P}^1; \tau, z) = \chi(E_{g,y}) \quad (\tau, z) \in \mathbb{H} \\
 & q = e^{2\pi i \tau}, y = e^{2\pi i z}
 \end{aligned}$$

B
 2
 ZJRKX

2. Morse algebra



$$\xi(\pi, \tau, z) \stackrel{r=0}{\longleftarrow} \xi^{\text{HEG}}(\pi, \tau, z, r)$$

$$E_{q,y} = y^{-D/2} \bigoplus_{\ell \in \mathbb{Z}} q^{\ell} (-y)^{\ell} \mathcal{T}_{\ell, \text{em}}, \quad r \in \mathbb{C}$$

$$u = e^{2\pi i r}$$

$$\xi^{\text{HEG}}(\pi, \tau, z) = (u)^{D/2} \sum_j (-u)^j \sum_{\ell \in \mathbb{Z}} q^{\ell} (-y)^{\ell} \dim H^j(\pi, \mathcal{T}_{\ell, \text{em}})$$

Thm [Kadomtshchikov 16]

If M is a parabolic or a K3 surface then ξ^{HEG} is an invariant

Properties:

- ξ is a regularized $U(1)$ -equivariant index of a Dirac operator on loop space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$\xi(\pi; \tau, z) = y^{-D/2} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

$$= \int \prod_{j=1}^D \frac{dx_j}{x_j} \frac{\text{Id } \text{ch}(\Lambda_y T^*)}{\mathcal{D}(\tau, z-x_j)} \quad \left[\text{Jacobi form modulus of wt } 0 \right]$$

$$\xi(\pi) = \chi(\Lambda_y T^*) \cdot \overline{\text{Atiyah-Singer}} \cdot \text{Id}(\pi) \cdot \text{ch}(\Lambda_y T^*)$$

Def [Thurston & Witten 87]

$$E_{q,y} = (-y)^{D/2} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y q^n T^*} \otimes \Lambda_{y^{-1} q^n T^*} \otimes S_{q^n T^*})$$

COMPLEX ELLIPTIC GENUS of M

$$\xi(\pi; \tau, z) = \chi(E_{q,y}) \quad (\tau, z) \in \mathfrak{h} \times \mathfrak{g}$$

$$q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}$$

2 More algebra

$M \rightsquigarrow \text{sVOA}(s) ?$





BN-MATH-VC1

$$\mathcal{E}(\pi, \tau, \varepsilon) \xleftarrow{r=0} \mathcal{E}^{\text{HEG}}(\pi, \tau, \varepsilon)$$

Def [Kadom/Tripathy 16]

$$E_{g,y} = y^{-D/2} \bigoplus_{\ell,m} q^{\ell} (-y)^m \mathcal{T}_{\ell,m}, \quad r \in \mathbb{C}, \quad u = e^{2\pi i r}$$

$$\mathcal{E}^{\text{HEG}}(\pi, \tau, \varepsilon) = (u)^{D/2} \sum_j (-u)^j \sum_{\ell,m} q^{\ell} (-y)^m \dim H^j(\pi, \mathcal{T}_{\ell,m})$$

Thm [Kadom/Tripathy 16]

If M is a 4-manifold or a K3 surface then \mathcal{E}^{HEG} is an invariant

Properties:

- \mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on M space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$\mathcal{E}(\pi; \tau, \varepsilon) = y^{-D/2} \sum_{\pi} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} \cdot \frac{(1-yq^{\varepsilon} e^{-x_j})(1-y^{-1}q^{\varepsilon} e^{x_j})}{(1-q^{\varepsilon} e^{-x_j})(1-q^{\varepsilon} e^{x_j})}$$

Fujimi formula

$$\chi_g(M) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch, Witten &]

$$E_{g,y} = (-y)^{D/2} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_{y^n} T^* \otimes S_{y^{-n}} T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(\pi, \tau, \varepsilon) = \chi(E_{g,y}) \quad (r, \varepsilon) = h \times \theta$$

$$q = e^{2\pi i r}, \quad y = e^{2\pi i \varepsilon}$$



2

ZJRKX

2. Morse algebra

$M \rightsquigarrow \text{sVOA}(s) ?$

basic ingredient: bc- $\beta\gamma$ system E



BN-MATH-VC1

$$\mathcal{E}(\pi, \tau, \epsilon) \xleftarrow{r=0} \mathcal{E}^{\text{HEG}}(\pi, \tau, \epsilon)$$

Def [Kadom/Tripathy 16]

$$E_{q,y} = y^{-D_2} \oplus_{\epsilon_m} q^{\epsilon(-y)^m} \mathcal{T}_{\epsilon_m}, \quad r \in \mathbb{C}$$

$$\mathcal{E}^{\text{HEG}}(\pi, \tau, \epsilon) = (uy)^{D_2} \sum_j (-u)^j \sum_{\epsilon_m} q^{\epsilon(-y)^m} \dim H^j(\pi, \mathcal{T}_{\epsilon_m})$$

Thm [Kadom/Tripathy 16]

If π is a operator or a KS surface then \mathcal{E}^{HEG} is an invariant

Properties:

- \mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$\mathcal{E}(\pi; \tau, \epsilon) = y^{-D_2} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

$$\chi_g(\pi) = \chi(\Lambda_y T^+) \xrightarrow{\text{Atiyah-Singer}} \int_{\pi} \text{Td}(\pi) \text{ch}(\Lambda_y T^+)$$

Def [Hirzebruch, Witten &]

$$E_{q,y} = (-y)^{D_2} \Lambda_y T^+ \otimes \hat{\otimes}_{n=1}^{\infty} (\Lambda_{y^n} T^+ \otimes \Lambda_{y^{-n}} T^+ \otimes S_{y^n} T^+ \otimes S_{y^{-n}} T^+)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(\pi, \tau, \epsilon) = \chi(E_{q,y}) \quad (\tau, \epsilon) = h \times \mathbb{R}$$

$$q = e^{2\pi i \tau}, y = e^{2\pi i \epsilon}$$



2

ZJRKX

2. Morse algebra

$M \rightsquigarrow \text{SVOA}(s) ?$

basic ingredient: bc- $\beta\gamma$ system E

F : cpx vector space with basis $(b_{n_1}, \dots, b_{n_j}, a_{m_1}, \dots, a_{m_k})$

$\begin{matrix} n_1, \dots, n_j \geq 0 \\ m_1, \dots, m_k > 0 \\ n_i, m_i \in \mathbb{Z} \end{matrix}$





BN-MATH-VC1

$$\mathcal{E}(M, \tau, \epsilon) \xleftarrow{r=0} \mathcal{E}^{HEG}(M, \tau, \epsilon)$$

Def [Kadom/Tripathy 16]

$$E_{q,y} = y^{-D/2} \bigoplus_{\substack{m \\ \text{cm}}} q^e (-y)^m \mathcal{T}_{cm}, \quad r \in \mathbb{C}, \quad u = e^{2\pi i r}$$

$$\mathcal{E}^{HEG}(M, \tau, \epsilon) = (u)^{D/2} \sum_j (-u)^j \sum_{\substack{m \\ \text{cm}}} q^e (-y)^m \dim H^j(M, \mathcal{T}_{cm})$$

Thm [Kadom/Tripathy 16]
 If M is a 4-manifold or a K3 surface then \mathcal{E}^{HEG} is an invariant

Properties

\mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

use $c(T) = \prod_{j=1}^D (1+x_j)$

$$\mathcal{E}(M; \tau, \epsilon) = y^{-D/2} \sum_{\substack{m \\ \text{cm}}} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

\uparrow $\text{Td} \quad \text{ch}(\Lambda_{-y} T^*)$ Fermi form

$$\chi_g(M) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Thirubuchari, Witten & ...]
 $E_{q,y} = (-y)^{D/2} \Lambda_y T^* \otimes \hat{\otimes}_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_n T^* \otimes \dots)$
 COMPLEX ELLIPTIC GENUS of M
 $\mathcal{E}(M; \tau, \epsilon) = \chi(E_{q,y}) \quad (\tau, \epsilon) = (h) \times \mathbb{R}$
 $q = e^{2\pi i \tau}, \quad y = e^{2\pi i \epsilon}$



2
ZJRKX

2. Heisenberg algebra

$M \rightsquigarrow$ sVOA(s)?

basic ingredient: bc- $\beta\gamma$ system E

F : cpx vector space with basis $(b_n, -b_{-n}, a_n, -a_{-n})_{n \in \mathbb{Z}}$
 $n \neq 0: \begin{cases} 2n_1 < 0 \\ 2n_2 > 0 \end{cases}$

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{h}}(\mathbb{C})$ is Lie algebra with basis $(a_n, b_n, 1)_{n \in \mathbb{Z}}$





BN-MATH-VC1

$$\mathcal{E}(\pi, \tau, \epsilon) \xleftarrow{r=0} \mathcal{E}^{\text{HEG}}(\pi, \tau, \epsilon)$$

Def [Kadomt'ipathy 16]

$$E_{g, -y} = y^{-D_2} \oplus_{\epsilon_m} q^{\epsilon} (-y)^m \mathcal{T}_{\epsilon_m}, \quad r \in \mathbb{C}, \quad u = e^{2\pi i r}$$

$$\mathcal{E}^{\text{HEG}}(\pi, \tau, \epsilon) = (u)^{D_2} \sum_j (-u)^j \sum_{\epsilon_m} q^{\epsilon} (-y)^m \dim H^j(\pi, \mathcal{T}_{\epsilon_m})$$

Thm [Kadomt'ipathy 16]

If M is a compact or a K3 surface then \mathcal{E}^{HEG} is an invariant

Properties:

- \mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$\mathcal{E}(\pi; \tau, \epsilon) = y^{-D_2} \sum_{\pi} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

\uparrow Td $\text{ch}(\Lambda_{-y} T^*)$ [Fermi form]

$$\chi_g(M) = \chi(\Lambda_y T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Thurston, Witten & ...]

$$E_{g, -y} = (-y)^{D_2} \Lambda_y T^* \otimes \hat{\otimes}_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_{y^n} T^* \otimes S_{y^{-n}} T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(\pi, \tau, \epsilon) = \chi(E_{g, -y}) \quad (\tau, \epsilon) \in \mathbb{H} \times \mathbb{R}$$

2. Hecke algebra

M mod SVOA(s)?

basic ingredient: bc- β system E

F : cpx vector space (with basis $(b_{n_1}, \dots, b_{n_k}, a_{m_1}, \dots, a_{m_k})_{\substack{n_i \in \mathbb{Z}, \sum n_i = 0 \\ m_i \in \mathbb{Z}, \sum m_i > 0 \\ n_i, m_i \in \mathbb{Z}}}$)

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C})$ in Lie algebra with basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$ with $[a_n, b_m] = \delta_{n, -m} 1$ all other $[a_n, a_m] = [b_n, b_m] = 0$



BN-MATH-VC1

$$\mathcal{E}(\pi, \tau, \varepsilon) \xleftarrow{r=0} \mathcal{E}^{\text{HEG}}(\pi, \tau, \varepsilon)$$

Def [Kadom/Tripathy 16]

$$E_{q,y} = y^{-D/2} \oplus_{\ell,m} q^{\ell} (-y)^m \mathcal{T}_{\ell,m}, \quad r \in \mathbb{C}$$

$$\mathcal{E}^{\text{HEG}}(\pi, \tau, \varepsilon) = (uy)^{-D/2} \sum_j (-u)^j \sum_{\ell,m} q^{\ell} (-y)^m \dim H^j(\pi, \mathcal{T}_{\ell,m})$$

Thm [Kadom/Tripathy 16]

If π is a operator or a K3 surface then \mathcal{E}^{HEG} is an invariant

Properties:

\mathcal{E} is a regularized $U(1)$ -equivariant index of a Dirac operator on Loop space

$$\text{use } c(T) = \prod_{j=1}^D (1+x_j)$$

$$\mathcal{E}(\pi; \tau, \varepsilon) = y^{-D/2} \prod_{j=1}^D \frac{x_j}{1-e^{-x_j}} (1-y e^{-x_j}) \prod_{n=1}^{\infty} \frac{(1-y q^n e^{-x_j})(1-y^{-1} q^n e^{x_j})}{(1-q^n e^{-x_j})(1-q^n e^{x_j})}$$

\uparrow Td $\text{ch}(\Lambda_y T^*)$ [Fujimi form]

$$\chi_y(\pi) = \chi(\Lambda_y T^*) \xrightarrow{\uparrow} \int \text{Td}(\pi) \text{ch}(\Lambda_y T^*)$$

2. Hecke algebra

π mod SVOA(s)?

basic ingredients: bc- $\beta\gamma$ system E

F cpx vector space with basis $(b_{n_1}, b_{n_2}, a_{m_1}, a_{m_2})_{n_i, m_i \in \mathbb{Z}}$

$F = \text{ind}_{\mathbb{N}}^{\mathbb{N}}(\mathbb{C})$ is Lie algebra with basis $(a_{n_1}, b_{n_2}, \mathbb{1})_{n_i \in \mathbb{Z}}$ with $[a_{n_1}, b_{m_2}] = \delta_{n_1, m_2} \mathbb{1}$ all other $[x_{n_1}, y_{m_2}] = 0$

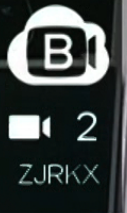
1. Refining $\chi(\pi)$

$$\chi(\pi) = \sum_{i,k} (-1)^{i+k} h^k \sum_n (-1)^n \chi(\Lambda^k T^*)$$

$$= \chi \left(\bigoplus_k (-1)^k \Lambda^k T^* \right)$$

$\Lambda_y E = \bigoplus_k (-1)^k \Lambda^k T^*$

with $y = -1, E = T^*$





BN-MATH-VC1

$$E(\pi, \tau, \rho) \xleftarrow{r=0} E^{HEG}(\pi, \tau, \rho)$$

Def [Kadom/Thiopathy 16]

$$E_{g, \tau} = \mathbb{C} \oplus_{\mathbb{C}^m} q^e (-y)^m \mathcal{V}_{em}, \quad r \in \mathbb{C}$$

$$u = e^{2\pi i r}$$

$$E^{HEG}(\pi, \tau, \rho) = (u)^{D/2} \sum_j (-u)^j \sum_{e, m} q^e (-y)^m \dim H^j(\pi, \mathcal{V}_{em})$$

Thm [Kadom/Thiopathy 16]

If M is a compact or a K3 surface then E^{HEG} is an invariant

$$\mathbb{C} = \text{span}_{\mathbb{C}}(\mathcal{L}) \supset \begin{cases} a_m \equiv 0, & m \leq 0 \\ b_n \equiv 0, & n \leq 0 \\ 1 = id \end{cases}$$

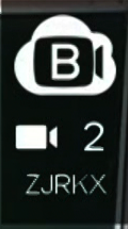
2. Hecke algebra

M mod SVOA(s) ?

basic ingredient: bc- $\beta\gamma$ system E

F : \mathbb{C}^n vector space (with basis $(b_n, -b_n, a_n, -a_n)_{n \in \mathbb{Z}}$)
 $n, m \in \mathbb{Z}$
 $2n_j \geq 0$
 $2m_k > 0$

$F = \text{ind}_{\mathfrak{h}_-}^{\mathfrak{h}}(\mathbb{C})$ \mathfrak{h} Lie algebra with basis $(a_n, b_n, 1)_{n \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, -n} \cdot 1$ all other $[x, y] = 0$



Hirzebruch χ_g -genus

$$\chi_g(M) = \chi(\Lambda_g T^*) \xrightarrow{\text{Atiyah-Singer}} \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch 88, Witten 88]

$$E_{g, \tau} = (-y)^{D/2} \Lambda_g T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{g, 2n} T^* \otimes \Lambda_{g, -2n} T^* \otimes S_{2n} T^* \otimes S_{2n} T)$$

COMPLEX ELLIPTIC GENUS of M



$$\begin{cases} a_m \equiv 0, & m \leq 0 \\ b_n \equiv 0, & n \leq 0 \\ 1 = \text{id} \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously

it's a genus

refine: $\chi(\Pi) \xleftarrow{z=0} \mathcal{E}(\Pi, \tau, z) \xleftarrow{r=0} \mathcal{E}(\Pi, \tau, z, r) \xrightarrow{\text{HEG}}$

Def [Kachru [Tripathy 16]]

$$E_{g, \tau} = y^{-D/2} \bigoplus_{l, m} q^l (-y)^m \mathcal{T}_{l, m}, \quad r \in \mathbb{C}, \quad u = e^{2\pi i r}$$

2. Mose algebra

M mod $\mathcal{SVOA}(s)$?

basic ingredient: bc- $\beta\gamma$ system E

F - cpx vector space (with basis
 $(b_n, -b_n, a_m, -a_m)_{n, m \in \mathbb{Z}}$)
 $n, m \in \mathbb{Z}$
 $n, m \geq 0$
 $m, n > 0$

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C})$ m. Lie algebra with
 basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, n} \cdot 1$ all other
 $[x_n, y_m] = 0$

Hirzebruch χ_g -genus.

$$\chi_g(\Pi) = \chi(\Lambda_g T^*) \xrightarrow{\text{Atiyah-Singer}} \int_{\Pi} \text{Td}(\Pi) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch 88, Witten 88]

$$E_{g, \tau} = (-y)^{D/2} \Lambda_g T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{g, 2n} T^* \otimes \Lambda_{g, 2n} T^*)$$

COMPLEX ELLIPTIC GENUS of M



$$\begin{cases} a_m \equiv 0, & m \leq 0 \\ b_n \equiv 0, & n \leq 0 \\ 1 = \text{id} \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously

idea: for $U \subset \mathbb{C}$ a holomorphic coordinate chart
 s.t.h. $\mathbb{E}_{g, \tau} / U = U \times \mathbb{E}$
 $\mathbb{E} \approx \mathbb{E}^{\text{std}}$

2. Mose algebra

M mod $\mathcal{SVOA}(S)$?

basic ingredient: bc- $\beta\gamma$ system E

F : $\mathbb{C}^p \times$ vector space (with basis
 $(b_n, -b_n, a_m, -a_m)$)
 $n, m \in \mathbb{Z}$
 $n \geq 1, m \geq 0$
 $n \leq -1, m < 0$
 $n, m \in \mathbb{Z}$

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C})$ \mathfrak{h} : Lie algebra with
 basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, n+1} - 1$ all other
 $[x_n, y_m] = 0$

Hirzebruch χ_g -genus

$$\chi_g(M) = \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

\uparrow
Atiyah-Singer

Def [Hirzebruch, Witten &]

$$\mathbb{E}_{g, \tau} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T \otimes S_{y^n} T^* \otimes S_{y^n} T)$$

COMPLEX ELLIPTIC GENUS of M



BN-MATH-VC1

$$\begin{cases} a_m \equiv 0, & m \leq 0 \\ b_n \equiv 0, & n \leq 0 \\ 1 = \text{id} \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously

idea: for $U \subset \mathbb{C}$ a holomorphic coordinate chart
 s.t.h. $(E_{g, \tau})|_U = U \times \mathbb{E}$

$$\mathbb{E} \simeq E^{\text{orb}}$$

$$S_{gp} T^*|_U \leftrightarrow \text{Sym}^*(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{-gp} T^*|_U \leftrightarrow \Lambda(\beta_m^{(1)}, \beta_m^{(D)})$$

2. Mose algebra

M mod $\mathcal{SVOA}(s)$?

basic ingredient: bc- $\beta\gamma$ system E

F - cpx vector space (with basis
 $(b_n, -b_n, a_m, -a_m)$)
 $n, m \in \mathbb{Z}$
 $n \geq 0, m \geq 0$
 $n < 0, m < 0$
 $n, m \in \mathbb{Z}$

$F = \text{ind}_{\mathfrak{h}_-}^{\mathfrak{h}_+}(\mathbb{C})$ m. Lie algebra with
 basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, n+1} \cdot 1$ all other
 $[x_n, y_m] = 0$

Hirzebruch χ_g -genus.

$$\chi_g(M) = \chi(\Lambda_g T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch & Witten &]

$$\mathbb{E}_{g, \tau} = (-y)^{\text{rk}} \Lambda_g T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{g, 2n} T^* \otimes \Lambda_{g, 2n-1} T^* \otimes S_{2n} T^* \otimes S_{2n-1} T^*)$$

COMPLEX ELLIPTIC GENUS of M



2

ZJRKX



$$\begin{cases} a_m \equiv 0, m \leq 0 \\ b_n \equiv 0, n \leq 0 \\ 1 = \text{id} \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously

idea for $U \subset \mathbb{R}$ a holomorphic coordinate chart
 s.t. $\mathbb{E}_{g, \gamma} | U = U \times \mathbb{E}$
 $\mathbb{E} \simeq E^{\text{asd}}$ $S_{\text{gp}} T^* | U \hookrightarrow \text{Sym}(b_m^{(1)}, b_m^{(2)})$
 $\Lambda_{\text{-gp}} T^* | U \hookrightarrow \Lambda(\beta_m^{(1)}, \beta_m^{(2)})$
 \leadsto construct a principal $SU(2)$ bdl of
 VOA modules out of this
 (2) this is not the setting in TQFT [Creutz/Holm]

2. Mose algebra

M mod $SVOA(s)$?

basic ingredient: bc- $\beta\gamma$ system E

F - cpx vector space (with basis
 $(b_n, -b_n, a_m, -a_m)_{n, m \geq 0}$
 $(m, n \geq 0)$
 $n, m \in \mathbb{Z}$)

$F = \text{ind}_{\mathfrak{h}_-}^{\mathfrak{h}}(\mathbb{C})$ m. Lie algebra with
 basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, n+1} \cdot 1$ all other
 $[x_n, y_m] = 0$

Hirzebruch χ_g -genus.

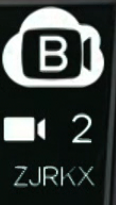
$$\chi_g(M) = \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

\uparrow
Atiyah-Singer

Def [Hirzebruch & Witten 88]

$$\mathbb{E}_{g, \gamma} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T^* \otimes S_{y^n} T^* \otimes S_{y^{-n}} T^*)$$

COMPLEX ELLIPTIC GENUS of M





BN-MATH-VC1

$$\begin{cases} a_m \equiv 0, m \leq 0 \\ b_n \equiv 0, n \leq 0 \\ 1 = id \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously

idea for $U \subset \mathbb{R}$ a holomorphic coordinate chart
 s.t. $\mathbb{R}E_{q,y}|_U = U \times \mathbb{R}E$

$$\mathbb{R}E \simeq E^{\text{odd}} \quad S_{\mathbb{R}^n} T|_U \simeq \text{Spin}(\beta_m^{(1)}, \beta_m^{(D)})$$

$$\Lambda_{-\mathbb{R}^n} T|_U \simeq \Lambda(\beta_m^{(1)}, \beta_m^{(D)})$$

\leadsto construct a principal $SU(D)$ bundle of
 VOA modules out of this

② this is not the setting in TQFT [Creutz/Hönl]
 ($m > 0, n \geq 0$) in $\mathbb{R}E$ we are missing the
 $\beta_m^{(1)}, \beta_m^{(D)}$

2. Mose algebra

M mod $SVOA(s)$?

basic ingredient: bc- $\beta\gamma$ system E

F : \mathbb{C}^n vector space (with basis
 $(b_n, -b_n, a_m, -a_m)_{\substack{n \geq 1 \\ m \geq 1}} \in \mathbb{Z}$)
 $F = \text{ind}_{\mathfrak{h}_-}^{\mathfrak{h}_+}(\mathbb{C})$ m. Lie algebra with
 $n, m \in \mathbb{Z}$
 basis $(a_n, b_m, 1)_{n, m \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{m, -n} \cdot 1$ all other
 $[x, y] = 0$

Hirzebruch χ_y -genus.

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch, Witten &]

$$\mathbb{R}E_{q,y} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T \otimes S_{y^n} T^* \otimes S_{y^{-n}} T)$$

COMPLEX ELLIPTIC GENUS of M



2

ZJRKX



a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{gp} T^* U \leftrightarrow \text{Sym}^n(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{-gp} T^* U \leftrightarrow \Lambda(\beta_n^{(1)}, \beta_n^{(D)})$$

→ construct a principal $SU(D)$ bundle of VOA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 99]

2. Mose algebra

M mod $SVOA(s)$?

basic ingredient: bc- $\beta\gamma$ system E

F : cpx vector space (with basis $(b_n, -b_n, a_n, -a_n, \mathbb{1})_{n \in \mathbb{Z}}$) $n \geq 2, n \geq 0$
 $n \leq -2, n \leq -1$
 $n, m \in \mathbb{Z}$

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C})$ \mathfrak{h} : Lie algebra with basis $(a_n, b_n, \mathbb{1})_{n \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{n+m,0} \cdot \mathbb{1}$ all other $[x_n, y_m] = 0$

Hirzebruch χ_g -genus:

$$\chi_g(M) = \chi(\Lambda_g T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch, Witten 88]

$$\mathbb{E}_{g, \eta} = (-\eta)^{\text{rk}} \Lambda_g T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{g, 2n} T^* \otimes \Lambda_{g, -2n} T^* \otimes S_{2n} T^* \otimes S_{2n} T^*)$$

COMPLEX ELLIPTIC GENUS of M



2

ZJRKX



a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{gp} T^* U \leftrightarrow \text{Sym}^n(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{-gp} T^* U \leftrightarrow \Lambda^n(b_m^{(1)}, b_m^{(D)})$$

→ construct a principal $SU(D)$ bundle of sVOA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 99]

Gluing $SU_n^{\text{ch}}(U) = E^{\otimes D}$ with U as above one obtains a sheaf of sVOAs called the CHIRAL de RHAR COMPLEX

2. Mose algebra

M mod sVOA(s) ?

basic ingredient: bc- $\beta\gamma$ system E

F : cpx vector space (with basis $(b_n, -b_n, a_n, -a_n)_{n \in \mathbb{Z}}$) $n \geq 2, n \geq 0$
 $n \leq -2, n \leq -1$
 $n, m \in \mathbb{Z}$

$F = \text{ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathbb{C})$ \mathfrak{h} : Lie algebra with basis $(a_n, b_n, 1)_{n \in \mathbb{Z}}$
 with $[a_n, b_m] = \delta_{n+m,0} \cdot 1$ all other $[x_n, y_m] = 0$

Hirzebruch χ_g -genus:

$$\chi_g(M) = \chi(\Lambda_g T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch, Witten 88]

$$\mathbb{E}_{g,y} = (-y)^{\text{rk}} \Lambda_g T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{g,1/n} T^* \otimes \Lambda_{g,y/n} T^* \otimes S_{1/n} T^* \otimes S_{y/n} T^*)$$

COMPLEX ELLIPTIC GENUS of M



2

ZJRKX



a holomorphic coordinate chart

$$U \cap U = U \times \mathbb{E}$$

$$S_{gp} T^* U \leftrightarrow \text{Sym}^n (b_{m_1}^{(1)}, \dots, b_{m_n}^{(n)})$$

$$\Lambda_{-gp} T^* U \leftrightarrow \Lambda (b_{n_1}^{(1)}, \dots, b_{n_n}^{(n)})$$

→ construct a principal $SU(D)$ bundle of sVOA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, \dots, b_0^{(n)}$

Thm [Malikov/Schubert/Vainshob 94]

Given $SU_n^{\text{ch}}(U) = E^{\otimes D}$ with U as above one obtains a sheaf of sVOAs called the CHIRAL de RHAR COMPLEX. It is filtered with associated graded $\mathbb{E}_{q,y}$

2. More algebra

M mod sVOA(s) ?

basic ingredient: bc- $\beta\gamma$ system E

F - cpx vector space (with basis $(b_{n_1}, \dots, b_{n_n}, a_{m_1}, \dots, a_{m_m})_{n_i, m_j \geq 0}$)
 $n_i, m_j \in \mathbb{Z}$

$F = \text{ind}_{\mathfrak{g}_-}^{\mathfrak{g}} (\mathbb{C})$ m. Lie algebra with basis $(a_{n_1}, b_{n_2}, 1)_{n_i, m_j \in \mathbb{Z}}$
 with $[a_{n_1}, b_{m_2}] = \delta_{n_1, m_2} \cdot 1$ all other $[x_{n_1}, y_m] = 0$

Hirzebruch χ_y -genus.

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch, Witten 88]

$$\mathbb{E}_{q,y} = (-y)^{\text{rk}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n}} T^* \otimes \Lambda_{y^{-2n}} T \otimes S_{y^n} T^* \otimes S_{y^n} T)$$

COMPLEX ELLIPTIC GENUS of M



2

ZJRKX



a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{\text{ym}} T^* U \leftrightarrow \text{Sym}(b_m^{(1)}, \dots, b_m^{(D)})$$

$$\Lambda_{\text{sym}} T^* U \leftrightarrow \Lambda(b_m^{(1)}, \dots, b_m^{(D)})$$

→ construct a principal $SU(D)$ bundle of VOA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, \dots, b_0^{(D)}$

Ima [Malikov/Schechtman/Vaintrob 14]

Given $SU_n^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above

one obtains a sheaf of sVOAs called the CHIRAL de RHAR COMPLEX. It is filtered with associated graded $\mathbb{E}_{q, -y}$

$$L_{\text{top}} \rightarrow \dots \rightarrow H^1(M, \mathbb{Q}_n^{\text{ch}})$$

2. Moore algebra

M mod sVOA(s) ?

basic ingredient: bc- $\beta\gamma$ system \mathbb{E}

F - cpx vector space (with basis $(b_m, -b_m, a_m, -a_m, \mathbb{1})_{m \in \mathbb{Z}}$) $\begin{matrix} m \geq 2 & 2m \geq 0 \\ m \leq -2 & 2m \leq 0 \end{matrix}$

$F = \text{ind}_{\mathfrak{h}_-}^{\mathfrak{h}_+}(\mathbb{C})$ m. Lie algebra with basis $(a_n, b_m, \mathbb{1})_{n, m \in \mathbb{Z}}$ with $[a_n, b_m] = \delta_{m, -n} \cdot \mathbb{1}$ all other $[x_n, y_m] = 0$

Hirzebruch χ_y -genus

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \sum_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch/Witten 82]

$$\mathbb{E}_{q, y} = \bigotimes_{n=1}^{\infty} (\Lambda_{y^{2n-1}} T^* \otimes \Lambda_{y^{2n}} T^* \otimes S_{y^{2n-1}} T^* \otimes S_{y^{2n}} T^*)$$

COMPLEX GENUS OF M



2

ZJRKX



BN-MATH-VC1

a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{\text{gr}} T^* U \leftrightarrow \text{Sym}^n (b_{\text{gr}}^{(1)}, b_{\text{gr}}^{(D)})$$

$$\Lambda_{\text{gr}} T^* U \leftrightarrow \Lambda (b_{\text{gr}}^{(1)}, b_{\text{gr}}^{(D)})$$

→ construct a principal $SU(D)$ bundle of sVA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 94]

Given $\mathcal{Q}_n^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above one obtains a sheaf of sVAs called the CHIRAL de RHAR COMPLEX. It is filtered with associated $\mathcal{E}_{q,y}$

$$\mathcal{E}_{q,y} \xrightarrow{\text{top}} \mathbb{E}_{q,y} \xrightarrow{\text{bottom}} \mathbb{H}^1(M, \mathcal{Q}_n^{\text{ch}})$$

Hirzebruch χ -genus.

$$\chi_g(M) = \chi(\Lambda_g T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_g T^*)$$

Def [Hirzebruch 88, Witten 88]

$$\mathbb{E}_{q,y} = (-y)^{\dim} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_{q^n} T^* \otimes S_{q^n} T)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M; \tau_1, \tau_2) = \chi(\mathbb{E}_{q,y}) \quad (\tau_1, \tau_2) = h \times g$$

$$q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$

Conclusion

$$\mathcal{E}(M; \tau_1, \tau_2) = y^{-\dim} \sum_j (-1)^j \dim \mathbb{H}^j(M, \mathcal{Q}_n^{\text{ch}}) q^{L^{\text{top}}} y^{L^{\text{bot}}}$$



a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{\text{ym}} T^* U \leftrightarrow \text{Sym}^2 (b_{\text{un}}^{(1)}, b_{\text{un}}^{(D)})$$

$$\Lambda_{\text{sym}} T^* U \leftrightarrow \Lambda^2 (b_{\text{un}}^{(1)}, b_{\text{un}}^{(D)})$$

→ construct a principal $SU(D)$ bundle of sVOA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 94]

Given $\mathcal{S}_D^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above

one obtains a sheaf of sVOAs called the CHIRAL de RHAR COMPLEX. It is filled

with associated graded $\mathbb{E}_{q, -y}$



Hirzebruch χ_y -genus.

$$\chi_y(M) = \chi(\Lambda_y T^*) \stackrel{\text{Atiyah-Singer}}{=} \int_M \text{Td}(M) \text{ch}(\Lambda_y T^*)$$

Def [Hirzebruch 88, Witten 88]

$$\mathbb{E}_{q, y} = (-y)^{\dim M} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\dim M} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T \otimes S_{y^n} T^* \otimes S_{y^{-n}} T)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M; \tau_1, \tau_2) = \chi(\mathbb{E}_{q, -y}) \quad (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$$

$$q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$

Conclusion.

$$\mathcal{E}(M; \tau_1, \tau_2) = y^{-\dim M} \sum_j (-1)^j \dim H^j(M, \mathcal{S}_D^{\text{ch}}) q^{L_0^{\text{top}}}$$

$$\neq H^j(M, \mathbb{E}_{1, -1})$$



a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{y^n} T^* U \leftrightarrow \text{Sym}^n(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{y^n} T^* U \leftrightarrow \Lambda^n(b_m^{(1)}, b_m^{(D)})$$

→ construct a principal $SU(D)$ bundle of
DFA modules out of this

⊗ this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Im [Malikov/Schectman/Vaintrob 11]

Given $\mathcal{Q}_n^{ch}(U) = \mathbb{E}^{\otimes D}$ with U as above

one obtains a sheaf of sVAs called the
CHIRAL de RHAR COMPLEX It is filtered

with associated graded \mathbb{E}_{q_1-y}
 $L_0^{top} \rightarrow \dots \rightarrow \mathbb{H}^1(M, \mathcal{Q}_n^{ch})$

Def [Malikov/Schectman/Vaintrob 11]

$$\mathbb{E}_{q_1-y} = (-y)^{D/2} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_y T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M, \tau_1, \tau_2) = \chi(\mathbb{E}_{q_1-y}) \quad (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$$

$$q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$

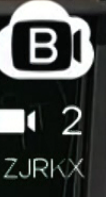
Conclusion

$$\mathcal{E}(M, \tau_1, \tau_2) = y^{-D/2} \sum_j (-1)^j \text{tr} \left(\frac{((-y)^{D/2} q^{L_0^{top}})}{\mathbb{H}^1(M, \mathcal{Q}_n^{ch})} \right)$$

$$\neq \mathbb{H}^1(M, \mathbb{E}_{1-y})$$

Def [WIT]

$$\mathcal{E}^{WIT}(M, \tau_1, \tau_2, \nu) = (ny)^{-D/2} \sum_j (-1)^j \text{tr} \left(\frac{((-y)^{D/2} q^{L_0^{top}})}{\mathbb{H}^1(M, \mathcal{Q}_n^{ch})} \right)$$





BN-MATH-VC1

a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{\text{ym}} T^* U \leftrightarrow \text{Sym}^n (b_{\text{sym}}^{(1)}, b_{\text{sym}}^{(D)})$$

$$\Lambda_{\text{sym}} T^* U \leftrightarrow \Lambda (b_{\text{sym}}^{(1)}, b_{\text{sym}}^{(D)})$$

→ construct a principal $SU(D)$ bundle of sVA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 94]

Given $\mathcal{S}_n^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above

one obtains a sheaf of sVAs called the CHIRAL de RHAR COMPLEX. It is filtered with associated graded $\mathbb{E}_{q,y}$

$$L_{\text{top}} \swarrow \searrow \rightarrow H^j(M, \mathcal{S}_n^{\text{ch}})$$

Def [Malikov/Schechtman 94]

$$\mathbb{E}_{q,y} = (-y)^{\text{Dh}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_y T^* \otimes S_{y^{-1}} T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M, \tau_1, \tau_2) = \chi(\mathbb{E}_{q,y}) \quad (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$$

$$q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$

Conclusion

$$\mathcal{E}(M, \tau_1, \tau_2) = y^{-\text{Dh}} \sum_j (-1)^j \chi_{\mathbb{E}_{q,y}}^j \left(\frac{(-y)^{\text{Dh}} q^{L_{\text{top}}}}{H^j(M, \mathcal{S}_n^{\text{ch}})} \right) \neq H^j(M, \mathbb{E}_{q,y})$$

Def [WIT]

$$\mathcal{E}^{\text{HEG}}(M, \tau_1, \tau_2, r) = (ny)^{-\text{Dh}} \sum_j (-1)^j \chi_{\mathbb{E}_{q,y}}^j \left(\frac{(-y)^{\text{Dh}} q^{L_{\text{top}}}}{H^j(M, \mathcal{S}_n^{\text{ch}})} \right)$$

Result [WIT]

For cpx top: $\mathcal{E}^{\text{HEG}} = \mathcal{E}^{\text{HEG}}; \text{ch}$



2

ZJRKX



BN-MATH-VC1

a holomorphic coordinate chart

$$U \times \mathbb{E}$$

$$S_{\text{ym}} T^* U \leftrightarrow \text{Sym}^2(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{\text{sym}} T^* U \leftrightarrow \Lambda^2(b_m^{(1)}, b_m^{(D)})$$

→ construct a principal $SU(D)$ bundle of sVA modules out of this

② this is not the setting in TQFT [Creutz/Holm] $(m > 0, n \geq 0)$ in \mathbb{E} we are missing the $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schectman/Vaintrob 14]

Given $\mathcal{S}_n^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above

one obtains a sheaf of sVAs called the CHIRAL de RHAR COMPLEX. It is filtered with associated graded $\mathbb{E}_{q,y}$

$$L_{\text{top}} \rightarrow \mathbb{E}_{q,y} \rightarrow \dots \rightarrow H^j(M, \mathcal{S}_n^{\text{ch}})$$

Def [Malikov/Schectman/Vaintrob 14]

$$\mathbb{E}_{q,y} = (-y)^{\text{Dh}} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n} T^* \otimes \Lambda_{y^{-n}} T^* \otimes S_y T^* \otimes \dots)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M, \tau_1, \tau_2) = \chi(\mathbb{E}_{q,y}) \quad (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$$

$$q = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$



2
ZJKX

Conclusion

$$\mathcal{E}(M, \tau_1, \tau_2) = y^{-\text{Dh}} \sum_j (-1)^j \text{tr} \left(\frac{(-y)^{\text{Dh}} q^{L_{\text{top}}}}{H^j(M, \mathcal{S}_n^{\text{ch}})} \right) \neq H^j(M, \mathbb{E}_{q,y})$$

Def [WIT]

$$\mathcal{E}^{\text{HEG}}(M, \tau_1, \tau_2, r) = (ny)^{-\text{Dh}} \sum_j (-1)^j \text{tr} \left(\frac{(-y)^{\text{Dh}} q^{L_{\text{top}}}}{H^j(M, \mathcal{S}_n^{\text{ch}})} \right)$$

Result [WIT]

For cpx tor $\mathcal{E}^{\text{HEG}} = \mathcal{E}^{\text{HEG}, \text{ch}}$
For K3 surfaces $\mathcal{E}^{\text{HEG}, \text{ch}}$ is an invariant different from \mathcal{E}^{HEG}



BN-MATH-VC1

a holomorphic coordinate chart

$$U \cong U \times \mathbb{E}$$

$$S_{\text{ym}} T^* U \cong \text{Sym}^2(b_m^{(1)}, b_m^{(D)})$$

$$\Lambda_{\text{sym}} T^* U \cong \Lambda(b_m^{(1)}, b_m^{(D)})$$

→ construct a principal $SU(D)$ bundle of
sVOA modules out of this

② this is not the setting in TQFT [Graber/Holm 17]
($m > 0, n \geq 0$) in \mathbb{E} we are missing the
 $b_0^{(1)}, b_0^{(D)}$

Thm [Malikov/Schechtman/Vaintrob 94]

Given $\mathcal{S}_n^{\text{ch}}(U) = \mathbb{E}^{\otimes D}$ with U as above
one obtains a sheaf of sVOAs called the
CHIRAL de RHAR COHOM. It is filtered
with associated graded

$$\mathcal{S}_n^{\text{ch}} \rightarrow H^j(M, \mathcal{S}_n^{\text{ch}})$$

Def [Malikov/Schechtman/Vaintrob 94]

$$\mathbb{E}_{g,y} = (-y)^{Dh} \Lambda_y T^* \otimes \bigotimes_{n=1}^{\infty} (\Lambda_{y^n T^*} \otimes \Lambda_{y^{-n} T^*} \otimes S_n T^*)$$

COMPLEX ELLIPTIC GENUS of M

$$\mathcal{E}(M, \tau_1, \tau_2) = \chi(\mathbb{E}_{g,y}) \quad (\tau_1, \tau_2) \in h \times \mathbb{R} \\ g = e^{2\pi i \tau_1}, y = e^{2\pi i \tau_2}$$

Conclusion

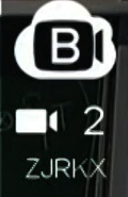
$$\mathcal{E}(M, \tau_1, \tau_2) = y^{-Dh} \sum_j (-1)^j \text{tr}_{H^j(M, \mathcal{S}_n^{\text{ch}})} ((-y)^{Dh} q^{L_0^{ch}}) \\ \subsetneq H^j(M, \mathbb{E}_{g,y}) \\ \neq H^j(M, \mathbb{E}_{g,y})$$

Def [W17]

$$\mathcal{E}^{\text{HEG}}(M, \tau_1, \tau_2, \tau) = (uy)^{-Dh} \sum_j (-1)^j \text{tr}_{H^j(M, \mathcal{S}_n^{\text{ch}})} ((-y)^{Dh} q^{L_0^{ch}})$$

Result [W17]

For cpx tri: $\mathcal{E}^{\text{HEG}} = \mathcal{E}^{\text{HEG}, ch}$
For K3 surfaces $\mathcal{E}^{\text{HEG}, ch}$ is an invariant different from \mathcal{E}^{HEG}
[Graber/Holm 14, Song 16/17]





Behtman/Vainstob 99
 $= E \otimes D$ with U as above

cap of sVAs called the
 COMPLEX. It is filtered

with associated graded $E_{q,y}$
 $L_{top} \rightarrow \dots \rightarrow L_{bot} \rightarrow H^*(M, \mathbb{Q})$

Conclusion: $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \langle \dots \rangle_{H^*(M, \mathbb{Q})}$
 $\neq H^*(M, E_{q,y})$

Def [W17]

$$\mathcal{E}^{HEG_{q,y}}(M, \tau, z, y) = (ny)^{-D/2} \sum_j (-1)^j \langle \dots \rangle_{H^*(M, \mathbb{Q})}^{((-y)^{2q})}$$

Result [W17]

For cpx top: $\mathcal{E}^{HEG} = \mathcal{E}^{HEG; d}$
 For K3 surfaces: $\mathcal{E}^{HEG; d}$ is an invariant different from \mathcal{E}^{HEG}
 use [Graber/Mohd/4, Song 16/17]

3 Interpretation in SCFT

large M non-linear sigma model $N=2$ SCFT



BN-MATH-VC1

Leichtman/Vainshtob 99

$= E \otimes D$ with U as above

cap of sLoAs called the COMPLEX. It is filtered with associated graded $E_{q,-y}$

$$L_{top} \rightarrow \mathbb{H}^*(M, \mathbb{Q}_n^{cn})$$



2 ZJRKX

Conclusion $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr} \mathbb{H}^j(M, \mathbb{Q}_n^{cn}) \binom{(-y)^{D/2}}{\mathbb{H}^j(M, \mathbb{Q}_n^{cn})}$

$\neq \mathbb{H}^*(M, E_{q,-y})$

Def [W17]

$$\mathcal{E}^{HEG, \mu}(M, \tau, z, \nu) = (\mu y)^{-D/2} \sum_j (-1)^j \text{tr} \mathbb{H}^j(M, \mathbb{Q}_n^{cn}) \binom{(-y)^{D/2} \mu y}{\mathbb{H}^j(M, \mathbb{Q}_n^{cn})}$$

Result [W17]

For cpx top: $\mathcal{E}^{HEG} = \mathcal{E}^{HEG, \mu}$
 For K3 surfaces: $\mathcal{E}^{HEG, \mu}$ is an invariant different from \mathcal{E}^{HEG}
 see [Graber/Mohdhan, Song 16/17]

3 Interpretation in SCFT

large M non-linear sigma model \rightarrow N=2 SCFT at central charge $c=3D$

\downarrow topological half twist

$\mathbb{H} \circ \mathbb{Z}_2 \circ L_{top} \circ \bar{\mathbb{Z}}_2$



BN-MATH-VC1

Behtman/Vainstob 99
 $= E \otimes D$ with U as above

of sVAs called the
 COMPLEX It is filtered

with associated graded $E_{q,y}$
 $L^{top} \rightarrow H^*(M, \mathbb{Q}_n^{ch})$

Conclusion $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \langle \tau, \mathbb{Z}^n \rangle_{\mathbb{Z}^n} \frac{((-y)^j}{\#(M, \mathbb{Q}_n^{ch})}$
 $\neq \#(M, E_{q,y})$

Def [WIT]

$$\mathcal{E}^{HEG}^M(M; \tau, z, y) = (ny)^{-D/2} \sum_j (-1)^j \langle \tau, \mathbb{Z}^n \rangle_{\mathbb{Z}^n} \frac{((-y)^j}{\#(M, \mathbb{Q}_n^{ch})} \langle \tau, L^{top} \rangle$$

3 Interpretation in SCFT

here M $\xrightarrow{\text{vanishing sigma model}}$ $\mathcal{N}=2$ SCFT at central charge $c=3D$
 \downarrow topological half twist

$$H \hookrightarrow \mathbb{Z}_2(L^{top}, \bar{\tau})$$

$$\downarrow$$

$$\mathcal{E}_{SCFT}(c)$$



BN-MATH-VC1

Leichtman/Vainshtob 99
 $= E^{\otimes D}$ with U as above

cap of sLoAs called the
 COMPLEX It is filtered

with associated graded $E_{q,-y}$
 $L_{top} \searrow \swarrow \rightarrow H^*(M, \mathbb{Q})$

Conclusion $\mathcal{Z}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{H^j(M, \mathbb{Q})} ((-y)^{D/2})$
 $\neq H^*(M, E_{q,-y})$

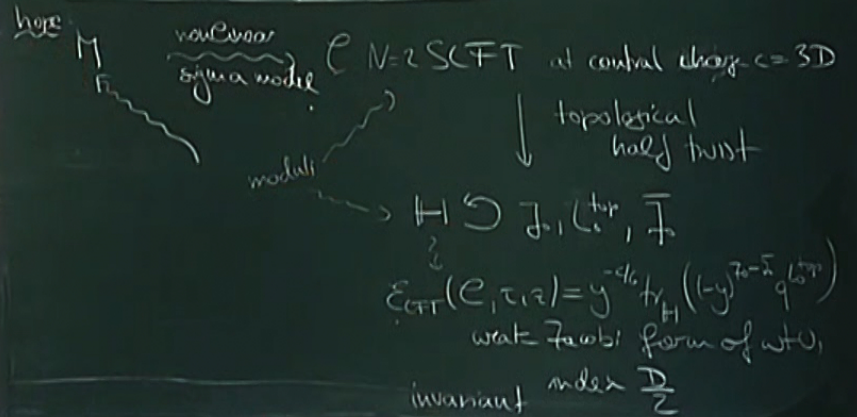
Def [W17]

$$\mathcal{Z}^{HEG, \text{top}}(M, \tau, z, y) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{H^j(M, \mathbb{Q})} ((-y)^{D/2} q^{L_{top}})$$

Result [W17]

For cpx tori $\mathcal{Z}^{HEG} = \mathcal{Z}^{HEG, \text{cl}}$
 For K3 surfaces $\mathcal{Z}^{HEG, \text{cl}}$ is an invariant different from \mathcal{Z}^{HEG}

3 Interpretation in SCFT





BN-MATH-VC1

Behtman/Vainstob 99
 $= E \otimes D$ with U as above
 cat of sVAs called the
 COMPLEX It is filtered
 with associated graded $E_{q,y}$
 $L_{top} \rightarrow \mathbb{H}^*(M, \mathcal{Q}_n^{ch})$

3 Interpretation in SCFT

here M non-linear sigma model \mathcal{L} $N=2$ SCFT at central charge $c=3D$
 \downarrow topological half twist
 moduli $\rightarrow \mathbb{H} \ni \mathbb{Z}_2, L_{top}, \bar{\mathbb{Z}}_2$
 $\mathcal{Z}_{CFT}(C, \tau, z) = y^{-D/2} \text{tr}_{\mathbb{H}} \left((-y)^{\mathbb{Z}_2} q^{L_{top}} \right)$
 wick \mathbb{Z}_2 form of wick
 invariant index $\frac{D}{2}$



2

ZJRKX

expect: $n \neq 1$, $\mathcal{Z}(M, \tau, z) = \mathcal{Z}_{CFT}(C, \tau, z)$

Conclusion: $\mathcal{Z}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}} \left((-y)^{\mathbb{Z}_2} q^{L_{top}} \right)$
 $\mathbb{H}^*(M, \mathcal{Q}_n^{ch}) \neq \mathbb{H}^*(M, \mathbb{E}_{1,2})$

Def [WIT]

$\mathcal{Z}^{HEG}^{ch}(M, \tau, z, y) = (ny)^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}} \left((-y)^{\mathbb{Z}_2} q^{L_{top}} \right)$

Realt [WIT]



BN-MATH-VC1

Behtman/Vainstob 997
 $= E \otimes D$ with U as above
 a family of sKAs called the
 COMPLEX It is filtered
 with associated graded $E_{q,-y}$
 $L_{top} \rightarrow H^*(M, \mathbb{Q})$

3 Interpretation in SCFT

here: M non-linear sigma model \in $N=2$ SCFT at central charge $c=3D$
 \downarrow topological half twist
 moduli $\rightarrow H^*(\mathbb{P}^1, L_{top}, \bar{L}_{top})$
 $\mathcal{E}_{CFT}(C, \tau, z) = y^{-D/2} \text{tr}_H \left((-y)^{7D-2} q^{L_{top}} \right)$
 with Fuchs form of $wt=0$
 invariant index $\frac{D}{2}$



2

ZJRKX

expect: $n \neq 1$, $\mathcal{E}(M, \tau, z) = \mathcal{E}_{CFT}(C, \tau, z)$
 true for complex tori
 and $M=K3$ if C is a K3-theory

$\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{H^j(M, \mathbb{Q})} \left((-y)^{7D-2} q^{L_{top}} \right)$
 $\neq \mathbb{H}^j(M, E_{1,y})$
 $\mathcal{E}^{HEG}^{(M)}(M, \tau, z, y) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{H^j(M, \mathbb{Q})} \left((-y)^{7D-2} q^{L_{top}} \right)$



BN-MATH-VC1

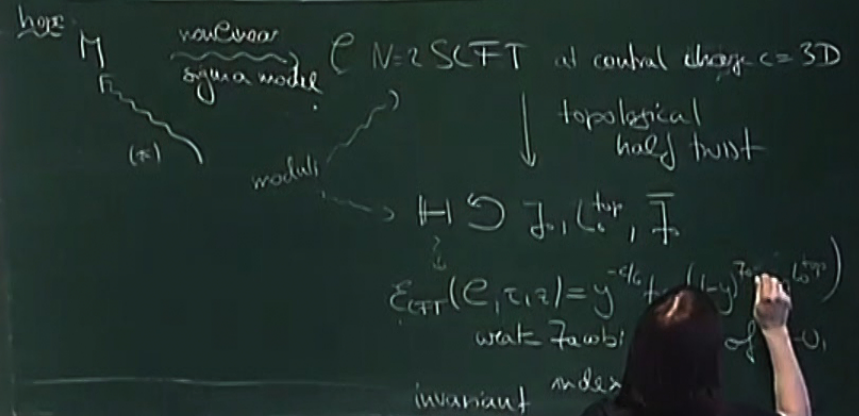
Behtman/Vaintrob 99
 $= E \otimes D$ with U as above
 a family of sheaves called the
 COMPLEX IT is filtered
 with associated graded $E_{q,y}$
 $L_{top} \rightarrow H^*(M, \mathcal{Q}_n^{top})$

expect $n \neq 1$, $\mathcal{E}(M; \tau, z) = \mathcal{E}_{CFT}(e; c, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a K3-theory

Def [Kadwin/Thirpathy 16]

$$\mathcal{E}_{CFT}^{HEG}(e; \tau, z, v) = (zy)^{-c/6} \text{tr}$$

3 Interpretation in SCFT



B
 2
 ZJKX



BN-MATH-VC1

Leichtman/Vainshtob 99
 $= E \otimes D$ with U as above
 a family of sLoAs called the
 COMPLEX IT is filtered
 with associated graded $E_{q,-y}$
 $L_{top} \rightarrow H^*(M, \mathbb{Q})$

expect $n \neq 1$, $\mathcal{E}(M; \tau, z) = \mathcal{E}_{CFT}(e; c, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a K3-theory

Def [Kadwin/Thipsting 16]
 $\mathcal{E}_{CFT}^{HEG}(\tau; \tau, z, v) = (uy)^{-d/2} \text{tr}$

3 Interpretation in SCFT

M \rightarrow \mathbb{R}^2 SCFT of central charge $c = 3D$
 topological half twist
 $H \otimes \mathbb{Z} \oplus L_{top} \oplus \bar{T}$
 $\mathcal{E}_{CFT}(e; \tau, z) = y^{-d/2} \text{tr}_{H \oplus \bar{T}} ((-y)^{J_0} \rho_{(1|1)}^{(1|1)})$
 writ. form of $\rho_{(1|1)}$
 invariant index \mathbb{D}_z

Def [WIT3]

$$\mathcal{E}_{CFT}^{HEG}(\tau; \tau, z, v) = (uy)^{-d/2} \sum_{\alpha} (-u)^\alpha \text{tr}_{E_{1, \alpha}} \rho_{(1|1)}^{(1|1)}$$

Result [WIT7]



BN-MATH-VC1

Leichtman/Vainshtob 99
 $= E \otimes D$ with U as above

cap of sLoAs called the
 COMPLEX It is filtered

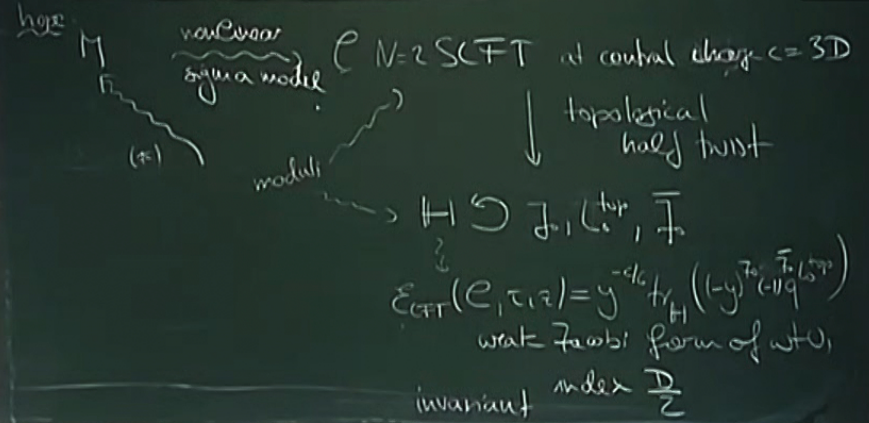
with associated graded $E_{q,-y}$
 $L_{top} \searrow \swarrow \rightarrow H^*(M, \mathcal{Q}_n^{ch})$

expect: in $(\neq 1)$, $\mathcal{E}(M, \tau, z) = \mathcal{E}_{CFT}(e, \tau, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a K3-theory

Def [Kadwin/Thipstry 16]

$$\mathcal{E}_{CFT}^{HEG}(e, \tau, z, v) = (uy)^{-c/6} \text{tr}_{\mathbb{H}}((-y)^{7_0} (-u)^{7_1} q^{L_0^{top}})$$

3 Interpretation in SCFT



Conclusion

$$\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{7_0} q^{L_0^{top}})$$

$\mathbb{H} = H^*(M, \mathcal{Q}_n^{ch})$
 $\neq H^*(M, E_{1,1})$

Def [WIT]

$$\mathcal{E}_{HEG}^{WIT}(M, \tau, z, v) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{7_0} q^{L_0^{top}})$$

Result [WIT]



BN-MATH-VC1

Behtman/Vainstob 99
 $= E^{\otimes D}$ with U as above
 def of sKAs called the
 COMPLEX. It is filtered
 with associated graded $E_{q,-y}$
 $L_{top} \rightarrow H^*(M, \mathcal{Q}_n^{ch})$

expect: in (*), $\mathcal{E}(M, \tau, z) = \mathcal{E}_{CFT}(E, \tau, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a KS-theory

Def [Kadwin/Tristram 16]
 $\mathcal{E}_{CFT}^{HEG}(E, \tau, z, v) = (uv)^{-c/6} \sum_{\mathbb{N}} (-y)^{7j} (-u)^{7i} q^{L_0^{top}}$
 not an invariant

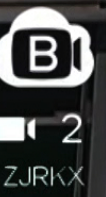
3 Interpretation in SCFT

hope: M non-linear sigma model \mathcal{E} N=2 SCFT at central charge $c=3D$
 topological half twist
 moduli $\rightarrow H^*(\mathcal{F}_1(L_{top}, \bar{\mathcal{F}}_0))$
 $\mathcal{E}_{CFT}(E, \tau, z) = y^{-c/6} \sum_{\mathbb{N}} (-y)^{7j} (-u)^{7i} q^{L_0^{top}}$
 writ \mathcal{F} as form of wt 0,
 invariant index $\frac{D}{2}$

Conclusion: $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_{\mathbb{N}} (-1)^j \text{tr}_{H^*(M, \mathcal{Q}_n^{ch})} ((-y)^{7j} q^{L_0^{top}})$
 $\neq H^*(M, E_{1,1})$

Def [WIT]
 $\mathcal{E}_{HEG}^{WIT}(M, \tau, z, v) = (uv)^{-D/2} \sum_{\mathbb{N}} (-u)^j \text{tr}_{H^*(M, \mathcal{Q}_n^{ch})} ((-y)^{7j} q^{L_0^{top}})$

Realt. [WIT]



ZJRKX



BN-MATH-VC1

Leichtman/Vaintrob 99
 $= E^{\otimes D}$ with U as above
 set of sLoAs called the
 COMPLEX It is filtered
 with associated graded $E_{q,-y}$
 $L_{top} \rightarrow \mathbb{H}^*(M, \mathcal{Q}_n^{ch})$

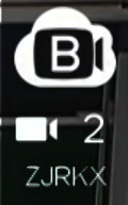
expect: $m(\pm 1)$, $\mathcal{E}(M, \tau, z) = \mathcal{E}_{CFT}(e; c, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a K3-theory

Def [Kadwin/Tripatay 16]

$$\mathcal{E}_{CFT}^{HEG}(e; \tau, z, r) = (uy)^{-c/6} \text{tr}_{\mathbb{H}}((-y)^{7r} (-u)^{7r} q^{L_0^{top}})$$

↑
not an invariant

Result [W17] For $\tau_2 = K3$ $\mathcal{E}_{CFT}^{HEG}(e; \tau, z, r) \xrightarrow{\text{genus-moduli}} \mathcal{E}_{CFT}^{HEG}(e)$



3 Interpretation in SCFT

hope: M non-convex sigma model \mathcal{E} N=2 SCFT at central charge $c=3D$
 (± 1) moduli \downarrow topological half twist
 $\mathbb{H} \ni \mathcal{F}_1(L_{top}, \bar{\mathcal{F}}_1)$
 $\mathcal{E}_{CFT}(e; \tau, z) = y^{-c/6} \text{tr}_{\mathbb{H}}((-y)^{7r} (-u)^{7r} q^{L_0^{top}})$
 writ \mathcal{F} as: form of wt 0,
 invariant index $\frac{D}{2}$

Conclusion: $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{7r} q^{L_0^{top}})$
 $\mathbb{H}^*(M, \mathcal{Q}_n^{ch})$
 $\neq \mathbb{H}^*(M, E_{1,1})$

Def [W17]

$$\mathcal{E}_{CFT}^{HEG}(M; \tau, z, r) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{7r} q^{L_0^{top}})$$

Result [W17]



BN-MATH-VC1

Leichtman/Vaintrob 99
 $= E^{\otimes D}$ with U as above
 a family of VOAs called the
 COMPLEX IT is filtered
 with associated graded $E_{q,-y}$
 \downarrow top \downarrow mod $H^*(M, \Omega_n^{ch})$

expect $m(\pm 1)$, $\mathcal{E}(M, \tau, z) = \mathcal{E}_{CFT}(E, c, z)$
 true for complex tori
 and $M = K3$ if E is a $K3$ -theory

Def [Kacchi/Thirupathy 16]

$$\mathcal{E}_{CFT}^{HEG}(E, \tau, z, r) = (uy)^{-c/c_2} \text{tr}_{\mathbb{H}}((-y)^{r_2} (-u)^{r_1} q^{L_0^{top}})$$

↑ not an invariant

Result [WIT] For $\tau = K3$ $\mathcal{E}_{CFT}^{HEG}(E, \tau, z, r) \xrightarrow{\text{gen-mod}} \mathcal{E}_{CFT}^{HEG}(E, \tau, z, r) \rightsquigarrow H^*(M, \Omega_n^{ch})$

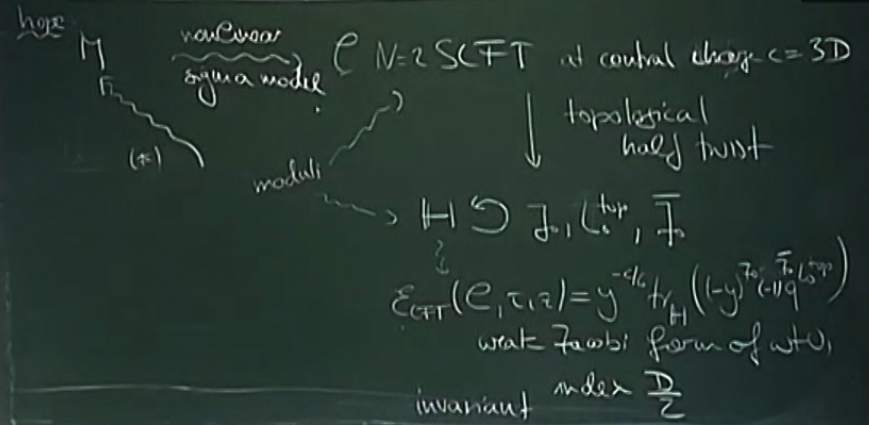
$$M = \begin{matrix} b_n = 0, n > 0 \\ 1 = id \end{matrix}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously



2
ZURICH

3 Interpretation in SCFT



Conclusion $\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{r_2} (-u)^{r_1} q^{L_0^{top}})$
 $\subset H^*(M, \Omega_n^{ch})$
 $\neq H^*(M, E_{1,1})$

Def [WIT]

$$\mathcal{E}_{CFT}^{HEG}(M, \tau, z, r) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}}((-y)^{r_2} (-u)^{r_1} q^{L_0^{top}})$$

Result [WIT]



BN-MATH-VC1

Verbitsman/Vainrob 99
 $= E^{\otimes D}$ with U as above
 cap of sVOAs called the
 COMPLEX IT is filtered
 with associated graded $E_{g,y}$

expect $m(\pm 1)$, $\mathcal{E}(\tau_1, \tau_2) = \mathcal{E}_{\text{CFT}}(e, c|z)$
 true for complex tori
 and $\tau = k3$ if \mathcal{E} is a $k3$ -theory

Def [Kadomtshipatly 16]

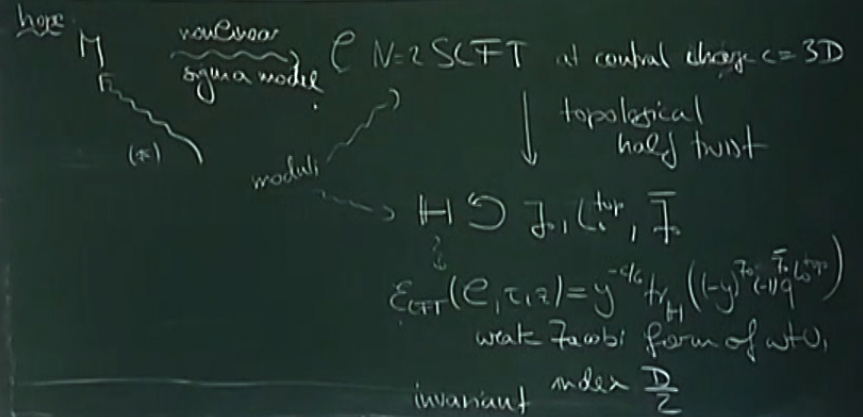
$$\mathcal{E}_{\text{CFT}}^{\text{HEG}}(e; \tau_1, \tau_2, r) = (uy)^{-c/6} \int_{\mathbb{H}} (-y)^{\tau_1} (-u)^{\tau_2} q^{L_0^{\text{top}}}$$

↑
not an invariant

Result [W17] For $\tau = k3$ $\mathcal{E}_{\text{CFT}}^{\text{HEG}}(e; \tau_1, \tau_2, r) \xrightarrow[\text{mod } \mathbb{H}]{\text{genus-1}}$ $\mathcal{E}_{\text{CFT}}^{\text{HEG}}(e)$
 $\rightsquigarrow H^*(M, \Omega_{\mathbb{H}}^{\text{ch}})$ is a genus space of states for $k3$ theories



3 Interpretation in SCFT



Def [W17]

$$\mathcal{E}_{\text{CFT}}^{\text{HEG}}(\tau_1, \tau_2, r) = (uy)^{-c/6} \int_{\mathbb{H}} (-u)^{\tau_1} \int_{\mathbb{H}} (-y)^{\tau_2} q^{L_0^{\text{top}}}$$

Result [W17]



BN-MATH-VC1

Behtman/Vaintrob 99
 $= E^{\otimes D}$ with U as above
 a set of sVOAs called the
 COMPLEX IT is filtered
 with associated graded $E_{q,-y}$

expect $m(\pm)$, $\mathcal{E}(M, \tau, z) = \mathcal{E}_{\text{CFT}}(e, \tau, z)$
 true for complex tori
 and $M = K3$ if \mathcal{E} is a $K3$ -theory

Def [Kadwin/Tripodi 16]

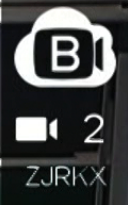
$$\mathcal{E}_{\text{CFT}}^{\text{HEG}}(e, \tau, z, r) = (uy)^{-c/6} \text{tr}_H \left((-y)^{J_0} (-u)^{\bar{J}_0} q^{L_0^{\text{top}}} \right)$$

↑
not an invariant

Result [W17] For $\tau = K3$ $\mathcal{E}_{\text{CFT}}^{\text{HEG}}(e, \tau, z, r) \xrightarrow[\text{mod } \mathbb{Z}]{\text{generic}}$ $\mathcal{E}_{\text{CFT}}^{\text{HEG}}$
 $\rightsquigarrow H^*(M, \Omega_{\mathbb{P}^1}^n)$ is a generic space of states for $K3$ theories

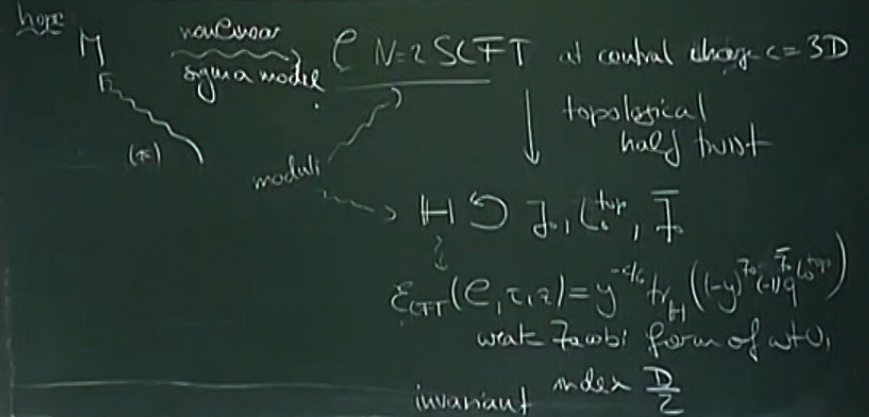
$$m_{\pm} \begin{cases} b_n = 0, n < 0 \\ 1 = \text{id} \end{cases}$$

F is a VOA built from free fields $a(x), b(x)$
 E arises by accompanying $a(x), b(x)$ by
 fermionic fields $\psi(x), \chi(x)$ analogously



ZJRKX

3 Interpretation in SCFT



Conclusion

$$\mathcal{E}(M, \tau, z) = y^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}(M, \Omega_{\mathbb{P}^1}^n)} \left((-y)^{J_0} (-u)^{\bar{J}_0} q^{L_0^{\text{top}}} \right)$$

$\neq \mathbb{H}(M, E_{1,1})$

Def [W17]

$$\mathcal{E}_{\text{CFT}}^{\text{HEG}}(M, \tau, z, r) = (uy)^{-D/2} \sum_j (-1)^j \text{tr}_{\mathbb{H}(M, \Omega_{\mathbb{P}^1}^n)} \left((-y)^{J_0} (-u)^{\bar{J}_0} q^{L_0^{\text{top}}} \right)$$

Result [W17]