

Title: A categorified Dold-Kan correspondence

Date: Aug 16, 2018 09:00 AM

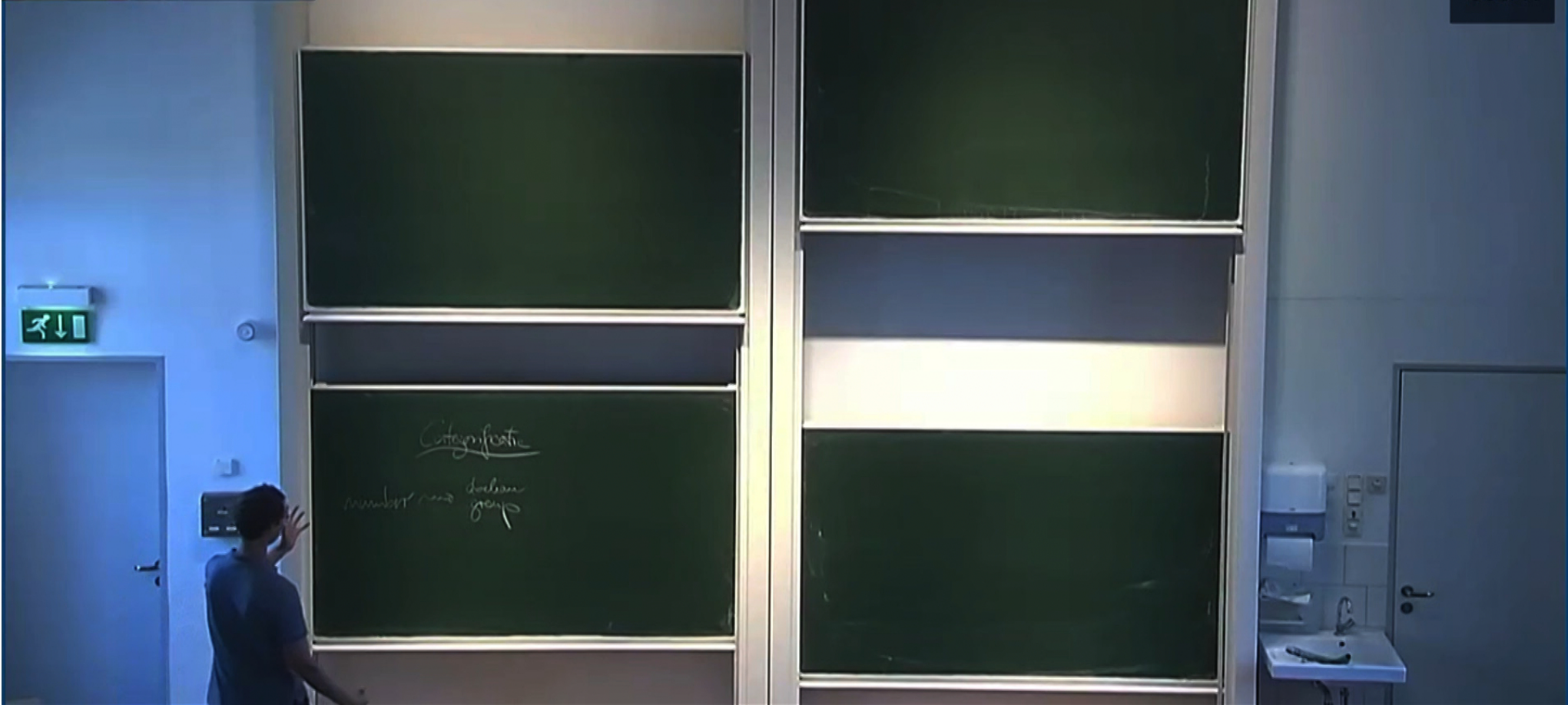
URL: <http://pirsa.org/18080077>

Abstract: Various recent developments, in particular in the context of topological Fukaya categories, seem to be glimpses of an emerging theory of categorified homotopical and homological algebra. The increasing number of meaningful examples and constructions make it desirable to develop such a theory systematically. In this talk, we discuss a step towards this goal: a categorification of the classical Dold-Kan correspondence



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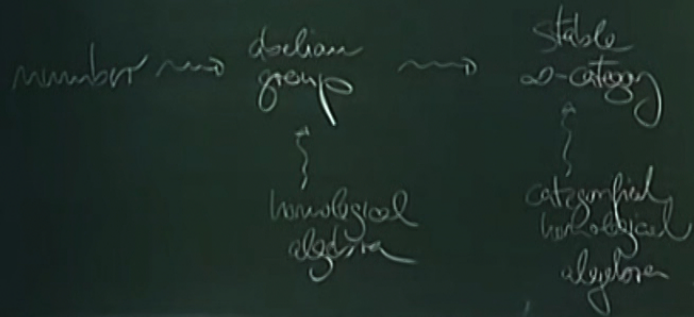


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Categorification



Motivato: Funkaya category = "categorical homology"

Sudul: categorical Poincaré-Lefschetz theory



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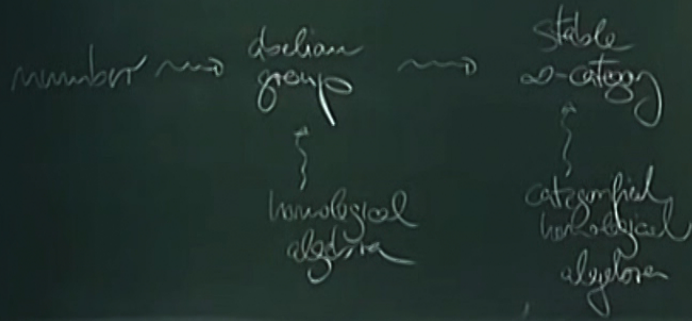
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Intuition - Fukaya category = "categorical homology"

Some rules of categorification

classical	categorical
abelian group A	the ∞ -category \mathcal{A}
$x \in A$	$X \in \mathcal{A}$
$y - x \in A$	$\text{map}(X \rightarrow Y) \in \mathcal{A}$
$\sum_{i=0}^n (x_i) \in A$	$\text{tot}(x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n) \in \mathcal{A}$
$C = A \oplus B$	$\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$

Categorification





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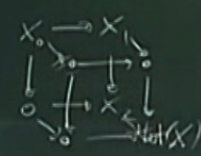
infinitesimal

$\text{number} \rightsquigarrow$ abelian group \rightsquigarrow stable ∞ -category
 \uparrow \uparrow
 homological algebra \uparrow categorical homological algebra

Deligne-Kazhdan correspondence

Motivation Fukaya category = "categorical homology"

Said categorical Picard-Lefschetz theory



isotonic abelian \mathcal{A}

Some rules of categorification

classical	categorical
abelian group A	stable ∞ -category \mathcal{A}
$x \in A$	$X \in \mathcal{A}$
$y - x \in A$	$\text{and}(X \rightarrow Y) \in \mathcal{A}$
$\sum_{i=0}^n (a_i) x_i \in A$	$\text{tot}(x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n) \in \mathcal{A}$
$C = A \oplus B$	$C = \langle \mathcal{A}, \mathcal{B} \rangle$

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infinitesimal

numbers \leadsto Lie group \leadsto stable ∞ -category

Dold-Kan correspondence

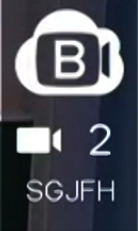
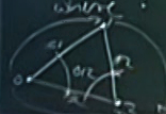
Classical (Dold, Kan, 1958)

$$C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$$

Categorical (D, 2017)

$$C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$$

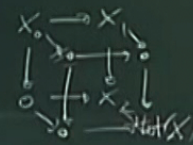
- where:
- St - $(\infty, 2)$ -category of stable ∞ -cats
 - $A \in \text{Cat}$ full ab 2-category
 - $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$



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Intuition: Fukaya category = "categorical homology"

Seidel: categorical Picard-Lefschetz theory



bicartesian abe in \mathcal{A}

Some rules of categorification

classical
abelian group A
 $x \in A$

$$y - x \in A$$

$$\sum_{i=0}^n \binom{n}{i} x_i \in A$$

$$C = A \oplus B$$

categorical
stable ∞ -category \mathcal{A}
 $X \in \mathcal{A}$

$$\text{and } (X \rightarrow Y) \in \mathcal{A}$$

$$\text{tot}(X \rightarrow X \rightarrow \dots \rightarrow X_n) \in \mathcal{A}$$

$$C \in \langle \mathcal{A}, \mathcal{B} \rangle$$



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an equivalence

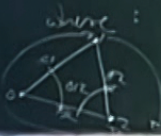
classical (Gold, Van, MSS)

$$C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$$

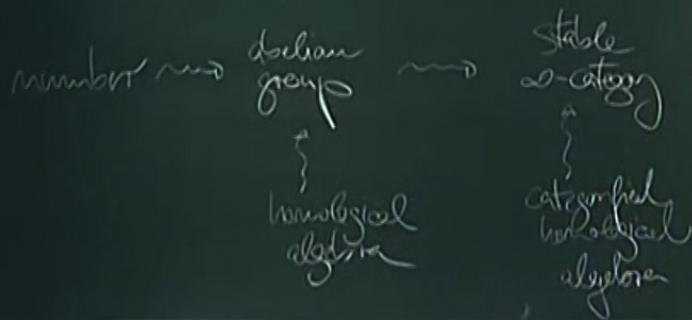
category (D, 2017)

$$C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$$

- where:
- St_A - $(\omega, 2)$ -category of stable ω -ats
 - $A = \text{Cat}$ full sub-2-category
 - $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$

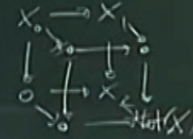


Category



motivation Fukaya category = "categorical homology"

Sichal categorical Picard-Lefschetz theory



isotonic abe in A

Some rules of categorification

classical

abelian group A

$x \in A$

$y - x \in A$

$\sum_{i=0}^n a_i x_i \in A$

$C = A \oplus B$

categorical

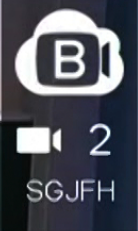
stable ω -category A

$X \in A$

$\text{and}(X \rightarrow Y) \in A$

$\text{tot}(X \rightarrow X \rightarrow \dots \rightarrow X_n) \in A$

$C = \langle A, B \rangle$





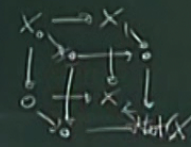
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an equivalence
 $C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$
 Categorical (D, 2017)
 $C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$
 where:
 • $\text{St} = (\infty, 2)$ -category of stable ∞ -cats
 • $A = \text{Cat}$ full sub-2-category
 • $\text{St}_A = \text{Fun}_{\text{c-l.}}(A^{\text{op}}, \text{St})$

with $d_n = 1$
 $\left\{ \begin{array}{l} d_0 \xrightarrow{d_1} d_1 \xrightarrow{d_2} d_2 \\ \text{with } d_n = 1, s_{n-1} = 1, d_n = -1 \end{array} \right\} \xrightarrow{d} \left\{ \begin{array}{l} B_0 \xrightarrow{d} B_1 \xrightarrow{d} B_2 \dots \\ d^2 = 0 \end{array} \right\}$

motivation: Fukaya category = "categorical homology"
 Seidel: categorical Picard-Lefschetz theory

 bicartesian ab in \mathcal{A} .

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Deligne-Kam correspondence

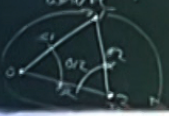
Classical (Deligne, Var, 1988)

$$C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$$

Categorical (D, 2017)

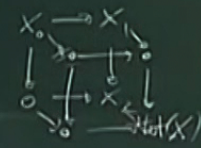
$$C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$$

- where:
- St - $(\infty, 2)$ -category of stable ∞ -cats
 - A - Cat full sub-2-category
 - $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$



Intuition: Fukaya category = "categorical homology"

Sichel: Categorical Picard-Lefschetz theory



bicartesian abe in \mathcal{A}

Classical

$$\subseteq C(A)_h = A_h /$$





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Deligne-Kam correspondence

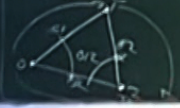
Classical (Deligne, Var, 1958)

$$C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$$

Categorical (D, 2017)

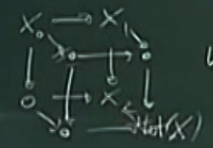
$$C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$$

- where:
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 - A - Cat full rebr 2-category
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Intuition Fukaya category = "categorical homology"

Sicdel Categorical Picard-Lefschetz theory



Classical

$$C(A)_n = \text{An}/(\text{idempotents}) \quad ; \quad d = \sum_{i=0}^n (-1)^i d_i$$



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Told-Kan correspondence
 Classical (Dold, Kan, 1958)
 $C: \text{Ab}_A \xrightarrow{\cong} \text{Ch}_2(\text{Ab}) \cdot N$
 Categorical (D, 2017)
 $C: \text{St}_A \xrightarrow{\cong} \text{Ch}_2(\text{St}) \cdot N$
 where:
 • St - (co, 2)-alg of stable ∞ -ats
 • $A \subset \text{Cat}$ full sub 2-category
 • $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$

$\text{Hom}_{\text{Ch}_2}(\text{St}_A, \text{St}_B) \cong \text{Hom}_{\text{Cat}}(A, B)$ | $d \geq 0$

Intuition Fukaya category = "categorical homology"
 Seidel: categorical Picard-Lefschetz theory
 $X_0 \rightarrow X_1$
 $\downarrow \quad \downarrow$
 $X_2 \rightarrow X_3$
 bicentric alge in A
 $\text{Hom}(X)$

Classical
 $C(A)_n = A_n / (\text{idempotents})$ | $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{\text{Ch}_2(\text{Ab})}(C(\mathbb{Z}A)^n, \mathbb{Z})$





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Told-Kan correspondence

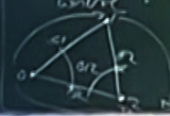
Classical (Dold, Kan, 1958)

$$C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot N$$

Categorified (D, 2017)

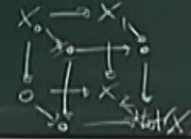
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 - $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$



Intuition Fukaya category = "categorical homology"

Sicdel categorical Picard-Lefschetz theory



bicartesian abe in \mathcal{A}

Classical

$$C(A)_n = \text{An}/(\text{idempotents}) \quad i \quad d = \sum_{i=0}^n (-1)^i d_i$$

$$N(B)_n = \text{Hom}_{\text{Ch}(\text{Ab})}(C(\mathbb{Z}A^{\text{op}})_1, S_1)$$



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Intuition Fukaya category = "categorical homology"
Seidel: categorical

Toled-Kuran correspondence
Classical (Dold, Kan, MSS)
 $C \cdot \text{Ab}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot N$
Categorical (D, 2017)
 $C \cdot \text{St}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{St}) \cdot N$
 where:
 • St_Δ - (co)2-category of stable ∞ -cats
 • $\Delta \subset \text{Cat}$ full sub 2-category
 • $\text{St}_\Delta = \text{Fun}_{\text{cat}}(\Delta^{\text{op}}, \text{St})$

$\text{Ch}_{\geq 0}(\text{Ab}) = \text{Ch}_{\geq 0}(\text{St})$ | $d^2 = 0$

Classical
 $C(A)_n = \text{Hom}_{\text{Ab}}(A^n, \mathbb{Z})$ i $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{\text{Ch}(\text{Ab})}(C(\mathbb{Z}S)_1, B)$
 $N(B)_n = \left\{ \begin{array}{l} x_{01} \in \beta_1, x_{02} \in \beta_2, x_{11}, x_{12} \in \beta_1, x_{21}, x_{22} \in \beta_2 \\ \cdot dx_{11} = x_{11} - x_{01} \\ \cdot dx_{12} = x_{12} - x_{02} + x_{01} \end{array} \right\}$



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Intravato Fukaya category = "categorical homology"

Classical

$$\mathbb{C} \quad C(A)_n = A_n / \langle \text{idempotents} \rangle \quad d = \sum_{i=0}^n C(i) d_i$$

$$\mathbb{N} \quad N(B)_n = \text{Hom}_{\mathcal{C}(A_2)}(C(\mathbb{Z}^n)_1, S_1)$$

$$N(B)_n = \left\{ \begin{array}{l} x_{02} \in \beta_2, x_{01}, x_{02}, x_{12} \in \beta_1, x_{01}, x_{12} \in \beta_3 \\ \cdot dx_{10} = x_{11} - x_1 \\ \cdot dx_{12} = x_{12} - x_{02} + x_{01} \end{array} \right\}$$

Told-Kan correspondence

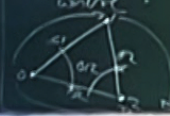
Classical (Dold, Kan, 1958)

$$\mathbb{C} \quad \text{Ab}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot \mathbb{N}$$

Categorified (D, 2017)

$$\mathbb{C} \quad \text{St}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{St}) \cdot \mathbb{N}$$

- where:
- St_Δ - $(\infty, 2)$ -category of stable ∞ -ats
 - $\Delta \subset \text{Cat}$ full sub 2-category
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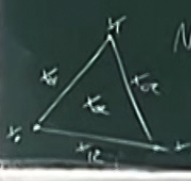


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Dold-Kan correspondence
 Classical (Dold, Kan, 1958)
 $C \cdot \text{Ab}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot N$
 Categorical (D, 2017)
 $C \cdot \text{St}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{St}) \cdot N$
 where:
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 • $\Delta \subset \text{Cat}$ full sub 2-category
 • $\text{St}_\Delta = \text{Fun}_{\text{cat}}(\Delta^{\text{op}}, \text{St})$

$\text{Hom}_{\text{cat}}(\Delta^{\text{op}}, \text{St}) \cong \text{Ch}_{\geq 0}(\text{St})$ | $d^2 = 0$

Classical
 $C(A)_n = A_n / (\text{idempotents})$ | $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{\text{CU}(\text{Ab})}(C(\mathbb{Z}\Delta^n)_1, \mathbb{S}_1)$
 $N(B)_n = \left\{ \begin{array}{l} x_{02} \in \mathbb{Z}, x_{01}, x_{02}, x_{12} \in \mathbb{B}_1, x_{01}, x_{11}, x_{22} \in \mathbb{B}_0 \\ \cdot dx_{10} = x_{11} - x_{01} \\ \cdot dx_{112} = x_{12} - x_{02} + x_{01} \end{array} \right.$



Categorical



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Gold-Kon correspondence
 Classical (Dold, Van, MSS)
 $C \cdot \text{Ab}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{Ab}) \cdot N$
 Categorical (D, 2017)
 $C \cdot \text{St}_A \xrightarrow{\cong} \text{Ch}_{20}(\text{St}) \cdot N$
 where:
 • St_A - (co, 2)-category of stable ∞ -cats
 • $A = \text{Cat}$ full sub 2-category
 • $\text{St}_A = \text{Fun}_{\text{cat}}(A^{\text{op}}, \text{St})$

$d = 0$

Classical
 $C(A)_n = \text{An}/(\text{degenerates})$; $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{\text{CU}(\text{Ab})}(C(\mathbb{Z}A^{\times n}), \mathbb{S}_1)$
 $N(B)_n = \left\{ \begin{array}{l} x_{01}, x_{02}, x_{01}x_{02}, x_{12} \in \mathbb{S}_1, x_{01}, x_{11}, x_{12} \in \mathbb{S}_2 \\ \cdot dx_{ij} = x_j - x_i \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$

Categorical
 $C(\text{St}) = \mathbb{S}_0 \leftarrow \mathbb{S}_1 \leftarrow \mathbb{S}_2$
 $\mathbb{S}_n = \text{St}_n / (\text{degenerates})$
 $d(\mathbb{S}_0) =$



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Dold-Kan correspondence
 Classical (Dold, Kan, 1958)
 $C \cdot \text{Ab}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot N$
 Categorical (D, 2017)
 $C \cdot \text{St}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{St}) \cdot N$
 where:
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 • $\Delta = \text{Cat}$ full sub 2-category
 • $\text{St}_\Delta = \text{Fun}_{\text{cat}}(\Delta^{\text{op}}, \text{St})$

$d = 0$

Classical
 $\subseteq C(A)_n = \text{Ab}/(\text{degenerates}) ; d = \sum_{i=0}^n (-1)^i d_i$
 $\underline{N} \cdot N(B)_n = \text{Hom}_{\text{CU}(\text{Ab})}(C(\mathbb{Z}\Delta^n), \mathbb{B}_1)$
 $N(B)_n = \left\{ \begin{array}{l} x_{012} \in \mathbb{B}_2, x_{01}, x_{02}, x_{12} \in \mathbb{B}_1, x_{01}, x_{11}, x_{12} \in \mathbb{B}_3 \\ \cdot dx_{ij} = x_j - x_i \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$

Categorical
 $\subseteq C(\mathcal{A}) = \mathbb{B}_0 \rightarrow \mathbb{B}_1 \leftarrow \mathbb{B}_2$
 $\mathbb{B}_n = \text{Ab}/(\text{degenerates})$
 $d(\mathbb{B}_{01}) =$



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Told-Kan correspondence

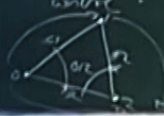
Classical (Dold, Kan, 1958)

$$C \cdot \text{Ab}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{Ab}) \cdot N$$

Categorified (D, 2017)

$$C \cdot \text{St}_\Delta \xrightarrow{\cong} \text{Ch}_{\geq 0}(\text{St}) \cdot N$$

- where:
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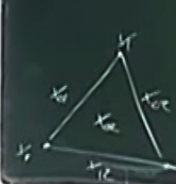


$$d(x_{i_1, i_2}) = x_{i_2} - x_{i_1} + x_{i_0} \quad | \quad d^2 = 0$$

Classical

$$C \cdot C(A)_n = \text{Ab}/(\text{deg. mem}) \quad | \quad d = \sum_{i=0}^{n-1} (-1)^i d_i$$

$$N \cdot N(B)_n = \text{Hom}_{\text{Ch}(\text{Ab})}(C(\mathbb{Z}^n), \mathbb{S})$$



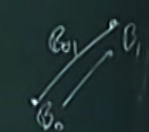
$$N(B)_n = \left\{ \begin{array}{l} x_{01} \in B_1, x_{02}, x_{12} \in B_1, x_0, x_1, x_2 \in B_0 \\ \cdot d x_{ij} = x_j - x_i \\ \cdot d x_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$$

Categorified

$$C \cdot C(\text{St}) = \mathbb{S}_0 \xrightarrow{d} \mathbb{S}_1 \xrightarrow{d} \mathbb{S}_2$$

$$B_{01} = \text{St}_n / (\text{degenerates})$$

$$d(B_{01}) = \text{Nat}(B_0 \rightarrow B_1)$$



$$d(B_{012}) = \begin{array}{ccc} B_{01} & \rightarrow & B_{02} \\ \downarrow & & \downarrow \\ 0 \cong B_{11} & \rightarrow & B_{12} \end{array}$$



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Told-Kan correspondence

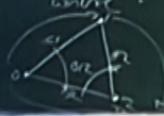
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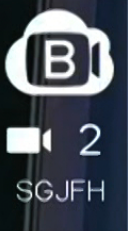
Categorified (D, 2017)

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 - $\Delta = \text{Cat}$ full sub 2-category
 - $\text{St}_\Delta = \text{Fun}_{\text{cat}}(\Delta^{\text{op}}, \text{St})$



$$d_0 = x_1 - x_0, d_1 = x_2 - x_1, d_2 = x_2 - x_0 \quad | \quad d^2 = 0$$

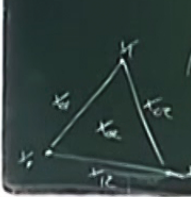


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Classical

$$C \cdot C(A)_n = A_n / (\text{idempotents}) \quad | \quad d = \sum_{i=0}^{n-1} (-1)^i d_i$$

$$N \cdot N(B)_n = \text{Hom}_{\text{Cat}(\text{Ab})}(C(\mathbb{Z}\Delta^n), \mathbb{B})$$



$$N(B)_n = \left\{ \begin{array}{l} x_{01} \in \beta_1, x_{02} \in \beta_2, x_{12} \in \beta_1, x_{01}, x_{11}, x_{02} \in \beta_0 \\ \cdot d x_{ij} = x_j - x_i \\ \cdot d x_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$$

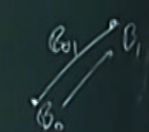
Categorified

$$C \cdot C(\text{St}) = \mathbb{B}_0 \xrightarrow{\beta_1} \mathbb{B}_1 \xrightarrow{\beta_2} \mathbb{B}_2$$

$$\mathbb{B}_n = A_n / (\text{degenerates})$$

$$d(\beta_{01}) = \text{det}(\beta_0 \rightarrow \beta_1)$$

$$d(\beta_{012}) = \text{det} \begin{pmatrix} \beta_{01} & \rightarrow & \beta_{02} \\ \downarrow & & \downarrow \\ 0 & \rightarrow & \beta_{12} \end{pmatrix}$$





BN-MATH-VC1

$$\begin{array}{l}
 \left. \begin{array}{l}
 \mathcal{B}_0 \xrightarrow{d} \mathcal{B}_1 \xrightarrow{d} \mathcal{B}_2 \\
 d^2 = 0
 \end{array} \right\} - \text{cd}
 \end{array}$$

Classical

$$\underline{C} \quad C(A)_n = A_n / (\text{degenerates}) \quad ; \quad d = \sum_{i=0}^{n-1} (-1)^i d_i$$

$$\underline{N} \quad N(\mathcal{B})_n = \text{Hom}_{\mathcal{C}(A_n)}(C(\mathbb{Z}A_n), \mathcal{B})$$

$$\begin{array}{l}
 \begin{array}{c}
 x_0 \\
 \swarrow \quad \searrow \\
 x_1 \quad x_2 \\
 \swarrow \quad \searrow \\
 x_3
 \end{array}
 \quad N(\mathcal{B})_n = \left\{ \begin{array}{l}
 x_{01} \in \mathcal{B}_1, \quad x_{01}, x_{02}, x_{12} \in \mathcal{B}_1, \quad x_{01}, x_{11}, x_{12} \in \mathcal{B}_0 \\
 \cdot d x_{10} = x_1 - x_0 \\
 \cdot d x_{012} = x_{12} - x_{02} + x_{01}
 \end{array} \right\}
 \end{array}$$

(((

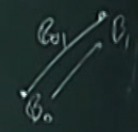
$$\underline{C} \quad C(A) = \mathcal{B}_0 \xrightarrow{d} \mathcal{B}_1 \xrightarrow{d} \mathcal{B}_2$$



$$\mathcal{B}_n = A_n / (\text{degenerates})$$

$$d(\mathcal{B}_{01}) = \text{det}(\mathcal{B}_0 \rightarrow \mathcal{B}_1)$$

$$d(\mathcal{B}_{012}) = \text{det} \begin{pmatrix} \mathcal{B}_{01} & \rightarrow & \mathcal{B}_{02} \\ \downarrow & & \downarrow \\ 0 & \approx & \mathcal{B}_{12} \end{pmatrix}$$

$$0 \approx \mathcal{B}_{11} \rightarrow \mathcal{B}_{12}$$




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 SGJFH



BN-MATH-VC1

$$S_0 \xrightarrow{a} S_1 \xrightarrow{a} S_2$$

$$d^2 = 0$$

$$N(\mathcal{B}_2) = \begin{cases} x_0 \rightarrow x_1 \rightarrow x_2 \\ \downarrow \quad \downarrow \\ S_0 \\ S_1 \end{cases}$$

Classical

$$C(A)_n = A_n / (\text{degenerates}) \quad ; \quad d = \sum_{i=0}^n (-1)^i d_i$$

$$N(\mathcal{B}_n) = \text{Hom}_{CU(A_n)}(C(\mathbb{Z}\Delta^n)_1, \mathcal{B}_1)$$

$$N(\mathcal{B}_2) = \left\{ \begin{array}{l} x_{01} \in \mathcal{B}_1, x_{02}, x_{12} \in \mathcal{B}_1, x_0, x_1, x_2 \in \mathcal{B}_0 \\ \cdot dx_{i,j} = x_j - x_i \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$$

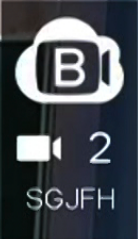
$$C(A) = S_0 \rightarrow S_1 \rightarrow S_2$$

$$\mathcal{B}_n = A_n / (\text{degenerates})$$

$$d(\mathcal{B}_{01}) = \det(\mathcal{B}_0 \rightarrow \mathcal{B}_1)$$

$$d(\mathcal{B}_{012}) = \det \begin{pmatrix} \mathcal{B}_{01} & \rightarrow & \mathcal{B}_{02} \\ \downarrow & & \downarrow \\ \mathcal{B}_0 & & \mathcal{B}_1 \end{pmatrix}$$

$$0 = \mathcal{B}_{11} \rightarrow \mathcal{B}_{12}$$





BN-MATH-VC1

$$S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$d^2 = 0$$

$$d(S_2)_2 = \begin{matrix} X_0 & \rightarrow & X_1 & \rightarrow & X_2 & & S_0 \\ \downarrow & & \downarrow & & \downarrow & & \\ X_{01} & \rightarrow & X_{02} & \rightarrow & X_{12} & & S_1 \\ & & \downarrow & & \downarrow & & \\ & & X_{11} & \rightarrow & X_{12} & & \end{matrix}$$

Classical

$$C(A)_n = A_n / (\text{degenerates}) \quad ; \quad d = \sum_{i=0}^n (-1)^i d_i$$

$$N(B)_n = \text{Hom}_{CU(A_n)}(C(\mathbb{R}^n)_1, S_1)$$

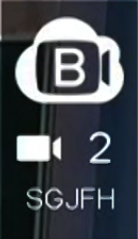
$$N(B)_2 = \left\{ \begin{array}{l} x_{02} \in B_2, x_{01}, x_{02}, x_{12} \in B_1, x_0, x_1, x_2 \in B_0 \\ \cdot dx_{i,j} = x_j - x_i \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right.$$

$$C(A)_2 = S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$S_{B_n} = A_n / (\text{degenerates})$$

$$d(B_{01}) = \det(B_0 \rightarrow B_1)$$

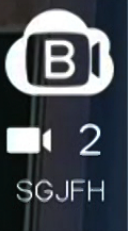
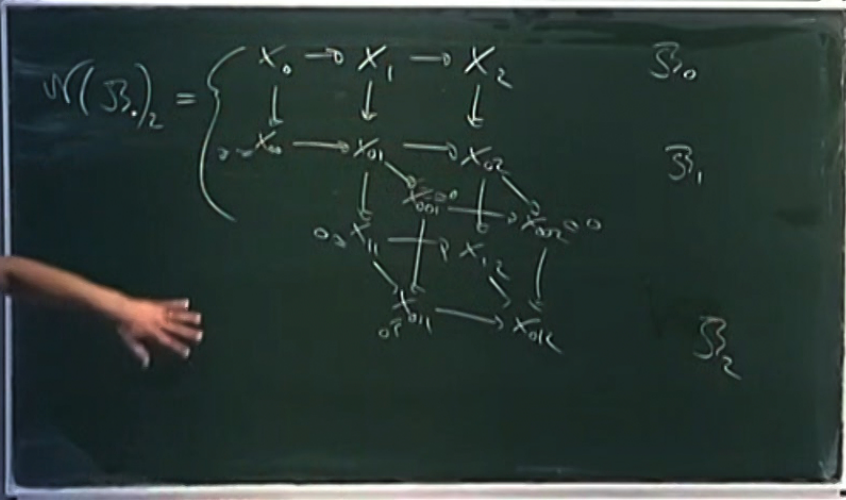
$$d(B_{012}) = \det \begin{pmatrix} B_{01} & \rightarrow & B_{02} \\ \downarrow & & \downarrow \\ 0 & \approx & B_{11} \end{pmatrix} \rightarrow B_{12}$$





BN-MATH-VC1

$$\begin{array}{c}
 \mathcal{S}_0 \leftarrow \mathcal{S}_1 \xrightarrow{d} \mathcal{S}_2 \\
 \left. \begin{array}{l} -1d_0 \\ d^2 = 0 \end{array} \right\}
 \end{array}$$



Classical

$\subseteq C(A)_n = A_n / (\text{degenerates})$; $d = \sum_{i=0}^{n-1} (-1)^i d_i$

$N(B)_n = \text{Hom}_{CU(A_n)}(C(\mathcal{S}_n)_1, \mathcal{S}_1)$

$N(B)_2 = \left\{ \begin{array}{l} x_{02} \in \mathcal{B}_2, x_{01}, x_{02}, x_{12} \in \mathcal{B}_1, x_0, x_1, x_2 \in \mathcal{B}_0 \\ \cdot d x_{ij} = x_j - x_i \\ \cdot d x_{012} = x_{12} - x_{02} + x_{01} \end{array} \right\}$

$\subseteq C(A) = \mathcal{S}_0 \leftarrow \mathcal{S}_1 \leftarrow \mathcal{S}_2$

$\mathcal{S}_n = A_n / (\text{degenerates})$

$d(\mathcal{B}_{01}) = \det(\mathcal{B}_0 \rightarrow \mathcal{B}_1)$

$d(\mathcal{B}_{02}) = \det(\mathcal{B}_{01} \rightarrow \mathcal{B}_{02})$

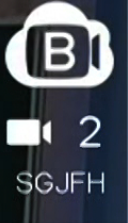
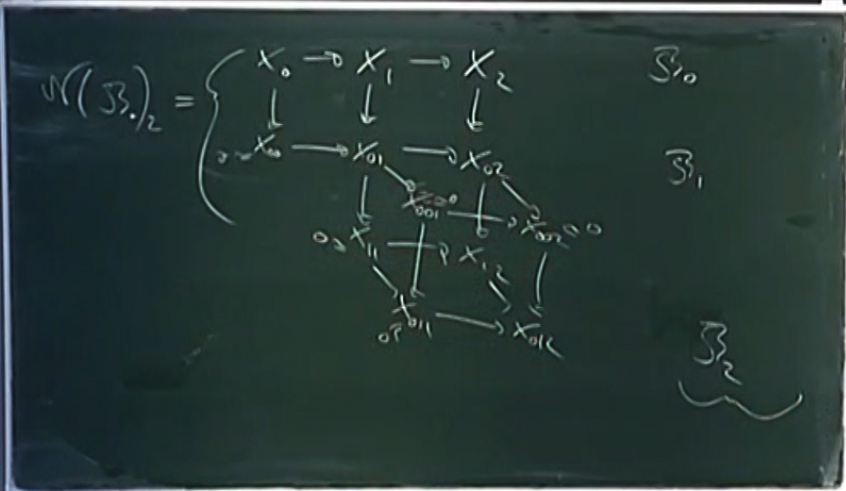
$0 \simeq \mathcal{B}_{11} \rightarrow \mathcal{B}_{12}$



BN-MATH-VC1

$$S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$d^2 = 0$$



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Classical

$\subseteq C(A)_n = A_n / (\text{degenerates}) \quad ; \quad d = \sum_{i=0}^n (-1)^i d_i$

$\underline{N} \cdot N(B)_n = \text{Hom}_{CU(A_n)}(C(\mathbb{Z}S)_1, B_1)$

$N(B)_n = \left\{ \begin{array}{l} x_{012} \in B_2, x_{01}, x_{02}, x_{12} \in B_1, x_0, x_1, x_2 \in B_0 \\ \cdot dx_{10} = x_1 - x_0 \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right\}$

$\subseteq C(A)_n = S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$

$B_n = A_n / (\text{degenerates})$

$d(B_{01}) = \det(B_0 \rightarrow B_1)$

$d(B_{012}) = \det \begin{pmatrix} B_{01} & B_{02} \\ \downarrow & \downarrow \\ 0 & B_{12} \end{pmatrix}$

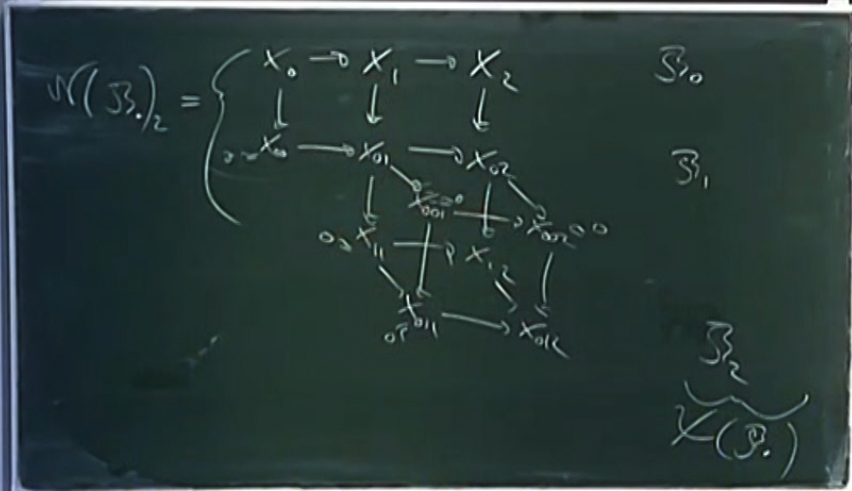
$0 \simeq B_{11} \rightarrow B_{12}$



BN-MATH-VC1

$$S_0 \xrightarrow{a} S_1 \xrightarrow{a} S_2$$

$$d^2 = 0$$



Classical

$C(A)_n = A_n / \text{Idempotents}$; $d = \sum_{i=0}^n (c_i) d_i$

$N(B)_n = \text{Hom}_{CU(A_n)}(C(\mathbb{Z}S^n)_1, B_1)$

$N(B)_2 = \left\{ \begin{array}{l} x_{012} \in B_2, x_{01}, x_{02}, x_{12} \in B_1, x_0, x_1, x_2 \in B_0 \\ \cdot dx_{i,j} = x_j - x_i \\ \cdot dx_{012} = x_{12} - x_{02} + x_{01} \end{array} \right\}$

$d(B_{01}) = \det(B_0 \rightarrow B_1)$

$d(B_{012}) = \det(B_0 \rightarrow B_1 \rightarrow B_2)$

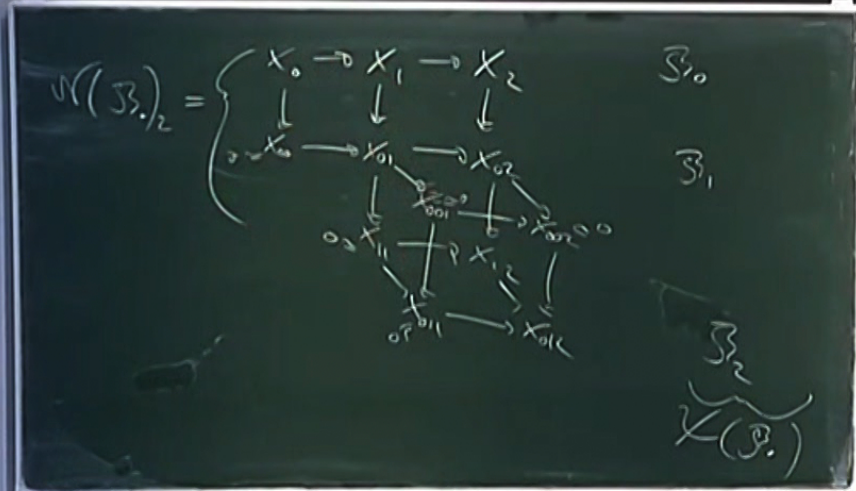
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BN-MATH-VC1

$$S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$d^2 = 0$$



2
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Classical

$$C(A)_n = A_n / \langle \text{degenerates} \rangle \quad ; \quad d = \sum_{i=0}^n (-1)^i d_i$$

$$N(B)_n = \text{Hom}_{CU(A_n)}(C(\mathbb{Z}S^n)_1, \mathbb{Z}B_1)$$

$$N(B)_2 = \left\{ \begin{array}{l} x_{012} \in B_2, \quad x_{01}, x_{02}, x_{12} \in B_1, \quad x_0, x_1, x_2 \in B_0 \\ \cdot d x_{ij} = x_j - x_i \\ \cdot d x_{i12} = x_{12} - x_{i2} + x_{i1} \end{array} \right\}$$

$$d(B_{01}) = \det(B_0 \rightarrow B_1)$$

$$d(B_{012}) = \det(B_{01} \rightarrow B_{02} \rightarrow B_{12})$$

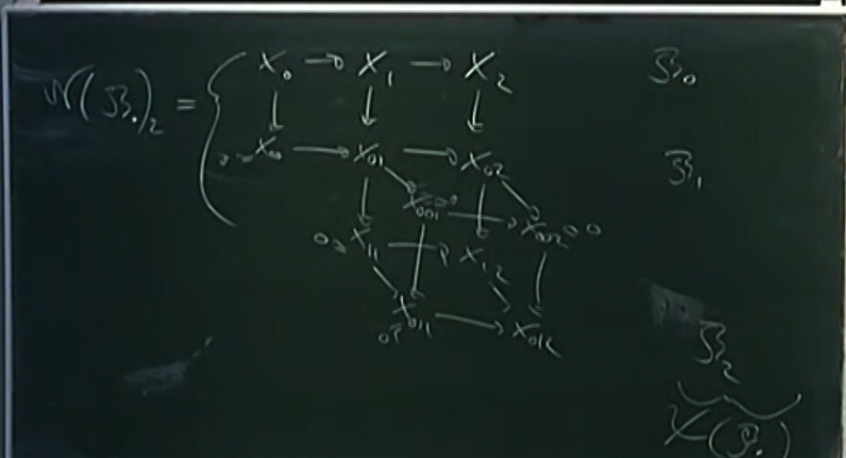
$$0 = B_{11} \rightarrow B_{12}$$



BN-MATH-VC1

$$S_0 \xleftarrow{d^1} S_1 \xleftarrow{d^2} S_2$$

$$d^2 = 0$$



Example: (1) $N(SSEIS) \sim S(S)$ (well-known)

" $N(S_1, 1) \quad K(S) = \mathcal{C}(S, S)$

(2) $N\left(\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right) = S(f)$

Classical

$\mathbb{C} \quad C(A)_n = A_n / (\text{idempotent}) \quad d = \sum_{i=0}^n (-1)^i d_i$

$\mathbb{N} \quad N(B)_n = \text{Hom}_{\mathcal{C}(A_2)}(C(\mathbb{Z}^n)_1, B_1)$

B

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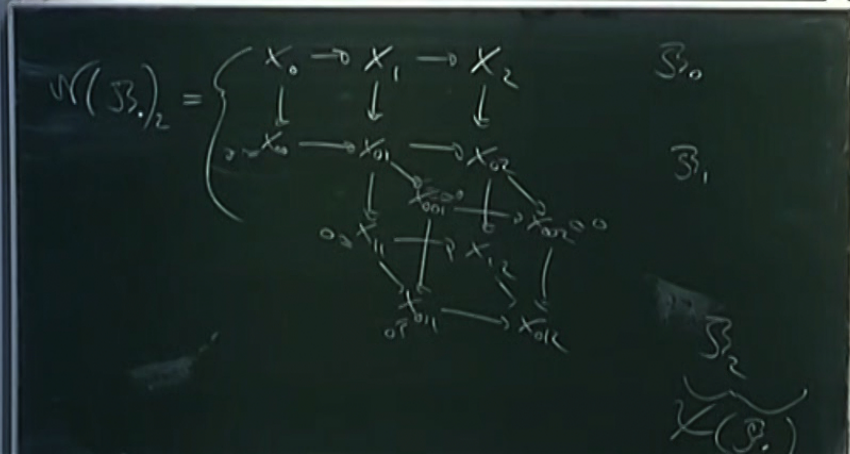
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BN-MATH-VC1

$$S_0 \xleftarrow{d} S_1 \xleftarrow{d} S_2$$

$$d^2 = 0$$



Example: (1) $N(SSEIS) \simeq S(S)$ Waldhausen
 S -categories
 $N(S, 1) \quad K(S) = \mathcal{C}(S, S)$
 (2) $N\left(\begin{smallmatrix} S_1 \\ \downarrow \\ S_2 \end{smallmatrix}\right) = S(f)$

Classical
 $\mathbb{C} \quad C(A)_n = A_n / (\text{idempotents}) \quad d = \sum_{i=0}^{n-1} (-1)^i d_i$
 $\mathbb{N} \quad N(\mathcal{B})_n = \text{Hom}_{\mathcal{C}(A_n)}(C(\mathbb{Z}^n)_1, \mathcal{B})$

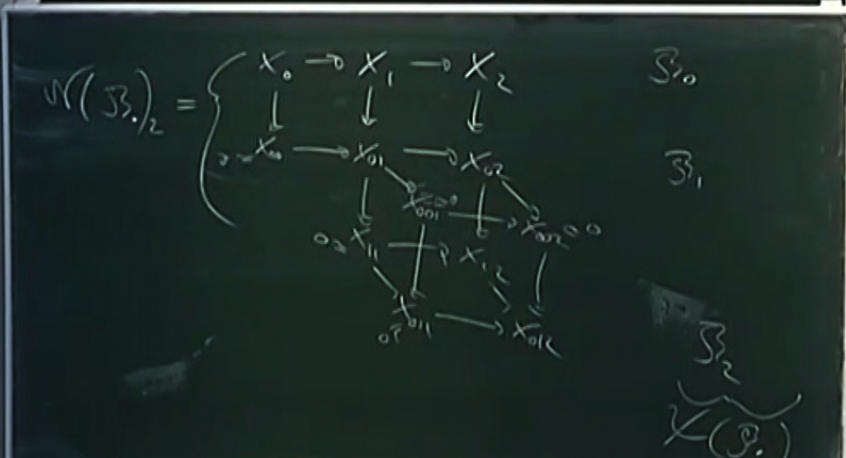
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BN-MATH-VC1

$$S_0 \xrightarrow{d_0} S_1 \xrightarrow{d_1} S_2$$

$$d^2 = 0$$



Example: (1) $N(SSE10) \approx S(S)$ (well-known, S -continuous)
 (2) $N(S_1, 1) \approx S(S) = \mathcal{C}(S, \mathbb{R})$
 (3) $N(SSE20) \approx$

Classical
 $C(A)_n = A_n / \text{Idempotent}$; $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{\mathcal{C}(A_n)}(C(\mathbb{Z}^n)_1, B_1)$

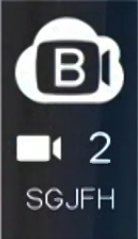
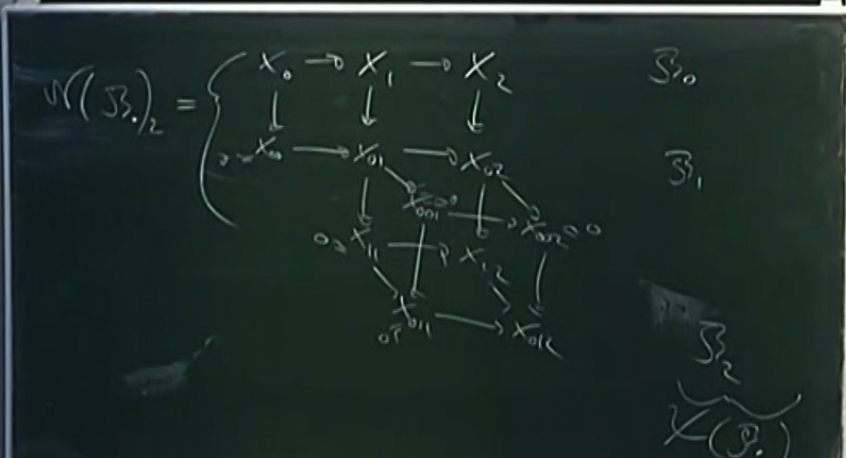
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BN-MATH-VC1

$$S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$d^2 = 0$$



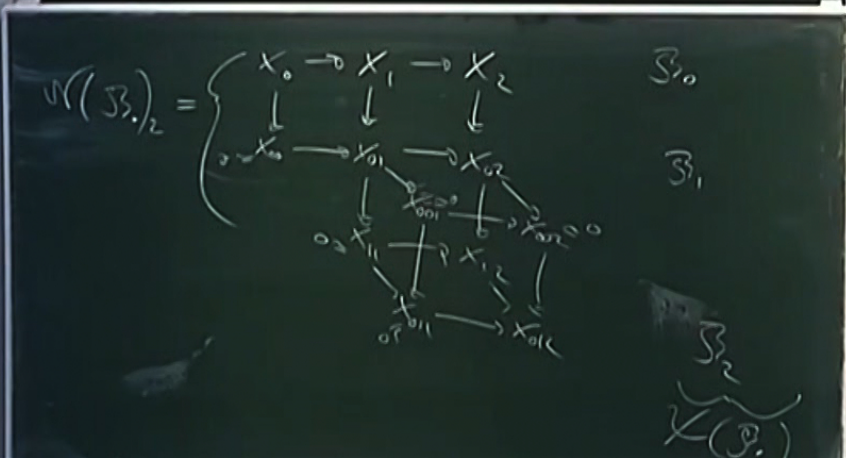
Example: (1) $\mathcal{N}(SSE10) \approx S.(S)$ (well-known, S -continuous)
 (2) $\mathcal{N}(S_1, 1) \approx S.(1)$
 (3) $\mathcal{N}(SSE20) \approx S^2.(S)$ Hesselholt - index

Classical
 $C(A)_n = A_n / (\text{idempotent})$ i $d = \sum_{i=0}^n (-1)^i d_i$
 $N(B)_n = \text{Hom}_{CU(A_2)}(C(ZD)_1, B_1)$



BN-MATH-VC1

$$\begin{array}{c}
 \left. \begin{array}{l}
 S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2 \\
 -1d_0 \\
 d^2 = 0
 \end{array} \right\}
 \end{array}$$



Example:

- $N(S_2)_1 \approx S_1(S_2)$ (Wahlweise S_1 -Geraden)

 $K(S_2)_1 = K(S_2) = \mathbb{C}(S_1(S_2))$
- $N(S_2)_2 \approx S_2(S_2)$

 $K(S_2)_2 = \mathbb{C}(S_2(S_2))$
- $N(S_2)_2 \approx S_2^2(S_2)$ Hesse'sche

 $K(S_2)_2 = \mathbb{C}(S_2^2(S_2))$

Classical

$\subseteq C(A)_n = A_n / \text{Idempotenten}$; $d = \sum_{i=0}^n (-1)^i d_i$

N : $N(B)_n = \text{Hom}_{\mathbb{C}(A_n)}(C(Z_0^*), B_1)$

$N(B)_2 = \left\{ \begin{array}{l} x_{01} \in B_1, x_{01}, x_{02}, x_{12} \in B_1, x_{01}, x_{12} \in B_2 \\ \cdot dx_{11} = x_{11} - x_1 \\ \cdot dx_{12} = x_{12} - x_{02} + x_{01} \end{array} \right\}$

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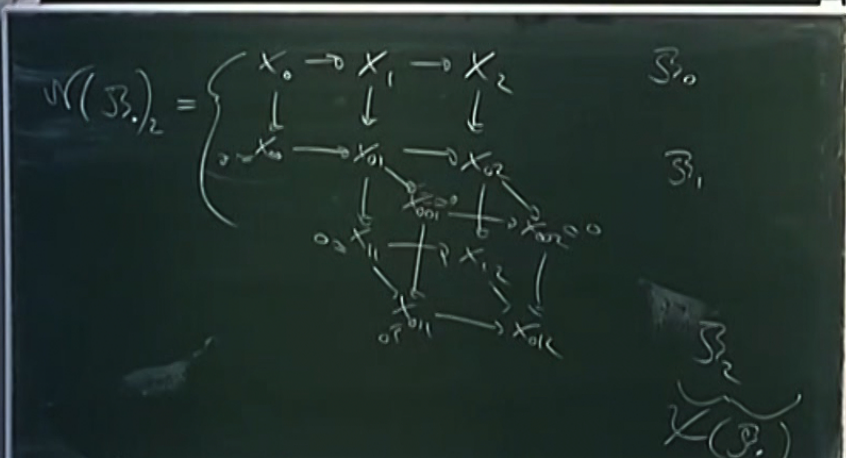
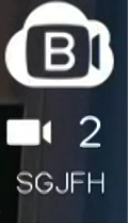


BN-MATH-VC1

$$\begin{array}{c}
 \mathcal{S}_0 \xrightarrow{a} \mathcal{S}_1 \xrightarrow{a} \mathcal{S}_2 \\
 \downarrow -id_0 \\
 d^2 = 0
 \end{array}$$

Example:

- (1) $N(\mathcal{S}[\mathcal{S}]) \approx \mathcal{S}(\mathcal{S})$ (Waldhausen \mathcal{S} -category)
- " $K(\mathcal{S}, 1) \quad K(\mathcal{S}) = \mathcal{S}(\mathcal{S})$
- (2) $N(\mathcal{S}, 1) \approx \mathcal{S}(f)$
- (3) $N(\mathcal{S}[\mathcal{S}]) \approx \mathcal{S}^1(\mathcal{S})$ (Hesschrift -triple)
- " $K(\mathcal{S}, 2) \quad K(\mathcal{S}) \approx \mathcal{S}^2(\mathcal{S})$



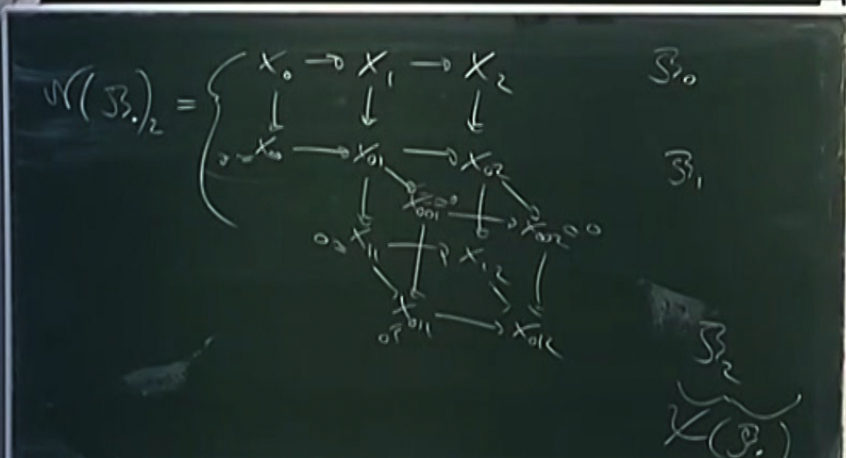
(4) $N(\mathcal{S}[\mathcal{S}]) = \mathcal{S}^{< \omega }(\mathcal{S})$



BN-MATH-VC1

$$S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2$$

$$d^2 = 0$$



Example: (1) $N(SSE_1) \approx S(S)$ Waldmaße
 $K(S,1) \quad K(S) = \Omega(S,1)$
 (2) $N(SSE_2) \approx S(f)$
 $K(S,2) \quad K(S) = \Omega^2(S,2)$

(1) $N(SSE_1) = S^{(1)}(S)$ T. Page
 Relative k -Log: $K(S) = \Omega^k(S,1)$

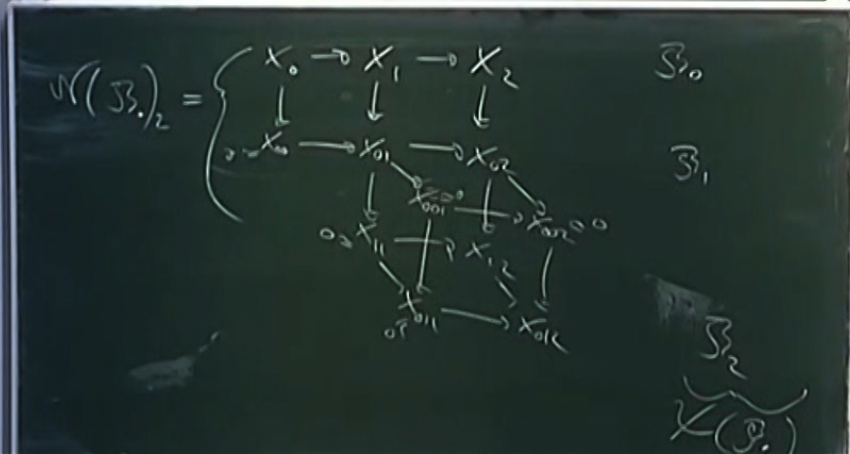
B
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BN-MATH-VC1

$$S_0 \xrightarrow{d_1} S_1 \xrightarrow{d_2} S_2$$

$$d^2 = 0$$



Example: (1) $N(SSE) \approx S(S)$ (Wahlweise S-Geraden)
 (2) $N(S_1) = S(f)$
 (3) $N(SSE) \approx S^2(S)$ Hessesche - Tripel

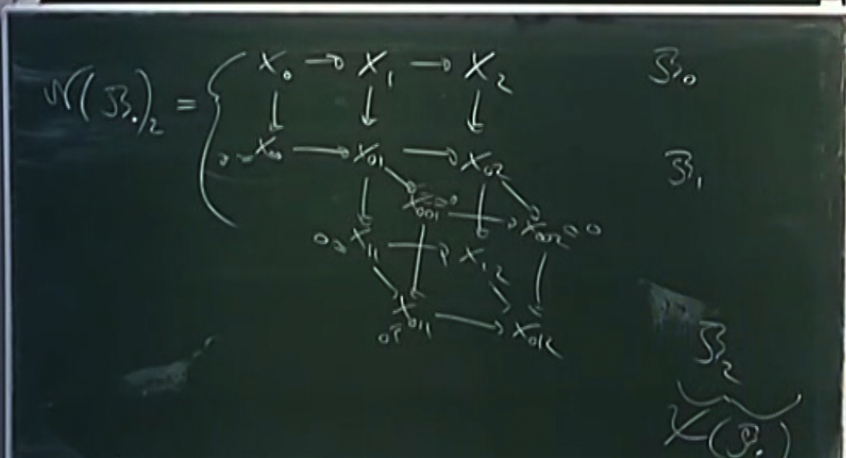
(4) $N(SSE) \approx S^{<u>2</u>}(S)$ T. Punkte
 Relative to k-Hog: $N(S) \approx \Omega^k N(S_1)$
 Hom filling (j.w./G. j. S.S.)

B
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 SGJFH



BN-MATH-VC1

$$\begin{array}{c}
 \mathcal{S}_0 \xleftarrow{d} \mathcal{S}_1 \xleftarrow{d} \mathcal{S}_2 \\
 \left. \begin{array}{l} -1d_0 \\ d^2 = 0 \end{array} \right\}
 \end{array}$$



Example:

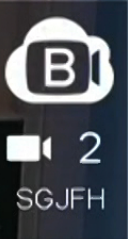
- (1) $N(\mathcal{S}EIS) \approx S(\mathcal{S})$ (Wahlweise S -Geraden)
- " $K(\mathcal{S}_1, 1)$ $K(\mathcal{S}) = \Omega(S(\mathcal{S}))$
- (2) $N\left(\begin{smallmatrix} \mathcal{S}_1 \\ \mathcal{S}_0 \end{smallmatrix}\right) \approx S(f)$

(iii) $N(\mathcal{S}E\omega) \approx S^{<\omega>}(\mathcal{S})$ T. Pogorzała

Relative to k -flag: $K(\mathcal{S}) = \Omega^k K(\mathcal{S}_1, 1)$

Hom filling (j.w./G. Jasso)

- higher Auslander-Reite flag (O. Jasso)





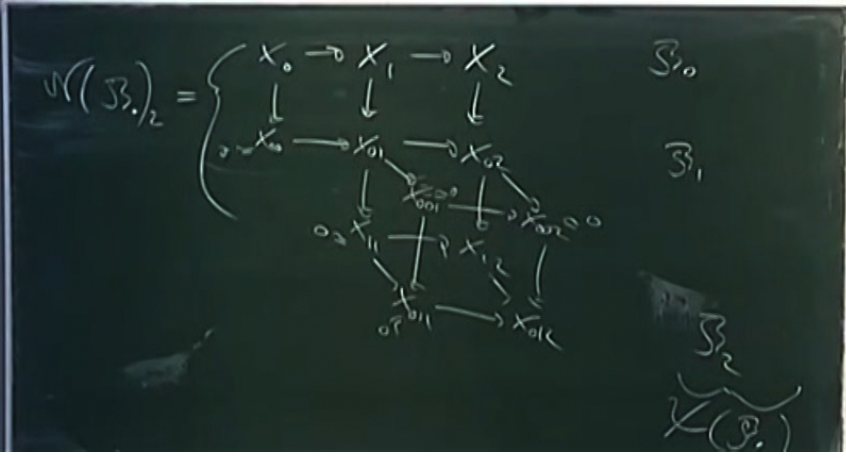
BN-MATH-VC1

$$S_0 \xleftarrow{d_1} S_1 \xleftarrow{d_2} S_2$$

$$d^2 = 0$$

(4) $N(S^{\leq n}) \cong S^{\leq n}(S)$ T. Poguntke
 Relative to k -Alg: $N(S) \cong \Omega^{\leq n}(S, \mu)$
 Home filling (j.w./G. Jasso).
 - higher Auslander-Reiter theory (O. Igusa)

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 SGJFH

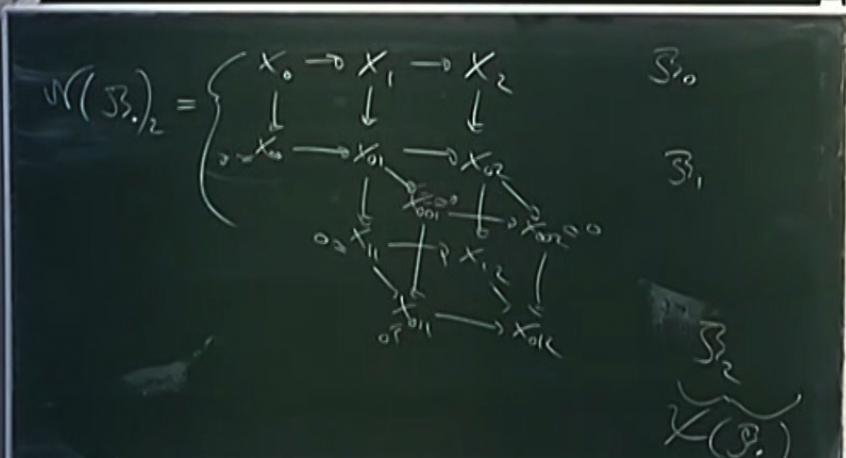


Foot: For $A \in \text{Alg}_k$ is k complex




BN-MATH-VC1

$$\begin{array}{c}
 \left. \begin{array}{l}
 S_0 \xrightarrow{d_1} S_1 \xrightarrow{d_2} S_2 \\
 d^2 = 0
 \end{array} \right\} - \text{id}_S
 \end{array}$$



Relation to k -theory: $K(S_1) \cong \Omega K(S_2)$
 Home filling (j.w./L. Jasso)
 - higher Ansterder-Peite theory (0. Jasso)

Fact: For $A \in \text{Alg}_k$ is k -complex
 Then (D. Jasso)
 (1) $A \in \text{St}_\Delta$

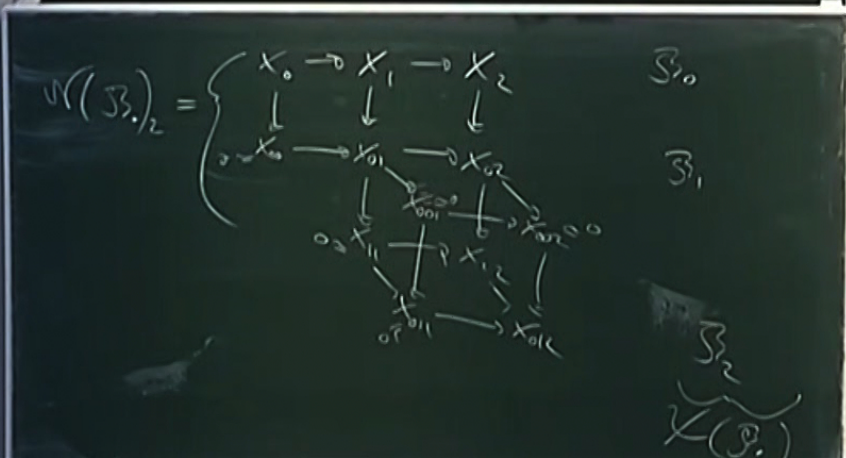

 2
 SGJFH



BN-MATH-VC1

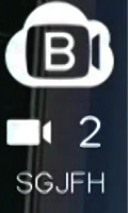
$$S_0 \xrightarrow{d_1} S_1 \xrightarrow{d_2} S_2$$

$$d^2 = 0$$



Relation to K -theory: $K(S_1) \cong \Omega^1 K(S_2)$
 Thom filling (j.w./L. Jasso)
 - higher Atiyah-Dixmier theory (0. Jasso)

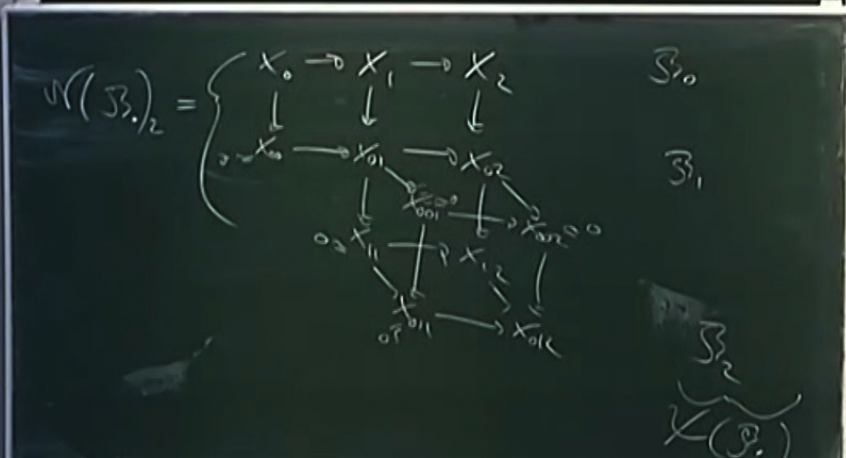
Fact: For $A \in \text{Alg}_n$ is K_n complex
 Then (A, Jasso)
 (1) $A \in \text{St}_A$ is K_n complex
 • $A_n \rightarrow A(K_n)$





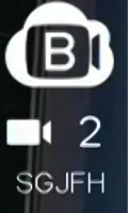
BN-MATH-VC1

$$\begin{array}{c}
 \mathcal{S}_0 \xleftarrow{d} \mathcal{S}_1 \xleftarrow{d} \mathcal{S}_2 \\
 d^2 = 0
 \end{array}$$



Relation to k -theory: $\mathcal{K}(\mathcal{S}_1) \cong \Omega^2 \mathcal{K}(\mathcal{S}_2)$
 Thom filling (j.w./L. Jasso)
 - higher Auslander-Reiter theory (0. Figure)

Fact: For $A \in \text{Alg}_k$ is k -algebra
 Then (\mathcal{S}_2)
 (1) $A \in \text{St}_A$ is k -algebra
 • $A_n \rightarrow A(\mathbb{N}^n)$
 if right-adjoint
 • $A_n \leftarrow \text{all } A(\mathbb{N}^n)$



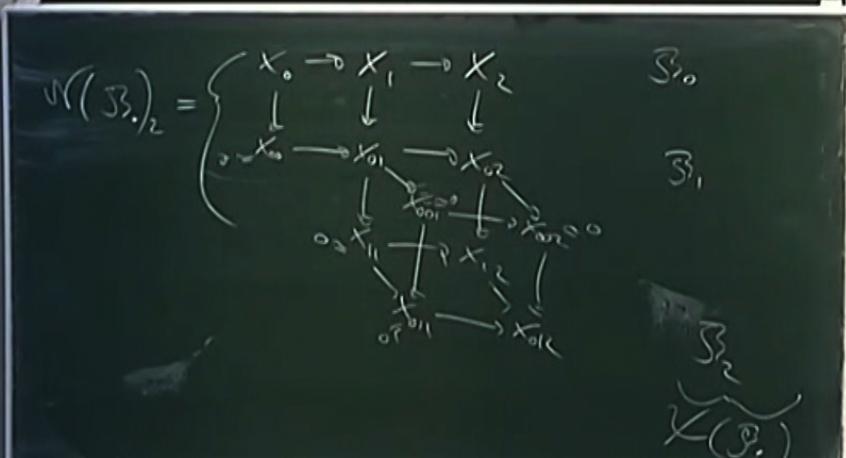
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BN-MATH-VC1

$$S_0 \xrightarrow{a} S_1 \xrightarrow{a} S_2$$

$$d^2 = 0$$



"

$$K(S_{3,1}) \quad K(S) = \mathcal{C}(S, \mathcal{B})$$

(2) $N(S_{3,1}) = S(f)$

(3) $N(S_{2,2}) \approx S^2(S)$ Hesseblatt
- Induziert

$$K(S_{2,2}) \quad K(S) \approx \mathcal{C}^2(S, \mathcal{C})$$

- (2) TFAE =
- (i) $\forall k. \mathcal{A}_k \cong \mathcal{A}(A_k)$
 - (ii) $\forall h. \mathcal{A}_h \cong \mathcal{A}(A_h)$
 - (iii) $\mathcal{A} \cong \mathcal{N}(0 \rightarrow \mathcal{B}_n \rightarrow \dots \rightarrow \mathcal{B}_0)$
 - (iv)

B

2

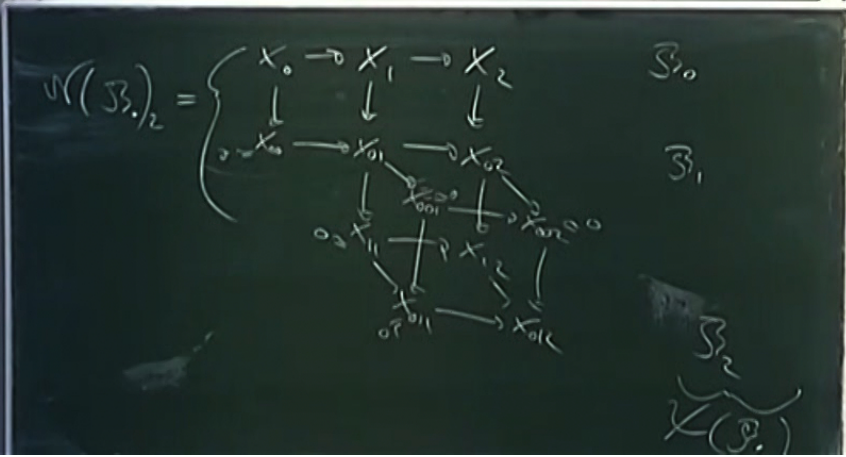
SGJFH



BN-MATH-VC1

$$S_0 \xrightarrow{a} S_1 \xrightarrow{a} S_2$$

$$d^2 = 0$$



"

$$K(S_{2,1}) \quad K(S_1) = \mathcal{C}(S_1(S_1))$$

(2) $N(S_{2,1}) = S_1(f)$

(3) $N(S_{2,2}) \approx S_1^2(S_1)$ Hesselholt
-modul-

$$K(S_{2,2}) \quad K(S_2) \approx S_2^2(K(S_2))$$

- (2) TFAE =
- (i) $\text{rk } A_n \approx A(\mathbb{N}_n)$
 - (ii) $\text{rk } A_n \approx \overline{A}(\mathbb{N}_n)$
 - (iii) $A \in \mathcal{N}(0 \rightarrow S_n \rightarrow \dots \rightarrow \mathbb{Z})$
 - (iv) $A \in \Delta$ $2k$ -Synd. dg.



B

2

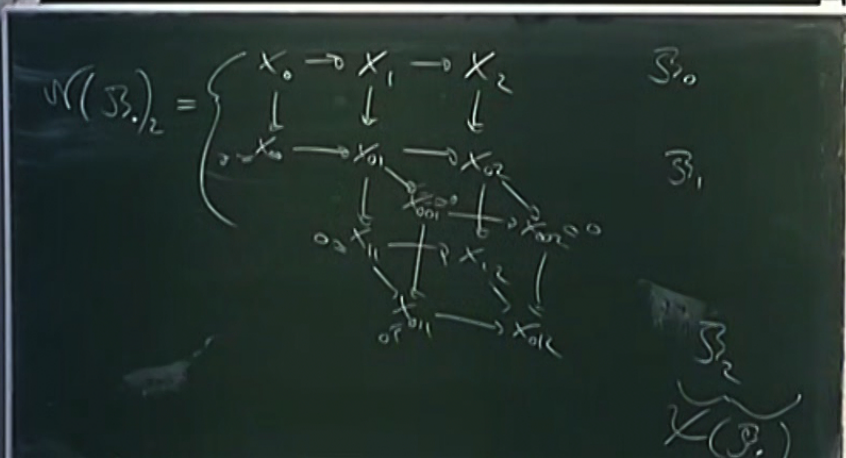
SGJFH



BN-MATH-VC1

$$\begin{matrix} S_0 & \xrightarrow{d_1} & S_1 & \xrightarrow{d_2} & S_2 \\ & & & & d^2=0 \end{matrix}$$

$-id_0$



$K(S_2, 1) \quad K(S_2) = \mathcal{N}(S_2)$

(2) $\mathcal{N}(S_2) = S_2$

(3) $\mathcal{N}(S_2) \approx S_2$ Hesselbrat
- Index
 $K(S_2, 2) \quad K(S_2) \approx S_2$

Thm. 2.10

(1) $\mathcal{A} \in \mathcal{S}T_A$ is outer Kan complex

$\sum_{i=1}^n \mathcal{S}(i) \cdot \mathcal{TFAE}$

(i) $u \cdot k \cdot \mathcal{A}_n \cong \mathcal{A}(k_n)$

(ii) $\mathcal{A} \cong \mathcal{N}(0 \rightarrow \mathcal{S}_n \rightarrow \dots \rightarrow \mathcal{P}_0)$

(iii) $\mathcal{A} \cong \mathcal{N}(0 \rightarrow \mathcal{S}_n \rightarrow \dots \rightarrow \mathcal{P}_0)$

(iv) $\mathcal{A} \cdot \Delta$ 2k-Segal obj.

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2

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