Title: Geometric Langlands: Comparing the views from CFT and TQFT

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Abstract: The goal of my talk will be to discuss the relation between two approaches to the geometric Langlands program. The ï¬•rst has been proposed by Beilinson and Drinfeld, using ideas and methods from conformal ï¬•eld theory (CFT). The second was initiated by Kapustin and Witten based on a topological version of four-dimensional maximally supersymmetric Yangâ€“Mills theory and its reduction to a two-dimensional topological sigma model. After discussing some issues complicating a direct comparison we will formulate a proposal for a precise relation between two main ingredients in the two approaches.

ometric Langlands: Comparing the views from CFT and TQFT

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2
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Geometric Langlands

roughly: (G: complex simple Lie group, $\check{G}$: Langlands dual group)

$$\boxed{\check{G} \text{ local systems}} \longleftrightarrow \boxed{\mathcal{D}-\text{modules on } \mathrm{Bun}_G}$$

$\check{G}$ local systems: Pairs $(\check{E}, \nabla)$,

(i) $\check{E}$: holomorphic/algebraic $\check{G}$-bundle

(ii) ∇ : holomorphic/algebraic connection on $\check{E}$.

$\mathcal{D}$-modules on Bun_G –
on open dense subsets of Bun_G represented by certain differential equatio
a flat connection away from singular loci.

It took a while to turn this into a precise conjecture (Arinkin-Gaitsgory):

$$\boxed{\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{Loc}_{\check{G}}(C))} \underset{\text{Eq. of categories}}{\longleftrightarrow} \boxed{\text{D-mod}(\mathrm{Bun}_G(C))}$$

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Kapustin-Witten (TQFT) approach

Top. twist of $d = 4$, $\mathcal{N} = 4$, G_c SYM $\rightsquigarrow$ TQFT_4^Ψ.
(G_c: compact real form of G; Ψ: parameter determining SUSY used to define top. twist)

- Dimensional reduction: TQFT_4^Ψ on $\Sigma \times C \rightsquigarrow \mathrm{TQFT}_2^\Psi$ on Σ, here $\Sigma = I \times \mathbb{R}$.
  TQFT_2^Ψ captures **part** of structure of TQFT_4^Ψ:
  - Category of boundary conditions $\mathcal{B}$
  - $I_{\mathcal{B}_1\mathcal{B}_2}$ – (Interval, ends decorated with boundary conditions $\mathcal{B}_1, \mathcal{B}_2$)
    $\mapsto \mathcal{H}(I_{\mathcal{B}_1\mathcal{B}_2}) \equiv \mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ – (space of states).
- Effective representation by $\mathcal{N} = (4,4)$ top. sigma model, target $\mathcal{M}_H(G, C)$.
  For particular values of Ψ:

$\mathcal{M}_H(G, C)$: Space of pairs $(\mathcal{E}, \nabla)$:
(i) $\mathcal{E}$: holomorphic/algebraic G-bundle
(ii) $\varphi \in H^0(C, \mathrm{End}(\mathcal{E}) \otimes K)$.

S-duality $\rightsquigarrow$ Homological mirror symmetry

$\mathcal{M}_H(G,C)$ has integrable structure (torus fibration) –

Hitchin map, $G = SL(N)$: $\pi((\mathcal{E},\varphi)) = (\mathrm{tr}(\varphi^2),\ldots,\mathrm{tr}(\varphi^N)) \in \mathfrak{B} = \bigoplus_{d=2}^{N} H^0(C,K^d)$
Fibre $\mathcal{F}_b = \pi^{-1}(b)$ over generic $b \in \mathfrak{B}$: Abelian variety (cplx. torus).

- S-duality of $\mathcal{N}=4$ SYM predicts key result (Hausel-Thaddeus, Donagi-Pantev)

> Torus fibrations of $\mathcal{M}_H(G,C)$ and $\mathcal{M}_H(\check{G},C)$ are dual to each other.

Physics interpretation: The $\mathcal{N}=(4,4)$ top. sigma models, targets $\mathcal{M}_H(G,C)$ and $\mathcal{M}_H(\check{G},C)$ are SYZ mirror to each other.

- This implies important relations between the categories of boundary conditions (homological mirror symmetry), in particular

Skyscraper $\mathcal{B}_\mu$ at $\mu \in \mathcal{M}_H(\check{G},C)$ $\quad\leftrightarrow\quad$ Fibre $\mathcal{F}_b$, $b=\pi(\mu)$, with flat line bundle.

Relation to geometric Langlands

There is a special boundary condition named "canonical coisotropic brane $\mathcal{B}_{\rm cc}$" such that the algebra

$$\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\rm cc}, \mathcal{B}_{\rm cc}) = \mathrm{Fun}_\hbar(\mathcal{M}_H(G,C)),$$

is the quantised algebra of functions on $\mathcal{M}_H(G,C)$.

- The spaces $\mathcal{H}(\mathcal{B}) := \mathrm{Hom}(\mathcal{B}_{\rm cc}, \mathcal{B})$ are naturally left $\mathcal{A}$-modules.
- Hitchin's Hamiltonians get quantised into elements of $\mathcal{A}$

$\rightsquigarrow$ $\mathcal{D}$-**module structure** on $\mathcal{H}(\mathcal{B})$ for any $\mathcal{B}$!

- NAH (Non-Abelian Hodge correspondence): Associates skyscraper $\mathcal{B}_{\check{\mathcal{E}},\nabla}$ to any local system $(\check{\mathcal{E}},\nabla)$.
- $\mathcal{H}(\mathcal{B}_{\check{\mathcal{E}},\nabla})$: $\mathcal{D}$-module associated to $(\check{\mathcal{E}},\nabla)$.

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Non-Abelian Hodge (NAH) correspondence

n a Higgs pair $(\mathcal{E}, \varphi)$,

- $\exists$ unique harmonic metric h on $\mathcal{E}$ satisfying $F_{\mathcal{E},h} + R^2[\varphi, \varphi^{\dagger_h}] = 0$, with $F_{\mathcal{E},h}$: curvature of the unique h-unitary connection $D_{\mathcal{E},h}$ having $(0,1)$-part $\bar{\partial}_{\mathcal{E}}$.

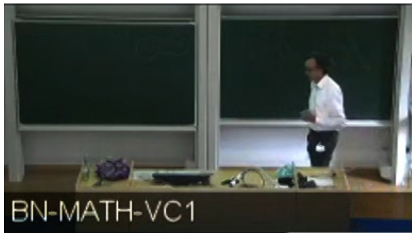
- $\exists$ two-parameter family of flat connections $\nabla_{\zeta,R} = \zeta^{-1}R\,\varphi + D_{\mathcal{E},h} + R\zeta\,\varphi^{\dagger_h}$.

- Decomposing $\nabla_{\zeta,R}$ into the $(1,0)$ and $(0,1)$-parts ∇' and ∇'' defines a pair $(\mathcal{F}, \nabla'_\epsilon)$, where $\mathcal{F}$: holomorphic bundle defined by ∇'', and the ϵ-connection $\nabla'_\epsilon = \epsilon\nabla' = \epsilon\partial_{\mathcal{E},h} + \varphi$, with $\epsilon = \zeta/R$, holomorphic in the complex structure defined by $\bar{\partial}_{\mathcal{F}}$.

Conversely, given a flat connection ∇, there exists a hermitian metric h and a Higgs pair $(\mathcal{E}, \varphi)$ allowing us to decompose ∇ as $\nabla = \varphi + D_{\mathcal{E},h} + \varphi^{\dagger_h}$.

Conformal limit: Limit $R \to 0$, $\zeta \to 0$, $\epsilon = \zeta/R$ fixed.

Thm: (Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke, Collier-Wentworth)
The conformal limit exists.



picture outlined by Kapustin and Witten has not yet been made rigorous

however, the ongoing work by Donagi-Pantev-Simpson (talk by Pantev at String-Math 2018))

e is, on the other hand an outline of a proof of GLC for $G = GL(2)$ (Gaitsgory).

Important ingredient: **Beilinson-Drinfeld construction**

Based on results of Feigin-Frenkel, Beilinson and Drinfeld first constructed the $\mathcal{D}$-modules associated to a particular type of local systems: **opers**

Example $\check{G} = PSL(2,\mathbb{C})$:

Pairs $(\mathcal{E}, \nabla)$, where $\mathcal{E} = \mathcal{E}_{\mathrm{op}}$, the unique extension $0 \to K^{1/2} \to \mathcal{E}_{\mathrm{op}} \to K^{-1/2} \to 0$ allowing a holomorphic connection ∇ of the form $\nabla = dz\big(\partial_z + \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)\big)$.

Corresponding $\mathcal{D}$-modules:

Eigenvalue equations for quantum Hitchin Hamiltonians, $\mathsf{H}_i f = E_i f$.

- Beilinson-Drinfeld: $\exists$ canonical isomorphism of algebras $\mathrm{Fun}\,\mathrm{Op}_{^L\mathfrak{g}}(C) \simeq \mathfrak{D}$. ($\mathfrak{D}$: commutative algebra of global DOs)
- Fixing an oper χ defines a homomorphism $\tilde{\chi} : \mathfrak{D} \to \mathbb{C}$. Corresponding $\mathcal{D}$-module $\mathfrak{D}_\chi$:

$$\mathfrak{D}_\chi = \mathfrak{D}/\ker\tilde{\chi} \cdot \mathfrak{D}.$$

- Picking generators H_i for $\mathfrak{D}$: $\mathsf{H}_i f = E_i f, \qquad E_i = \tilde{\chi}(\mathsf{H}_i)$.

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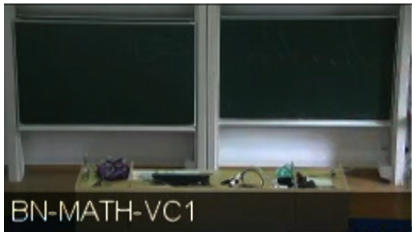

CFT construction

-based approach of Beilinson and Drinfeld constructs for each oper an object in the category of $\mathcal{D}$-modules on Bun_G as **conformal blocks**$^{*)}$ of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at the critical level $k = -h^\vee$.

$^{*)}$ Invariants under $\mathfrak{g}_{\text{out}} = H^0(C \setminus \{P_1, \dots, P_n\}, \mathcal{E} \underset{G}{\times} \mathfrak{g})$ in $\hat{\mathfrak{g}}_k$-modules $\mathcal{R}_i$ associated to points P_i

- Ward-identities characterising the conformal blocks equip the sheaves of conformal blocks over Bun_G with a $\mathcal{D}$-module structure,

$$\hat{\mathfrak{g}}_k\text{-action on conf. blocks} \quad \leftrightarrow \quad \text{DO's on } \mathrm{Bun}_G.$$

- The universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}}_k)$ has a large **center** at $k = -h^\vee$, isomorphic to the space of ${}^L\mathfrak{g}$-opers on the formal disc (Feigin-Frenkel).

  $\Rightarrow$ Given oper ρ, may define conformal blocks with representations $\mathcal{R}_{\rho_i}$ associated to restriction ρ_i of ρ to discs around P_i.

- $\mathcal{D}$-module structure includes eigenvalue equations $D_i f = E_i f$ for the quantised Hitchin Hamiltonians, with eigenvalues E_i parameterising the choice of opers.

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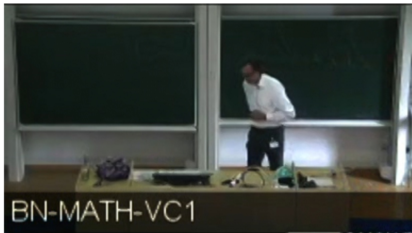

Generalisation to irreducible local systems

sider **meromorphic opers**, opers with a very specific type of singularity at a divisor $\mathbb{D}$ on C.

Example: $\check{\mathfrak{g}} = \mathfrak{sl}_2$. Consider pairs $(\check{\mathcal{E}}_{\rm op}, \nabla)$, $\nabla = dz\big(\partial_z + \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)\big)$, with

$$t(y) = -\frac{\lambda(\lambda+2)}{4(y-u)^2} + \sum_{k=-1}^{\infty} t_n (y-u)^n, \quad u \in \mathbb{D},$$

where $\lambda \in \mathbb{Z}_+$ and t_n satisfy a polynomial equation ensuring that holonomy of ∇ is trivial in $\mathrm{PSL}(2,\mathbb{C}) \rightsquigarrow$ **apparent singularities** at $y = u \in \mathbb{D}$.

> **Thm** (Yoshida): *Any local system for $G = \mathrm{PSL}(2,\mathbb{C})$ on $C = C_{g,n}$ can be represented by a meromorphic oper with at most $3g-3+n$ singularities.*

General case: For Lie algebra $\check{\mathfrak{g}}$ replace λ by weight λ of **dual** algebra $\mathfrak{g}$.

Resulting picture:

- $\mathrm{Loc}_{\check{G}}(C)$ stratified by singularity type of oper representing $(\check{E}, \nabla)$ (Arinkin '16)
- Structure of corresponding $\mathcal{D}$-module **depends sensitively** on singularity type.

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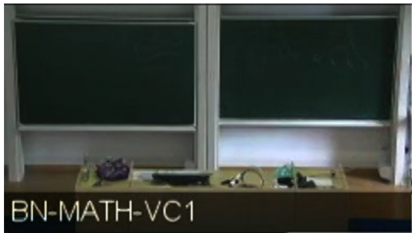

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Example

$g = 0$, n punctures, $\mathbb{D} = \sum_{k=1}^{d} w_k$: $\mathcal{D}$-module $\rightsquigarrow$ eigenvalue equations

$$\mathsf{H}_s \mathbf{G} = E_s \mathbf{G}\,, \qquad \mathsf{k}_l \mathbf{G} = \kappa_l \mathbf{G}\,,$$

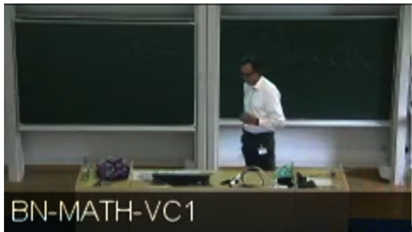
for vector-valued function $\mathbf{G}(\mathbf{z}, \mathbf{w}|\mathbf{x}) \in (\mathbb{C}^2)^{\otimes m}$, $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (w_1, \ldots, w_d)$, $\mathbf{x} = (x_1, \ldots, x_n)$, with

$$\mathsf{H}_s = \sum_{s' \neq s} \frac{\mathcal{J}_s^a \mathcal{J}_{s'}^{a'}}{z_s - z_{s'}} \eta_{aa'} + \sum_{k=1}^{d} \frac{\mathcal{J}_s^a \sigma_k^{a'}}{z_s - w_k} \eta_{aa'}\,,$$

$$\mathsf{k}_l = \sum_{s} \frac{\sigma_l^a \mathcal{J}_s^{a'}}{w_l - z_s} \eta_{aa'} + \sum_{k \neq l}^{d} \frac{\sigma_l^a \sigma_k^{a'}}{w_l - w_k} \eta_{aa'}\,,$$

with σ_r^a being the 2×2-matrices representing $\mathfrak{sl}_2$ on the k-th tensor factor in $(\mathbb{C}^2)^{\otimes d}$, $\mathcal{J}_r^a$: DO representing $\mathfrak{sl}_2$ on variable x_r, $a = 1, 2, 3$.

Can we understand the stratified structure from the point of view of TQFT?



an we understand the stratified structure from the point of view of TQFT_2 ?

uzzle: The SYZ mirror symmetry picture looks perfectly smooth w.r.t. $(\check{E}, \nabla)$

How to see the stratified structure?

Proper answer will require construction of Fukaya category / family Floer theory defining A-model on $\mathcal{M}_H(G, C)$. (cf. remarks by Pantev, talk at String-Math 2018)

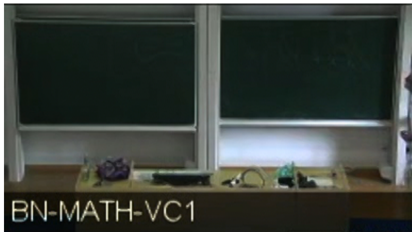
A preliminary question: Does anything special happen if $(\check{E}, \nabla)$ is an oper?

Observation 1a): $(G = \mathrm{SL}(2))$ [DFKMMN]: Conformal limit of NAH relates

$$\text{opers} \quad \left(\check{\mathcal{E}}_{\mathrm{op}}, dz\left(\epsilon\partial_z + \left(\begin{smallmatrix} 0 & u \\ 1 & 0 \end{smallmatrix}\right)\right)\right) \quad \text{to Higgs pairs} \quad \left(K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \left(\begin{smallmatrix} 0 & u \\ 1 & 0 \end{smallmatrix}\right)\right).$$

Such Higgs pairs represent **natural origin*** $\mathcal{F}_{(u,0)}$ of torus fiber $\mathcal{F}_u$.

(* in the sense of fibrewise duality: dual of a torus with trivial flat line bundle)



Can we understand this structure from the point of view of TQFT$_2$?

Observation/Conjecture 1b): Conformal limit of NAH relates stratification of $\mathrm{Loc}_{\check{G}}(C)$ by oper singularity type to stratification of $\mathcal{M}_H(\check{G}, C)$ by Segre invariant

$$S(\check{\mathcal{E}}) = \deg(\check{\mathcal{E}}) - 2 \max_{\mathcal{L} \hookrightarrow \check{\mathcal{E}}} (\deg(\mathcal{L})).$$

Indeed, representing the bundle $\check{\mathcal{E}}$ as extension

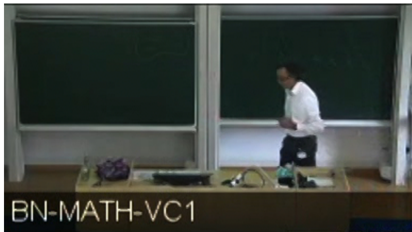
$$0 \to \mathcal{L} \to \check{\mathcal{E}} \to \det(\check{\mathcal{E}}) \otimes \mathcal{L}^{-1} \to 0,$$

noting that $\epsilon \partial_y + A(y)$ can be represented as

$$A(y) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \quad \text{with} \quad \mathrm{c} \in H^0(C, KL^{-2}\det(\check{\mathcal{E}})),$$

there is a canonical way to transform $A(y)$ to oper form with $\mathbb{D}$: divisor of zeros of c.

Very similar representation exists for $(\check{E}, \varphi) \in \mathcal{M}_H(\check{G}, C)$ (integrability, SOV).



Can we understand this structure from the point of view of TQFT_4 ?

Observation/Conjecture 2:

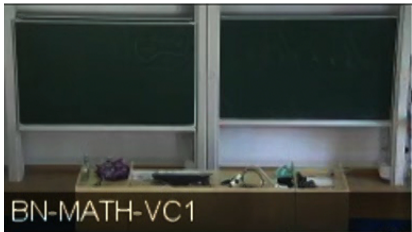
- There $\exists$ deformation of GLC, "quantum geometric Langlands correspondence" GLC_κ with parameter κ describing deformation away from critical level $h = -h^\vee$.
  (Feigin, Frenkel, Stoyanovsky; Gaitsgory, Lurie...)

- GLC_κ is conjecturally related to TQFT_4^Ψ for certain $\Psi = \Psi(\kappa)$.
  (Kapustin-Witten, Gaiotto-Witten)

- This deformation is related to a hyperkähler rotation of $\mathcal{M}_H(G, C)$.
  (Gaiotto-Witten, J.T.)

Observation 3:

The R-dependence of TQFT_4^Ψ is inessential.



ervation/Conjecture 4:

to see “dependence” of $\mathcal{D}$-module on bundle $\mathcal{E}$, and how to see $\hat{\mathfrak{g}}_k$-symmetry?

- There exists a boundary condition $\mathcal{B}_{\mathcal{E}}$ (from Hitchin's second fibration) such that

$$\mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathcal{E}}) \simeq \text{Space of conformal blocks on } C.$$

(Balasubramanian-J.T., related to results of Gaiotto, Frenkel-Gaiotto)

- $\mathcal{B}_{\mathcal{E}}$ can be used to explain TQFT$_4$-origin of $\hat{\mathfrak{g}}_k$-symmetry. (Frenkel-Gaiotto)
- This leads to the following proposal (Balasubramanian-J.T.)

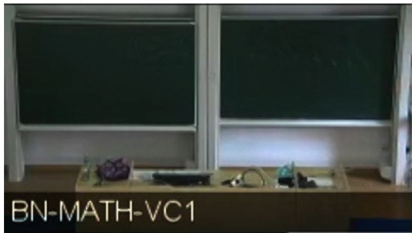
$$\mathrm{Hom}_{\mathcal{M}_H(G)}(\mathrm{B}_{\mathrm{cc}}, \mathcal{F}_{(u,0)}) \simeq \left[\text{Fiber of } \mathfrak{D}_{\rho_u} \text{ over } \mathcal{E} \in \mathrm{Bun}_G^{\mathrm{vs}}\right].$$

Right side independent of $\mathcal{E}$ ($\mathcal{D}$-module structure $\rightsquigarrow$ flat connection over $\mathrm{Bun}_G^{\mathrm{vs}}$).

Observation/Conjecture 5:

There are dual Hecke functors moving among strata

(In CFT-picture related to braided tensor categories of degenerate/ dual degenerate representations).



No conclusion, many questions....

many of which will require higher algebra ...

- How to deal with the stratified/stacky nature of the various moduli spaces relevant here?
- How to exploit the higher categorical structures of TQFT_4 that are not (easily) seen in the description as TQFT_2?

... some of which may require extensions/generalisations of the present mathematical formalism of TQFT...

- How to deal with TQFT_n assigning infinite-dimensional vector space manifolds?

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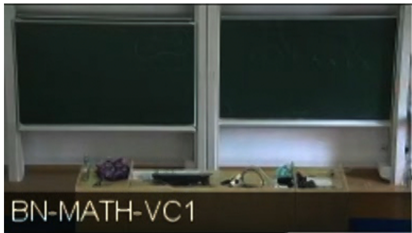

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